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Surgery on Poincaré
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0. Introduction

The object of this paper is to present a theory of surgery on normal and Poincaré spaces, with some applications to Poincaré geometry and the topology of manifolds.

Other approaches to Poincaré surgery have been made by Norm Levitt, and Lowell Jones. Levitt's approach is to use sophisticated manifold theory to obtain a Poincaré embedding theorem and then apply the usual manifold program. He has also studied special cases of the transversality problem. However the understanding of manifolds does not seem to be complete enough yet to finish this program. Jones also uses manifold theory in the form of "patch" structures. These are a high-powered sort of handlebody structure on Poincaré spaces. Again sophisticated manifold theory is used to manipulate the patches and produce "stable" (after $X(CP^2)^j$) existance and transversality theorems. An unstable existance theorem for patch structures is given in section 5.

The present approach to the problem was outlined by W. Browder in the spring of 1969. He remarked "a problem in homotopy theory should have a homotopy theoretical solution", and suggested the use of fibrations as a tool in the construction. Recently he has been able to do 0 and 1- surgery under very general conditions, and has obtained interesting consequences which are not accessible with our techniques. The main missing ingredient in the program was "removing cells above the middle dimension", lemma 3.7. That this technique should have remained hidden so long is not surprising in view of the fact that most of the theory of Poincaré spaces required for its application also had yet to be developed (sections 1,2).

The central problem of surgery may be phrased as "given a degree 1 normal map, what are the obstructions to it being normally cobordant to a homotopy equivalence?" We rephrase this as follows. Take the mapping cylinder of the map, then it has a spherical fibration (normality), with a distinguished map of a sphere to the Thom space of this fibration (degree 1). Generally we call such an object a <u>normal space</u> (2.4), and ask; given a normal space with part of its boundary Poincaré, when is it normally bordant to a Poincaré space, holding the Poincaré boundary fixed. When applied to the mapping cylinder the resulting Poincaré space is a normal Poincaré bordism to the identity map, answering the original question. Stating the problem in this context has other advantages, since for example bordism classes of normal spaces are geometric representatives for homology with coefficients in the MSG spectrum.

The main result is the normal surgery lemma 3.1, which states that a normal pair (X,Y) with $\pi_1 Y = \pi_1 X$ is normally bordant to a Poincaré pair. This implies the Poincaré version of Wall's "important special case", and is applied in section 4 to show the obstructions encountered are exactly the surgery obstructions in the groups $L_n(\pi)$. Section 4 also gives a version of the more usual type of surgery on a Poincaré space.

Section 3 is devoted to a proof of 3.1, of which the central step is 3.7. Heavy use is made of a theory of stable maps which induce cup and cap products on homology. The homotopy information in these maps is much stronger than the corresponding algebra, and is crucial for the argument. Section 1 is a development of the properties of such maps.

Section 2 gives the definitions and elementary properties of Poincaré and normal spaces. The power of our homotopy-theoretic approach is illustrated by the large number of characterizations of Poincaré complexes given. For example it is easily shown that in a fibration with fiber, base, and total space dominated by finite complexes, the total space satisfies duality if and only if both the fiber _and_ the base do also. Here, as everywhere in this paper, duality means Poincaré duality with universal local coefficients. The torsion of a Poincaré space (torsion of the chain equivalence giving duality) is defined and some formulas derived.

The applications fall into two categories, local, and global. By local we mean questions concerning the structure of a particular Poincaré space, whereas global questions concern them en masse (for example bordism). Section 5 is concerned with local applications. The first is transversality, extending partial results of Levitt and Jones. Given a map from a Poincaré space to a space containing the disc bundle of a spherical fibration, the obstruction to making the map transversal is a surgery obstruction over a group manufactured from the fundamental groups present (5.2). Next a Poincaré embedding theorem is given from which a great deal of Poincaré surgery can be recovered. Finally the question of patch structures on Poincaré and normal spaces is considered. Normal spaces always have patch structures, and a Poincaré space of dimension n has a topological patch structure with no face of dimension less than 4 iff it has a cover by n-3 open sets so that the normal fibration restricted to each has a topological reduction. Using the theory of (topological) category of spaces one sees that a 4-connected Poincaré space, or $X \times CP^2$, has a smooth patch structure.

Section 6 is devoted to global applications. It is an outline of special cases from a more general treatment. In particular the constructions of large simplicial sets are omitted. Familiarity with the author's thesis will allow easy reconstruction of the omissions. The first subject is the long exact sequence $\to L_n(\pi_1 X) \to \Omega_n^{\text{Poincaré}}(X) \to H_n(X;MSG) \to L_{n-1}(\pi_1 X) \to$, which is immediate from the definitions (recall $H_n(X;MSG) = \Omega_n^{\text{normal}}(X)$). This was deduced by Levitt for the case X a point. The sequence for X a point arises from a fibration of spectra $L \to \Omega \to MSG$. Now if $B_{(\Omega^P,MSG)G_n}$ is the classifying space for S^{n-1} fibrations with an Ω^P Thom class which reduces to the natural MSG class, we find that the obvious map $B_{\widetilde{TOP}_n} \to B_{(\Omega^P,MSG)G_n}$ is a homotopy equivalence for $n \geq 3$. When $n = \infty$ this is a result of Levitt and Morgan.

Next, if X is a Poincaré space and Y is the normal space of dimension 5 with $\sigma Y = 1 \in L_4(0) = Z$, then define the <u>L-index</u> of X by $I(X) = \sigma(X \times Y) \quad L_n(\pi_1 X)$. This has been independently discovered by Wall in the manifold case, and solves a problem of Novikov. It is easily seen that if X is normal with Poincaré boundary, $8\sigma(X) = I(\partial_P X)$. This expresses 8 times the surgery obstruction of a normal map as the difference of the L-indexes of the domain and range. It also shows that for any space K, $8(H_*(K;Z))$ is representable by Poincaré spaces. (Complete results are obtained for special $\pi_1 K$). The L-index also gives a map of spectra $\Omega^P \xrightarrow{I} \mathbb{L}$. There is a (difficult) product structure on $\mathbb{L}$ for which this is a ring map. This gives (via the Levitt-Morgan characterization) a homotopy equivalence $B_{\widetilde{TOP}_n} \xrightarrow{\simeq} B_{(\mathbb{L},Q)G_n}$, where $Q = B[\text{fiber } \mathbb{L} \xrightarrow{8} \mathbb{L}]$. This replaces the more difficult construction of the same map previously announced by the author.

The author apologizes for the many errors of spelling and grammar. This version is somewhat preliminary, and will be rewritten to attempt to increase clarity before publication.

1. Thom spaces, duality, and geometric products.

In [1]Atiyah gave an arithmetic for manipulating Thom spaces of vector bundles, which was extended to spherical fibrations in [14]. We extend it to fairly general fibrations to take advantage of the definition of a "normal fibration" for a complex (Levitt, [6]). Normal fibrations and S-W duality are extended to the equivariant theory of the universal cover of a space dominated by a finite complex. Actually since the Thom space is just the cofiber of a map it depends only on the homotopy class of the map on the total space. The context of fibrations, however, is useful for the Whitney sum operation, and for later discussions of smooth, PL and TOP bundles.

Next this arithmetic of Thom spaces is applied to construct "geometric products". Given a geometric representative $S^k \to X$ or $X \to S^k$ of a homology or cohomology class, we construct maps of spaces which realize cup or cap product with the class on homology and cohomology. These constructions again generalize ones given in [1]. We close with a lemma on the normal fibration of a fiber space.

To avoid loosing the fundamental group in stable homotopy theory, we work with the equivariant theory of a cover. Let π be a (discrete) group, then a <u>π-space</u> will be a space on which π acts freely, except at the basepoint if it has one, and which has the equivariant homotopy type of a CW complex of the same description. A <u>π-complex</u> is a countable polyhedron, locally finite except perhaps at the basepoint if it has one, and a π action free on the complement of the basepoint. The basepoint should also have an equivariant regular neighborhood; the complex then can be represented as $(K,*) = (K_0/K_1,*)$ where (K_0,K_1) is a locally finite π-free pair, and K_0/K_1 is the union of K_0 and the cone on K_1.

As usual $X\vee Y$ denotes disjoint union with basepoints identified, $X\wedge Y = X\times Y/X\vee Y$, and $\Sigma^n X = S^n \wedge X$. $X_+ = X\vee(\text{point})$, and is X disjoint union a basepoint, provided it does not already have one. If $f: X \to Y$ is a map we denote the mapping cylender $X\times I \cup_{f\times\{1\}} Y$ by Y_f. This notation (suggested by Kervaire) emphasizes the fact that Y and Y_f are canonically homotopy equivalent. $X \simeq Y$ denotes equivariant homotopy equivalence.

Fibrations are in the sense of Hurewicz. Any map may be considered a fibration by application of the path space construction, which is equivariant and very functional. If η, ν are fibrations over X, the <u>Whitney sum</u> $\eta\oplus\nu$ is formed by pulling η back over the mapping cylender of ν and taking the total space of this union with the mapping cylender of the restriction to the total space of ν. The fiber is the join $F(\eta\oplus\nu) = F(\eta)*F(\nu)$ of the fibers, and $\eta\oplus\nu$ can be described as a subset of the join of total spaces $E(\eta)*E(\nu)$. The first description, however, does not require ν to be a fibration.

If $\xi: E(\xi) \to X$ is a fibration, then the the Thom space of ξ is the cofiber of ξ; $T(\xi) = X_\xi/E(\xi)$, with the cone point as basepoint.

1.1 Proposition. If ξ, η are fibrations over X,Y respectively, then $T(P_1^*\xi \oplus P_2^*\eta) \simeq T\xi \wedge T_\eta$, and if $X = X_1 \cup X_2$, $T\xi \simeq T(\xi|X_1) \cup$ $\cup T(\xi|X_1 \cap X_2) T(\xi|X_2)$.

Here $P_1: X \times Y \to X$, so $P_1^*\xi$ is a bundle over $X \times Y$. The natural maps are actually homeomorphisms.

Next we need the rudiments of equivariant Spanier-Whitehead duality (see [13] for $\pi = 1$ case). Let $(K,L) = (K_0/K_1, L_0/L_1)$ be a pair of finite dimensional π-complexes, then $(K_0;K_1,L_0)$ is a collared countable locally finite triad, so embeds in $R^{n-3} \times [0,\infty)^2$ with $K_0 \cap R^{n-3} \times \{0\} \times [0,\infty) = K_1$, and $K_0 \cap R^{n-3} \times [0,\infty) \times \{0\} = L_0$, for n sufficiently large. Let U be a regular neighborhood, $\partial_1 U = U\ R^{n-3} \times \{0\} \times [0,\infty)$ $\partial_2 U = U \cap R^{n-3} \times [0,\infty) \times \{0\}$, and $\partial_0 U = \partial U - \mathrm{int}(\partial_1 U \cup \partial_2 U)$. π acts freely on the 4-ad$(U;\partial_0 U,\partial_1 U,\partial_2 U)$, so the dual $D^n(K,L) = U/\partial_0 U$ is a π-complex. If $n > 2 \dim K + 2$, then embeddings are isotopic so D^n is well-defined. There is a cofibration sequence $D^n(K,L) \to D^n(K) \to D^n(L)$, and $D^n(K,L) = D^n(K/L)$.

1.2 Proposition. If K is a finite dimensional π-complex, $D^m D^n K \simeq \Sigma^{m-n} K$, where $m > 2n+2 \geq 4 \dim K + 6$.

Proof. $D^n K = U/\partial_0 U$, so the corresponding pair is $(U,\partial_0 U)$. A regular neighborhood of U will be $I^{m-n} \times U$, and the corresponding ∂_0 is $\partial(I^{m-n} \times U) - (\mathrm{int}\ \partial_0 U) \times I^{m-n}$. Thus $D^m D^n K = I^{m-n} \times U/$ $S^{m-n-1} \times U\ I^{m-n} \times \partial_1 U \simeq \Sigma^{m-n} K$. ///

$\underline{D}^n$ is an equivariant homotopy functor on nonempty $(\frac{n}{2} - 2)$- dimensional π-complexes whose quotients K/π are finite complexes (<u>π-finite complexes</u>). Given $K \to L$, embed (L_0,L_1) in $R^{n-2}\times[0,\infty)$ with neighborhood U, then $L \simeq U$ $R^{n-2}\times[-1,0]$. Because they are π-finite the map is proper and can be approximated by an embedding $(K_0,K_1)\subset R^{n-2}\times[-1,0]$ U $\subset R^{n-2}\times[-1,\infty)$ with regular neighborhood V. Now appropriate collapsing gives a π-map $U/\partial_0 U \to V/\partial_0 V$, the <u>dual</u> of the map $K \to L$.

This clearly extends to a functor on <u>proper maps</u> of countable complexes, a good setting for non-compact manifolds. We are less interested in non-compactness than homotopy, however, so we extend the theory to a homotopy functor on the category of π-spaces dominated by a π-finite complex. If X is dominated, consider $X = \underleftarrow{\lim}$ (dominating π-finite n-complexes)(n large), and define $D^{2n+2}X = \underrightarrow{\lim}\, D^{2n+2}$(dominating n-complexes). The maps in the limit are given by composing the projections and retractions given in the data. That X is such a limit follows from Wall's theory of finiteness for CW complexes, thought of as the theory of complexes dominating a given space.

Further since X is also $\underrightarrow{\lim}$(dominating complexes), DX itself is dominated.

Now if (K,L) is a finite π-pair the <u>normal fibration</u> ([6]) $v^n(K,L)$ is the inclusion $\partial_0 U \to U$ defined above, U is a regular neighborhood of $(K_0;L_0,K_1)\subset R^{n-3}\times[0,\infty)^2$, so $\nu^n(K,L)$ is a fibration over K_0. We will be primarily interested in the free case, and obtain a bundle over $(K_+)_0 = K$. Clearly $T\nu^n(K,L) = D^n(K,L)$.

Extending the normal fibration to the category of dominated complexes is a little more difficult, but fruitful.

<u>1.3 lemma</u>. If $f\colon (K,L) \to (M,N)$ is a map of free finite pairs, then a left inverse for f induces a fiber map $\nu^n(K,L) \to \nu^n(M,N)$ over f.

<u>Proof</u>. Let U be a regular neighborhood of $(M,N)\subset R^{n-2}\times[0,\infty)$, and approximate f by an embedding with neighborhood $V\subset U$. Now if r $(M,N) \to (K,L)$ has $rf\sim 1_{(M,N)}$, then approximating $r\colon (M,N) \to (V,\partial V)$ by an embedding with neighborhood U', uniqueness of regular neighborhoods tells us that $U-U'\approx\partial_0\times[0,1)$. The inclusion $\partial_0\subset U-U'$ then induces the desired fiber map. ///

We remark that this map is degree 1 as a map of (manifold) pairs $(K,\nu^u(K,L)\ L_{\nu^{n-1}(L)}) \to (M,\nu^n(M,N)\ N_{\nu^{n-1}(N)})$. The lemma can be improved to a characterization of degree 1 fiber maps of normal fibrations in terms of stable retractions of covers.

Now if (X,Y) is a free dominated pair, define $\nu^n(X,Y) = \varinjlim(\nu^n(K,L),\ (K,L)\ \pi\text{-finite dominating }(X,Y))$. In this case the limit is over commutative diagrams

$$\begin{tikzcd} (K,L) \arrow[r] \arrow[dr] & (K',L') \arrow[d] \\ & (X,Y) \end{tikzcd}$$

where the map $(K,L) \to (K',L')$ is also a retraction, and so induces a map of normal fibrations. To see this limit is justified requires more careful use of [15]. For example if (K,L) and (K^1,L^1) both dominate (X,Y), then there is a common dominator (although no least common dominator unless X is finite). Also there is a retraction $(K,L) \longrightarrow (X,Y)$ which is a homology isomorphism except in one dimension, where the kernel is projective. Using two such which differ in different dimensions and a common dominator for comparison purposes, an infinite number of cells can be added in each of four dimensions to $\nu^n(K,L)$ to obtain $\nu^n(X,Y)$.

Now that $\nu^n(X,Y)$ is defined (for large n), we give an application generalizing the main theorem of [1].

1.4 Proposition. If (X,Y) is a free dominated π-pair, ξ a S^{k-1} fibration over X, and $(-\xi)$ is a $S^{\ell-1}$ fibration such that $\xi \oplus (-\xi) \simeq X \times S^{k+\ell-1}$, then $D^{n+\ell+k} T\xi \simeq T\nu^n(X) \oplus (-\xi)$ and $D^{n+\ell+k}(T\xi/T(\xi)Y)) \simeq T\nu^n(X,Y) \oplus (-\xi)$.

Proof. Suppose (K,L) is finite and dominates (X,Y), then $D^n T\xi = D^n(K_\xi, E(\xi))$. Since $(K_{\xi\oplus(-\xi)} E(\xi \oplus (-\xi)) \simeq (K \times D^{\ell+k}, K \times S^{\ell+k-1})$, a regular neighborhood of this pair will be given by $(U \times D^{\ell+k}, U \times S^{\ell+k-1})$, where U^{n-1} is a regular neighborhood of K. Embedding $(K_\xi, E(\xi)) \subset (U \times D^{\ell+k}, U \times S^{\ell+k-1})$ (assuming ℓ is large) this becomes a regular neighborhood of this pair with appropriate boundary $\simeq \partial U_{(-\xi)}\ E_{(-\xi)}$. This last is exactly the definition of $\nu^n(K) \oplus (-\xi)$, so $D^{n+\ell+k} T\xi \simeq T(\nu^n(K) \oplus (-\xi))$ as desired. The relative version is obtained by either relativising the proof, or from the cofibration sequence. The statement for (X,Y) results from passing to the limit. ///

Geometric products: If $f: X \longrightarrow Y$ we can form $X \longrightarrow X \times Y$ and $X \longrightarrow X_+ \wedge Y$, which on H_* are essentially "$\cap[f]$". We generalize to stable maps $\Sigma^n X \to Y$, and maps where the bundle over X is not trivial; $T\xi \to Y$.

Suppose η is a fibration over X so that both $E(\eta)$ and X are dominated by π-finite complexes. Then from a map $T\eta \xrightarrow{f} \Sigma Y$ we construct a stable map $T\eta \to T(X \times Y) \overset{X_+ \wedge T\eta \;\; id \times f}{=} X_+ \wedge \Sigma Y$. Suspend until Y is highly connected, then $E(\eta) \to X \to T\eta$ is a cofibration which induces an exact sequence

$$
\begin{array}{c}
[\Sigma X, \Sigma Y] \to [\Sigma E(\eta), \Sigma Y] \to [T\eta, \Sigma Y] \to [X, \Sigma Y]. \\
\wr\!\wr \\
[E(\eta), Y]
\end{array}
$$

$$
\begin{tikzcd}
E(\eta) \arrow[r] \arrow[d] & X \times Y \arrow[d] \\
X \arrow[r, dash] \arrow[d] & X \arrow[d] \\
T(\eta) \arrow[r, "\cap f"] & X_+ \wedge \Sigma Y
\end{tikzcd}
$$

Y highly connected implies that the end terms are zero. Thus the map $T\eta \to \Sigma Y$ defines $E(\eta) \to Y$. Cross with η and this gives a fiber map $E(\eta) \to X \times Y$, whose map of Thom spaces is the desired object. Regard $X \times Y$ as the pullback of the Y-bundle over a point, then 1.1 shows $T(X \times Y) \simeq \Sigma Y \wedge X_+$.

1.5 Definition. If $f: T\eta \to \Sigma Y$ is a map as above, then the geometric product is the stable map $\cap f: T\eta \to \Sigma Y \wedge X_+$.

Further if ρ is another fibration over X, then f induces a product $\cap f: T(\eta \oplus \rho) \to \Sigma Y \wedge T\rho$. This is obtained by Whitney sum of ρ with the fiber map constructed above. Note that the original map $T\eta \to \Sigma Y$ can be recovered by composing $\cap f$ with projection on the first factor.

<u>Example</u>. If η is a fibration with fiber, base, and total space dominated by π-finite complexes, η is stabily trivial iff the the inclusion of the fiber $\Sigma F(\eta) \to T\eta$ has a left inverse.

<u>Proof</u>. The stable fiber map $\eta \to X \times F(\eta)$ constructed above is an equivalence. Milnor and Spanier developed an early form of the geometric product (see [1]) to prove a special case of this example.///

Dually, suppose we are given a map of dominated π-spaces $Y \to T\xi$, where now ξ is a S^{k-1} fibration with $S^{\ell-1}$ inverse $(-\xi)$. Then the dual is $D^n g$: $D^n T\xi \to D^n Y$, or by 1.4, $T\nu^{n-k-\ell}(X) \oplus (-\xi) \to \Sigma D^{n-1} Y$. Apply the above to obtain a stable fiber map $\nu(X) \oplus (-\xi) \to X \times D^{n-1} Y$. Add ξ to each side to get $\nu^n(X) \to \xi \oplus (X \times D^{n-1} Y)$, and apply T to obtain $D^n X_+ \to T\xi\ D^n Y$.

<u>1.6 Definition</u>. If g: $Y \to T\xi$ is a map as above, ξ a sphere fibration, then the geometric product is the map $g\cap$: $D^n X_+ \to T\xi\ D^n Y$

We can also define the product with a map with "Z coefficients" by starting with a bundle ξ on X/π, and a map $Y \to T\xi$. Dualize to $T\nu(X/\pi) \oplus (-\xi) \simeq DT\xi \to DY$. Denote the projection $X \to X/\pi$ also by π, and observe $\nu(X) = \pi^*\nu(X/\pi)$. Thus there is a map $T\nu(X) \oplus (-\pi^*\xi) \to T\nu(X/\pi) \oplus (-\xi) \to DY$, yielding as above $g\cap$: $DX_+ \to T(\pi^*\xi) \wedge DY$.

Another variant is the relative product. Given ξ on a pair (X,Z), and $g: Y \to T\xi/T(\xi|Z)$, then the construction using the relative form of 1.4 gives $g\cap$: $D(X/Z) \to T\xi \wedge DY$. Other versions will appear in the next section.

When both products are defined so they can be composed, they " commute ".

1.7 Proposition. Suppose X, Y are π-spaces dominated by π-finite complexes, X free, ω a finite complex, and ξ a spherical fibration on X/π. Given maps $g:\omega \to T\xi$ and $f: \Sigma^n X \to Y$, then the following diagram commutes;

$$\begin{tikzcd}
D^m X_+ = T\nu^m(X) \arrow[r, "\cap f"] \arrow[d, "g\cap"] & Y \wedge D^m X_+ \arrow[d, "Y\wedge(g\cap)"] \\
T(\pi^*\xi)\wedge D^m W \arrow[r, "(\cap f)\wedge D^m W"] & Y\ T(\pi^*\xi)\ D^m W
\end{tikzcd}$$

The composition is denoted $(g\cap f)\cap$.

Proof. The top $\cap f$ is obtained by adding $\nu^m(X)$ to the unadorned product, the bottom is ξ added. Consider the corresponding diagram of bundle maps constructed in the definition, and commutativity is clear.///

Normal fibrations do not behave very well with respect to cofibrations, because the fiber tends to change wildly. The situation with fibrations is somewhat better. It is easily seen for example that $\nu(X\times Y) = P_1^*\nu(X) \oplus P_2^*\nu(Y)$, and $\nu(X/\pi) = \nu(X)/\pi$. The following generalizes the situation of differentiable fiber bundles.

1.8 Proposition. If $(F_0,F_1) \xrightarrow{i} (X_0,X_1) \xrightarrow{r} Y$ is a fibration with (F_0,F_1), (X_0,X_1) and Y all dominated by finite complexes, then there is a fibration $\nu(i_0,i_1)$ over X_0 with $i^*\nu(i_0,i_1) \simeq \nu(F_0,F_1)$, and $\nu(i_0,i_1) \oplus r^*\nu(Y) \simeq \nu(X_0,X_1)$.

Proof. First assume that Y is finite, by pulling the fibration back over a retraction. If (F_0,F_1) were also finite, we could approximate $(X_0,X_1) \xrightarrow{(r,0)} Y\times R^{n-2}\times[0,\infty)$ by a fiber map which is an embedding in each fiber. The boundary of a regular neighborhood of the image would clearly give the desired fibration. The general case is obtained by going through the construction of $\nu(F_0,F_1)$ over each cell in Y. Choose a pair dominating (F_0,F_1) and start constructing (X_0',X_1') by taking a copy of this pair over each 0-simplex of Y, mapped to the appropriate fiber of (X_0,X_1). When done over the k-skeleton of Y, this gives for each (k+1) -cell a retraction $(X_0,X_1)S^k \to (X_0,X_1)|S^k \subsetneq (F_0,F_1)XD^{k+1}$. Lift the projection to (F_0,F_1) through the given retraction, and form the mapping cylinder to define (X_0',X_1') over D^{k+1}.

Note that the inverse image of each simplex is collared in the inverse of a simplex containing the first. Now using this we can approximate ν': $(X_0',X_1') \to Y\times R^n$, some large n, by a map which commutes with ν' and projection on Y, and which is an embedding. Let $\nu^n(r')$ be the boundary of a regular neighborhood of the image, then the desired $\nu^n(i_0,i_1)$ is the direct limit of $\nu^n(r')$ over all such constructions.///

2. Normal and Poincaré spaces.

Poincaré spaces occupy a special place in homotopy theory. For example they are characterized by having the fiber of the normal fibration dominated by a finite complex. After a review of homology from the appendix, the equivalent definitions of Poincaré spaces are given. Next normal spaces are introduced, a few properties given, and normal maps are defined. Normal spaces are useful because the mapping cylender of a normal map is a normal space. Finally the torsion of a Poincaré space is defined and some properties developed.

Suppose X is a space and $\omega: \pi_1 X \to Z/2 = \{-1,1\}$ is a homomorphism. We regard the chains $C_*(\tilde{X})$ as a left $Z[\pi_1 X]$ complex and define

$$H^*(X;B) = H(\hom_{Z[\pi]}(C_*(\tilde{X}),B))$$

$$H^{\omega}_*(X;B) = H(B \otimes_{Z[\pi]} C_*(\tilde{X})) .$$

Here B is a left $Z[\pi]$ module, and the notation H^{ω} indicates the use in the tensor product of the right module structure $b\alpha = \bar{\omega}(\alpha)b$, $\bar{\omega}$ the antiinvolution $\bar{\omega}(\Sigma n_g g) = \Sigma n_g \omega(g) g^{-1}$ on $Z[\pi_1 X]$. The definition of the relative group is similar.

There are two products in algebra which do not come from geometric products. First if (X,Y) is finite, then the collapsing map $(D^n,S^{n-1}) \to (D^n X, D^{n-1} Y_+) \to (D^n X/Y,*)$ gives an element $[S] \in H_n(D^nX/Y,*;Z)$. The product with x gives an isomorphism $[S]\cap \,: H^*(X,Y;Z) \xrightarrow{\sim} H^0{}_{n-*}(D^nX/Y,*;\, Z)$. $\cap[S]$ also induces an isomorphism with $Z[G]$ coefficients, G any quotient of π_1X. Second, for an S^{n-1}-bundle ξ over X define $\omega_1(\xi) = \omega:\pi_1X \to Z/2 = \{-1,1\}$ by whether or not the bundle pulled back to S^1 is trivial. A <u>local orientation</u> for ξ is an element $U \in H^n(T\tilde{\xi};Z_\omega)$ whose product $\cap U:H^0_{n+*}(T\tilde{\xi},*;Z[\pi_1X]) \to H^\omega_*(X;Z[\pi_1X])$ is an isomorphism. There are exactly two such orientations.

<u>2.1 Definition</u>. A pair (X,Y) is a <u>Poincaré pair</u> iff it is dominated, and (X,Y), (Y,ϕ) satisfy one of the following equivalent conditions.

1) the fiber of $\nu^n(X,Y)$ is dominated by a finite complex
2) $\nu^n(X,Y)$ is a spherical fibration.
3) there is a spherical fibration ξ and a map α: $S^k \to T\xi/T\xi|Y$ whose geometric product $\alpha\cap$: $D^n(\tilde{X}/\overline{Y}) \longrightarrow T\tilde{\xi}\wedge S^{n-k}$ is a homotopy equivalence, where $\overline{Y}$ is the cover induced from $\tilde{X}$.
4) there is a homomorphism ω: $\pi_1X \to Z/2$ and a class $[X,Y] \in H^\omega_m(X,Y;Z)$ whose product induces an isomorphism $[X,Y]\cap:H^*(X,Y;Z[\pi_1X]) \xrightarrow{\sim} H^\omega{}_{m-*}(X;Z[\pi_1X])$.

The <u>dimension</u> of the pair is n-(dim fiber $\nu^n(X,Y)$) in (1) and (2), k-(dim ξ) in (3), and m in (4). The class [X,Y] in (4) is called a <u>fundamental class</u> for (X,Y). We show the various conditions are equivalent.

$\underline{(3) \Rightarrow (2)}$ The geometric product comes from a bundle map $\nu^n(\tilde{X},\tilde{Y}) \to \tilde{\xi} \oplus S^{n-k-1}$, which is a fiber equivalence because $\tilde{X}$ is simply connected and the Thom map is an equivalence.

$\underline{(2) \Rightarrow (3)}$ The bundle map $\nu(Y) \to \nu(X,Y)|Y$ gives $S \to T\nu(X,Y)/T(\nu(X,Y)|Y)$. The associated geometric product is the identity map $D^n(\tilde{X}/\tilde{Y}) = T\nu^k(\tilde{X},\tilde{Y}) \oplus S^{n-k-1}$.

$\underline{(3) \Rightarrow (4)}$ Let $\omega = \omega_1(\xi)$, U a local orientation for ξ, of dimension k, and $[S]$ the homology class of the sphere $S^n \to D^n(X/Y)$. Then $[X,Y] = [S] \cap U$, and the product is given by

$$H^*(X,Y;\Lambda) \xrightarrow[\sim]{\cap[S]} H^0_{n-*}(D^n(\tilde{X}/\tilde{Y});\Lambda) \xrightarrow{\sim} H^0_{n-*}(T(\tilde{\xi} \oplus S^{n-k})) \xrightarrow[\sim]{\cap U} H^{\omega}_{(n-k)-*}(X;\Lambda$$

Here $\Lambda = [\pi_1 X]$.

$\underline{(4) \Rightarrow (2)}$ Let U be a neighborhood of (X,Y), then using duality in U gives a class U inducing a Thom isomorphism for $\nu^n(\tilde{X},\tilde{Y})$. Since $\pi_1\tilde{X} = \{1\}$, a simple spectral sequence arguement (Browder [3], section 1.4) shows the fiber of $\nu^n(\tilde{X},\tilde{Y})$ is a sphere.

$(2) \Leftrightarrow (1)$ Clearly (2) implies (1). For the converse we make use of a lemma.

<u>2.2 Proposition</u> If $(F_0,F_1) \xrightarrow{i} (X_0,X_1) \xrightarrow{r} Y$ is a fibration with everything dominated by finite complexes, then (X_0,X_1) is a Poincaré pair iff (F_0,F_1) and Y are Poincaré.

<u>Proof</u>. According to proposition 1.8 there is a "normal bundle along the fibers" $\nu(i)$ over X_0 with $i^*\nu(i) = \nu(F_0,F_1)$, and $\nu(i) \oplus r^*\nu(Y) \simeq \nu(X_0,X_1)$. In particular the fiber of the normal fibration of (X_0,X_1) is the join of those of (F_0,F_1) and Y, and is a sphere iff the two components are.///

Now to show $(2) \Longrightarrow (1)$, we note that 2.2 applied to $\nu^n(X,Y)$ and $(X,\nu^n(X,Y))$ shows both the fiber F and the pair (CF,F) are Poincaré. Since $\pi_1 F = \{1\}$, this can only happen if $F \simeq S^k$, some k.

We remark here that Y must be assumed Poincaré because we are using universal coefficients, as in Wall [16], [17]. With Z coefficients [3] or consistant coefficients [11] this is a consequence of the assumption on (X,Y) (see also 2.5 below). Universal coefficients are a result of our use of the sphere in duality. Browder has developed a theory of Poincaré embeddings in homotopy spheres which promises to give a geometric setting for the algebra of [11]. The present setting, however, is adequate for our purposes.

Example. Manifolds are Poincaré spaces, since they are well known to embed in Euclidean space stability with a normal disc bundle.

Implicit in the equivalence $(2) \Longleftrightarrow (3)$ is a uniqueness result for the normal fibration as a spherical fibration. The next proposition makes this explicit and sharpens it a little.

2.3 Proposition Suppose (X,Y) is a Poincaré pair with fundamental class [X,Y] of dimension n. If ξ is a S^{k-1} fibration with $\omega_1(\xi) = \omega_1(\nu(X,Y))$, a map $\alpha\colon S^{n+k} \to T\xi/T(\xi|Y)$, and a local orientation U such that $U \cap [\alpha] = [X,Y]$, then there is a (unique) stable bundle isomorphism $\xi \simeq \nu(X,Y)$ which takes α to the canonical collapsing map.

Proof. As above the geometric product gives a bundle map; the dual of α is $D^m(T\xi/T(\xi|Y)) \simeq T\nu(X,Y) \oplus (-\xi) \to S^{m-n-k}$, which gives a map $\nu(X,Y) \oplus (-\xi) \to S^{m-n-k-1}$ of the total space. Cross with the projection and add ξ to each side to obtain a stable bundle map $\nu(X,Y) \to \xi$. By construction the composition $S \to T\nu(X,Y)/T(\nu|Y) \to T\xi/T(\xi|Y)$ is α. Now using Poincaré duality for $\tilde{X}$ and $U_\xi \cap [\alpha] = [X,Y]$ establishes $T\nu(\tilde{X},\tilde{Y}) \to T\tilde{\xi}$ is a homotopy equivalence. Therefore the bundle map $\nu(X,Y) \to \xi$ is a homotopy equivalence of total spaces, hence a bundle isomorphism.///

The main question we investigate (in the next section) is, if the structure of a Poincaré space is weakened, when can a Poincaré space be reconstructed from the data? To that end we introduce normal spaces, which will also be useful in other contexts.

2.4 Definition A normal pair is a dominated pair (X,Y) with a sphere bundle ξ and either

(1) a map $S \longrightarrow T\xi/T(\xi(Y))$, or

(2) a fiber map $\nu(X,Y) \longrightarrow \xi$.

As in the proof of definition 2.1, these are equivalent. This structure is inherited by the boundary, however.

2.5 Proposition If (X,Y) is a normal pair, Y is normal. If (X,Y) satisfies the conditions of 2.1 and $\pi_1(Y) \to \pi_1(X)$ is injective, then Y satisfies these conditions, so (X,Y) is Poincaré.

<u>Proof</u>. For the first part the fiber map is given by the composition $\nu^{n-1}(Y) \to \nu^n(X,Y)|Y \to \xi|Y$, the sphere map is the boundary in stable homotopy. For the second statement the fiber map of Y as a normal space gives a map of cofibrations

$$\begin{tikzcd}
D^{n-1}(\overline{Y}_+) \arrow[r] \arrow[d] & D^n(\tilde{X},\overline{Y}) \arrow[r] \arrow[d] & D^n(\tilde{X}_+) \arrow[d] \\
T(\xi/\overline{Y}) \arrow[r] & T\tilde{\xi} \arrow[r] & T\xi/T(\tilde{\xi}/\overline{Y}) .
\end{tikzcd}$$

The center map is the geometric product (3) of 2.1, so a homotopy equivalence. The product is the Thom map of a bundle isomorphism $\nu(\tilde{X},\overline{Y}) \to \tilde{\xi}$. Add $(-\xi)$ to each side to obtain $\nu(\tilde{X},\overline{Y}) \oplus (-\xi)$ $X{\times}S$. The dual of the Thom map of this equivalence is $D(X_+) \xrightarrow{\sim} D(T\nu(\tilde{X},\overline{Y}) \oplus (-\tilde{\xi})) = D(D(T\tilde{\xi}/T(\tilde{\xi}|\overline{Y}))$ $= T\tilde{\xi}/T(\tilde{\xi}|\overline{Y})$, the right-hand map, also a homotopy equivalence. Therefore the Thom map of $\nu(\overline{Y}_+) \to \tilde{\xi}|Y$ is a homotopy equivalence. If $\pi_1 Y \to \pi_1 X$ is injective (for each component of Y) $\overline{Y}$ has simply-connected components, so the bundle map itself is an equivalence.///

This is a geometric version of I.2.2 of [3]. Note we have exact commutativity rather than up to sign. Algebraically this is due to non-standard sign conventions suggested by the geometry (see the appendix).

A <u>triad</u> is just a space with two subspaces, denoted $(X;Y_1,Y_2)$. A <u>Poincaré triad</u> is a triad $(X;Y_1,Y_2)$ such that

$(X, Y_1 \cup Y_2)$, $(Y_1, Y_1 \cap Y_2)$, $(Y_2, Y_1 \cap Y_2)$ are all Poincaré pairs. There are refinements of duality in such a situation, for example the product $[X,Y] \cap : H^*(X, Y_1) \xrightarrow{\simeq} H^{\omega}_{n-*}(X, Y_2)$ is an isomorphism. This is H_* applied to the center map in the map of cofibrations

$$\begin{tikzcd}
D(Y_{1+}) \arrow[r] \arrow[d] & D(X, Y_1) \arrow[r] \arrow[d] & D(X_+) \arrow[d] \\
T(\xi | Y_1)/T(\xi | Y_1 \cap Y_2) \arrow[r] & T\xi/T(\xi | Y_2) \arrow[r] & T\xi/T(\xi | Y_1 \cup Y_2)
\end{tikzcd}$$

The map on each end is a homotopy equivalence since $(Y_1, Y_1\ Y_2)$, $(X, Y_1 \cup Y_2)$ are both Poincaré.

Next we have the "sum theorem", see [3] §I.3, or [17] 2.7.

<u>2.6 Proposition</u> Suppose $(X,Y) = (X_1,Y_1) \cup (X_2,Y_2)$ and denote $(X_1 \cap X_2, Y_1 \cap Y_2)$ by (X_0, Y_0). Then (X,Y) is a normal space iff $(X_1, Y_1 \cup X_0)$ and $(X_2, Y_2 \cup X_0)$ are normal, and the induced structures on (X_0, Y_0) agree. If $(X_1; Y_1, X_0)$ and $(X_2; Y_2, X_0)$ are Poincaré triads, (X,Y) is Poincaré. If (X,Y) and $(X_2; Y_2, X_0)$ are Poincaré and $\pi_1 X_1 \to \pi_1 X$ and $\pi_1 Y_1 \to \pi_1 Y$ are injective, then $(X_1; Y_1, X_0)$ is also Poincaré.

<u>Proof</u>. From the definition there is an inclusion of total spaces $\nu(X_i, Y_i \cup X_0) \subseteq \nu(X,Y)$, which induces a bundle map $\nu(X_i, Y_i \cup X_0) \to \nu(X,Y)|X_i$.

Further the union $\nu^n(X_1,Y_1 \cup X_0) \cup_{\nu^{n-1}(X_0,Y_0)} \nu^n(X_2,Y_2 \cup X_0) \to \nu^n(X,Y)$ is a homotopy equivalence. Thus if there is a bundle map $\nu(X,Y) \to \xi$, the restriction to X_i gives a normal structure for each piece, and bundle maps on each piece which agree on $\nu^{n-1}(X_0,Y_0)$ glue together to give a bundle map for $\nu(X,Y)$. This gives the statement about normal spaces.

The Poincaré assertions follow also, since if the bundle maps on each piece are isomorphisms they glue together to give an isomorphism. If they are isomorphisms over (X_0,Y_0) and $(X_2,Y_2 \cup X_0)$ then exersion shows that $\nu(X_1,Y_1 \cup X_0) \to \xi|X_1$ is a homology isomorphism on the cover of X_1 induced from the universal cover of X. If this cover has simply-connected components ($\pi_1 X_1 \to \pi_1 X$ injective) then this is also an isomorphism. The same considerations applied to Y shows $\nu(Y_{1+})$ is also spherical, so the pair (X_1,Y_1) is Poincaré. Clearly there are many other such statements possible, with appropriate restrictions on the fundamental groups.///

Next we investigate normal maps.

2.7 Definition A map $f: (W,X,\xi) \to (Y,Z,\rho)$ of normal spaces is a normal map if an isomorphism $b: \xi \simeq f^*\rho$ is given such that the diagram

$$\begin{array}{ccc} S & = & S \\ \downarrow & & \downarrow \\ T\xi/T(\xi|X) & \xrightarrow{Tf^* \circ Tb} & T\rho/T(\rho|Z) \end{array}$$

commutes. Further if the pairs are oriented (local orientations U_ξ and U_ρ are given) then f is <u>degree 1</u> if $U_\xi = b^*U_\rho$.

Since there are exactly two choices for U_ξ, f is either degrees 1 or -1. If the pairs are Poincaré and the map degree 1, then $f_*[W,X] = [Y,Z]$, since $f_*[X,\omega] = f_*([S]\cap U_\xi) = f_*([S]\cap f^*U_\rho) = [S]\cap U_\rho = [Y,Z]$.

<u>2.8 Proposition</u> Suppose $f\colon (W,X,\xi) \to (Y,Z,\rho)$, $b\colon \xi \simeq f^*\rho$ is a normal map of normal spaces.

(1) The mapping cylender $(Y_f;W,Y,Z_{f|X})$ is a normal 4-ad

(2) If f is a homology equivalence of Poincaré pairs, with $Z[\pi_1 Y]$, $Z[\pi_1 Z]$ coefficients, the mapping cylender is a Poincaré 4-ad.

<u>Proof</u>. Let ρ_f denote ρ pulled back over Y_f (and identified with ξ via b over W). The mapping cylender of the diagram in the definition gives $(S\times I, S\times\{0,1\}) \to (T\rho_f/T(\rho_f|Z_f), T\rho/T(\rho|Z)\cup T\xi/T(\xi|X))$. Collapsing the second member of each pair to a point gives Y_f a normal structure which restricts to the given one on the boundary. (2) is included primarily for reference, since in this case $Y_f \simeq (W,X)\times(I,\{0,1\})$, which is clearly Poincaré.///

Note also that if (W,X), (Y,Z) are oriented and f degree 1, then the mapping cylender has a (unique) orientation which restricts to the given ones on the boundary.

To close this section we discuss the torsion of a Poincaré pair. Recall [9] that if Λ is a ring (associative with unit),

$K_1(\Lambda)$ is the abelianization of $\varinjlim GL_n(\Lambda)$, and for a group π $Wh(\pi) = K_1(Z[\pi])/\{\pm\pi\}$. For a groupoid with finite components (eg $\pi_1 X$, X not corrected), we define $Wh(\cup\pi_i) = \Sigma Wh(\pi_i)$. Suppose $f: X \to Y$ is a homology equivalence of π-finite π-complexes, then $f_*: C_*X \to C_*Y$ is a chain equivalence of free based $Z[\pi]$ modules, the basis being canonical up to actron of $\{\pm\pi\}$. Thus there is a Whitehead torsion $\tau(f) \in Wh(\pi)$ defined.

Now if (X,Y) is a finite Poincaré pair, then $\nu(X,Y)$ is a sphere bundle, hence has a finite complex structure. However, for any (X,Y) $\nu(X,Y)$ is defined as part of the boundary of a regular neighborhood, so $\nu(X,Y)$ and $D(X,Y)$ have finite complex structures from the definition. The torsion of a Poincaré pair is the discrepancy between the two structures on $\nu(X,Y)$.

<u>2.9 Definition</u> If (X,Y) is a finite Poincaré pair of dimension n, the torsion <u>$\tau(X)$</u> $\in Wh(\pi_1 X)$ is $(-1)^{m-n}$ times the torsion of the geometric product (identity map) $D^m(\tilde{X},\tilde{Y}) \to T\nu^m(\tilde{X},\tilde{Y})$. The torsion $\tau(X,Y_1)$ of a Poincaré <u>triad</u> $(X;Y_0,Y_1)$ is $(-1)^{m-n}$ times the torsion of the product $D^m(\tilde{X},\tilde{Y}_0) \to T\tilde{\nu}/T(\tilde{\nu}|\tilde{Y}_1)$.

Another way to view this is as the torsion of the chain map which induces the product $\cap[X,Y]: H^*(X,Y;Z[\pi]) \to H^{\omega}_{n-*}(X;Z[\pi])$.

To describe the relationships between the various torsions (eg $\tau(X,Y)$ and $\tau(X)$ some involutions on $Wh(\pi)$ are needed. If $R: \Lambda \to \Lambda$ is an automorphism, an automorphisms of $GL_n(\Lambda)$ is defined by $A \mapsto (R^n)^{-1} A R^n$, which induces an automorphism $\overline{R}$ of $K_1(\Lambda)$. In terms of matrices this replaces (a_{ij}) by $(R(a_{ij}))$.

If R is an antiautomorphism ($R(\alpha\beta) = R(\beta)R(\alpha)$) then the automorphism is $(a_{ij}) \mapsto (R(a_{ji}))$, again inducing $\bar{R}$ on $K_1(\Lambda)$. In particular if $\omega:\pi \to Z/2 = \{-1,1\}$ is a homomorphism, as at the beginning of the section, then $(\Sigma n_g g) \mapsto \Sigma n_g \omega(g) g^{-1}$ is an autiinvolution of $Z[\pi]$ which induces an involution $\bar{\omega}$ on $Wh(\pi)$. We denote $\bar{O}(\tau)$ by $\bar{\tau}$. Also $\Sigma n_g g \mapsto \Sigma n_g \omega(g) g$ is an involution of $Z[\pi]$ inducing an involution $\tilde{\omega}$ on $Wh(\pi)$. Clearly $\bar{\omega} = \bar{O}\tilde{\omega} = \tilde{\omega}\bar{O}$.

Now some rules for manipulating torsions of π-complexes.

(1) If

$$\begin{array}{ccccc} W & \longrightarrow & X & \longrightarrow & X/\omega \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ Y & \longrightarrow & Z & \longrightarrow & Z/Y \end{array}$$

is a homology equivalence of cofibrations of finite π-complexes, $\tau(\beta) = \tau(\alpha) + \tau(\gamma)$.

(2) If $f_i: (X_i, X_1 \cap X_2) \to (Y_i, Y_1 \cap Y_2)$, $i = 1,2$, are homology equivalences which agree on $X_1 \cap X_2$, then $f_1 \cup f_2$ is homology equivalence and $\tau(f_1 \cup f_2) = \tau(f_1) + \tau(f_2) - \tau(f_1 \cap f_2)$.

(3) $f: X \to Y$ homology equivalence, then $D^n f: DY \to DX$ has torsion $\tau(D^n f) = (-1)^n \overline{\tau f}$.

(4) $f: X \to Y$ homology equivalence, ξ a S^{k-1} bundle over Y, then $Tf^*: Tf^*\xi \to T\xi$ has torsion $\tau(Tf^*) = (-1)^k \widetilde{\omega_1(\xi)}(\tau f)$.

(5) If $f: \eta \to \mu$ is a fiber map over X, total spaces with π-finite complex structures and ξ a S^{k-1} bundle over X, then $f \oplus \xi: \eta \oplus \xi \to \mu \oplus \xi$ has torsion $\tau(f\oplus\xi) = (-1)^k \widetilde{\omega_1(\xi)}(\tau f)$.

Rules (1) and (2) are from the sum theorem for short exact sequences of free based chain equivalences. The third and fourth result from special choices of CW structure for DX and $T\xi$ which display Df and Tf* on the chain level as of the form $Rf_{\#}R^{-1}$ where R is $\overline{O}$ or $\widetilde{\omega_1(\xi)}$ respectively. The signs come from dimension shifts. Lastly (5) results from application of (1), (2), and (4) to the definition of $\oplus$.

<u>2.10 Proposition</u> Suppose (X,Y) is a Poincaré pair of dimension n, and $i:\pi_1 Y \to \pi_1 X$ the inclusion. Then $\tau(X) = (-1)^n \overline{\omega}(\tau(X,Y))$ and $i_*\tau(Y) = \tau(X) + (-1)^{n-1}\overline{\omega}(\tau(X))$ where $\omega = \omega_1(\nu(X,Y))$. Further if $(X;Y_0,Y_1)$ is a Poincaré triad, $\tau(X,Y_0) = \tau(X,Y_1) - \tau(Y_0 \cup Y_1) + \tau(Y_0 \cap Y_1)$.

<u>Proof</u>. Let ξ^k denote $\nu(X,Y)$ as a S^{k-1} fibration, and let $(-\xi)$ be a $S^{\ell-1}$ fibration inverse for ξ. The collapsing map $S^{n+k} \to T\xi^k/T(\xi|Y)$ gives rise to a bundle map $\gamma^{m-k-\ell}(X,Y) \oplus (-\xi) \to X \times S^{m-n-k-1}$, the precursor to the geometric product. Now this is an equivalence, since the Thom map gives the geometric product which is an equivalence. The two products are obtained by

$$\begin{array}{ccc} & \nu^{m-k-\ell}(X,Y)\oplus(-\xi) \quad X\times S^{m-n-k-1} & \\ \swarrow\!\!\swarrow \oplus\xi & & \searrow\!\!\searrow T \\ \nu^m(X,Y)\to\xi^{m-n} & & T\nu^{m-k-n}(X,Y)\oplus(-\xi)\to X_+\wedge S^{m-n-k} \\ \Big\Downarrow T & & \Vert \\ & & D^m(T\xi^k/T(\xi|Y)) \\ & & \Big\Downarrow D^{2m-n-k} \\ D^m\omega(X,Y)\to T\xi^{m-n} & & D^m(X_+)\to T\xi^{m-n}/T(\xi|Y). \end{array}$$

By definition the bottom left map has torsion $(-1)^{m-n}\tau(X)$, which is the same as the map above it by (1). By (5) the top map has torsion $(-1)^{m-n-k}\tilde{\omega}(\tau(X))$, which again is the same as the middle right map. Applying (3) shows $\tau(X,Y)=(-1)^{m-n}(-1)^{2m-n-k}\overline{\omega}[(-1)^{m-n-k}\tilde{\omega}(\tau(X))] = (-1)^n\overline{\omega}(\tau(X))$, as claimed. Now apply (1) to the diagram

$$\begin{array}{ccccc} D^{n-1}(Y_+) & \to & D^n(X,Y) & \to & D^n(X_+) \\ \downarrow & & \downarrow & & \downarrow \\ T(\xi|Y) & \to & T\xi & \to & T\xi/T(\xi|Y) \end{array}$$

to get $i_*\tau(Y) = \tau(X) - \tau(X,Y) = \tau(X) + (-1)^{n-1}\overline{\omega}(\tau(X))$

Finally apply (1) to the diagram

$$\begin{array}{ccccc} D^{n-1}(Y_1,Y_1\cap Y_0) & \to & D^n(X,Y_0) & \to & D^n(X_+) \\ \downarrow & & \downarrow & & \downarrow \\ T(\xi|Y_1)/T(\xi|Y_1\cap Y_0) & \to & T\xi/T(\xi|Y) & \to & T\xi/T(\xi|Y_0\cup Y_1) \end{array}$$

to obtain $[\tau(X_1 Y_0) = \tau(X)] - \tau(Y, Y_1 \cap Y_0)$. The symmetric formula and application of (2) to the Y gives the final formula. ///

These formulas can easily be applied to get relations among the various torsions appearing in the constructions in 2.6 and 2.9.

Note that using the ideas of Gersten [4] on torsion of self-equivalences, all of this can be extended to Poincaré spaces dominated by finite complexes. This refinement becomes useful only in one application, so rather than carry along another obstruction the problem will be treated separately when it arises.

3. Normal surgery

The objective in this section is to present a proof of the normal surgery lemma.

3.1 Theorem Suppose $(X; Y_0, Y_1, \xi)$ is a normal triad, with $(Y_0, Y_0 \cap Y_1)$ Poincaré. If $\pi_1 Y_1 \simeq \pi_1 X$ and $\dim X \geq 5$, then there is an $[\frac{n-1}{2}]$-connected normal map

$$(W; Y_0, Z) \to (X; Y_0, Y_1)$$

with $(W; Y_0, Z)$ a Poincaré triad, with $\tau(W, Z) = 0 \in Wh(\pi_1 X)$.

Note that the rest of the torsions of the triad can be calculated in terms of $i_* \tau(Y_0)$, using 2.10. The connectivity of the map can be improved slightly, but is immaterial in most of the applications. Also it seems very likely it holds for all n; it is known for $n \neq 3, 4$.

The proof proceeds by inductively making the geometric product $D(X,Y) \to T\xi$ highly connected. Once it is connected to $\frac{1}{2}\dim(X,Y)$ a sort of "duality" implies it is a homotopy equivalence, so X is Poincaré. It can be made connected to just below the middle with no π_1 or dimension restrictions (3.5), the middle is a little more complicated.

The major step in the proof is lemma 3.7, which is used to remove excess cells from a space above the middle dimension. The section begins with a discussion of kernels, which are used to keep track of progress in the construction.

<u>3.2 Definition</u> Suppose $(X;Y_0,Y_1;\xi)$ is a normal triad of dimension n. Let $\tilde{X}$ be the universal cover and $\overline{Y}_i$ the induced cover, and form the geometric product (denoted in section 2 by $\cap[S]$) P^k: $D^{n+k}(\tilde{X},\overline{Y}_0) \to T\xi^k/T(\xi|Y_1)$. For a $Z[\pi_1 X]$ - module B define the <u>kernels</u> of the triad;

$$K^{\omega}_*(X,Y_1;B) = H_{*+k+1}(P^k;B)$$

$$K^*(X,Y_1;B) = H^{*+k+1}(P^k;B) .$$

This indexing convention is used because it is suggestive of dimensions encountered later, and agrees with previous conventions for manifolds.

By definition there is an exact sequence
$\to K^{\omega}_j(X,Y_1) \to H^{n-j}(X,Y_0) \to H^{\omega}_j(X,Y_1) \to K^{\omega}_{j-1}(X,Y_1) \to$,
and similarly for K^*. The map of cofibrations

$$\begin{array}{ccccc} D^{n+k}(X,Y_0) & \to & D^{n+k}(X) & \to & D^{n+k}(Y_0) \\ \downarrow & & \downarrow & & \downarrow \\ T\xi/T(\xi|Y_1) & \to & T\xi/T(\xi|Y_1 \cup Y_0) & \to & T(\xi|Y_0)/T(\xi|Y_1 \cap Y_0) \end{array}$$

gives an exact sequence $\to K^{\omega}_*(X,Y_1) \to K^{\omega}_*(X,Y_0 \cup Y_1) \to K^{\omega}_{*-1}(Y_0, Y_0 \cap Y_1) \to$. There is also a Meyer-Vietoris sequence for K_* in situations like that in 2.6.

The kernels of a normal space also satisfy a sort of duality.

3.3 Proposition If $(X;Y_0,Y_1;\xi)$ is an n-dimensional normal triad, then for any left $Z[\pi,X]$ module B there is a natural isomorphism $K_*^{\omega}(X,Y_0;B) \simeq K^{n-*-1}(X,Y_1;B)$. In particular if (X,Y) is a normal pair of dimension n with Y Poincaré and $K_j(X;Z[\pi_1 X]) = 0$ for $j \leq [\frac{n-1}{2}]$, then $K_*(X) = 0$ for all $*$, and (X,Y) is Poincaré.

Proof. ($[\frac{n-1}{2}]$ is the greatest integer less than or equal to $\frac{n-1}{2}$). Let P_1^k: $D^{n+k}(\tilde{X},\overline{Y}_0) \to T\xi/T(\xi|\overline{Y}_1)$ be the geometric product, then $K_*^{\omega}(X,Y_1) = H_{*+k+1}^{\omega}(P_1^k)$. Apply D^m to obtain $D^m(P_1^k)$: $T\nu^{m-j-k}(X,Y) + (-\xi)^j \to \Sigma^{m-n-k}(\tilde{X}/\overline{Y}_0)$ with $K_*^{\omega}(X_1 Y_1) \simeq H^{m-k-*}(D^m P_1^k)$. But by the Thom isomorphism this has the same cohomology as P_0^{m-n-k} : $D^{m-k}(X,Y_1) \to T\xi^{m-n-k}/T(\xi|Y_0)$. By definition $H^{m-k-*}(P_0^{m-n-k}) = K^{n-*-1}(X,Y_0)$, so $K_*^{\omega}(X,Y_1) \simeq K^{n-1-*}(X,Y_0)$.

To prove the last statement note that if (X,Y) is normal and Y Poincaré, the exact sequence for (X,Y) shows $K_*(X) \simeq K_*(X,Y)$. If $K_j(X) = 0$ $j \leq k$ then $K^j(X) = 0$ also. Applying duality we see $K_j(X) = 0$, $j \geq n-1-k$. If $k = [\frac{n-1}{2}]$, this gives $K_j(X) = 0$ for all j. ///

The shift down one dimension in the duality of the kernels can be illustrated in a special case. If f: $W \to X$ is a normal map of Poincaré spaces then the kernels of f in the sense of [8], [10] satisfy duality in the same dimension as W,X. However the mapping cylender $(X_f;W,X)$, a normal space of one dimension higher, has these same kernels.

<u>3.4 Definition</u> A normal pair (X,Y) is <u>j-Poincaré</u> if $K_i^{\omega}(X;Z[\pi_1 X]) = 0,\ i \leq j$.

The last part of 3.3 can be restated as; if Y is Poincaré and (X,Y) is $[\frac{n-1}{2}]$ -Poincaré then (X,Y) is Poincaré. The next proposition is " surgery below the middle dimension" .

<u>3.5 Lemma</u> If (X,Y) is a normal pair of dimension $n \geq 3$, there is a normal map $f:(W,Y) \to (X,Y)$ with $f[\frac{n}{2}] + 1$ -connected and (W,Y) $[\frac{n-1}{2}] - 1$ Poincaré.

<u>Proof</u>. Suppose (X,Y) is (j-1)-Poincaré, $j \leq [\frac{n-1}{2}]-1$, then we find an (n-j)- connected normal map $f:(W,Y) \to (X,Y)$. Since any normal pair is j-Poincaré for some j (eventually all the groups are zero), the proposition follows by induction. W is constructed by taking the fiber of a map of X to a wedge of spheres and adjusting using another induction. We begin by describing some maps to spheres.

<u>3.6 Proposition</u> Suppose (X,Y) is a π-pair with basepoint, then

(1) (Hurewicz theorem) if X,Y are simply connected and $\pi_j(X,Y) = 0\ j < k$, then $H_k(X,Y;Z[\pi]) \cong [\vee_\pi D^k, \vee_\pi S^{k-1}; X,Y]_\pi$

(2) (Hopf theorem) if A is a free $Z[\pi]$-module of dimension n (possibly countable) and $H^j(X,Y;A) = 0$ for $j > k$, then

$$H^k(X,Y;Z) \simeq \lim_{m\to\infty} [\Sigma^m X, \Sigma^m Y; \vee^n \vee_\pi S^{m+k}, *]_\pi$$

(3) (Freudental suspension theorem) if K is an n-connected π-space with base point, then $[X,Y;K,*]_\pi \to [S^1\wedge X, S^1\wedge Y; S^1\wedge K, *]_\pi$ is surjective if $\dim(X-Y) \leq 2n+1$, bijective if $\dim(X-Y) \leq 2n$.

Proof. Here $\vee_\pi S^k$ has the obvious π-action permuting the components. (1) is exactly the usual theorem since the groups are isomorphic to the integral nonequivariant ones. If (X,Y) is dominated and $n < \infty$ then (2) is the SW dual of (1). In the general case the proof is the usual one [13, p.431] with minor modifications for equivariance [14]. Finally (3) follows easily from the usual suspension theorem [13,p.458] applied cell by cell and extended equivariantly to the translates of the cell. We use here the fact that the action is cellular and free except at the basepoint. ///

Returning to the proof of 3.5, we are supposing $K_i(X) = 0$, $i \leq j - 1$. If $P^k\colon D^{n+k}(X,Y) \to T\xi$ is the geometric product, 3.6 (1) implies $K_j^\omega(X) = H_{j+k+1}(P^k) \simeq [\vee_\pi D^{j+k+1}, \vee_\pi S^{j+k}; T\xi, D^{n+k}(X,Y)]_\pi$. Since (X,Y) is dominated $K_j(X)$ is finitely generated ([6]). Thus there is a π-map $\rho\colon \vee^m \vee_\pi S^{j+k} \to D^{n+k}(X,Y)$ with a mullhomotopy of $P^k\circ\rho$ so that the induced homomorphism $H_{j+k+1}(\vee^m \vee_\pi D^{j+k+1}, \vee^m \vee_\pi S^{j+k}) = Z[\pi]^m \to K_j^\omega(X)$ is surjective. The dual of ρ is a map $D^P(\rho)\colon \Sigma^{P-n-k}(X/Y) \to \vee^m \vee_\pi S^{P-k-j}$. 3.6 (3) applies to desuspend $D^P(\rho)$ to $\alpha\colon X/Y \to \vee^m \vee_\pi S^{n-j}$, because $2(n-j-1) \geq 2n - 2 - 2[\frac{n-1}{2}] + 2 \geq n$.

Define F as the fiber over the basepoint of $\alpha:\tilde{X} \to V^m V_\pi S^{n-j}$ made a fibration. F is a free π-complex, which is the first step in the construction of W.

First, since α is a map of pairs, there is a canonical lifting of the inclusion $Y \subset X$ to $Y \subset F$, giving a map $(F,Y) \xrightarrow{i} (X,Y)$. We define a "normal" structure for $(F,Y;i^*\xi)$ by lifting the structure of $(X,Y;\xi)$ ((F,Y) will not be normal because it is not dominated). Note there is a cofibration $F \xrightarrow{i} X \xrightarrow{W} V^m(V_\pi\Sigma^{n-j}F_+)$ (the geometric Wang sequence), and the composition $(V^mV_\pi\Sigma^{n-j}(i))$ W is the geometric product 1.5 with α. Consider the diagram

$$\begin{tikzcd}
& D^{m+k}(T\xi^k/T\xi|Y) \arrow[rr, "\cap\alpha"] \arrow[d, "\cap[S]"] \arrow[dl, dashed] & & V\Sigma^{n-j}D(T\xi/T\xi|Y) \arrow[d, "V\Sigma(\cap[S])"] \\
\Sigma^{m-n}F_+ \arrow[r, "\Sigma i"] & \Sigma^{m-n}X_+ \arrow[r, "EW"] \arrow[rr, bend right, "\cap\alpha"'] & V\Sigma^{m-j}F_+ \arrow[r, "V\Sigma i"] & V\Sigma^{m-j}X_+
\end{tikzcd}$$

The first three terms in the bottom line form the Wang sequence. In the stable range (k large) to lift $[S]\cap$ to $\Sigma^{m-n}F_+$ (dotted arrow) it is sufficient to find a nullhomotopy of $W\circ([S]\cap)$. We have a nullhomotopy of $(V\Sigma Ti^*)\circ W\circ([S]\cap)$ since this is $(\cap\rho)\circ([S]\cap)$, which by 1.7 is $(V\Sigma[S]\cap)\circ(\cap\rho) = ([S]\cap\rho)$ A nullhomotopy of $[S]\cap\rho = P^k\cdot\rho$ was part of the data, so the geometric product formed from it is trivial also. Now notice $i;\ F \to X$ is (n-j-1)-connected, and since $n \geq 3$ $\pi_1 F \cong \pi_1 X$.

By the Thom isomorphism $V\Sigma^{m-j}{}_i$ is m+n-2j-1-connected, and dim $D^{m+k}(T\xi/T(\vec{\xi}\,|Y)) = m$. Thus if $m+1 \leq n+m-2j-1$ the nullhomotopy can be lifted to a nullhomotopy of $W\wedge([S]\wedge)$, and the dotted arrow exists. The dimension restriction is $j \leq [\frac{n}{2}] - 1$.

Next an expression for C_*F will be found. Let $C^c_*(\quad)$ denote <u>cellular</u> chains in this paragraph. Any reasonable manipulation of C^c_* can be realized as the cellular chains of an equivalent complex, by [6]. The Wang sequence shows $C^c_*\tilde{F} \simeq C^c_*(V^m V_\pi \Sigma^{n-j}(\tilde{F}_+)) \oplus_{w_\#} C^c_*\tilde{X}$ (see the appendix for details on the algebraic mapping cone). Let $A_{n-j} = Z[\pi]^m$, then $C^c_*(V^m V_\pi \Sigma^{n-j}\tilde{F}_+) = A_{n-j} \otimes_Z C^c_*F$. Since $\pi_1 F = \pi_1 X$, $H_1 C^c_*\tilde{F} = 0$, and so this is chain equivalent to $B_* \oplus A_{n-j}$, where $B_* = 0$ for $* \leq n-j+1$. $w_\# \simeq \tilde{w}_\# \oplus \rho_\#$, $\rho_\# : C^c_{n-j}(\tilde{X}) \longrightarrow A_{n-j}$. Thus $C^c_*\tilde{F} \simeq B_* \oplus_{\tilde{w}_\#} (A_{n-j} \oplus_{\rho_\#} C^c_*(\tilde{X}))$.

Finally recall the diagram

$$\begin{tikzcd}
 & D^{m+k}(T\xi^k/T\xi\,|Y) \arrow[d, "\cap[S]"] \arrow[dl, dashed] \\
\Sigma^{m-n}\tilde{F}_+ \arrow[r] & \Sigma^{m-n}\tilde{X}_+
\end{tikzcd}$$

Since (X,Y) is $(j-1)$-Poincaré $H_{m-n+*}(\cap[S];\ Z[\pi]) = 0$ for $* \geq n-j$. The existance of the lift of $\cap[S]$ then implies $\Sigma^{m-n}\tilde{X}_+ \longrightarrow \Sigma^{m-n}\tilde{X}/\tilde{F}$ deforms into the $(m-j)$-skeleton of $\Sigma^{m-n}\tilde{X}/\tilde{F}$.

All the hypotheses have been checked for application of the following lemma to F to eliminate the B_*.

<u>3.7 lemma</u> Suppose $F \to X$ is a map, X the universal cover and $\bar{F}$ the induced cover, such that

1) $\dim X = n$, and $k \geq [\frac{n-1}{2}]$, $k \geq 2$,

2) $C_j^c \bar{F} \simeq A_j \oplus C_j^c \tilde{X}$, $j \geq k$, A_j free $Z[\pi_1 X]$-modules of countable rank

3) for some m $\Sigma^m \tilde{X}_+ \to \Sigma^m \tilde{X}/\bar{F}$ deforms into a $(m+k)$-skeleton.

Then there is (a unique) $F' \to F$ such that $C_*^c \bar{F}' = \begin{cases} C_*^c F, & * < k \\ C_*^c X, & * \geq k. \end{cases}$

<u>Proof</u>. Note if F' exists X/F' is a $(m+k)$ skeleton of X/F, so these conditions are actually necessary and sufficient. The uniqueness follows from observing that given such a F' it maps into the construction to be given at each stage, ending up with a homotopy equivalence. The chain complex hypothesis is included to give control of <u>which</u> skeleton is used.

The proof is by downward induction on k. If $k > n$, then we can take F' to be the $(k-1)$-skeleton of F. Suppose the statement is true for $k+1$, then we may apply it to obtain F' F with $C_*^c F' = C_* X$, $* \geq k+1$. There is a diagram

$$\begin{tikzcd} & \Sigma^m X/F' \arrow[d] \\ \Sigma^m X_+ \arrow[ur] \arrow[r] & \Sigma^m X/F \end{tikzcd}$$

Since the lower map deforms into a $(m+k)$ skeleton, and $\Sigma X/F'$

is a (m+k+1)-skeleton, we see $\Sigma^m X_+ \to \Sigma^m X/F'$ also deforms into a (m+k) skeleton. Thus F' satisfies all the hypotheses above, and additionally the corresponding A_* vanish above k.

Thus assume $A_* = 0$, $* \geq k + 1$. Now $C_*^c(X,F) = \begin{cases} 0 & *>k+1 \\ A_k & *=k+1 \end{cases}$

so $H^*(X,F;A_n) = 0$, $* > k+1$. The identity $C_{k+1}^c(X,F) \to A_k$ gives an element in $H^{k+1}(X,F;A_k)$ which by the Hopf theorem 3.6 (2) is represented by a map $\rho:(\Sigma^m\tilde{X}/\bar{F},*) \to (V^j V_\pi S^{m+k+1},*)$, where j is the (possibly countably infinite) rank of A_k. But $\Sigma^m X_+ \to \Sigma^m \tilde{X}/\bar{F}$ deforms into a (m+k)-skeleton so $\rho = \delta\alpha$, $\alpha\colon \Sigma^{m+1}F_+ \to V^j V_\pi S^{m+k+1}$. Because $k \geq [\frac{n-1}{2}]$, by the Feudental theorem 3.6 (3) this desuspends to $\alpha:\bar{F} \to V^j V_\pi S^k$. Let F_1 be the fiber of α over the basepoint. We will be able to apply the induction hypothesis to F_1.

There is the Wang sequence $\bar{F}_1 \to \bar{F} \xrightarrow{w} V^j V_\pi\ \Sigma^k(\bar{F}_{1+})$. Now $C_*^c(V^j V_\pi \Sigma^k(\bar{F}_{1+})) = A_k \otimes_Z C_* F_1$, and since $H_1\bar{F}_1 = H_1\bar{F} = 0$ this is chain equivalent to $A_k \oplus B_*$, $B_* = 0$, $* \leq k + 1$. Also $w_\# = \tilde{w}_\# + \alpha_\#$, $\alpha_\#:C_k F \to A_k$ the projection on A_k. Thus $C_*\bar{F}_1 \simeq B_* \oplus_{\tilde{w}_\#} (A_k \oplus_{\alpha_\#} C_* F)$, and the new A_k can be cancelled with the old to give $C_*^c\bar{F}_1$ with $C_k^c F_1 = C_k^c X$. The proof will be completed by verifying the hypotheses of the theorem for F_1, k + 1, and $C_*^c F_1 = B_* \oplus_{\tilde{w}_\#} C_* F = B_* \oplus_{\tilde{w}_\#} C_* X$, $X \geq k + 1$. We lack only condition (3).

First consider the fibration sequence

$$X/F_1 \longrightarrow X/F \longrightarrow V^j V_\pi \Sigma^{k+1}(F_{1+}) \longrightarrow \Sigma X/F \ .$$

As above $\dim X/F = k + 1$, and the third term is k-connected. The $(k+1)$-skeleton of the third term is $V^j V_\pi S^{k+1}$, with inclusion i. The map $X/F \to V^j V_\pi \Sigma^{k+1}(F_{1+})$ is $i \circ \rho$, by construction. This gives a formula for $\Sigma^{m+1}X_+ \to \Sigma^{m+1}X/F$;

$$\begin{tikzcd}
\Sigma^{m+1}X_+ \arrow[rr, "="] \arrow[d, "\Sigma^{m+1}P_1"] & & C_+\Sigma^m X \arrow[d, "h"] & \underset{\Sigma X}{\cup} \arrow[d, "\Sigma^m P"] & C_-\Sigma^m X \arrow[d, "C\Sigma^m P"] \\
\Sigma^{m+1}X/F_1 \arrow[r] & (V^j V_\pi \Sigma^{m+k+1}F_{1+}) \cup_{i \circ \rho} & (\Sigma^m X/F)\times I & \underset{\Sigma X/F}{\cup} & C\Sigma^m X/F \ .
\end{tikzcd}$$

Here $P: X \to X/F$ is the projection, and h is the homotopy obtained above of $\Sigma^m \rho \circ P \sim *$. This homotopy was obtained by composing $\Sigma\rho \circ P$ with the deformation of $\Sigma^m X_+$ into the $(m+k)$-skeleton of $\Sigma^m X/F$. Putting in the whole deformation, therefore, gives a deformation of $\Sigma^{m+1}P_1$ into a $(m+k+1)$-skeleton of $V^j V_\pi S^{m+k+1} \cup_{\Sigma\rho P} C\Sigma^m X/F$, hence into a $(m+k+1)$-skeleton of $\Sigma^{m+1}(X/F_1)$. This dimension is actually one better than what we need.

Finally, applying the induction hypothesis for $k + 1$ to F_1, there is $F' \to F_1$ with $C_*^c F' = \begin{cases} C_*^c X, & * \geq k+1 \\ C_*^c F, & * \leq k \end{cases}$. Recall, however, that $C_k^c F_1 = C_k^c X$, and $C_*^k F_1 = C_*^c F$, $* < k$. Thus F' is the space needed for the lemma. ///

Before resuming the proof of 3.5 we give a few amusing corollaries of 3.7.

3.8 Corollary Suppose X is j-connected of dimension n, Y is k-connected of dimension m, and $f: X \longrightarrow Y$ is a map with fiber F. If $i, j \geq 1$ and $\min(2k, j+k+1) \geq m, [\frac{n-1}{2}]$, then the natural map $X/F \longrightarrow Y$ has a right inverse $\nu: Y \longrightarrow X/F$, and there is a cofibration

$$\Sigma^{-1}Y \longrightarrow Z \longrightarrow X \longrightarrow Y$$

iff the map $X \longrightarrow X/F$ stably factors through ν.

Specializing to $Y = S^m$, so $k = m - 1$, we get

3.9 Corollary Suppose X 1-connected of dimension n and $\rho: X \rightarrow S^m$ is a map, $m \geq \frac{n+1}{2}$. Then there is a cofibration sequence $S^{m-1} \rightarrow Y \rightarrow X \xrightarrow{\rho} S^m$ iff $\rho \wedge 1_{X_+}: X \longrightarrow \Sigma^m X_+$ stabily factors through $\rho\wedge *: S^m \longrightarrow \Sigma^m X_+$.

This last condition may be interpreted as a very strong way of saying that the element $\rho^*[S^m] \epsilon H^m(X)$ has $x \cap \rho^*[S^m] = 0$ for all $x \epsilon H_k(X)$, $k > m$. $\rho\wedge 1_{X_+}$ is the geometric product 1.5 which induces $\rho^*[S^m]$ on homology.

Proof of 3.8 According to Serre there is a filtration of a complex equivalent to C_*X with graded associated $C^c_*F \otimes C^c_*Y$. F is $\min(j,k-1)$-connected so below $(k+1) + \min(j+1,k)$ we have $C_*X \simeq C_*F \oplus_h C_*Y$, and $C_*X/F \simeq C_*Y$. Since $\dim Y = m \leq \min(j+k+1, 2k)$ this shows Y is an m-skeleton of X/F, and the retraction exists as asserted. The stable factoring of $X \longrightarrow X/F$ through this retraction is thus a deformation into an m-skeleton, so the extra cells, which are of dimension $\geq \min\{2k, j+k+1\} + 1 \geq [\frac{n+1}{2}]$ can be stripped off by lemma 3.7. This constructs Z. Since $2k \geq m$ Y desuspends to $\Sigma^{-1}Y$, and the map of this to Z is constructed using the relative Hurewicz theorem.///

Proof of 3.9 Let F be the fiber of $\rho: X \to S^m$, then $i: F \to X$ is (m-1) connected. Now the Wang sequence gives $X \to X/F \simeq \Sigma^m F_+ \xrightarrow{\Sigma^m i} \Sigma^m X_+$. $\Sigma^m i$ is (2m-1)-connected, so since $n \leq 2m-1$, or $m \geq \frac{n+1}{2}$, $X \to X/F$ deforms to an m-skeleton iff the composition $X \to \Sigma^m X_+$ does. This composition is $\rho \wedge 1_{X_+}$, and the inclusion $S^m \to \Sigma^m X_+$. Now this corollary follows from 3.8.///

Finally, for the last time, we return to the proof of 3.5. Just before the statement of lemma 3.7 we had verified all the hypotheses for its application. Thus starting with a representative $\rho: V^m V_\pi S^{j+k} \to D^{n+k}(X,Y)$ of $K_j^W(X)$, and dualizing to obtain $D\rho: (X,Y) \to V^m V_\pi S^{n-j}$, we apply 3.7 to extend this to a cofibration $V^m V_\pi S^{n-j-1} \xrightarrow{\alpha} (W,Y) \xrightarrow{f} (X,Y) \xrightarrow{D\rho} V^m V_\pi S^{n-j}$ f has the desired connectivity, so it only remains to see how Poincaré it is. (W,Y) is dominated (by [6] for example) so the lifting of the geometric product of (X,Y) defined above gives (W,Y) a normal structure with f a normal map. Take the dual of the cofibration and add the geometric products to get

$$\begin{tikzcd} V^m V_\pi S^{n+j} \arrow[r, "\rho"] \arrow[dr, "*"'] & D^{n+k}(X,Y) \arrow[r, "Df"] \arrow[d, "\cap[S]"] \arrow[dr, dashed] & D^{n+k}(W,Y) \arrow[r, "D\alpha"] \arrow[d, "\cap[S]"] & V^m V_\pi S^{k+j+1} \\ & T\xi^k & Tf^*\xi \arrow[l, "Tf^*"'] \end{tikzcd} .$$

Temporarily denote $H_{*+k+1}(\cap[S]\circ Df)$ by $\tilde{K}_*(X)$, then there is an exact sequence

$$---\to H_{*+1}(VS^j)\to \tilde{K}_*X \longrightarrow K_*W \to H_*(VS^j) \quad .$$

Thus $\tilde{K}_*X = K_*W$, $* \neq j,\ j-1$, and
$0 \to \tilde{K}_jX \to K_jW \quad Z[\pi]^m \to \tilde{K}_{j-1}X \to K_{j-1}W \to 0$ is exact. The segment

$$\begin{tikzcd}
D^{n+k}(X,Y) \arrow[d, "\cap[S]"'] \arrow[dr, "\cap[S]\circ Df"] & \\
T\xi^k & Tf^*\xi \arrow[l, "Tf^*"']
\end{tikzcd}$$

of the diagram also gives an exact sequence. By the Thom isomorphism $H_{*+k+1}(Tf^*) \cong H_{*+1}(f) = H_{*+1}(VS^{n-j})$, so this sequence is $--\to K_*X \to \tilde{K}_*X \to H_{*+1}(VS^{n-j}) \to K_{*-1}X \to$. Thus $K_*(X) = \tilde{K}_*(X)$, $* \neq n-j-1,\ n-j-2$, and
$0 \to K_{n-j-1}X \to \tilde{K}_{n-j-1}X \to Z[\pi]^m \to K_{n-j-2}X \to \tilde{K}_{n-j-2}X \to 0$ is exact. Therefore if $2j \leq n-3$, $\tilde{K}_*X = K_*X$ for $* \leq j$ and we get $K_*W \cong K_*X = 0$, $* \leq j-1$, and

$$Z[\pi]^m \xrightarrow{\rho_*} K_jX \longrightarrow K_jW \to 0$$

is exact. ρ was originally chosen so ρ_* onto, however, so $K_jW = 0$ as well, and (W,Y) is j-Poincaré.

This all started as an induction on j, supposing (X,Y) $(j-1)$-Poincaré. The construction of (W,Y) completes the induction step, and therefore the proof of 3.5. ///

The middle dimension

The proof of 3.1 can now be completed. This done by considering several cases and showing in each how to extend the proof of 3.5 to the middle dimension. There is a similarity in outline between these arguements and those in the manifold case [17 §4].

Suppose then that $(X;Y_0,Y_1;\xi)$ is a normal triad of dimension n, with $(Y_0,Y_0\cap Y_1)$ Poincaré and $\pi_1Y_1 \longrightarrow \pi_1X$ an isomorphism. By applying 3.5 we may also assume $(Y_1,Y_0\cap Y_1)$ is $[\frac{n}{2}]$ - 2-Poincaré, and $(X,Y_0\ Y_1)$ is $[\frac{n-1}{2}]$- 1-Poincaré.

All homology is with $Z[\pi]$ coefficients, $\pi = \pi_1X = \pi_1Y_1$.

Case 1; $n = 2k + 1 \geq 5$ In this case the kernels vanish except for the sequence $0 \to K_kY_1 \longrightarrow K_kX \longrightarrow K_k(X,Y_1) \longrightarrow K_{k-1}Y_1 \longrightarrow 0$. If $K_k(X,Y_1)$ can be killed without disturbing $K_{k-1}X = K_{k-2}Y_1 = 0$, the whole would vanish. The exact sequence shows the new $K_{k-1}Y_1$ vanishes, as its "dual" K_kY_1 does also, which forces $K_kX = 0$.

Now by [15] $K_k(X,Y_1)$ as a sole nonvanishing relative homology group is finitely generated stably free over $Z[\pi]$. Replacing $(X;Y_0,Y_1)$ by $(XVS^{k+1};Y_0,Y_1VS^{k+1})$ replaces $K_k(X_1Y_1)$ by $K_k(X,Y_1) \oplus Z[\pi]$. Thus we may assume it is free. It can also be based in such a way that killing this basis will make $\cap[S]: D^{n+j}(X,Y_0) \to T\xi^j/T(\xi|Y_1)$ a simple homotopy equivalence.

As in the proof of 3.5 (just after 3.6) this basis is represented by a map. $\alpha: \Sigma^j X/Y_0 \to V^m V_\pi S^{j+k+1}$, together with a nullhomotopy of $D\alpha\cap[S]$. By 2.6(c) since dim $X/Y_0 \leq 2k+1$ α desuspends to $\alpha: (X,Y_0) \to (V^m V_\pi S^{k+1},*)$. Let $F \to X \to VVS^{k+1}$ and $G \to Y_1 \to VVS^{k+1}$ denote the fibers over the basepoint. As in the proof of 3.5 we verify the hypotheses of 3.7 for F and G.

The first step is to pull the geometric products of X, Y_1 back to F,G. Refer to the appropriate diagram in the proof of 3.5. There is a cofibration sequence of such diagrams which somewhat abbreviated looks like

$$\begin{tikzcd}
 & D^{m+j}(T\xi/T\xi|Y_0\cup Y_1) \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & D^{m+j}(T\xi/T\xi|Y_0) \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & D^{m+j}(T\xi|Y_1/T\xi|Y_0\cap Y_1) \arrow[dl, dashed] \arrow[dd] & \\
\Sigma F \arrow[rr] \arrow[dr] & & \Sigma F/G \arrow[rr] \arrow[dr] & & \Sigma G \arrow[dr] & & \\
 & \Sigma^{m-2k-1}X_+ \arrow[rr] \arrow[dr] & & \Sigma^{m-2k-1}X/Y_1 \arrow[rr] \arrow[dr] & & \Sigma^{m-2k}Y_{1+} \arrow[dr] & \\
 & & V\Sigma F \arrow[rr] & & V\Sigma F/G \arrow[rr] & & V\Sigma G .
\end{tikzcd}$$

We wish to construct the dotted lifts, and as before it is sufficient to construct nullhomotopies of the compositions down and forward. The composition of the center projection with an m-connected map $(V\Sigma^{m-k}i_F/i_G)$ is trivial. Now however, since $\pi_1 X = \pi_1 Y$ dim $D^{m+j}(T\xi^S/T\xi|Y_0) \leq m - 2$ (note the connectivity is slightly better than needed). Thus the nullhomotopy can be pulled back to $V\Sigma$ F/G, showing the middle dotted lift exists. Similarly dim $D^{m+j}(T\xi^j/T\xi|Y_0 \cup Y_1) \leq m - 2$, so there is a compatible lift on the left. Finally the cofiber of these two provide a lift for Y_1.

The calculations of $C_*^c F$, $C_*^c G$ apply as before, so lemma 3.7 applies to remove the extra cells. This ends with $(W;Y_0,Z)$, $Y_0 \cap Z = Y_0 \cap Y_1$, and cofibrations

$$\begin{array}{ccccccc}
\cup^m \cup_\pi S^k & \longrightarrow & Z & \longrightarrow & Y_1 & \xrightarrow{\alpha} & \vee^m \vee_\pi S^{k+1} \\
\| & & \downarrow & & \downarrow & & \| \\
\cup^m \cup_\pi S^k & \longrightarrow & W & \longrightarrow & X & \xrightarrow{\alpha} & \vee^m \vee_\pi S^{k+1}
\end{array} .$$

It is easily seen directly (since $T\xi/T\xi|Y_1$ does not change) that there is an exact sequence $0 \rightarrow Z[\pi]^m \xrightarrow{D\alpha} K_k(X,Y_1) \rightarrow K_k(W,Z)$ Thus the geometric product $D(W,Y_0) \rightarrow T\xi^*/T(\xi|Z)$ is a simple homotopy equivalence. Reference to the exact sequences at the end of the proof of 3.5 show we have not disturbed the lower K, so all kernels vanish. Since $\pi_1 X = \pi_1 Y_1 = \pi_1 W = \pi_1 Z$ $(k \geq 2)$, this implies $(W;Y_0,Z)$ is a Poincaré triad as required.

<u>Case 2; $n = 2k + 2 > 6$</u> In this case both X and Y_1 are (k-1)-Poincaré, so the remaining kernels are

$0 \to K_{k+1}X \to K_{k+1}(X,Y) \to K_kY_1 \to K_kX \to K_k(X,Y_1) \to 0$.

The first step is to eradicate $K_k(X,Y_1)$, leaving the middle three terms of the sequence nonzero. $K_{k+1}(X,Y)$ is then killed to complete the surgery.

$K_k(X,Y_1)$ is finitely generated, so we represent generators as usual by a map $\alpha\colon (\tilde{X},\bar{Y}_0) \to \vee^m \vee_\pi S^{k+2}$, together with a nullhomotopy of $D\alpha \cap [S]$. This is exactly the situation treated in the odd dimensional case, except the connectivity is slightly better. Thus there is a new triad $(X';Y_0,Y_1')$, also (k-1)-Poincaré, and an exact sequence

$Z[\pi]^m \xrightarrow{D\alpha} K_k(X,Y_1) \to K_k(X',Y_1') \to 0$. Since $D\alpha$ was chosen to be surjective, $K_k(X',Y_1') = 0$.

Now we have $0 \to K_{k+1}(X,Y_1) \to K_kY_1 \to K_kX \to 0$ exact. As lone nonvanishing homology groups they are all finitely generated stably free. $K_{k+1}(X,Y_1)$ can be made free as above by replacing $(X \vee S^{k+2}; Y_0 Y_1 \vee S^{k+2})$. Choose a basis for $K_{k+1}(X,Y_1)$ which if killed will make the geometric product $D(X,Y_0) \to T\xi/T(\xi|Y_1)$ a simple homotopy equivalence. Represent this basis by a map $\alpha\colon (X,Y_0) \to \vee^m \vee_\pi S^{k+1}$. α desuspends this far because $\pi_1 Y = \pi_1 X$ implies $\dim X/Y_0 \leq 2k$.

Again we are in the situation considered in the odd dimensional case. This time the connectivity is slightly worse, but still enough for the arguement to work. This produces $(W;Y_0,Z)$ as above, with $D(W,Y_0) \to T\xi/T(\zeta|Z)$ a simple equivalence. The exact sequences show W and Z are still (k-1)-Poincaré, thus Poincaré, and the surgery is complete.

The reason the proof fails for $n < 5$ is fiberings over S^2 are required. This may change the fundamental group. It seems likely, however, that the result is true for all n. For example for $n = 2, 1$ it follows from Steenrod representability in those dimensions.

4. Poincaré surgery and the obstruction.

In this section we show that the obstruction to a map $(X^n,Y;\xi) \to (K,L)$ from a normal pair to a pair to be cobordant to a map from a Poincaré pair is exactly the nonsimplyconnected surgery obstruction $L_{n-1}(\pi_1 L \to \pi_1 K)$ defined in [17§9]. This gives the solution to geometric problems of Poincaré complexes in terms of "familiar" obstructions on the one hand (chapter 5), and on the other allows use of Poincaré complexes in the investigation of these obstructions (chapter 6).

Also in this section is a lemma on the "restoration" of Poincaré duality which allows a fairly vigerous sort of surgery on Poincaré complexes.

Suppose (K,L) is a cw pair with a homomorphism $\omega\colon \pi_1 K \to \mathbb{Z}/2$. ω induces an involution $\bar{\omega}$ on $Wh(\pi_1 L) \xrightarrow{i_*} Wh(\pi_1 K)$, (2.9). Suppose further that H is a pair of subgroups $H_1 \subseteq Wh(\pi_1 K)$, $H_2 \subseteq Wh(\pi_1 L)$ with $\bar{\omega}(H_j) = H_j$, and $i_*(H_2) \subseteq H_1$.

4.1 definition. Given (K,L,ω,H) as above, define $L_n^H(K,L)$ as bordism classes of maps $f\colon (X;Y_0,Y_1) \to (K;K,L)$, where $(X;Y_0,Y_1)$ is a normal triad of dimension $n+1$, $\omega_1(\xi) = \omega \circ \pi_1 f$ $(Y_0,Y_0\ Y_1)$ is Poincaré, and $\tau(Y_0) \in H_1$, $\tau(Y_0 \cap Y_1) \in H_2$.

Here bordism means via the same type of object one dimension higher.

Recall that Wall's definition [17,§9] is bordism classes of objects $(M;\partial_0 M,\partial_1 M) \xrightarrow{f} (N;\partial_0 N,\partial_1 N) \xrightarrow{g} (K;K,L)$ where M is a manifold of dimension n, N a simple Poincaré triad with

$\omega_1(\nu_N) = \omega\circ\pi_1 g$, f a degree 1 normal map, and $\partial_0 f$ a homotopy equivalence with Whitehead torsion in H. There is a map from this defintion to 4.1; the mapping cylinder of f is a normal space with the required properties, by 2.8. To show this induces an isomorphism of definitions for $n \geq 5$, a bordism is constructed from a normal space as in 4.1 to a mapping cylinder of a normal map. It is sufficient to consider the absolute case $L = \phi$.

Let $f\colon (X,Y) \to K$ be a map, X normal of dimension n+1 and Y Poincaré with torsion $\tau(Y)\in H$. Y has a smooth 2-skeleton; $Y \simeq A \cup_g B$ where A is a smooth n-manifold with boundary with the homotopy type of a 2-skeleton of Y, B is a Poincaré space with boundary, and $g\colon\partial B \to \partial A$ is a homotopy equivalence. This was announced in [16], and will be proved in section 5. Also since a bordism is harmless here we could start with a smooth 1-skeleton [16] and do surgery to reduce the fundamental group to that of Y, obtaining a bordism to a Poincaré space with such a decomposition.

B is made simple as follows; if $h\colon W \to \partial B$ is a homotopy equivalence, then W and (B_h,W) are Poincaré. A little calculation shows $\tau(B_h) = (-1)^{n-1}\overline{\tau(h)} + \tau(B)$. Let h have $\tau(h) = (-1)^n\overline{\tau(B)}$, then $\tau(B_h) = 0$, and the torsion is concentrated in $g\circ h\colon W \to \partial A$.

Next let $C \subseteq X$ be a homotopy 2-skeleton with $C \cap Y = A$. $(X;B,C)$ is a normal triad with $(B,B\cap C)$ Poincaré, and $\pi_1 C = \pi_1 X$.

By the surgery lemma 3.1 there is a normal map $h\colon (W;B,Z) \longrightarrow (X;B,C)$ with $(W;B,Z)$ Poincaré and $\tau(W,Z) = 0$. Since $\tau(B) = 0$, all the torsions are zero.

Since C and A are 2-skeletons the lift given by the smooth structure of A,

$$\begin{tikzcd} & B_O \arrow[d] \\ A \arrow[ur, "\nu_A"] \arrow[r, "\xi|A"'] & B_G \end{tikzcd}$$

extends to a lift of $\xi|C$. Now relative smooth transversality applied to the normal triad $(C;A,Z)$ gives a degree 1 normal map $(D;A,E) \longrightarrow (C;A,Z)$ from a manifold triad. In particular we have constructed a degree 1 normal map $(E,\partial A) \xrightarrow{j} (Z,\partial B)$ from a smooth manifold to a simple Poincaré space. The mapping cylinders of the two normal maps constructed above give a normal bordism from (X,Y) to $(W;B,Z) \cup_j (D;A,E)$. However Poincaré boundary is allowed in the bordisms in the definition of L, so amalgamating W and D into the bordism, we obtain a bordism of (X,Y) to the mapping cylinder of j.

If applied to something which is already a mapping cylinder this does not give back the same map. However using the surgery lemma for manifolds it is not hard to see it is L-bordant to the original.

This shows the maps from the Wall definition of L to 4.1 is surjective. The same argument with more notation gives a relative version which extends such a splitting on a boundary to the inside of a bordism. This shows the equivalence relations are also the same, and the two definitions are isomorphic. ///

The groups $L_n^H(K,L)$ only depend on the homomorphism $\pi_1 L \to \pi_1 K$ and $\omega: \pi_1 K \to Z/2$. They lack only excision to be a cohomology theory, and there is even a limited excision theorem. They are periodic period 4 above dimension 4 via cartesian product with CP^2. For an account of their properties see [17], or [11] for a treatment more relevant to the purposes here. Many of these properties can be proved directly using 3.1.

4.2 Corollary Suppose $(X;Y_0,Y_1;\xi)$ is a normal triad of dimension $n \geq 5$, or 4 if $Y_1 = \phi$, with Y_0 Poincaré. Then there is an obstruction $\sigma(X) \in L_{n-1}(X,Y_1)$ to finding a normal map $(W;Y_0,Z) \to (X;Y_0,Y_1)$ with W Poincaré. $\sigma(X)$ is an invariant of the bordism class of the map $(X;Y_0) \to (K(\pi_1 X,1),K(\pi_1 Y_0,1))$, and vanishes if ξ has a reduction to B_{TOP}.

Proof. The "bordism class of the map" is of course by bordisms of normal triads inducing Poincaré bordisms of Y_0. Also any L_n^H can be used where H contains the torsions of $(Y_0,Y_0 \cap Y_1)$, and then $(W;Y_0,Z)$ will have torsions in H.

The obstruction $\sigma(X)$ is the identity map $(X;Y_0,Y_1) \to (X;X,Y_1)$. The existance of the normal map and invariance are simple consequences of the definition and the surgery lemma. If ξ has a topological structure the desired normal map results from topological transversality if the dimension is not 4. In dimension 4 the obstruction to transversality is essentially the possible nonexistence of an almost parallizable toplogical manifold of index 8. There is such a Poincare space however, $(\#^8(CP^2))$ so there is Poincare transversality for topological bundles in dimension 4. ///

Next a form of Poincaré surgery below the middle dimension.

4.3 Proposition. Let (X,Y) be a Poincaré pair of dimension n. Suppose $(K;L_0,L)$ is a finite complex triad with a map $\phi\colon (L_0,L_0\cap L_1) \to (X,Y)$ so that $\phi^*\nu(X,Y)$ extends to a bundle ξ on K. If $\dim(K-L_0) \leq \frac{n}{2}$, $\dim(L_1-L_0) \leq \frac{n-1}{2}$, and $n \geq 5$ (4 if $L_1-L_0=\phi$), then there is a Poincaré 4-ad of the form $(X\cup_o K;X,Y\cup_o L_1,Z)$.

This formulation is essentially due to Browder, who observed that if $\phi\colon S^k \to X$ is an element on which one wants to do surgery, then X/S^k has the right homotopy type for the desired bordism. Thus what remains to be done is to find an appropriate "other end" to make it a Poincare triad. The content of 4.3 is to assert the existence of such an "other end", Z.

Proof. 4.3 follows from a double application of the $L_1-L_0 = \phi$ case. This in turn follows from the case $(K,L_0) = (D^{k+1},S^k)$ by induction on the skeleta of K.

Consider $\phi: S^k \to (X,Y)$ with $k \leq \frac{n}{2} -1$, and an extension of $\phi^*\nu(X,Y)$ over D^{k+1}. This gives an extension ζ of $\nu(X,Y)$ over X/S^k, giving it the structure of a normal space. $(X/S^k;X,X/S^k)$ is a normal triad of dimension n+1. The kernels of X/S^k are

$$K^{\omega}_{*}(X/S^k;Z[\pi]) = \begin{cases} 0, *\neq k, n-k-1 \\ \mathbb{Z}[\pi], *=k, n-k-1 \end{cases}, \text{ where } \pi = \pi_1(X/S^k).$$

$K^{\omega}_{n-k-1}(X/S^k)$ is generated by the dual of j. If we take the fibers of this particular map in the proof of 3.5, the resulting object will Poincaré, the desired Z (compare with 3.9). When $k = [\frac{n}{2}]-1$ the considerations of the proof of 3.1 are required to find a lift of the geometric product. The restriction $k \leq [\frac{n}{2}]-1$ is required so that k and n-k-1 are not equal or adjacent, and will not interact. The more delicate middle-dimensional situation will be considered in section 5.///

<u>4.4 Corollary</u>. If $(X;\xi)$ normal of dimension $n \geq 4$, π is a finitely presented quotient of $\pi_1 X$ through which $\omega_1(\xi)$ factors, and $\cap[X]: H^*(X;\mathbb{Z}[\pi]) \to H^{\omega}_{n-*}(X;\mathbb{Z}[\pi])$ is an isomorphism, then there is a normal bordism $(W;X,Y)$ with $\pi = \pi_1 W = \pi_1 Y$, which is algebraically Poincaré with $\mathbb{Z}[\pi]$ coefficients. Thus Y is Poincaré.

<u>Proof</u>. From the definition of normal, X is dominated, and so $\pi_1 X$ is finitely presented. Let $\{\alpha_i\}$ be normal generators for ker $(\pi_1 X \to \pi)$, and consider $\vee S^1 \xrightarrow{\vee\alpha_i} X$. By hypothesis $\{\alpha_i\} \subset \ker(\pi_1 X \to \pi) \subset \ker\, \omega_1(\xi)$, so $(\vee\alpha_i)^*\xi$ is trivial. Now

algebraically the properties of X/VS^1 are the same as required in 4.3, so the same considerations give Y Poincaré, with $(X/VS^1;X,Y)$ normal. ///

This is very similar to a theorem of Browder, who however has $\pi = \{1\}$, and uses algebraic properties of Z to replace the assumption that X is dominated by a finite complex. In fact he is often able to conclude that X is dominated. This hints at the existance of a general geometric theory of spaces satisfying duality with $\mathbb{Z}[\pi]$ coefficients, but except for $\pi = \{1\}$ (Browder) and $\pi = \pi_1(X)$ (sections 1,2 above) it is not clear what this will be.

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