70-23,630

QUINN,III, Frank Stringfellow, 1946-
A GEOMETRIC FORMULATION OF SURGERY.

Princeton University, Ph.D., 1970
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

70-23,630

QUINN,III, Frank Stringfellow, 1946-
A GEOMETRIC FORMULATION OF SURGERY.

Princeton University, Ph.D., 1970
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

for Barbara

Introduction

The object of this work is to give a treatment of surgery "geometric" in several senses. First in the sense of defining spaces of surgery maps, structures, etc, so the usual groups appear as homotopy groups and long exact sequences appear, as homotopy sequences of fibrations. Second, geometric in the sense that the geometry of manifolds and spaces is used as much as possible to prove the results, avoiding in particular the algebraic characterization of the obstruction groups by Wall (Surgery of Compact Manifolds). The first object is realized fairly completely and has considerable advantages for some types of applications (4.2). The attempt to remove the algebra is also successful, with the major exception of the properties of the Periodicity map of Wall.

The first chapter is concerned with set and homotopy theory, mainly definitions and fixing notation. Since many of the constructions are of the form "take the space of all manifolds," and the justification for such constructions is rather vague, the set theory is handled fairly carefully. This has several interesting consequences, for example a "universal disjoint union" operation. At the close in 1.4 a simple example is given. A space is defined whose homotopy groups are the unoriented bordism groups.

Next is a treatment of homology and duality, the definition of surgery maps. and the statement of the surgery lemma. The version of homology used is essentially Wall's homology of the universal cover with group ring coefficients. It is made into a coherent theory by defining it on the category of spaces with reference maps to a fixed space, and taking homology with respect to covers pulled back from the universal cover of the reference space. This allows a treatment of non-simply-connected duality essentially the same as that of Browder in Surgery on simply connected manifolds. Our treatment of Poincaré spaces is perhaps more elaborate than necessary, but was written when an approach making much stronger use of the properties of such spaces was planned. Technical difficulties arose in this approach, which was designed to give information about G/TOP, and was rendered unnecessary by the recent calculation of that space.

The first section of the third chapter contains the definition of a sequence of functors $\mathbb{L}_j^s(K,\partial K)$ and $\mathbb{L}_j^h(K,\partial K)$ for $j \in \mathbb{Z}$ and $(K,\partial K)$ a topological pair with an orientation homomorphism $\pi_1 K \longrightarrow \mathbb{Z}_2$. $\mathbb{L}_j(K,\partial K)$ is essentially the Δ-set (simplicial set) of surgery maps of dimension j over $(K,\partial K)$, $\mathbb{L}^h$ defined with homotopy equivalences and $\mathbb{L}^s$ with simple homotopy equivalences. $\mathbb{L}_j(K,\partial K)$ depends only on $\pi_1(\partial K) \longrightarrow \pi_1 K$, and $\mathbb{L}_j(K,\partial K) \simeq \Omega\, \mathbb{L}_{j-1}(K,\partial K)$ so $\mathbb{L}_j$ is an infinite loop space. $\pi_m \mathbb{L}_j(K,\partial K) = L_{j+m}(\pi_1 \partial K \longrightarrow \pi_1 K)$ is the obstruction group of Wall. Section 3.2 defines two special maps of $\mathbb{L}$ spaces, the pullback and the assembly. If $(M,\partial M)$ is a triangulable manifold, then the assembly is a map from the mapping space $\Delta(M,\partial M;\mathbb{L}_j(K,\partial K)) \longrightarrow \mathbb{L}_{j+m}(K\times M), \partial K\times M)$ obtained by taking the disjoint union (over common faces) of the images of the simplices of a map $(M,\partial M) \longrightarrow \mathbb{L}_j$. The pullback is defined for $M \longrightarrow E \longrightarrow B$ a block fibration with manifold fibers, and is a map $\mathbb{L}_j B \longrightarrow \mathbb{L}_{j+m} E$ which takes the pullback of the fibration over each surgery map of the first space. Wall's periodicity map is a special case of the pullback.

Section 3.3 introduces the notion of transversality for a surgery map, and shows it is equivalent to an embedding problem. The splitting theorem of Cappell is then applied to give a transversality theorem. This is used to give proofs of geometric analogues of the restricted excision and Meyer-Vietoris theorems of Cappell. Lastly transversality and the assembly map are used to calculate in terms of mapping spaces the $\mathbb{L}$ spaces for a class of groups defined by Waldhausen. If $\pi_1 M$ is a Waldhausen group, then $\Delta(M, \partial M; \mathbb{L}_j)(pt)) \longrightarrow \mathbb{L}_{j+m}(M)$ has a natural right inverse, and it is a homotopy equivalence if M is a $K(\pi, 1)$.

The last chapter contains the application of the $\mathbb{L}$ spaces to the structure of manifolds. Given a Poincaré space X of dimension m the space of manifold structures $S^h\ (X, \partial X)$ in a category $\mathcal{C}$ = TOP, PL, or DIFF homotopy equivalent to X are defined. S^s, the same space using simple homotopy equivalences is also defined. The usual exact sequence then appears as the homotopy sequence of a homotopy fibration,

$$S\ \ (X, \partial X) \longrightarrow \Delta(X, \partial X; G/\mathcal{C}\ ) \longrightarrow \mathbb{L}_m(X)\ .$$

A geometric proof, avoiding the algebra of Wall, is given. The results of 3.2 are then applied to this fibration in special cases.

We then close with section 4.2 which applies the theory to the

problem: given a map $f:M \longrightarrow N$ of closed manifolds with fiber the homotopy type of a finite complex, when is it homotopic to a block fibration over N? The answer (a rather formal one) is given in terms of mapping spaces and the fiber of the assembly map of 3.2. More complete solutions in special cases, and a number of corollaries including part of an assertion of Sullivan about topological bundles, are given. A few problems suggested by this work are also discussed.

This thesis was written under the direction of W. Browder, to whom I am indebted for many helpful suggestions and discussions. I would also like to express my gratitude to G. T. Whyburn and E. E. Floyd for their earlier direction and instruction, and to M. A. Kervaire for teaching me surgery. Thanks are due many friends for helpful discussions, about half the people listed in the Bibliography, and D. Sullivan and S. Cappell in particular. I am grateful to Miss Florence Armstrong for her fast and accurate typing of the manuscript.

Contents

1. Set and homotopy theory

1.1 Foundational remarks

Fix a set $\mathcal{S}$ with at least 2^I elements, where I is the unit interval, and an injective function $\mathcal{C} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$. Henceforth manifolds, topological spaces, and complexes will usually be understood to have underlying point sets contained in $\mathcal{S}$. Since they all have the order of I, all separable metric spaces or countable complexes are isomorphic to ones contained in $\mathcal{S}$. Moreover, the classes of things with this restriction will be sets, as desired for the constructions.

The set of subsets of $\mathcal{S}$ has operations "product" and "disjoint union" induced on it by $\mathcal{C}$ and a choice of two distinct elements $@, @' \in \mathcal{S}$: if $X, Y \subset \mathcal{C}\,\mathcal{S}$ then

$$X \pi Y = \mathcal{C}(X \times Y)$$

$$X \amalg Y = \begin{cases} \mathcal{C}(X \times \{ @ \} \cup Y \times \{@'\}) & \text{if } X \neq \varphi \neq Y \\ X \cup Y & \text{if one is empty.} \end{cases}$$

— can be elaborated to "union over a subspace"; if φ is a function from some subset of X to Y, then define $X \amalg_{\varphi} Y = (X \setminus \text{domain}(\varphi)) \amalg Y$. In case X and Y have topologies, manifold structures, etc., and φ is an appropriate function, then the sets $X \pi Y$, $X \amalg_{\varphi} Y$ are to be endowed with the induced topology, manifold structure, or whatever.

$\amalg$ and π also induce operations on functions between subsets of $\mathcal{S}$ via the canonical bijection with the categorical product and pushout of sets. Further properties of π and $\amalg$ will be investigated in section 1.3.

1.2. n-ads and maps

The notion of an n-ad enters strongly in the constructions to be made. Our use of the term is essentially that of Wall [27].

Let Δ^n denote the standard n-simplex with (ordered) vertices $\{V_o, \ldots, V_n\}$, and let $\partial_j\Delta^n$ be the base containing all but the j^{th} vertex. If $\alpha \subset \{0,1,\ldots,n\}$ define $\partial_\alpha \Delta^n = \Delta^n \cap (\bigcap_{j\in\alpha} \partial_j\Delta^n)$. Consider the collection of all $\partial_\alpha\Delta^n$ as a category, with morphisms the natural inclusions. If $\mathcal{C}$ is another category, then the category $\mathrm{Fun}(\Delta^n, \mathcal{C})$ with objects (covariant) functors $\Delta^n \longrightarrow \mathcal{C}$ and morphisms natural transformations of functors is defined. Fun $(\Delta^n, \mathcal{C})$ will be called the category of $\mathcal{C}$-(n+2)-lattices (see J. H. C. Whitehead and others in [ ], vol. IV, pp. 104-227).

There are numerous functors on categories of lattices formally induced from the structure of Δ^n. Some useful ones are given below (the notation varies from that of Wall [27]). Suppose $\alpha \subset \{0,\ldots,n\}$ and $F \in \mathrm{Fun}(\Delta^n, \mathcal{C})$.

1) The inclusion $\partial_\alpha\Delta^n \xrightarrow{\subset} \Delta^n$ induces ∂_α: Fun $(\Delta^n, \mathcal{C}) \longrightarrow$ Fun $(\Delta^{n-|\alpha|}, \mathcal{C})$. This corresponds to taking the face $\partial_\alpha F$ and all lower faces which map into it.

2) δ_α: Fun $(\Delta^n, \mathcal{C}) \longrightarrow$ Fun $(\Delta^{n-|\alpha|}, \mathcal{C})$ is induced by regarding $\partial_\alpha\Delta^n$ as an (n+2)-lattice with $\partial_\beta(\partial_\alpha\Delta^n) = \partial_{\alpha\cup\beta}\Delta^n$, and the inclusion as a map of (n+2)-lattices. Some of the faces coincide, however.

$(\partial_\beta(\partial_\alpha\Delta^n) = \partial_\gamma(\partial_\alpha\Delta^n)$ iff $\beta \cup \alpha = \gamma \cup \alpha)$ and $(n-|\alpha| + 2)$-ads are obtained after forgetting the duplications. δ_α corresponds to just omitting all faces $\partial_\beta F$ from F with $\beta \cap \alpha \neq \varphi$. The identity $\partial_\alpha\Delta^n \longrightarrow \partial_\alpha\Delta^n$ induces a natural transformation of functors $\partial_\alpha \longrightarrow \delta_\alpha$ (is a morphism of $\mathcal{C}$ $(n - |\alpha| + 2)$-lattices $\partial_\alpha F \longrightarrow \delta_\alpha F$).

3) If for every $\beta \subset \{0,\ldots,n-2\}$ the pullback P_β of the morphisms $F(\partial_{\beta \cup (n-1)} \Delta^n \longrightarrow \partial_\beta \Delta^n)$ and $F(\partial_{\beta \cup (n)}\Delta^n \longrightarrow \partial_\beta \Delta^n)$ exists, then we can define $G \in \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ by $\partial_\beta G = \partial_\beta F$, $\partial_{\beta \cup (n-1)} G = P_\beta$, with β as above, and the naturally induced morphisms. This operation corresponds to combining the last two faces of F.

Naturally if $\mathcal{C}$ has products or coproducts, etc, the corresponding categories of lattices will have similar structures (the product of an n-lattice and an m-lattice will be an $(n+m-1)$-lattice, etc.). In general the same symbol used in $\mathcal{C}$ will be used for such operations, and the precise meaning will depend on the context.

Lattices are a little too general to be useful when working with manifolds, so we add a few restrictions to get n-ads. A more general definition is possible, but the category theory involved is unenlightening. Suppose $\mathcal{C}$ has a (forgetful) functor to the category of sets whose values we won't distinguish from the original, an operation ∂ on the objects of $\mathcal{C}$ such that $\partial\partial M = \varphi$, and a notion of subobjects which are subsets. In this case a <u>$\mathcal{C}$ n-ad</u> is a $\mathcal{C}$ n-lattice with each $\partial_\alpha M \longrightarrow M$ the inclusion of a subobject, $\partial_{\alpha\cup\beta} M = \partial_\alpha M \cap \partial_\beta M$ for $\alpha \cup \beta \subset \{0,\ldots n-2\}$,

$\partial M = \bigcup_j \partial_j M$, and each $\partial_j M$ a $\mathcal{C}$ (n-1)-ad. The category of n-ads is the corresponding full subcategory of $\mathcal{C}$ -n-lattices.

Examples are categories of manifolds; topological, piecewise linear, and smooth. In the smooth case corners on the boundary must be allowed so the boundaries of an n-ad will fit together. A manifold with corners is one with charts diffeomorphic to open sets in sets of the form $\{X \in \mathbb{R}^m \mid \lambda_j(x) \geq 0,\ j = 1, \ldots, k$, where $\lambda_j : \mathbb{R}^m \longrightarrow \mathbb{R}$ is linear$\}$. Nothing new is introduced by this, however, since the classical "straightening the angle" trick (Cerf [6]) shows that such a thing has a unique natural differentiable structure.

Another example is $\mathcal{C}$ = topological or CW pairs with $\partial(X,Y) = (Y, \varphi)$ and objects subspaces. A topological or CW n-lattice, however, is homotopy equivalent to an n-ad by a generalization of the mapping cylinder construction.

1.3. Δ-objects

In 1.2 objects over a category were defined which have internal structure similar to that of Δ^n. Now objects with the external structure of the collection of all Δ^n will be introduced. The theory of Δ-objects is due to Rourke and Sanderson [15], and is essentially the same as the theory of simplicial objects (May [10]).

Let Δ be the category with objects Δ^n, $n \geq 0$, and morphisms ∂^n_α (here $\alpha \subseteq \{0, \ldots, n\}$ and we regard $\partial^n_\alpha : \Delta^{n-|\alpha|} \xrightarrow{\subset} \Delta^n$ as a map). If $\mathcal{C}$ is a category, then $\mathrm{Fun}^-(\Delta, \mathcal{C})$, the category of contravariant functors $\Delta \longrightarrow \mathcal{C}$, is called the category of $\mathcal{C}$ Δ-objects. When $\mathcal{C}$ is a category of things, we will also call an object of $\mathrm{Fun}^-(\Delta, \mathcal{C})$ a Δ-thing (Δ-set, Δ-group, etc). Our primary interest is in Δ-sets, often with some additional structure. If X is a Δ-set, the elements of $X(\Delta^n)$ are called the n-simplices of X.

In the category of Δ-sets, mapping "spaces" are defined. Let X and Y be Δ-sets; then $\Delta(X,Y)$ is the Δ-set defined by $\Delta(X,Y)(\Delta^n) = \{\Delta\text{-maps } Y \times \Delta^n \longrightarrow X\}$, with face maps given by restriction. For our purposes $Y \times \Delta^n$ denotes the categorical product, although the general theory seems to require something more elaborate. If P is a polyhedron, then $\Delta(P,X)$ is the Δ-set with n-simplices the set {some triangulation of $P \times \Delta^n$ in which the faces $P \times \partial_\alpha \Delta^n$ are subcomplexes together with a Δ-map of this triangulation into X}. Again face maps are defined by restriction. Mapping spaces of Δ-set and polyhedral lattices and n-ads are defined in the straightforward manner.

The set of path components of a Δ-set X is the set $X(\Delta^o)$ divided by the relation $a \sim b$ iff there is a map $\varphi: I \longrightarrow X$ (I is here regarded as a polyhedron, distinct from Δ') with $\varphi(0) = a$ and $\varphi(1) = b$. Similarly the path components $\pi_o(X,Y)$ of a Δ-set pair (X,Y) is the set of path components of X not intersecting those of Y. Now if $* \in Y$ we can define the loop space $\Omega^n(X,Y,*) = \Delta(I^n, \partial I^n, *; X, Y, *)$ of a pair and the relative homotopy sets $\pi_n(X\ Y,*) = \pi(\Omega^n(X,Y,*))$.

A simplicial object (May [10]) is a Δ-object by forgetting the degeneracy operators. All the homotopy theory of simplicial sets is valid also for Δ-sets which satisfy an extension condition. If Λ^n_j is $\cup_{k \neq j} \partial_k \Delta^n$, then a Δ-set $X: \Delta \to \langle \text{sets} \rangle$ satisfies the extension condition, or is said to be <u>Kan</u>, if each Δ-map $\Lambda^n_j \to X$ extends over Δ^n, for any $n \geq j$. For example the Whitehead theorem that a map is a homotopy equivalence iff it induces isomorphisms of all homotopy groups is valid for Kan Δ-sets.

Unfortunately many of the Δ-sets we will want to define do not at first satisfy the extension condition. To remedy this we will routinely include in our constructions an application of the functor Ex^∞ (Kan [7]), which essentially adds simplices until the extension condition is satisfied. $Ex' X = \Delta(*, X)$, $Ex^n X = Ex'(Ex^{n-1}(X))$, and $Ex^\infty X$ is the direct limit of the natural inclusions $\ldots \subset Ex^n X \subset Ex^{n+1} X \subset \ldots$. Ex^∞ preserves homotopy and many other things, and since a review of the definitions reveal that a Δ-map $\Lambda^n_j \to Ex^m X$ extends to a map $\Delta^n \to Ex^{m+1} X$, $Ex^\infty X$ satisfies the extension condition.

One important consequence of the Kan condition is that it implies a stronger extension condition: if $K \supset L$ are Δ-sets, $|L|$ is a retract of $|K|$ (i.e. there is a continuous map of geometric realizations $|K| \to |L|$ which is the identity on $|L|$), and $L \to X$ is a Δ-map, then it extends to a Δ-map $K \to X$. This is the key lemma in proving the Whitehead theorem. Our Δ-sets will satisfy a "slow countable extension condition," namely that if K is as above and <u>countable</u>, then any Δ-map $L \to X$ is homotopic to one which extends to all of K. When X satisfies this condition the inclusion $X \subset Ex^{\infty}(X)$ induces isomorphisms of homotopy groups. Thus the homotopy type of $Ex^{\infty}(X)$ is uniquely specified by X, and for purposes of investigating homotopy groups we can use those of X.

1.4. An example

We now take up an example which embodies many of the major points of later constructions. Let $\mathcal{C}$ be a category of compact manifolds (all contained in the set $\mathcal{S}$ of 1.1 and hence a small category), or more generally a small cobordism category in the sense of Strong [17]. Define the Δ-set $\hat{\Omega}_k^{\mathcal{C}}$, for $k \in \mathbb{Z}$, by $\hat{\Omega}_k^{\mathcal{C}}(\Delta^n)$ is the set of $\mathcal{C}$ (n+2)-ads of dimension $n+k$, and face maps are induced by taking faces ∂_α of the objects.

Here a deviation from the theory of n-ads must be introduced. Note that although the intersection of all the faces of Δ^n is empty, we have not assumed this for (n+2)-ads. Since for example in $\hat{\Omega}_k^{\mathcal{C}}$ we want closed manifolds as vertices of the simplices, this assumption must be added. In general when forming a Δ-set with simplices n-ads of some sort, we require that the faces of the object which form its vertices as a simplex are all disjoint.

Now set $\Omega_k^{\mathcal{C}} = Ex^{\infty}(\hat{\Omega}_k^{\mathcal{C}})$.

1.4.1. Proposition

1) $\Omega_k^{\mathcal{C}}$ is a Δ-monoid with operation induced by $\amalg$, with identity the empty manifold. It is homotopy associative and commutative, and has a homotopy inverse.

2) There is a natural map of Δ-monoids $\Omega_k^{\mathcal{S}} \longrightarrow \Omega(\Omega_{k-1}^{\mathcal{S}}, *)$ which is a homotopy equivalence.

3) The homotopy groups $\pi_j(\Omega_k^{\mathcal{S}}, *)$ are the (unoriented) bordism groups $\mathfrak{N}_{j+k}(\mathcal{S})$.

A few comments before an indication of the proof. $\Omega_k^{\mathcal{S}}$ has another monoid structure induced by π, which induces the standard bordism ring structure in the homotopy groups. This is omitted since we will not investigate products in our other constructions. Part 2) implies that $\Omega_k^{\mathcal{S}}$ is an infinite loop space, and in fact since an n-ad of negative dimension is empty, $\Omega_{k-1}^{\mathcal{S}}$ is a classifying space for $\Omega_k^{\mathcal{S}}$, $k \leq 0$. Finally 3) indicates the usefulness of the construction. Most sequences f groups arising in topology can be realized as homotopy groups of some aturally defined Δ-set, and exact sequences usually arise as the homopy sequence of fibrations of these sets. The spaces, however, contain uch more information (k-invariants) than the groups, and a formulation ing spaces often gives stronger results than the corresponding groups. is will be our approach to surgery.

oof. $\amalg$ induces an operation on $\hat{\Omega}_k^{\mathcal{S}}(\Delta^n)$, all n, with $\emptyset$ as ntity element by definition, and which commutes with boundary operators ce the faces of an n-ad are subsets. From the definition of Ex^∞, $\Omega_k^{\mathcal{S}}$ arly has the same structure. The homotopy inverse is, as usual with riented bordism, the identity map. The homotopy statements are erred for a while.

Next $\hat{\Omega}_k^{\mathcal{S}}$ will be shown to satisfy the slow countable extension condition. Suppose $\Lambda^n_j \longrightarrow \hat{\Omega}_k^{\mathcal{S}}$ is a Δ-map then each in $(\partial_\ell \Lambda^n_j)$ is a manifold (n+1)-ad. If they were disjoint except for the necessarily common faces then we could take any set with I points in the complement in $\mathcal{S}$ of the image and give it the structure of $M \times [0,1)$, where M is obtained by gluing the faces together over the common subfaces. Now $M \times [0,1)$ union the original set has an obvious compact manifold (n+2)-ad structure, which is our desired extension $\Delta^n \longrightarrow \hat{\Omega}_k^{\mathcal{S}}$. Thus extension problems arise from non-disjointness problems.

In fact the above argument shows that if K is a Δ-set with strictly fewer points than $\mathcal{S}$, $K \supset L$ with $|K|$ a retract of $|K|$, and a Δ-map $L \longrightarrow \Omega_k^{\mathcal{S}}$ which is nonsingular (i.e. has all image manifolds mutually disjoint except for common faces), then the map extends to a Δ-map $K \longrightarrow \hat{\Omega}_k^{\mathcal{S}}$. The restriction on the size of K insures that im(L) does not fill up $\mathcal{S}$, and there is always room to choose a set (or a set for each point of K) with I points disjoint from it all to use in defining an extension.

To apply this, we must make a Δ-map $f: L \longrightarrow \hat{\Omega}_k^{\mathcal{S}}$ nonsingular. Take an injection of sets $i: L \longrightarrow \mathcal{S}$, and define $g: L \longrightarrow \hat{\Omega}_k^{\mathcal{S}}$ inductively on the skeleta of L by: if σ is an n-simplex of L, $g(\sigma) = \not{c}(\mathrm{int}(f(\sigma)), i(\sigma)) \cup (\cup_j g(\partial))$, with the canonically induced manifold structure. Since i is an injection, g is nonsingular. It is also homotopic to f by a remark we will state as a lemma, since it will be used again.

1.4.2. Lemma: Suppose $f, g : L \longrightarrow \hat{\Omega}_k^{\mathfrak{S}}$ are Δ-maps with $f(\sigma^n)$ isomorphic to $g(\sigma)$ as $\mathfrak{S}$ (n+2)-ads for each simplex $\sigma^n \in L$. If L has strictly fewer points than $\mathcal{S}$, f and g are homotopic.

Proof: Using the restriction on the size of L make f nondegenerate and disjoint from the images of both f and g in $\mathcal{S}$. The resulting map $\bar{f}$ is homotopic to both f and g since we can choose sets disjoint from everything with the structure of $\operatorname{int}(f(\sigma) \times I)$ for all $\sigma \in L$, then glue the images of $\bar{f}(\sigma)$ on one end, $f(\sigma)$ or $g(\sigma)$ on the other using the isomorphism hypothesis, and inductively defined lower faces along the edges to get a map of $L \times \Delta' \longrightarrow \hat{\Omega}_k^{\mathfrak{S}}$. □

The slow countable extension now follows, since a countable set is much smaller than $\mathcal{S}$. In fact this shows the slow extension condition is satisfied with respect to Δ-sets with fewer points than $\mathcal{S}$.

These considerations also suggest how to prove the homotopy statements in part 1) of the proposition. The commutivity statement, for example, would follow from homotopy of two maps $\hat{\Omega}_k^{\mathfrak{S}} \times \hat{\Omega}_k^{\mathfrak{S}} \longrightarrow \hat{\Omega}_k^{\mathfrak{S}}$ whose images on each simplex are isomorphic. $\hat{\Omega}_k^{\mathfrak{S}} \times \hat{\Omega}_k^{\mathfrak{S}}$ is far too big, however, being closer to the order of $2^{\mathcal{S}}$. To avoid this we do it all over again with a set $\underline{\mathcal{S}}$ which is bigger than $\hat{\Omega}_k^{\mathfrak{S}}$, and an injection $\mathcal{S} \longrightarrow \underline{\mathcal{S}}$ which commutes with the product maps $\mathcal{C}$ and $\underline{\mathcal{C}}$. From this we define a new Δ-set $\underline{\hat{\Omega}}_k^{\mathfrak{S}}$, much bigger than the old one, with a canonical inclusion $\hat{\Omega}_k^{\mathfrak{S}} \longrightarrow \underline{\hat{\Omega}}_k^{\mathfrak{S}}$. Lemma 1.4.2 now applies to the composition of the two maps with the inclusion, giving a homotopy in $\underline{\hat{\Omega}}_k^{\mathfrak{S}}$. A countable Δ-set can always be deformed from $\underline{\hat{\Omega}}^{\varepsilon}_{k}$ into $\hat{\Omega}^{\varepsilon}_{k}$, so the inclusion induces isomorphism of homotopy groups, and passes to a homotopy equivalence after application of Ex^{∞}. Application of a homotopy inverse gives the desired homotopies in Ω^{ε}_{k}, completing the proof of statement 1) of Proposition 1.4.1.

The following strengthened version of Lemma 1.3.2 is also obtained: two maps $L \longrightarrow \hat{\Omega}^{\varepsilon}_{k}$ with isomorphic images of each simplex are homotopic as maps $L \longrightarrow \Omega^{\varepsilon}_{k}$.

The map in 2) is an optical illusion; take a simplex of $\hat{\Omega}^{\varepsilon}_{k}$ and look at it differently and it becomes one of $\Omega(\hat{\Omega}^{\varepsilon}_{k-1}, *)$. An n-simplex of $\hat{\Omega}^{\varepsilon}_{k}$ is a $\mathcal{E}$ (n+k)-ad of dimension $n+k$, which is an $(n+4)$-ad with $\partial_{n+1}M = \partial_{n+2}M = 0$, which in turn is the image of a Δ-map $\varphi: \Delta^{n} \times I \longrightarrow \hat{\Omega}^{\varepsilon}_{k-1}$ with the product triangulation on $\Delta^{n} \times I$ and $\varphi(\Delta^{n} \times \{0\} = \varphi(\Delta^{n} \times \{1\} = \emptyset$. This gives an inclusion $\hat{\Omega}^{\varepsilon}_{k} \xrightarrow{\subset} \Omega(\hat{\Omega}^{\varepsilon}_{k-1}, *)$ which passes to the desired map on application of Ex^{∞}.

Note, however, that the reverse of this procedure shows that any map in $\Omega(\hat{\Omega}^{\varepsilon}_{k-1}, *)$ defined on the product triangulation of $\Delta^{n} \times I$ lies in the image of $\hat{\Omega}^{\varepsilon}_{k}$. Thus an application of disjointness and gluing on finite complexes produces an inverse to the map on homotopy groups, so the final map $\Omega^{\varepsilon}_{k} \longrightarrow \Omega(\Omega^{\varepsilon}_{k-1}, *)$ is a homotopy equivalence as asserted in 2).

Finally, we evaluate the homotopy groups. By the slow extension condition we can use $\hat{\Omega}^{\varepsilon}_{k}$, and by making nonsingular and gluing we can represent homotopy elements by Δ-maps $\Delta^{n} \longrightarrow (\hat{\Omega}^{\varepsilon}_{k}, \emptyset)$. This is an (n+1)-ad with all faces empty; hence a closed manifold of dimension n+k. A homotopy between two such similarly becomes a compact manifold with the two original ones as boundary. This is the definition of the bordism group $\Omega_{n+k}(\xi)$, as asserted in statement 3). The proof of 1.4.1 is complete. □

This example has been treated in some detail since similar considerations are often skipped over in the literature. This kind of construction is central to our approach, however, as it seemed desirable to include a treatment. Propositions similar to 1.4.1 will be stated without proof in later chapters, since the proofs are essentially the same.

2. Poincaré complexes

2.1 Homology and duality

In this section the versions of (singular) homology and cohomology to be used are defined, and extensions of standard Poincaré duality lemmas are stated in the context. The format of Browder [2] is followed fairly closely, with allowances for the fundamental group.

Suppose (X, Y) is a connected and locally 1-connected topological pair, and $\omega : \pi_1 X \longrightarrow \mathbb{Z}_2 = \{-1, 1\}$ is a homomorphism. ω defines an anti-involution on $\mathbb{Z}(\pi_1 X)$ by

$$\overline{\Sigma n_i g_i} = \Sigma\, \omega(g_i)\, n_i g_i^{-1} .$$

Let $(\widetilde{X}, \overline{Y})$ denote the universal cover of X and the corresponding cover of Y, then the singular chain complex $C_*(\widetilde{X}, \overline{Y})$ has a natural right $\Lambda = \mathbb{Z}(\pi_1 X)$-complex structure induced by the covering transformations. If B is a right Λ-module, define

$$H^*(X, Y; B) = H(\hom_\Lambda (C_*(\widetilde{X}, \overline{Y}); B))$$
$$H^t_*(X, Y; B) = H(C_*(\widetilde{X}, \overline{Y}) \otimes_\Lambda B) .$$

The t on H^t_* signifies the use of the left module structure $\lambda b = b\overline{\lambda}$ on B in forming the tensor product, a notation introduced by Wall.

To obtain a coherent theory, it is often necessary to use covers other than the universal one.

<u>2.1.1. Lemma</u>: Suppose ω factors $\pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$, that B is a right $\Gamma = \mathbb{Z}(G)$ module, and that $(\ddot{X}, \ddot{Y})$ is the cover of (X, Y) corre-

sponding to G , then there are natural isomorphisms of right Γ-modules:

$$C_*(\widetilde{X}, \overline{Y}) \otimes_\Lambda B \cong C_*(\ddot{X}, \ddot{Y}) \otimes_\Gamma B$$

$$\hom_\Lambda(C_*(\widetilde{X}, \widetilde{Y}), B) \cong \hom_\Gamma(C_*(\ddot{X}, \ddot{Y}), B) .$$

Proof: First note that $C_*(\widetilde{X}, \overline{Y}) \otimes_\Lambda \Gamma = C_*(\ddot{X}, \ddot{Y})$, which is clear from the definition of $\otimes_\Lambda$ and properties of covering spaces. The top isomorphism now follows from associativity of the tensor product, while the second follows from adjointness of hom and $\otimes$. $\square$

From this we see, for example, that if (X,Y) is a pair, $\Pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$ a homomorphism, and B a right $\mathbb{Z}(G)$ module, then the exact homology and cohomology sequences for the pair $(\ddot{X}, \ddot{Y})$ give exact sequences

$$\ldots \longrightarrow H_n^t(Y;B) \longrightarrow H_n^t(X;B) \longrightarrow H_n^t(X,Y;B) \longrightarrow H_{n-1}^t(Y;B) \longrightarrow$$

$$\ldots \longrightarrow H^n(X,Y;B) \longrightarrow H^n(X,B) \longrightarrow H^n(Y;B) \longrightarrow H^{n-1}(X\ Y;B) \longrightarrow .$$

In $H_n^t(Y;B)$, the homomorphism $\Pi_1 Y \longrightarrow \mathbb{Z}_2$ is the composition $\Pi_1 Y \longrightarrow \Pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$.

The next definition generalizes and combines most of the good features of previous ones (Wall [25], [27], Browder [2], Spivak [18]). Let (X,Y) be a connected and locally 1-connected topological pair with the homotopy type of a finite CW pair, with a homomorphism $\Pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$, and a right $\mathbb{Z}(G)$-module B .

2.1.2. Definition. (X,Y) is a ("compact") B-Poincaré pair of dimension is if there is a fundamental class

$$[X,Y] \in H_n^t(X,Y;\mathbb{Z}) \quad \text{such that the cap product}$$

$$[X,Y]\cap : H^{\ell}(X;B) \longrightarrow H^t_{n-\ell}(X,\hat{Y};B)$$

$$[X,Y]\cap : H^{\ell}(X,Y;B) \longrightarrow H^t_{n-\ell}(X;B)$$

are isomorphisms, all ℓ. Further, (X,Y) is G-simple if it is a $\mathbb{Z}(G)$-Poincaré pair, and the cap product on the cellular chains of the covers (as finite complexes) induced by cap product with $\xi \in [X,Y]$

$$\hom(C^c_*(\ddot{X}),\ \mathbb{Z}G) \xrightarrow{\xi\cap} C^c_{n-*}(\ddot{X},\ddot{Y})$$

$$\hom(C^c_*(\ddot{X},\ddot{Y}),\ \mathbb{Z}G) \xrightarrow{\xi\cap} C^c_{n-*}(\ddot{X})$$

are simple equivalences of finitely generated free based $\mathbb{Z}(G)$ complexes. (X,Y) is called simple if it is $\Pi_1 X$ simple.

Standard arguments show that if $\Pi_1 X \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \mathbb{Z}_2$ and (X,Y) is a G_1 (simple) Poincaré pair, then it is a G_2 (simple) Poincaré pair.

The advantage of this coherent introduction of the fundamental group is that the exact sequence arguments of the simply-connected case apply with little change. An example is the structure of the boundary.

2.1.3. Lemma: If $\pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$ is a homomorphism, B a right $\mathbb{Z}G$ module, and (X,Y) a (G-simple) B-Poincaré pair of dimension n, then $(Y,\emptyset)$ is a (G-simple) B-Poincaré pair of dimension $n-1$, with fundamental class $\partial[X,Y]$.

Proof: The duality statement follows from application of the 5-lemma to a map of exact sequences, which by Theorem I.1.5 of [2] is commutative up to sign;

$$\begin{array}{ccccccc} \longrightarrow H^q(X,Y;B) & \longrightarrow & H^q(X;B) & \longrightarrow & H^q(Y;B) & \longrightarrow \\ \downarrow \cap[X,Y] & & \downarrow \cap[X,Y] & & \downarrow \cap\partial[X,Y] & \\ \longrightarrow T^t_{n-q}(X;B) & \longrightarrow & H^t_{n-q}(X,Y;B) & \longrightarrow & H^t_{n-q-1}(Y;B) & \longrightarrow \end{array}$$

The simplicity statements follow from the sum theorem for Whitehead torsion applied to the sequence of chain complexes

$$O \longrightarrow C_*(\ddot{Y}) \longrightarrow C_*(\ddot{X}) \longrightarrow C_*(\ddot{X}, \ddot{Y}) \longrightarrow O .$$

A Poincaré n-ad is essentially an n-ad with the homological structure of a manifold n-ad, as defined in 1.2. However, we will need to be able to use different reference groups for the different faces. For convenience in the next chapter, this is done using the fundamental group of another n-ad. Thus suppose K is an n-ad, with an orientation homomorphism $\pi_1 K \longrightarrow \mathbb{Z}_2$.

2.1.4. Definition: An n-ad X with a map $f:X \longrightarrow K$ is a (simple) K-Poincaré n-ad if, for n=1 $(X, \emptyset)$ is a (simple) $\pi_1 K$ Poincaré pair and for $n > 1$ $(X, \cup_j \partial_j X)$ is a (simple) $\pi_1 K$ Poincaré pair with orientation class V, and each $\partial_j X$ is a (simple) $\partial_j K$ Poincaré (n-1)-ad with respect to fundamental classes V_j such that $\partial V = \Sigma(-1)^j V_j$.

In case $\partial_j K = K$ all j, then such an X is called a $\pi_1 K = G$ (simple) Poincaré n-ad.

Now we can state the alterating duality theorem for triads.

2.1.5. Proposition: If $(X;\partial_0 X, \partial_1 X)$ is a triad dominated by a finite CW triad, $\pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$, and B a right $\mathbb{Z}G$ module, then the following are equivalent.

1) $(X;\partial_0 X, \partial, X)$ is a (G-simple) B-Poincaré triad with respect to $V \in H_n^t(X, \partial_0 X \cup \partial_1 X; \mathbb{Z})$

2) $(\partial_0 X, \partial_{\{0,1\}} X)$ is a (G-simple) B-Poincaré pair with respect to ∂V, and the homomorphisms

$$V \cap : H^q(X, \partial_1 X;B) \longrightarrow H^t_{n-q}(X, \partial_0 X;B)$$

$$V \cap : H^q(X, \partial_0 X;B) \longrightarrow H^t_{n-q}(X, \partial_1 X;B)$$

are isomorphisms for all q (are induced by a simple equivalence of $\mathbb{Z}G$ chain complexes).

Proof: The 5-lemma applied to

$$\begin{array}{ccccccc}
\longrightarrow H^q(X, \partial_0 X; B) & \longrightarrow & H^q(X;B) & \longrightarrow & H^{q-1}(\partial_0 X;B) \longrightarrow \\
\downarrow V\cap & & \downarrow V\cap & & \downarrow \partial_0 V \cap \\
\longrightarrow H^t_{n-q}(X, \partial_1 X;B) & \longrightarrow & H^t_{n-q}(X, \partial_0 X \cup \partial_1 X;B) & \longrightarrow & H^t_{n-q-1}(\partial_0 X \cup \partial_1 X; \partial_1 X;B) \longrightarrow \\
& & & & H^t_{n-q-1}(\partial_0 X, \partial_{\{0,1\}} X;B)
\end{array}$$

and

$$\begin{array}{ccccccc}
\longrightarrow H^q(X, \partial_1 X;B) & \longrightarrow & H^q(X, \partial_0 X \vee \partial_1 X;B) & \longrightarrow & H^{q-1}(\partial_0 X, \partial_{\{0,1\}} X;B) \longrightarrow \\
\downarrow V\cap & & \downarrow V\cap & & \downarrow \partial_0 V \cap \\
\longrightarrow H^t_{n-q}(X, \partial_0 X;B) & \longrightarrow & H^t_{n-q}(X;B) & \longrightarrow & H^t_{n-q-1}(\partial_0 X;B) \longrightarrow
\end{array}$$

which according to Browder [2] commute up to sign.

There is also the "sum theorem," with statement essentially identical to Theorem I.3.2 of [2]. Suppose $(X_1, Y_1) \cup (X_2, Y_2) = (X, Y)$, $(X_1, Y_1) \cap (X_2, Y_2) = (A, B)$, $\pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$, B a right $\mathbb{Z}G$ module, and $V \in H^t_n(X, Y; \mathbb{Z})$.

<u>2.1.6. Proposition.</u> Any two of the following imply the third.

1) (X, Y) is a (G-simple) B-Poincaré pair with respect to V.

2) (A, C) is a (G-simple) B-Poincaré pair with respect to $\partial_0 X$.

3) $(X_i, Y_i \cup A)$ are (G-simple) B-Poincaré pairs with respect to $j_\wedge V$ for $i = 1, 2$, and j_i the inclusion $(X_i, Y_i \cup A) \subset (X, Y_i \cup X_{i\pm 1})$.

<u>Proof:</u> Again the 5-lemma applied to the sign commutative diagram (Coefficient B)

$$\begin{array}{ccccc} \to H^{q-1}(C) & \longrightarrow & H^q(X) & \longrightarrow & H^q(X_1) \oplus H^q(X_2) \to \\ \downarrow \partial_0 V \cap & & \downarrow V \cap & & \downarrow j_1 V \cap \oplus j_2 V \cap \\ \to H^t_{n-q}(A, C) & \to & H^t_{n-q}(X, Y) & \to & H^t_{n-q}(X_1, Y_1 \cup A) \oplus H^t_{n-q}(X_2, Y_2 \cup A). \end{array}$$

The simplicity statement is from the sum theorem for Whitehead torsion applied to the corresponding short exact sequence of chain complexes.

The situation of 2.1.6 is variously called a <u>splitting</u> of (X, Y), or a <u>codimension O embedding</u> of $(X_1; Y_1, X_1 \cap X_2)$ (or $(X_2; Y_2, X_1 \cap X_2)$) in (X, Y). More generally a <u>splitting</u> of a pair (X, Y) is a (G-simple) homotopy equivalence with another pair which decomposes as above. A wider class of embeddings is similarly defined.

2.1.7. Definition. If (X,Y) is a (G-simple) B-Poincaré pair with respect to $\pi_1 X \longrightarrow G \longrightarrow \mathbb{Z}_2$, and (M,N) a topological pair, then a (G-simple) embedding of (M,N) in (X,Y) is a splitting $(X,Y) \xrightarrow{\sim} (X_1,Y_1) \amalg (X_2,Y_2)$ and a homotopy equivalence (G-simple) $(M,N) \xrightarrow{\sim} (X_1,Y_1)$.

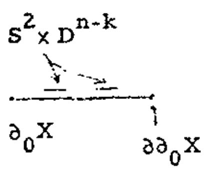
This can be used to show manifolds satisfy simple duality.

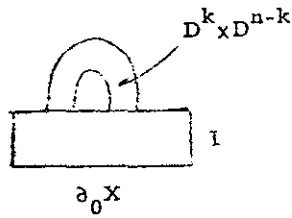
2.1.8. Definition. An elementary handlebody (differentiable, PL, topological, or Poincaré) is a triad $(X;\partial_0 X, \partial_1 X)$ with $\partial_0 X$ a compact manifold or Poincaré space of dimension n, a codimension 0 embedding $S^k \times D^{n-k} \subset \operatorname{int}(\partial_0 X)$, and an isomorphism

$$(X;\partial_0 X, \partial_1 X) \cong ((\partial_0 X \times I) \cup_{S^k \times D^{n-k} \times \{1\}} (D^{k+1} \times D^{n-k} \times \{1\});$$

$$\partial_0 X \times \{0\},\ \partial\partial_0 X \times I \cup (\partial_0 X \times \{1\} - \operatorname{im}(S^k \times D^{n-k}) \times \{1\}) \cup_{S^k \times \partial D^{n-k}} D^{k+1} \times \partial D^{n-k} \times \{1\}.$$

picture:





Now in the manifold case it is a classical argument that this construction yields a manifold triad. Using Proposition 2.1.5, we see that the situation is similar for Poincaré spaces.

2.1.9. Lemma: If $\partial_0 X$ is a (simple) Poincaré space, then an elementary handlebody $(X;\partial_0 X, \partial_1 X)$ is a (simple) Poincaré triad.

Proof: $H^*(X;\partial_0 X;B) \approx H^*(D^{k+1}, S^k;B) \approx B$

$$H_*^t(X;\partial_1 X;B) \approx H_*^t(D^{n-k+1}, S^{k-1};B) \approx B ,$$

and the cap product $([\partial_0 X] \times I + [D^{k+1} \times D^{n-k}]) \cap$ is an isomorphism. The alternating duality lemma 2.1.5 now applies to give the result. □

The group, and in particular the orientation homomorphism with respect to which it is simple is the same as $\partial_0 X$ if $k \neq 0$, but in the 1-handle case $k = 0$ the group may have to be enlarged if one component of $S^0 \times D^n$ is attached by an orientation-reversing isomorphism.

A handlebody is a triad $(X;\partial_0 X, \partial_1 X)$ with a finite filtration $\{(X^k;\partial_0 X, \partial_1 X^k)\}$ such that for each k $(\overline{X^k - X^{k-1}}; \partial_1 X^{k-1} \cap (\overline{X^k - X^{k-1}}), \partial_1 X^k \cap (\overline{X^k - X^{k-1}}))$ is an elementary handlebody. Inducting using 2.1.9, a handlebody is a simple Poincaré space. The lemma is actually necessary for the definition, in order to know the upper boundary of each elementary handlebody in the filtration is a $\pi_1 X$-simple Poincaré space and thus can be the base of an elementary handlebody.

The theorem that every manifold (dimension ≥ 6 in the topological case) has a handlebody structure yields the following corollary.

2.1.10 Corollary: A compact manifold pair $(M, \partial M)$ is a $\pi_1 M$-simple Poincaré space, where the orientation homomorphism $\pi_1 M \longrightarrow \mathbb{Z}_2$ is the first Stiefel-Whitney class of M.

2.2. Surgery maps

The use of the term "surgery map" here is entirely standard, namely a degree one normal map from a manifold to a Poincaré space. The only difference is that the n-ad case is treated, and a reference n-ad is fixed and Poincaré n-ad is understood to mean with respect to this n-ad. Mostly for reference, then, the definitions are given and the main results to be used are stated. Standard references here are Browder [2, Chap. 1], and Wall [27, Chaps. 1,2].

Let $\mathcal{C}$ denote one of the categories of manifolds, diff, Pl, or top, and let K be a topological n-ad with an orientation homomorphism $\pi_1 K \longrightarrow \mathbb{Z}_2$.

2.2.1. Definition: A map of n-ads $M \longrightarrow X \longrightarrow K$, with M and X K-simple) Poincaré spaces is degree 1 if $(\partial_\alpha f)_*([\partial_\alpha M]) = [\partial_\alpha X]$ for every $\alpha \subset \{0,\ldots,n-2\}$.

The main property of a degree one map is that the homomorphism defined by duality and the map in cohomology is a right inverse for the map in homology. It is thus a split surjection in homology. Similarly it is a split injection in cohomology (with a canonical cokernel). If $M \longrightarrow X \longrightarrow K$ is a degree one map define

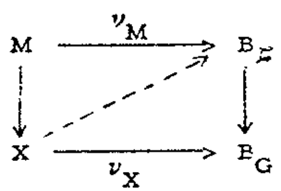
$$K^t_n(M;B) = \ker(H^t_n(M;B) \longrightarrow H^t_n(X;B)) = H^t_{n+1}(X,M;B)$$

$$K^n(M;B) = \operatorname{coker}(H^n(X;B) \longrightarrow H^n(M;B)) = H^{n+1}(X,M;B) .$$

Relative groups $K^t_*(M,Z;B)$ are also defined with respect to any union of faces $Z = \bigcup_{j\in\alpha} \partial_j M$, and the K-groups satisfy Poincaré duality in the sense that $K^*(M, \bigcup_\alpha \partial_j M; B)$ is carried isomorphically to $K^t_{m-*}(M, \bigcup_{\bar{\alpha}} \partial_j M; B)$ by the duality isomorphism of M.

These groups have many other properties, and in fact a detailed investigation of these properties leads to an algebraic characterization of surgery [27, Chaps. 5-8]. The approach taken here is completely geometric, and except for one theorem (the periodicity theorem) is independent of this algebra.

2.2.2. Definition: A normal map of a ζ n-ad to a Poincaré n-ad over K is an n-ad map $M \longrightarrow X$ together with a reduction of the stable normal fibration of X to B_ζ such that the diagram



commutes.

Note that if $M \longrightarrow X$ is a normal map, then the orientation homomorphism of M is induced from that of X, and hence from K.

2.2.3. Definition: A (ζ) surgery map over K is a degree 1 normal map from a ζ n-ad to a Poincaré n-ad over K.

2.2.4. Theorem: (the surgery lemma) Suppose $f : (M; \partial_0 M, \partial_1 M) \longrightarrow (X; \partial_0 X, \partial_1 X) \longrightarrow (K; \partial_0 K, \partial_1 K)$ is a surgery map of a ζ triad to a (simple) Poincaré triad over K with $\partial_0 f$ a (simple) homotopy equivalence and $\dim M \geq 6$.

a) If $\pi_1\partial_1 X \longrightarrow \pi_1 X$ is an isomorphism, then there is a surgery map $N \longrightarrow X \times I \longrightarrow K \times I$ with

1) $\partial_0 N \longrightarrow X \times \{0\}$ isomorphic to $M \longrightarrow X$

2) $\partial_1 N \longrightarrow \partial_0 X \times I$ isomorphic to $\partial_0 M \times I \longrightarrow \partial_0 X \times I$

3) $\partial_2 N \longrightarrow X \times \{1\}$ a (simple) homotopy equivalence.

b) If $\pi_1\partial_1 K \longrightarrow \pi_1 K$ is an isomorphism, then there is a surgery map $N \longrightarrow Y \longrightarrow K \times I$ with

1) $\partial_0 N \longrightarrow \partial_0 Y \longrightarrow K \times \{0\}$ isomorphic to $M \longrightarrow X \longrightarrow K$

2) $\partial_1 N \longrightarrow \partial_0 Y \longrightarrow \partial_0 K \times I$ isomorphic to $\partial_0 M \times I \longrightarrow \partial_0 X \times I \longrightarrow \partial_0 K \times I$

3) $\partial_2 N \longrightarrow \partial_0 Y \longrightarrow K \times \{1\}$ a (simple) homotopy equivalence .

The second part of this assertion follows from part a) by showing under the conditions of b) a cobordism can be found to a map satisfying a). This involves low-dimensional surgery on Poincaré spaces, which actually proves a bit more which we will need. Rather than complicate the statement further, we will refer to the proof ([27, Chaps. 4 and 9]) when necessary.

This theorem and the s-cobordism theorem are essentially the only geometric facts about Σ -manifolds we will need.

3. Surgery spaces

3.1. Definition and basic properties

In this section the spaces $\mathbb{L}_m(K)$ for K a topological n-ad are defined, and the basic properties explored. These include naturality, the infinite loop space structure, fibration sequences of an n-ad, dependence on the fundamental group lattice, and the relation between $\mathbb{L}^h$ and $\mathbb{L}^s$.

Let K be an n-ad with an orientation homomorphism $\pi_1 K \longrightarrow \mathbb{Z}_2$.

3.1.1. Definition. If $m \in \mathbb{Z}$, $\mathbb{L}^h_m(K)$) is defined to be the Δ-set with k-simplices (K-simple) compact topological surgery maps of (n+k+3)-ads of dimension $m+k$,

$$f : M \longrightarrow X \longrightarrow S_{k+1}(\Delta^k * K)$$

with $\partial_{k+1} f$ a (K-simple) homotopy equivalence. The first k faces of f serve as its boundaries as a k-simplex, so we also require $\partial_{\{0,\dots,k\}} X = \emptyset$.

We note that as in the example in 1.4, we should require all sets to be subsets of $\mathcal{S}$, apply Ex^∞, etc. The considerations worked out in detail in 1.4, however, show that it is sufficient to make a blanket statement at the beginning that this is to be done uniformly, and not mention it again.

As with the example 1.4.1, we have

3.1.2. Proposition

(i) $\mathbb{L}_m(K)$ is an Abelian h-space with operation disjoint union, and identity the empty surgery map.

2) The natural homotopy equivalence $\mathbb{L}_m(K) \longrightarrow \Omega(\mathbb{L}_{m-1}(K), *)$ gives $\mathbb{L}_m(K)$ the structure of an infinite loop space.

3) The homotopy groups $\pi_j(\mathbb{L}_m(K))$ are the Wall surgery obstruction groups [27, Chap. 9 for L^s] $L_{m+j}(K)$. □

In 3.1.1 $\mathbb{L}$ means either $\mathbb{L}^s$ or $\mathbb{L}^h$. When it is true of a functor (here of s or h) independently of the value of the argument, the argument is often left out.

This construction is plainly a covariant functor on the category of n-ads with orientation homomorphism. The induced morphisms are given simply by composition with the reference map of each simplex.

A homotopy fibration (of Δ-sets) is a pair of maps $A \xrightarrow{a} B \xrightarrow{b} C$ with a homotopy of ba to the point map, such that the resulting map of A into the fiber of b is a homotopy equivalence. The homotopy $ba \sim *$ is usually more or less canonically defined by the problem, and will seldom be mentioned explicitly. In this case being a homotopy fibration is equivalent, by Whitehead's theorem, to the homotopy groups fitting into a long exact sequence. A homotopy fibration sequence is a sequence of maps, each successive pair of which is a homotopy fibration.

The next property of $\mathbb{L}$ is the fibration sequence of an n-ad. If K is an n-ad, $0 \leq j \leq n-2$, then the cofibration sequence $\partial_j K \xrightarrow{i} \delta_j K \xrightarrow{j} K$ (3.2) induces maps of $\mathbb{L}$. A natural map $\partial_j : \mathbb{L}_m(K) \longrightarrow \mathbb{L}_{m-1}(\partial_j K)$ is defined by taking the boundary of each simplex lying over $\partial_j K$ (the $(k+j+1)^{st}$ boundary of a k-simplex).

3.1.2. Proposition. The sequence

$$\cdots \longrightarrow \mathbb{L}_m(\delta_j K) \xrightarrow{i^*} \mathbb{L}_m(\delta_j K) \xrightarrow{j^*} \mathbb{L}_m(K) \xrightarrow{\partial j} \mathbb{L}_{m-1}(\partial_j K) \longrightarrow \cdots$$

is a homotopy fibration sequence.

Proof: This proposition is essentially [27, 9.6] which says that the corresponding sequence of groups is exact. The proof is an easy geometric argument which also gives the necessary homotopies to zero. □

3.1.3. Proposition. If $K_1 \longrightarrow K_2$ is a morphism of n-ads with orientation homomorphism inducing isomorphism of fundamental group lattice $\pi_1 K_1 \xrightarrow{\simeq} \pi_1 K_2$, then the induced Δ-map $\mathbb{L}_m(K_1) \longrightarrow \mathbb{L}_m(K_2)$ is a homotopy equivalence.

Proof: The fibration sequence 3.1.2 reduces the proposition to the 1-ad case, for suppose inductively that it holds for n-1 ads, then in the diagram

$$\begin{array}{ccccc}
\mathbb{L}_m(\partial_j K_1) & \longrightarrow & \mathbb{L}_m(\delta_j K_1) & \longrightarrow & \mathbb{L}_m(K_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}_m(\partial_j K_2) & \longrightarrow & \mathbb{L}_m(\delta_j K_2) & \longrightarrow & \mathbb{L}_m(K_2)
\end{array}$$

the top and bottom rows are fibrations, and the first two vertical maps homotopy equivalences. The 5-lemma now applies to show the last vertical map is a homotopy equivalence. Now let M be the mapping cylinder (a 2-ad) of the map $K_1 \longrightarrow K_2$ of 1-ads; then the fibration sequence $\mathbb{L}_m(K_1) \longrightarrow \mathbb{L}_m(K_2) \longrightarrow \mathbb{L}_m(M)$ shows the problem to be equivalent to the contractibility of $\mathbb{L}_m(M)$. 2.2.4 b) is exactly the statement that $\pi_j \mathbb{L}_m(M) = 0$

for $j + M \geq 6$. This is one of the places where we regret our imprecise statement of 2.2.4, since Wall's theory using a more sensitive version shows [27, 9.4.1] that $\pi_s \mathbb{L}_0(K_1) \longrightarrow \pi_s \mathbb{L}_0(K_2)$ is an isomorphism. Given this, however, we can complete the proposition using obstruction theory. We can assume K_2 is a $K(\pi_1 K_1, 1)$.

Let $M \xrightarrow{f} X \longrightarrow K_2$ be a 4-dimensional surgery map representing an element of $\pi_4 \mathbb{L}_0(K_2)$ (i.e. a 4-simplex with empty boundaries). According to Wall [25], X is homotopy equivalent to a complex dominated by a 2-complex attached to the upper boundary of a handlebody with 0 and 1 handles on ∂M. Surgery on the inverse images of the handles gives a homotopy of f to a map which is an isomorphism over a neighborhood of these handles. Removing a regular neighborhood of these handles gives a cobordism of f to a map with image dominated by a two-dimensional complex. The obstructions to pulling the reference map $X \longrightarrow K_2$ back to K_1 lie in $H^j(X, \pi_j(K_2, K_1))$. Since $\pi_1 K_1 \longrightarrow \pi_1 K_2$ is an isomorphism and $\pi_2 K_2 = 0$, the coefficient groups are zero for $j \leq 2$. Thus the problem with image dominated by a 2-complex can be pulled back, giving an inverse to $\pi_4 \mathbb{L}(K_1) \longrightarrow \pi_4(\mathbb{L} K_2)$.

Since representatives of π_0, π_1, and π_2 have images already dominated by a 2-complex, they are all isomorphic. If $f: M \longrightarrow X \longrightarrow K$ represents an element of $\pi_3 \mathbb{L}_0(K_2)$ with $\partial X \neq \emptyset$ then $H^3(X) = 0$ and it can be pulled back. If $\partial X = \emptyset$, then again by [25] X is equivalent to a complex dominated by a 2-complex with a 3-cell attached. f can be made an isomorphism over this cell, which can then be removed to give a problem with nonempty boundary. This constructs an inverse for the map on π_3.

To pass from the case $K_2 = K(\pi_1 K_1, 1)$ to general case apply $\mathbb{L}_0$ to $K_1 \to K_2 \to K(\pi_1 K_2, 1)$. □

Next we consider the relation between the functors $\mathbb{L}^h$ and $\mathbb{L}^s$. Denote the fiber of the natural map by $A_{m+1}(K) \to \mathbb{L}^s{}_m(K) \to \mathbb{L}^h{}_m(K)$. The homotopy of the fiber was determined by Rothenberg by geometric arguments, and a (mostly) algebraic version is given by Shaneson in [17]. We give enough geometry to reduce the proof to facts about Whitehead torsion. Not wanting to become involved with stable algebra, we would just refer to [17] for the proof except that some of the constructions are interesting for other reasons.

Given an element in $\mathbb{L}^h{}_m(K)$, it fails to come from an element in $\mathbb{L}^s{}_m(K)$ to the extent that the range Poincaré space fails to be K-simple, and the homotopy equivalence boundary of the map fails to be K-simple. We can concentrate the obstruction in either place. If $f: M \to X \to K$ is a simplex of $\mathbb{L}^h{}_m(K)$, with the face carried by homotopy equivalence denoted by $\partial_h f$, we can use a collar of $\partial_h M$ in M to replace X with the mapping cylinder of $\partial_h M \to X$. The map $\partial_h f$ is replaced by the identity, so the obstruction to this problem coming from $\mathbb{L}^s$ is just the torsion of the duality map of $(X, \partial_h X)$. Conversely if K is an n-ad, and $m-n \geq 6$, the identity map $\mathbb{L}^n{}_m(K) \to \mathbb{L}^h{}_m(K)$ is homotopic to a map for which the Poincaré complex in each image is a smooth manifold. Since these satisfy simple duality, the entire obstruction lies in the torsion of

$\partial_h f$. The construction of the homotopy is by induction over the skeleta of $\Sigma^h_m(K)$, and over the faces of each simplex.

Suppose X is a Poincaré space over K with part of its boundary smooth, then it is equivalent to one with a smooth neighborhood of the 1-skeleton with respect to the smooth face, $(X, \partial_s X)$. Do smooth surgery to reduce the fundamental group of this neighborhood to that of X, covered by topological surgery in the domain, and take a thickening of a 2-skeleton of the result rel $\partial_s X$. By the dimension restriction, π_1 of the upper boundary of this thickening is the same as that of X, so by the surgery lemma 2.2.4 a surgery problem $M \longrightarrow X$ can be deformed to one which is a homotopy equivalence on the complement of the thickening. Incorperating this into the face which is a homotopy equivalence, we have a cobordism of a surgery map rel faces where it is already smooth, to one with smooth target.

Now using the second construction in which the obstruction is the torsion of a map, the torsion must satisfy a duality formula $\tau = (-1)^n \tau^*$. Here $*: Wh(\pi_1 K) \longrightarrow Wh(\pi_1 K)$ is induced by the standard involution of Wh and the antiinvolution $-$ on $\mathbb{Z}(\pi_1 K)$. An h-cobordism may be glued on the homotopy face to change the torsion by $\tau + (-1)^{n+1}\tau^*$, any τ, so noting that with the $\mathbb{Z}_2$ module structure induced by $*$ on $Wh(\pi_1 K)$, $H^n(\mathbb{Z}_2; Wh(\pi_1 K)) = \{\tau \in Wh(\pi_1 K) \mid \tau = (-1)^n \tau\} / \{\tau + (-1)^n \tau^*\}$ the following proposition is very reasonable.

2.2.4. Proposition: $\pi_j A_m(K) \cong H^{m+j}(\mathbb{Z}_2; Wh(\pi_1 K))$

Actually $A_m(K)$ is homotopy equivalent to $\Omega^m \hom_{\mathbb{Z}_2}(\mathbb{Z}_2(C_*(K(\mathbb{Z}_2,1)), Wh(\pi_1 K))$, if m is large enough. Since we won't have occasion to need even 3.1.4, we won't go into this.

Some remarks on the constructions above: First, the first characterization of the obstruction together with the final answer gives some information on the torsions which may appear in the duality map of a Poincaré space, and how it can vary under certain restricted types of cobordisms. Second, these two constructions give several characterizations of the surgery spaces. The second construction shows that we could have used just surgery maps of smooth manifolds to smooth manifolds. The first construction, together with topological transversality (which requires a dimension condition at the present time) shows we could have defined $\mathbb{L}$ as the cobordism space of Poincaré spaces (over K) with one face a topological manifold, and an extension of the normal bundle of this face to a reduction of the whole normal bundle to B_{top}. Here it is essential that the objects be Poincaré spaces, K-simple for $\mathbb{L}^s(K)$. The definition 3.1.1 was chosen because it contains both of these, and seems close to the widest most general and natural form of the space.

3.2. Some special maps

Here we use geometry to construct two maps of $\mathbb{L}$ spaces which in some cases gives strong information about their structure. The first, the pullback, is essentially a mystery since the only calculations have been done using the algebraic characterization. The second, the assembly map, is slightly better understood. A geometric calculation in some interesting special cases is made in 3.3, it turns up in the structure sequence in 4.1, and is discussed a little more in 4.2 in connection with the fibration problem.

Suppose $\pi: E \longrightarrow B$ is a block fibration over a CW n-ad B, with fiber a compact manifold k-ad M^m. If $N \longrightarrow X \longrightarrow B$ is a surgery map over B, then we can form the pullback fibrations

$$\begin{array}{ccccc} \pi^* N & \longrightarrow & \pi^* X & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & X & \longrightarrow & B \end{array}$$

Since the fiber is a compact manifold, this is a surgery map over E, with dimension raised by m. ($\pi^* N$ is clearly a manifold. That $\pi^* X$ is a Poincaré space and the map degree 1 is an easy spectral sequence argument.)

3.2.1. Definition. The map $P_\pi: \mathbb{L}_j(B) \longrightarrow \mathbb{L}_{j+m}(E)$ induced by taking pullbacks is called the pullback map.

Two special cases of the pullback are worthy of note: finite covers and products. If $\pi: \overline{B} \longrightarrow B$ is a finite cover, then the pullback P_π is sometimes called the transfer. Wall has shown algebraically that the pullback defined by the universal cover over $K(\mathbb{Z}_p, 1)$, p odd, is surjective in homotopy $\mathbb{L}(\mathbb{Z}_p) \longrightarrow \mathbb{L}(0)$. On the other hand if the fibration is a product $E = B \times M$, the pullback is just the product map [27, Chap. 9] $\mathbb{L}_j(B) \xrightarrow{\times M} \mathbb{L}_{j+m}(B \times M)$. The other theorem concerning the pullback is

3.2.2. Proposition: (periodicity) If B is an n-ad, $j-n \geq 4$, then the product $P_{\times \mathbb{C}P^2} : \mathbb{L}_j(B) \longrightarrow \mathbb{L}_{j+4}(B)$ is a homotopy equivalence. $\square$

Again the only known proof is by the algebraic characterization of Wall [27, Chaps. 5-8]. This is the only result we will make extensive use of that does not yet have a geometric proof. It seems to be a very important problem to get a better geometric understanding of this map.

Now assume, dually, that $F \longrightarrow E \xrightarrow{\pi} M$ is a fibration with a (polyhedral) manifold k-ad as base, and fiber a space with orientation homomorphism which factors through the image of $\pi_1 F$ in $\pi_1 E$. We construct a fibration $\mathbb{L}_j(\pi) \longrightarrow M$ with fiber $\mathbb{L}_j(F)$. First, the action of ΩM on F induces an action on $\mathbb{L}_j(F)$, and the tensor product of this action with the universal ΩM bundle over M gives the correct fibration. We need a more geometric description, however. Define $\mathbb{L}_j(\pi) = \bigcup_{\sigma^\ell \in M} \mathbb{L}_{j+\ell}(\pi^{-1}(\sigma^\ell))$, and for a simplex $a \in \mathbb{L}_{j+\ell}(\pi^{-1}\sigma^\ell) \subset \mathbb{L}_j(\pi)$ with ℓ the least such, define the projection of a in M to be σ^ℓ. $\mathbb{L}_j(\pi) \longrightarrow M$ is a block fibration over this triangulation of M with fiber $\mathbb{L}_j(F)$.

So far M could have been a Δ-set, but now consider the Δ-set of sections of this bundle, $\Gamma_M(\mathbb{L}_j(\pi))$. If $S: M \longrightarrow \mathbb{L}_j(\pi)$ is a section, then over each simplex $\sigma^\ell \in M$ we get a surgery map $N \longrightarrow X \longrightarrow \pi^{-1}(\sigma^\ell)$ of dimension $j+\ell$. Take the disjoint union of these surgery maps (over common faces) to obtain a map of spaces $\overline{N} \longrightarrow \overline{X} \longrightarrow E$. If M is a compact manifold, then by a classical gluing theorem $\overline{N}$ is also. Similarly the Poincaré sum theorem (in a generalization of its application to 2.1.10) show that $\overline{X}$ is a Poincaré space (simple for $\mathbb{L}^s$) over E, with respect to some orientation homomorphism $w: \pi_1 E \longrightarrow \mathbb{Z}_2$ extending the homomorphism on the image of $\pi_1 F$. A spectral sequence, or the sum theorem for surgery maps, shows that $\overline{N} \longrightarrow \overline{X}$ is a surgery map of dimension $m+j$. Similarly we can assemble the image of a P-simplex $\Delta^p \times M \longrightarrow \mathbb{L}_j(\pi)$ to get a surgery map of dimension $j+m+p$ over $E \times \Delta^p$.

Sections in different components of $\Gamma_M(\mathbb{L}_j(\pi))$ may assemble to give surgery maps with different orientation homomorphisms. Therefore given $\omega: \pi_1 E \longrightarrow \mathbb{Z}_2$ extending the homomorphism on the image of $\pi_1 F$, denote by $\Gamma_M(\mathbb{L}_j(\pi))_\omega$ the components which assemble to give that homomorphism.

<u>3.2.3. Definition:</u> The map $A_{\pi,\omega}: \Gamma_M(\mathbb{L}_j(\pi))_\omega \longrightarrow \mathbb{L}_{j+m}(E)$ obtained by assembling the images of the sections is called the <u>assembly</u> map.

An important special case is again when π is a product $F \times M \longrightarrow M$. In this case $\mathbb{L}_j(\pi)$ is also a product $\mathbb{L}_j(F) \times M \longrightarrow M$, and the space of sections is just the space of maps $\Delta(M, \mathbb{L}_j(F))$. Since $\mathbb{L}_j(F)$ depends only on $\pi_1 F$, π need only be a product on π_1 to decompose as $\mathbb{L}_j(F) \times M$.

The bundle $\mathbb{L}_j(\pi)$ has a canonical section (the base point of each fiber), and by $\Gamma_{(M,\partial_0 M)} \mathbb{L}_j \pi$ we mean sections which agree with this one on $\partial_0 M$.

P_π and A_π are clearly natural with respect to morphisms in the category on which they are defined (bundles with fiber or base a fixed manifold), and with respect to boundary maps in either the base or fiber.

Further, they are natural with respect to each other in the following sense.

Suppose $\pi: E \longrightarrow M^m$ is a fibration with manifold base, and $\xi: Y \longrightarrow M$ is a map of manifolds which is a block fibration with manifold fiber N^n on $\xi^{-1}(M - \partial_0 M) \longrightarrow M - \partial_0 M$. Take the pullback

$$\begin{array}{ccc} X & \xrightarrow{\xi^*\pi} & Y \\ \Big\downarrow{\scriptstyle \pi^*\xi} & & \Big\downarrow{\scriptstyle \xi} \\ E & \xrightarrow{\pi} & M \end{array}$$

Now $\pi^*\xi$ is a bundle map covering ξ, which induces a bundle map $b: \mathbb{L}_j(\xi^*\pi) \longrightarrow \mathbb{L}_j(\pi)$ covering ξ. The diagram

$$\begin{array}{ccc} \Gamma_{(Y,\partial_0 Y)}\, \mathbb{L}_j(\xi^*\pi) & \xrightarrow{A\xi^*\pi} & \mathbb{L}_{j+n+m}(\delta_0 X) \\ \Big\downarrow{\scriptstyle \Gamma_{(\xi\partial_0\xi)}^{b}} & & \Big\downarrow{\scriptstyle P_{\pi^*\xi}} \\ \Gamma_{(M,\partial_0 M)}\, \mathbb{L}_j(\pi) & \xrightarrow{A_\pi} & \mathbb{L}_{j+m}(\delta_0 E) \end{array}$$

commutes.

This is a little clearer if $E = F \times M$. The diagram becomes

$$\begin{array}{ccc} \Delta(Y, \partial_0 Y; \mathbb{L}_j F) & \longrightarrow & \mathbb{L}_{j+n+m}(F \times \delta_0 Y) \\ \Big\downarrow{\scriptstyle \Delta(\xi, 1)} & & \Big\downarrow{\scriptstyle P_{1 \times \xi}} \\ \Delta(M, \partial_0 M; \mathbb{L}_j F) & \longrightarrow & \mathbb{L}_{j+m}(F \times \delta_0 X) \end{array}$$

In the cases we will be able to calculate $\mathbb{L}$ spaces in terms of the assembly map, this constitutes a calculation of the pullback.

3.3. Transversality and splitting

If K contains a homotopy disc bundle, we can define a notion of transversality of a surgery map to this bundle. Transversality is very useful in constructing maps on $\mathbb{L}$ spaces, and a sufficient condition for transversality due to Cappell results in several computations for $\mathbb{L}$ spaces.

Since we are mostly interested in trivial line bundles, the definition can be considerably simplified by use of topological disc bundles. Let $K = K_1 \amalg_{S\xi} D\xi$ where $S\xi$ and $D\xi$ the topological sphere and disc bundles over an n-ad K_0. Suppose $f: M \longrightarrow X \longrightarrow K$ is a topological surgery map of n-ads with a distinguished face $\partial_h f$ which is a homotopy equivalence.

3.3.1. Definition: The surgery map $f: M \longrightarrow X \longrightarrow K$ is transversal to the bundle ξ if X has a Poincaré splitting as $X_1 \cup_{S\xi^*} D\xi^* \longrightarrow K_1 \cup D\xi$ where ξ^* is the pullback of ξ over the inverse image X_0 of K_0, f is transversal to the bundle ξ^*, and on the homotopy face the induced maps $\partial_h f^{-1}(\partial_h X_0) \longrightarrow \partial_h X_0$ and $(\partial_h M - \partial_h f^{-1}(D\xi^*)) \longrightarrow \partial_h X_1$ are homotopy equivalences.

Similarly for a simple surgery problem simple transversality is defined by requiring the splitting of X and the resulting homotopy equivalences be simple.

At the end of 3.1 we saw that the subset of $\mathbb{L}_j(K)$ with manifolds as target Poincaré space is homotopy equivalent to all of $\mathbb{L}_j(K)$ for $j+k \geq 5$ (K a k-ad). Now in these dimensions if $M \longrightarrow N \longrightarrow K$ is such a surgery map, N can be made transversal to ξ, and M can be made transversal to the resulting submanifold of N. This is not transversality as a surgery map since the homotopy face may not be split into homotopy equivalences. If $\partial_h M$ could be split into homotopy equivalences over the submanifold of $\partial_h N$, then relative transversality would apply to give a transversal surgery map.

Obtaining transversal surgery maps is thus seen to be equivalent to an embedding problem. The standard codimension ≥ 3 embedding theorem [27, Chap. 11] shows the problem to be trivial and uninteresting in this case. If $\dim D\xi = 2$, then essentially nothing is known. In codimension 1, however, there is the splitting theorem of Cappell [4]:

3.3.2. Proposition: Suppose $f: M \longrightarrow N$ is a simple homotopy equivalence of manifold pairs, and $N \supset P$ is a two-sided codimension one submanifold with $\partial P \subset \partial N$ and $\pi_1 P \longrightarrow \pi_1 N$ injective. If $f \mid \partial M$ is split into homotopy equivalences over the embedding $\partial P \subset \partial N$ and $m \geq 6$, then the splitting may be extended to a homotopy splitting of M.

Notice that we have lost a little here in starting with a simple homotopy equivalence and getting only a homotopy splitting. Our reaction to this will be to assume the appropriate Whitehead groups are zero. In simple applications (see [17]) it can easily be allowed for, but gets too complicated for what we will do here.

Now suppose as above that $K = K_1 \amalg_{K_0 \times \{0,1\}} K_0 \times I$, K_1 and K_0 n-ads, and consider $K_2 \times I$ as a trivial D' bundle over $K_2 \times \frac{1}{2}$.

<u>3.3.3. Lemma</u>: If $\pi_1 K \longrightarrow \pi_1 K$ is injective and $j - n \geq 5$, then the identity of $\mathbb{L}^s_j(K)$ is homotopic to a map with the image of each simplex homotopy transversal to $K_0 \times \frac{1}{2}$, leaving fixed those already transversal.

<u>Proof</u>: Call the subcomplex of transversal maps A, then this is just the statement that A is a deformation retract of $\mathbb{L}^s_j(K)$. We show $\pi_m(\mathbb{L}^s_j(K), A) = 0$, all m. The usual assembly process and the comments above show that a relative homotopy element is equivalent to a surgery map $M \xrightarrow{f} N \xrightarrow{g} K \times D^m$ with N a manifold transversal to $K_0 \times \frac{1}{2}$ and $\partial_n M \longrightarrow \partial_n N \longrightarrow K \times S^{m-1}$ transversal as a surgery map to $K_0 \times \frac{1}{2}$. To apply 3.3.2 to the homotopy faces of this map it is only necessary to show that under the circumstances we can also assume $\pi_1 \partial_h g^{-1}(K_0 \times \frac{1}{2} \times D^m) \longrightarrow \pi_1 \partial_h N$ is injective. Given this relative transversality as above yields a transversal surgery map showing the relative homotopy element was trivial.

There is a diagram

$$\begin{array}{ccc} \partial_h N & \longrightarrow & K \times D^m \\ \downarrow & & \downarrow \\ \partial_h g^{-1}(K_0 \times \frac{1}{2} \times D^m) & \longrightarrow & K_0 \times \frac{1}{2} \times D^m \end{array}$$

By assumption the right map is a π_1 injection. Surgery can be done to make the top map a π_1 injection, for if S^1 is embedded in $\partial_h N$ it has trivial normal bundle iff $\omega(s^1) = 0$. By definition of $\mathbb{L}$, ω factors through $\pi_1 K$, and so is zero on the kernel of $\pi_1 \partial_h N \longrightarrow \pi_1 K$. By the same argument the bottom map can be made a π_1 injection, and by doing surgery on $\partial_h N$ by $S^1 \subset \partial_h^{g-1}(K_0 \times D^m) \subset \partial_h N$, the result of the surgery on $\partial_h^{g-1}(K_0 \times D^m)$ is realized as a transversal inverse image of $K_0 \times D^m$ in N. This surgery does not disturb $\pi_1 \partial_h N$ since an S^1 trivial in $K_0 \times D^m$ is also trivial in $\partial_h N$ by the first injectivity. The effect on $\partial_h N$ therefore is connected sum with $S^2 \times S^{n-2}$. The left map is now a π_1-injection as desired.

All this surgery on $\partial_h N$ can be covered by surgery of $\partial_h M$. 3.3.2 now applies to split the boundary, and completes the proof. □

The main usefulness of transversality is in the following construction. Let $K \supset \xi$ be a k-ad containing an n-dimensional topological disc bundle as above, and denote the pullback of the given maps by

$$\begin{array}{ccc}
 & R_j(K, \xi) & \\
\swarrow & & \searrow \\
\mathbb{L}_j(K_1, S\xi) & & \mathbb{L}_{j-n}(K_0) \\
\partial \searrow & & \swarrow \; P_{S\xi} \\
 & \mathbb{L}_{j-1}(S\xi) &
\end{array}$$

Here $\mathbb{L}_j(K_1, S\xi)$ denotes $\mathbb{L}_j$ of the (n+1)-ad $(K_1, \partial_0 K_1, \ldots, \partial_{n-2} K_1, S\xi)$. The homotopy pullback is formed by making one map a fibration, and taking the pullback fibration over the other. The set theoretic pullback is the subset of the product each factor of which is mapped to the same element of the range. Here it is important to note that they have the same homotopy type, since ∂ is already essentially a fibration. Given a map of a complex in $\mathbb{L}_j(K_1, S\xi)$ and a homotopy of its projection, we can lift the homotopy after it has been moved a little to make it disjoint (see 1.4) by gluing the images on the boundary of the images of the complex in $\mathbb{L}_j(K_1, S\xi)$.

Thinking of $R_j(K, \xi)$ as the set-theoretic pullback, there is a natural map $c: R_j(K, \xi) \longrightarrow \mathbb{L}_j(K)$ gotten by gluing an element of $\mathbb{L}_j(K, S\xi)$ and $P_{S\xi}$ of an element of $\mathbb{L}_{j-n}(K_0)$ together on the boundary. Each simplex of the image of this map is transversal to the bundle ξ, by definition. Conversely there is a map d from the transversal subset of $\mathbb{L}_j(K)$ to $P_j(K, \xi)$ given by the inverse image of K_0, and the complement of an open tubular neighborhood of this inverse image which is mapped by a bundle map to ξ. $dc = 1$, and a homotopy $1 \sim cd$ is obtained by an expansion along the fibers of ξ until a transversal surgery map is transversal to the whole disc bundle $D\xi$.

<u>3.3.4. Theorem</u>: Suppose $K = K_1 \amalg_{K_0 \times \{0,1\}} K_0 \times I$ is an n-ad, $\pi_1 K_0 \longrightarrow \pi_1 K$ is injective, and $Wh(\pi_1 K_i) = Wh(\pi_1 K) = 0$, then $c: R_j(K, K_0 \times I) \longrightarrow \mathbb{L}_j(K)$ is a homotopy equivalence, for $j - n \geq 5$.

<u>Proof</u>: The argument above shows that c is a homotopy equivalence with the subset of $\mathbb{L}_j(K)$ transversal to $K_0 \times \frac{1}{2}$. 3.3.3 implies that

under the conditions of the theorem this subset is a deformation retract of $\mathbb{L}_j(K)$. $\square$

This form of transversality is necessary for our most elaborate applications. It can also be used to give quick proofs of some very useful results which can be proved by easier arguments (compare [4]).

<u>3.3.5. Theorem</u>: Suppose $K = K_0 \cup K_1$ is an n-ad, $Wh(\pi_1 K) = Wh(\pi_1 K_i) = Wh(\pi_1(K_0 \cap K_1)) = 0$, and $\pi_1(K_1 \cap K_2) \longrightarrow \pi_1 K$ is injective, then for $j - n \geq 5$ we have

1) (excision) $\mathbb{L}_j(K_2, K_1 \cap K_2) \longrightarrow \mathbb{L}_j(K, K_1)$ is a homotopy equivalence.
2) (Meyer-Vietoris) $\mathbb{L}_j(K_1 \cap K_2) \longrightarrow \mathbb{L}_j(K_1) \times \mathbb{L}_j(K_2) \longrightarrow \mathbb{L}_j(K_1 \cup K_2)$ is a homotopy fibration sequence.

<u>Proof</u>: First we define the maps in the "Meyer-Vietoris" sequence. The first map is the product of the natural inclusion in one factor and (-1) times the inclusion in the other. The second map is the sum of the natural inclusions.

To start both proofs, set $K_3 = K_1 \cap K_2$, and replace K by $K_1 \times \{0\} \cup K_2 \times \{1\} \cup K_3 \times I$. Denote $K_1 \times \{0\} \cup K_2 \times \{1\}$ by $K_1 \amalg K_2$, then we are in the situation of 3.3.4 with K_1 replaced by $K_1 \amalg K_2$, and K_0 replaced by K_3. Thus we have a homotopy pullback diagram

$$\begin{array}{ccc} & \mathbb{L}_j(K) & \\ \swarrow & & \searrow \\ \mathbb{L}_j(K_1 \amalg K_2, K_3 \amalg K_3) & & \mathbb{L}_{j-1}(K_3) \\ \partial \searrow & & \swarrow (1,-1) \\ & \mathbb{L}_{j-1}(K_3 \amalg K_3) & \end{array}$$

Now clearly $\mathbb{L}_j(A_1 - A_2) = \mathbb{L}_j(A_1) \times \mathbb{L}_j(A_2)$, so the fiber of the bottom left map is, by the fibration sequence, the natural inclusion of $\mathbb{L}_j(K_1) \times \mathbb{L}_j(K_2)$. By definition of a homotopy pullback, however, this is also the fiber of the top right map, which establishes the Meyer-Vietoris sequence.

On the other hand, the lower right map is clearly the fiber of the map $\mathbb{L}_{j-1}(K_3 - K_3) \longrightarrow \mathbb{L}_{j-1}(K_3)$ induced by projection $K_3 \amalg K_3 \longrightarrow K_3$. Therefore the upper left map is the fiber of $\partial + \partial : \mathbb{L}_j(K_1, K_3) \times \mathbb{L}_j(K_2, K_3) \longrightarrow \mathbb{L}_{j-1}(K_3)$. Now take the natural map of fibration sequences

$$
\begin{array}{ccccc}
* & \longrightarrow & \mathbb{L}_j K & \longrightarrow & \mathbb{L}_j K \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}_j(K_1, K_3) & \longrightarrow & \mathbb{L}_j(K_1, K_3) \times \mathbb{L}_j(K_2, K_3) & \longrightarrow & \mathbb{L}_j(K, K_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}_j(K_1, K_3) & \xrightarrow{\ \partial\ } & \mathbb{L}_{j-1}(K_3) & \longrightarrow & \mathbb{L}_{j-1}(K_1)
\end{array}
$$

Since the top and bottom row are also fibrations, the 5-lemma implies the center is too. But now comparison of the center with the product sequence $L_j(K_1, K_3) \longrightarrow \mathbb{L}_j(K_1, K_3) \times \mathbb{L}_j(K_2, K_3) \longrightarrow \mathbb{L}_j(K_2, K_3)$ shows the center splits, and establishes the excision theorem. □

Theorem 3.3.4 is the statement that under certain conditions an $\mathbb{L}$ space is the pullback of some maps of other $\mathbb{L}$ spaces. If we could build K from simply-connected spaces by operations like the above, then we could build up $\mathbb{L}(K)$ from $\mathbb{L}$ (pt) by pullback. Our next object is to do just that, using the assembly map 3.2.3 to keep track of things, since the function spaces turn out to have the same formal properties.

We begin with the formal properties of the assembly. Let M be a triangulable manifold n-ad, K a k-ad, and $h: N \longrightarrow M$ a proper embedding as a sub-polyhedral n-ad with normal disc bundle $\xi: D\xi \longrightarrow N$.

3.3.6. Lemma: The natural diagram

$$\begin{array}{ccccc} & & \Delta(M, \partial_0 M; \mathbb{L}_j K) & & \\ & \swarrow^{\Delta(h,1)} & & \searrow^{\Delta(\bar{h},1)} & \\ \Delta(N, \partial_0 N; \mathbb{L}_k K) & & & & \Delta(M - D^0\xi, S\xi; \mathbb{L}_j K) \\ & \searrow^{\Delta(S\xi,1)} & & \swarrow_{\Delta(\partial,1)} & \\ & & \Delta(S\xi, \partial_0 S\xi; \mathbb{L}_j K) & & \end{array}$$

is a homotopy pullback. The assembly $\Delta(M, \partial_0 M; \mathbb{L}_j K) \longrightarrow \mathbb{L}_{j+m}(K \times \delta_0 M)$ factors through $R_{j+m}(K \times \delta_0 M, K \times \xi)$.

Proof: The diagram is a set-theoretic pullback. and $\Delta(\partial, 1)$ is a fibration so it is a homotopy pullback also. That the assembly factors through P is just the statement that surgery maps assembled from a map $M \longrightarrow \mathbb{L}_j K$ are transversal to $K \times N$, which is clear from the construction of the reference map. $\Box$

Next we define what we mean by groups built up from zero. If $K = K_1 \amalg_{K_0 \times \{0,1\}} K_0 \times I$, then by Van Kampen's theorem $\pi_1 K$ is the free product of the groupoid $\pi_1 K_1$ (lattice for n-ads $n > 1$) amalgamated over the homomorphisms $a, b: \pi_1 K_0 \rightrightarrows \pi_1 K_1$. The requirement that $\pi_1 K_0 \longrightarrow \pi_1 K$ be injective is equivalent to the homomorphisms a, b being injective. Thus an <u>injective free product</u> is a free product of a groupoid amalgamated over injections.

Let Wald_0 be the class of groupoids with one element in each component, and the empty groupoid. Inductively define Wald_n to be the class of groupoids L which are injective free products of $L_0 \rightrightarrows L_1$, with $L_0, L_1 \in \text{Wald}_{n-1}$, and $\text{Wald} = \cup_n \text{Wald}_n$. The name stems from the investigation of this class of groupoids by Waldhausen [21] in conjunction with his work on Whitehead groups. A subgroupoid of a groupoid in Wald is also in Wald, and an extension of one by another is again a Waldhausen groupoid.

We can finally state the theorem.

<u>3.3.7. Theorem</u>: Suppose K is a k-ad with $Wh(\pi_1 K \times G) = 0$ for each $G \in \text{Wald}$, and M is a manifold n-ad.

a) If $\pi_1 \delta_0 M$ is a lattice of Waldhausen groups, then the assembly $\Delta(M, \partial_0 M; \mathbb{L}_j K)_p \longrightarrow \mathbb{L}_{j+m}(K \times \delta_0 M)$ is a natural retraction (has a natural right inverse) when $j - k \geq 5$.

b) If $\delta_0 M$ is a $K(\pi_1 \delta_0 M, 1)$ of Waldhausen groups, then the assembly is a homotopy equivalence, $j - k \geq 5$.

<u>Proof:</u> Here p denotes the product orientation homomorphism $\pi_1(K \times \delta_0 M) \longrightarrow \pi_1 K \longrightarrow \mathbb{Z}_2$. a) is well known in the simply connected case and is easy to see in special cases using b). b) is a generalization

of Shaneson's calculation [16],[17] (essentially $M = S'$, $K = pt$), and of Wall's results in [26], which concern groups built up by extensions by $\mathbb{Z}$, $\partial_0 M = \emptyset$, and $K = pt$. The Whitehead group hypothesis can be weakened in several ways, for example to vanishing on products with Wald_n groups if $\pi_1 \delta_0 M$ is a lattice of Wald_n groups. These hypotheses can be stated in terms of the structure of $\pi_1(K)$ alone, [1], [21], and are satisfied for example by the Waldhausen groupoids themselves.

In case $\pi_1 \delta_0 M$ is a Wald_n lattice, the proposition follows by induction from the groupoid case (M a 2-ad) using the boundary fibration sequence and naturality.

To start the induction on n, we show that the proposition holds for $n = 0$, M a 2-ad. Beginning with b), M is contractible and ∂M a homology sphere. Let $(M, \partial M) \longrightarrow (D^m, S^{n-1})$ be a homology equivalence, then since $\mathbb{L}_j(K)$ is a loop space $\Delta(D^m, S^{m-1}; \mathbb{L}_j K) \to \Delta(M, \partial M; \mathbb{L}_j K)$ is a homotopy equivalence. The loop relations, which state that the assembly map $\Omega^m \mathbb{L}_j K \longrightarrow \mathbb{L}_{j+m} K$ is a homotopy equivalence, shows b) for Wald_0 groupoids.

Now for a), if $\partial M \neq \emptyset$ we can define a map $\emptyset : (M, \partial M) \longrightarrow (D^m, S^{m-1})$ which is an isomorphism on $\emptyset^{-1}(\text{int } D^m)$. Naturality of the assembly gives a diagram

$$\begin{array}{ccc} \Delta(M, \partial M; \mathbb{L}_j K) & \xrightarrow{A_m} & \mathbb{L}_{j+m}(K \times \delta_0 M) \\ \downarrow & & \downarrow \\ \Delta(D^m, S^{n-1}; \mathbb{L}_j K) & \longrightarrow & \mathbb{L}_{j+m}(K \times \delta_0 D^m) \end{array}$$

The bottom and right maps are homotopy equivalences, and composing with a homotopy inverse gives a retraction.

If $\partial M = \emptyset$, then the inclusion $D^m \subset M$ yields a fibration sequence

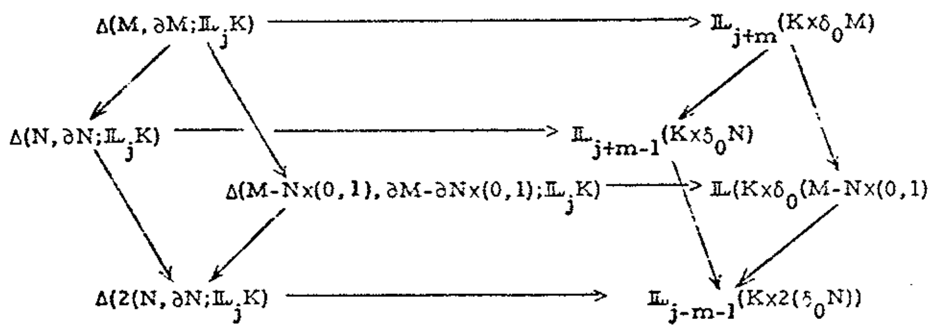
$$\begin{array}{ccccc}
\Delta M\text{-int}\,D^m, \delta^{m-1}; \mathbb{L}_j) & \longrightarrow & \Delta(M, \mathbb{L}_j) & \longrightarrow & \Delta(D^m, \mathbb{L}_j) \\
\big\downarrow \uparrow & & \big\downarrow & & \\
\mathbb{L}_{j+m} & \xrightarrow{\ \sim\ } & \mathbb{L}_{j+m} & &
\end{array}$$

and the retraction is given by the bounded case.

It is well defined up to homotopy, and natural with respect to natural maps of $\mathbb{L}_j(K)$, and boundary when extended to lattices.

Now suppose 3,3,7 holds for lattices of Wald_{n-1} groupoids, and consider a connected 2-ad M with $\pi_1 \delta_0 M \in \mathrm{Wald}_n$. Let $\pi_1 \delta_0 M$ be the free product over injections $L_0 \rightrightarrows L_1$, $L_0, L_1 \in \mathrm{Wald}_{n-1}$. Suppose we could find a 2-sided codimension 1 submanifold $N \times I \subset M$ such that $\pi_1 N = L_0$, $\pi_1(M - N \times (0,1)) = L_1$, and the inclusions on the ends of $N \times I$ induce the homomorphisms $L_0 \rightrightarrows L_1$ above. Since $\pi_1 \delta_0 N \longrightarrow \pi_1 \delta_0 M$ is injective, the transversality theorem 3.3.4 applies. Take the natural (assembly) map of the mapping space pullback 3.3.6 to the $\mathbb{L}$ space pullback given by 3.3.4.

By hypothesis since $\pi_1 N$, $\pi_1(M-N)$ are Wald_{n-1} groupoids, there are natural right inverses for the lower three horizontal maps. By naturality they commute, and induce a map of pullbacks. This is easily seen to be a retraction, as required by a) for Wald_n groups.



If $\delta_0 N$ and $\delta_0(M-N \times (0,1))$ are $K(L_i, 1)$'s, then the hypotheses for b) is satisfied on the lower maps so they and thus the top map are homotopy equivalences. This gives b) for Wald_n groups.

Thus we have reduced 3.3.7 to a problem of splitting a manifold, given a splitting of its fundamental group. To proceed requires a lemma.

<u>3.3.8. Lemma</u>: If $K_0 K_1$ are $K(\pi_1(K_i), 1)$'s $f, g: K_0 \longrightarrow K_1$ are injections on π_1, then $K = K_1 \amalg_{f\times\{0\}\cup g\times\{1\}} K_0 \times I$ is a $K(\pi_1 K, 1)$. Thus each Waldhausen lattice has a $K(L, 1)$ which is a finite complex.

This follows easily from inspection of the universal cover $\widetilde{K}$. By the injectivity of f, g, it is composed of copies of $\widetilde{K}_0 \times I$ joining disjoint copies of $\widetilde{K}_1$. A huge Meyer-Vietoris sequence shows it has the homology of a point, hence is contractible. $\square$

Note that 3.3.8 implies 3.3.7 is non-vacuous. Given $G \in$ Wald, embed a finite $K(G, 1)$ in $\mathbb{R}^\ell$, some ℓ. A regular neighborhood M satisfies the hypotheses of 3.3.7 b), so $\mathbb{L}_{j+\ell}(G) = \Delta(M, \partial M; \mathbb{L}_j(\mathrm{pt}))$.

We must now split manifolds. First the dimension may have to be raised. Let $n \geq 0$, then periodicity and adjointness give a commutative diagram

$$
\begin{array}{ccc}
\Delta(M, \partial_0 M; \mathbb{L}_j(K)) & \xrightarrow{\ A\ } & \mathbb{L}_{j+m}(K \times \delta_0 M) \\
\downarrow \ \Delta(1, \times(\mathbb{C}P^2)^n) & & \downarrow \ \times(\mathbb{C}P^2)^n \\
\Delta(M, \partial_0 M; \Omega^{4n}\, \mathbb{L}_j(K)) & \longrightarrow & \mathbb{L}_{j+m+4n}(K \times \delta_0 M) \\
\downarrow \wr & \nearrow A & \\
\Delta(M \times D^{4n}, \partial_0 M \times D \cup M \times S^{4n-1}; \mathbb{L}_j(K)) & &
\end{array}
$$

The hypotheses on $\pi_1 \delta_0 M$ are preserved by this operation.

3.3.7 b) can now be completed. By 3.3.8 let K be a finite complex $K(\pi_1 \delta_0 M, 1)$ with a splitting corresponding to the presentation of $\pi_1 \delta_0 M$ as an injective free product, $K = K_1 \cup_{K_0 \times \{0,1\}} K_0 \times [0,1]$. If we multiply M by D^{4n} with $4n > 2 \dim K$, $M \times D^{4n}$ is a regular neighborhood of K by the s-cobordism theorem. The regular neighborhood over $K_0 \times I$ is a collar on a thickening of K_0 embedded in codimension 1 in $M \times D^{4n}$. This gives the desired splitting of $M \times D^{4n}$, and the induction hypothesis applies to the pieces to show the bottom assembly map in the diagram is a homotopy equivalence. Since the periodicity maps are homotopy equivalences the top map is one also.

To complete 3.3.7 a), a splitting of M by a transversal inverse image of $K_0 \times \frac{1}{2}$ which induces isomorphism on fundamental groups must be found. If $m \geq 6$ this can easily be done by modifying the standard embedding theorem to correct the cobordism arising in the similar construction at the bottom of page 37, or directly by very low dimensional surgery ambiently on the inverse image. For low dimensional M the retraction is defined by multiplying by D^8 as above, and applying the high dimensional case. Naturality of the retraction with respect to natural maps of $\mathbb{L}_j(K)$ (periodicity in this case) implies that it is well defined and has the required naturality properties. □

We close with a simple calculation.

3.3.9. Corollary: Suppose G is a 3-dimensional knot group, and $G \longrightarrow \mathbb{Z}$ is the quotient by the commutation subgroup. Then $\mathbb{L}_6(G) \xrightarrow{\simeq} \mathbb{L}_6(\mathbb{Z}) = \mathbb{L}_6(pt) \times \mathbb{L}_5(pt)$ is a homotopy equivalence.

Proof: Let c be a PL loop in S^3 with $\pi_1(S^3 - c) = G$, and let M be the complement of an open tubular neighborhood of c. There is a homology isomorphism $(M, \partial M) \longrightarrow (S^1 \times D^2, S^1 \times S^1)$. By [21], G is a Waldhausen group and M is a $K(G,1)$. Since $\mathbb{L}_5(pt)$ is a loop space, the following are all homotopy equivalences.

$$\mathbb{L}_8(G) \xleftarrow{\simeq} \Delta(M, \partial M; \mathbb{L}_5 pt) \xrightarrow{\simeq} \Delta(S^1 \times D^2, S^1 \times S^1; \mathbb{L}_5\, pt)$$

$$\xrightarrow{\simeq} \Delta(S^1, \mathbb{L}_7\, pt) \simeq \Omega(\mathbb{L}_7\, pt) \times \mathbb{L}_7\, pt \simeq \mathbb{L}_8(pt) \times \mathbb{L}_7(pt).$$

Periodicity applied to both sides brings the dimensions down to those claimed. In Chapter 4 we will see that $\mathbb{L}_5(pt) \simeq \Omega G/TOP$, so further periodicity gives $\mathbb{L}_5(G) \simeq G/TOP \times \Omega G/TOP$. □

Corollary 3.3.9 was conjectured in the 1969 Princeton thesis of S. Cappell. A special case is treated there, and the corresponding statement for 4-dimensional knots is shown to be false.

4. Structures on Poincaré spaces

In this chapter we study the question of when a Poincaré space is homotopy equivalent to a manifold, and the problem of classifying such structures when one exists. This is the problem surgery was evolved to answer ([2], [27, Chap. 10], [20]). This theory fits very well into our geometric setting (see [13] for the simply connected case), and solves some problems which cannot be effectively attacked with the group formulation.

4.1. The structure sequence

Suppose X is an (X-simple) Poincaré n-ad of dimension m, and $\partial_0 X$ is a manifold $(n-1)$-ad in the category $\mathcal{C}$ = diff, PL, or top. Define Δ-sets $\underline{S^h_{\mathcal{C}}(X, \partial_0 X)}$ and $\underline{S^s_{\mathcal{C}}(X, \partial_0 X)}$ with k simplices homotopy equivalences (simple for S^s) $M \longrightarrow X \times \Delta^k$ of $(n+k+2)$-ads with $M \in \mathcal{C}$ and $\partial_{k+3} M \longrightarrow \partial_0 X \times \Delta^k$ a $\mathcal{C}$ isomorphism. Under the same conditions define $\underline{NM_{\mathcal{C}}(X, \partial_0 X)}$ as the Δ-set with k-simplices $\mathcal{C}$ - surgery maps $M \longrightarrow X \times \Delta^k$ of $(n+k+2)$-ads which is a $\mathcal{C}$ -isomorphism $\partial_{k+3} M \longrightarrow \partial_0 X \times \Delta^k$.

Since a homotopy equivalence is a surgery map, there is a natural forgetful map $S(X, \partial_0 X) \longrightarrow NM(X, \partial_0 X)$. Moreover since a $\mathcal{C}$ isomorphism is a homotopy equivalence, there is a natural map $NM(X, \partial_0 X) \longrightarrow \mathbb{L}_m(\delta_0 X)$.

4.1.1. Definition: The sequence $S(X, \partial_0 X) \longrightarrow NM(X, \partial_0 X) \longrightarrow \mathbb{L}_m(\delta_0 X)$

is called the structure sequence of $(X, \partial_0 X)$.

Before considering the naturality properties of the structure sequence, which are many, we prove the main theorem concerning it.

4.1.2. Theorem: If X is a Poincaré n-ad of dimension m, $\partial_0 X$ is a manifold $(n-1)$-ad, and $m-n \geq 4$, then the structure sequence of $(X, \partial_0 X)$ is a homotopy fibration over the identity component of $\mathbb{L}_m \delta_0 X$.

Proof: This is a geometric version of the structure sequence [27, Chap. 10]. The considerations there construct a map $\mathbb{L}_{m+1}(\delta_0 X) \longrightarrow S(X, \partial_0 X)$, so the long exact sequence of groups implies as usual that the maps form a fibration sequence. This map is constructed using the algebraic characterization of the obstruction groups. To avoid the algebra, which we do not find illuminating, we give an alternate proof by constructing this map directly.

If the structure sequence is to be a fibration, then $S(X, \partial_0 X) \longrightarrow NM(X, \partial_0 X)$ will be a homotopy principal $\Omega\, \mathbb{L}_m(\delta_0 X) = \mathbb{L}_{m+1}(\delta_0 X)$-fibration. We actually construct the action of $\mathbb{L}_{m+1}(\delta_0 X)$ on $S(X, \partial_0 X)$ by constructing a homotopy of the projection $S(X, \partial_0 X) \times \mathbb{L}_{m+1}(\delta_0 X) \longrightarrow \mathbb{L}_{m+1}(\delta_0 X)$ to a map in which the image of each simplex is a normal cobordism from its homotopy equivalence factor to another homotopy equivalence. A map from the product to $S(X, \partial_0 X)$ is then given by taking the second homotopy equivalence. The construction will also show that any two actions constructed in this way are homotopic.

<u>4.1.3. Lemma</u>: Suppose M^m is a manifold pair $m \geq 5$ and $f: N^{m+1} \longrightarrow Y$ a map of manifold triads such that

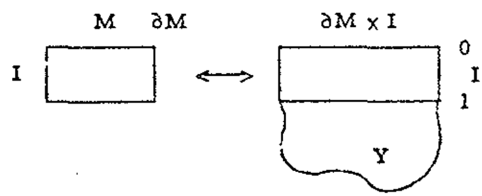
1) $\partial_0 Y = \partial M \times I$, and $\partial_0 f$ is a normal cobordism from $1_{\partial M}$ over $\partial M \times \{0\}$ to another homotopy equivalence.

2) $\partial_1 f$ is a homotopy equivalence.

3) the projection $\partial_0 Y \longrightarrow \partial M$ extends to a map $(Y, \partial_0 Y) \longrightarrow (M, \partial M)$ making f a surgery map over M.

Then $N \longrightarrow Y \longrightarrow M$ is cobordant, as a surgery map over M and rel $\partial_0 f$, to a map of triads $N' \longrightarrow M \times I \longrightarrow M$ which is a normal cobordism from 1_M to a homotopy equivalence.

<u>Proof</u>: First make f a product $\partial_0 f \times 1_I$ on a collar $\partial_0 N \times I$ of the boundary. f is then $1_{\partial M} \times 1_I$ over $M \times \{0\} \times I \subset \partial_0 Y \times I$, so we can glue a copy of $1_{M \times I}$ to f over $1_{\partial M \times I}$. After a little homotopy, the reference map $Y \longrightarrow M$ can be extended over $Y \amalg_{\partial M \times I} M \times I$ by the projection $M \times I \longrightarrow M$.



Call this object Y', then $\pi_1 Y' \longrightarrow M$ is surjective since we glued on a copy of M. Kill the kernel as usual by surgery on some copies of $S' \subset \operatorname{int} Y'$, and cover as in [27, Chap. 9] by surgery on $N \cup M \times I$.

Call it Y' again, then $\pi_1 M = \pi_1(M \times \{0\} \amalg_{\partial M} \partial M \times I \times \{0\}) = \pi_1 Y'$, so since the rest of $\partial Y'$ is covered by homotopy equivalence, the surgery lemma applies to $(Y', M \times \{0\} \amalg_{\partial M} \partial M \times I \times \{0\})$. The surgery lemma provides a cobordims as a surgery map of f' to a homotopy equivalence, keeping the rest of $\partial Y'$ fixed. Altering perspective a little, this cobordism is just a cobordism of f' as a surgery map over M to the map over the cobordism of the boundary $M \times \{0\} \amalg_{\partial M} \partial M \times I \times \{0\}$. This last is a normal cobordism of 1_M to a homotopy equivalence of M which is $\partial_0 f$ on the boundary, which is the conclusion of the lemma. □

Using the lemma it is simple to complete the proof of 4.1.2. First, by a fibration over the identity component, we mean that if A is the component of $NM(X, \partial_0 X)$ whose image lies in the identity component $\mathbb{L}_m(\delta_0 X)_e$, then $S(X, \partial_0 X) \subset A$, and $S(X, \partial_0 X) \longrightarrow A \longrightarrow \mathbb{L}_m(\delta_0 X)_e$ is a homotopy fibration. If $A = \emptyset$, then $S(X, \partial_0 X) = \emptyset$, and the theorem is trivially satisfied. On the other hand if $A \neq \emptyset$, then a path from the image of an element to $\emptyset$ gives a surgery map over $(\delta_0 X, \delta_0 X)$, and the surgery map given by the surgery lemma in this circumstance provides an element of $S(X, \partial_0 X)$.

Thus suppose $S(X, \partial_0 X) \neq \emptyset$, and suppose inductively that a homotopy of the projection $S(X, \partial_0 X) \times \mathbb{L}_{m+1}(\delta_0 X) \longrightarrow \mathbb{L}_{m+1}(\delta_0 X)$ has been constructed to a map m_{k-1} such that the image of any j-simplex has the correct form for $j \leq k - 1$. Consider the image of a k-simplex. In the first factor (in $S(X, \partial_0 X)$) use a mapping cylinder cobordism to replace the homotopy equivalence with the identity map of the domain. We are now almost in the situation of 4.1.3. The part of the boundary of the image which form its faces as a k-simplex have the correct form by hypothesis. Glue a copy of $1_{\partial_0 X \times \Delta^k} \times I$ on a collar neighborhood of its boundary as a k-simplex and hold the side $1_{\partial_0 X \times \Delta^k} \times \{0\}$ fixed, incorporating the rest in the face mapped by homotopy equivalence. Now the smallest faces of the image as an object over $\delta_0 X$ (i.e. over $\partial_{\{1,\ldots,j,j+1,\ldots,n-2\}} X$) satisfy the hypotheses of the lemma. Glue on the resulting cobordisms to maps of the correct form, and add the corrected part of the boundary to the part being held fixed. Now the next faces satisfy 4.1.3. Applying 4.1.3 inductively to successively higher dimensional faces we get a cobordism of the image to an object which itself satisfies 4.1.3, and thence to a normal cobordism of the correct form. A choice of such a cobordism for each k-simplex can easily be used to construct a homotopy of M_{k-1} to M_k satisfying the conditions on the k-skeleton.

There is a canonical homotopy of $S(X, \partial_0 X) \longrightarrow NM(X, \partial_0 X) \longrightarrow \mathbb{L}_m \delta_0 X$ to the trivial map. To show the sequence is a homotopy fibration, it therefore suffices to show the resulting homomorphisms $\pi_j(NM(X, \partial_0 X), S(X, \partial_0 X)) \longrightarrow \pi_j(\mathbb{L}_m(\delta_0 X), *)$ are isomorphisms. An element in the left group is just a surgery map $M \longrightarrow X \times D^j$ of (n+1)-ads which is an isomorphism on $\partial_0 X \times D^j$, and a homotopy equivalence on $X \times S^{j-1}$. The homomorphism is injective since a homotopy of such an element to 0 in the right side provides a cobordism of it to 0, and by the surgery lemma a normal cobordism rel the boundary to a homotopy equivalence. This last, however, is just a homotopy to zero in the left pair.

To prove surjectivity, since the result is only expected to hold for $j \geq 1$, we take a relative loop space,

$$\Delta(I, \{0,1\}; NM(X, \partial_0 X), S(X, \partial_0 X) \longrightarrow \Omega\, \mathbb{L}_m \delta_0 X = \mathbb{L}_{m+1}(\delta_0 X) .$$

The action above was just a factoring of the projection of $S(X, \partial_0 X) \times \mathbb{L}_{m+1}(\delta_0 X)$ through this map, so it is surjective in homotopy. □

We make one further refinement in the structure sequence. The $B_{\mathcal{L}}$ reductions of the normal bundle give a map $NM(X, \partial_0 X) \longrightarrow \Delta(X; B_{\mathcal{L}})$ whose image under the composition with $B_{\mathcal{L}} \longrightarrow B_G$ is just the single point $\{\nu_X\}$. If $NM(X, \partial_0 X) \neq \emptyset$, then we can subtract one reduction off of all the others, translating the classifying map over to the base point, and lift to the fiber $NM(X, \partial_0 X) \longrightarrow \Delta(X, \partial_0 X; G/\mathcal{L})$. $\partial_0 X$ is taken to the base point since all reductions agree there by assumption. This map is a homotopy equivalence with homotopy inverse easily constructed using transversality. For transversality in the topological category we need $m - n \geq 6$ here [8], but we will always be able to avoid this restriction using periodicity.

Thus if $S(X, \partial_0 X) \neq \emptyset$ (and hence $NM(X, \partial_0 X) \neq \emptyset$), we get a homotopy fibration sequence

$$S_{\mathcal{L}}(X, \partial_0 X) \longrightarrow \Delta(X, \partial_0 X; G/\mathcal{L}) \longrightarrow \mathbb{L}_m(\delta_0 X)$$

for $m - n \geq 4$.

4.1.4. Proposition: (naturality of the structure sequence) If X is an (X-simple for S^s, $\mathbb{L}^s$) Poincaré n-ad of dimension m with $\partial_0 X$ given a manifold structure, $S(X, \partial_0 X) \neq \emptyset$, and $m - n \geq 4$, then the following diagrams commute.

a) boundary

$$\begin{array}{ccccc}
S(X,\partial_0 X) & \longrightarrow & \Delta(X,\partial_0 X; G/\mathcal{C}) & \longrightarrow & \mathbb{L}_m(\delta_0 X) \\
\downarrow & & \downarrow & & \downarrow \\
S(X) & \longrightarrow & \Delta(X; G/\mathcal{C}) & \longrightarrow & \mathbb{L}_m(X) \\
\downarrow \partial & & \downarrow & & \downarrow \partial \\
S(\partial_0 X) & \longrightarrow & \Delta(\partial_0 X; G/\mathcal{C}) & \longrightarrow & \mathbb{L}_{m-1}(\partial_0 X)
\end{array}$$

is a fibration square.

b) pullback $\xi : \epsilon \longrightarrow X$ (block) fibration, compact manifold fibers

$$\begin{array}{ccccc}
S(X,\partial_0 X) & \longrightarrow & \Delta(X,\partial_0 X; G/\mathcal{C}) & \longrightarrow & \mathbb{L}\,(\delta_0 X) \\
\downarrow P_\xi & & \downarrow \Delta(\xi,1) & & \downarrow P_{\delta_0 \xi} \\
S(E,\partial_0 E) & \text{---} & \Delta(E,\partial_0 E; G/\mathcal{C}) & \longrightarrow & \mathbb{L}(\delta_0 E)
\end{array}$$

c) change of category if $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a natural inclusion (one of DIFF $\longrightarrow$ PL $\longrightarrow$ TOP) then there is a fibration square

$$\begin{array}{ccccc}
S_{\mathcal{C}_1/\mathcal{C}_2}(X,\partial_0 X) & \longrightarrow & \Delta(X,\partial_0 X; \mathcal{C}_1/\mathcal{C}_2) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
S_{\mathcal{C}_1}(X,\partial_0 X) & \longrightarrow & \Delta(X,\partial_0 X; G/\mathcal{C}_1) & \longrightarrow & \mathbb{L}_m(\delta_0 X) \\
\downarrow & & \downarrow & & \downarrow \\
S_{\mathcal{C}_2}(X,\partial_0 X) & \longrightarrow & \Delta(X,\partial_0 X; G/\mathcal{C}_2) & \longrightarrow & \mathbb{L}_m(\delta_0 X)
\end{array}$$

d) assembly: suppose $\pi : E \longrightarrow B$ is a homotopy fibration with fiber X over a manifold B^b, and is a block fibration $\partial_0 X \longrightarrow \partial_0 E \longrightarrow B$ and $X \longrightarrow \partial_{n-1} E \longrightarrow \partial_0 M$. Then

$$\begin{array}{ccccc}
\Gamma_B(S(\pi, \partial_0 \pi)) & \longrightarrow \Delta(E, \partial_0 E \cup \partial_{n-1} E; G/\mathcal{L}) & \longrightarrow & \Gamma_B(\mathbb{L}_m \delta_0 \pi) \\
\Big\downarrow A\pi & \Big\downarrow 1 & & \Big\downarrow A_{\delta_0 \delta_{n-1} \pi} \\
S(E, \partial_0 E \cup \partial_{n-1}) & \longrightarrow \Delta(E, \partial_0 E \cup \partial_{n-1} E; G/\mathcal{L}) & \longrightarrow & \mathbb{L}_{m+b}(\delta_0 \delta_{n-1} E)
\end{array}$$

We omit the proof, which is just a long verification of the definitions, and give some remarks.

First, since $\mathbb{L}_m(\delta_0 X)$ does not depend on the category, c) gives smoothing and triangulating theory; $S_{\mathcal{L}_1/\mathcal{L}_2}(X, \partial_0 X) \xrightarrow{\simeq} \Delta(X, \partial_0 X; \mathcal{L}_1/\mathcal{L}_2)$. To obtain this we have used immersion theory ([9] in the topological case) to do surgery, and transversality (which we regard as the main import of [8] in the topological category) to evaluate $NM_{\mathcal{L}}(X, \partial_0 X)$.

Second, d) gives a geometric formulation of the fiber of the assembly map; the classifying space of the fiber of $\Gamma_B S(\pi, \partial_0 \pi) \longrightarrow S(E, \partial_0 E \cup \partial_{n-1} E)$. This shows that it is interesting rather than leading at once to calculations. Section 4.2 will explore this point a little further.

If M is closed and simply connected, the structure sequence actually splits. Let D be a disc embedded in M, and set $M_o = M - \text{int } D$. Thinking of D as a framed embedding of a point, then for $m \geq 5$ the codimension 3 embedding theorem [27] implies any homotopy equivalence $N \to M$ can be split into homotopy equivalences of pairs $(N_o, \partial N_o) \to (M_o, \partial M_o)$ and $(\overline{N - N_o}, \partial N_o) \to (D, \partial D)$. More generally a homotopy equivalence of (k+2)-ads $N \to \Delta^k \times M$ can be split into homotopy equivalences over $\Delta^k \times M_o$ and $\Delta^k \times D$, extending a splitting given on the boundary. Taking the first factor definesa map of Δ-sets $S_{\mathcal{C}}(M) \to S_{\mathcal{C}}(M_o)$, which fits into a commutative diagram

$$\begin{array}{ccccc} S_{\mathcal{C}}(M) & \longrightarrow & \Delta(M, G/\mathcal{C}) & \longrightarrow & \mathbb{L}\, m(*) \\ \downarrow & & \downarrow & & \downarrow \\ S_{\mathcal{C}}(M_o) & \longrightarrow & \Delta(M_o, G/\mathcal{C}) & \longrightarrow & \mathbb{L}\, m(*,*) \end{array}$$

The middle map is the restriction, which has fiber $\Omega^m(G/\mathcal{C})$. $\mathbb{L}\, m(*,*) = *$ since $\pi_1 * = \pi_1 *$. If $\mathcal{C}$ = TOP, or PL, then the fiber of the left vertical map is also trivial. A homotopy inverse is constructed by: on the 0-skeleton if $m = 5$, ∂N_o is a homotopy 4-sphere, thus h-cobordant to S^4. The map $N_o \to M_o$ extends to $N_o \cup$ (h-cobordism ∂N_o to S^4) $\to M_o$, then add the cone over the boundary of both, and extend linearly. For higher dimensional skeleta, and $m > 5$, after a map has been defined on the k-skeleton, the missing part in an element of the k+1 skeleton has as boundary a homotopy sphere, hence a sphere by the Poincaré conjecture, and again adding the cone over this boundary and extending the map linearly defines an element of the k-skeleton of $S_{\zeta}(M)$.

The fibers of the diagram are therefore

$$\begin{array}{ccccc}
* & \longrightarrow & \Omega^m(G/\zeta) & = & \Omega^m(G/\zeta) \\
\downarrow & & \downarrow & & \downarrow \wr \\
S_{\zeta}(M) & \longrightarrow & \Delta(M,G/\zeta) & \longrightarrow & \mathbb{L}_m(*) \\
\downarrow \wr & & \downarrow & & \downarrow \\
S_{\zeta}(M_0) & \underset{\sim}{\longrightarrow} & \Delta(M_0,G/\zeta) & \longrightarrow & *
\end{array}$$

For $\pi_1 M = 0$ and ζ = PL or TOP, this gives a splitting of the structure sequence, and calculates $S_{\zeta}(M) = \Delta(M_0, G/\zeta)$, $\mathbb{L}_5(*) = \Omega^5(G/TOP) = \Omega^5(G/PL)$. The exact periodicity of G/TOP under $\times\, \mathbb{C}P^2$ will allow us to improve dimension restrictions in many cases [8].

This shows that the map $\Delta(X, \partial_0 X; G/\mathrm{top}) \longrightarrow L_m(\delta_0 X)$ is itself an assembly map, being essentially $\Delta(X, \partial_0 X; \mathbb{L}_0(*)) \longrightarrow \mathbb{L}_m(* \times \delta_0 X)$. Thus the results of 3.3 concerning the assembly can be applied directly to the structure sequence.

4.1.5. Corollary: Suppose M^m is a manifold n-ad with $\pi_1 \delta_0 M \in$ Wald, and $m - n \geq 4$, then

a) $S_{TOP}(M, \partial_0 M) \longrightarrow \Delta(M, \partial_0 M; G/\mathrm{top}) \longrightarrow \mathbb{L}_m(\delta_0 M)$ is naturally split by the right inverse for the assembly given by 3.3.7 a).

b) if M is a $K(\delta_0 M, 1)$, then $S_{TOP}(M, \partial_0 M)$ is contractible.

Proof: First replace M by $M \times D^8$, and use adjointness in the mapping space to obtain a diagram

$$\begin{array}{ccccc} S_{TOP}(M \times D^8, \partial_0 M \times D^8 \cup M \times S^7) & \longrightarrow & \Delta(M, \partial_0 M; \Omega^8 G/TOP) & \longrightarrow & \Omega^8 \mathbb{L}_m(\delta_0 M) \\ & & \Big\downarrow \Delta(1, E_{D^8}) & & \Big\downarrow \\ & & \Delta(M, \partial_0 M; \mathbb{L}_8(*)) & \xrightarrow{A_M} & \mathbb{L}_{m+8}(* \times \partial_0 M) \end{array}$$

The two vertical maps are homotopy equivalences, and the bottom map has a natural retraction by 3.3.7 a). The top sequence is isomorphic to the one in 4.1.5 a) by periodicity (exact on G/TOP) so it is also naturally split. In case b), the lower map is a homotopy equivalence by 3.3.7 b), so the fiber is contractible. □

Note that change of categories now calculates $S_{\mathcal{L}}(M, \partial_0 M)$ in case b) as $\Delta(M, \partial_0 M; TOP/\mathcal{L})$. In particular since TOP/PL is a $K(\mathbb{Z}_2, 3)$, $S_{PL}(M, \partial_0 M) \xrightarrow{\sim} \Delta(M, \partial_0 M; K(\mathbb{Z}_2 3))$ for M a Waldhausen $K(\pi_1 \delta_0 M, 1)$.

4.2. The fibration problem

Now we present a problem which this formulation of surgery is particularly well adapted to solve. Suppose $f: M^m \longrightarrow N^n$ is a map of closed manifolds $F \longrightarrow E \xrightarrow{\pi} N$ the homotopy equivalent fibration. The question is, if F has the homotopy type of a finite complex, when is f homotopic to a fibration of some kind. When $m = n$ and $F = pt$, this is the homotopy equivalence problem solved in 4.1 by a technique essentially introduced in part II of [20]. The solution given below, for $\dim F \geq 5$, is similar to the techniques of part I of [20]. This problem has also been considered in the case $N = S^n$ by A. J. Casson [5].

First we elaborate the problem to include the relative version. Suppose M is a j-ad, N a k-ad $j \geq k$, and $f: M \longrightarrow N$ is a map of k-ads after forgetting the last j-k faces of M. The fiber of f is a k-lattice of (k-j+1)-ads, with $F_\gamma =$ fiber $(\partial_\sigma M \longrightarrow \partial_\alpha N)$, $\alpha \subset \{0, \ldots, k-2\}$. Say that <u>f has fiber F</u> if all the morphisms in the lattice are homotopy equivalences $F_\alpha \xrightarrow{\sim} F_\beta \xrightarrow{\sim} F$ (of (k-j+1)-ads).

Now suppose $f: M \longrightarrow N$ is a map of Poincaré k-ads, with M a j-ad, $\pi: \epsilon \longrightarrow N$ the equivalent fibration which has fiber a (j-k+1)-ad F with the homotopy type of a finite complex (j-k+1)-ad.

4.2.1. Lemma: F is a Poincaré $(j-k+1)$-ad of dimension $m-n$. Further if $V \subset N$ is a submanifold k-ad with f transversal to V, $f^{-1}(V) \longrightarrow \pi^{-1}(V)$ is a surgery map in a natural way.

Proof: This is another spectral sequence argument, which we will write out this time. By induction it is sufficient to let $k = 2$, $j = 3$, so F is also a 2-ad. Let $\pi_1 F \longrightarrow \pi_1 E \longrightarrow \mathbb{Z}_2$ induce the antiinvolution of $\mathbb{Z}(\pi_1 F)$, and define $v \in H^t_{m-n}(F, \partial F; \mathbb{Z})$, where $F = \pi^{-1}(x)$, by: let U be a disc neighborhood of x, then the projection $U \longrightarrow x$ induces a map $\pi^{-1}(U) \longrightarrow F$ which is homotopy equivalent to the projection $F \times U \longrightarrow F$. There is a Thom isomorphism

$$\begin{array}{ccc} H^t_*(F,\partial F;\mathbb{Z}) & \xrightarrow{\simeq} & H^t_{*+n}(\pi^{-1}U, \pi^{-1}\partial U \cup \pi^{-1} U \cap \partial F;\mathbb{Z}) \\ & & \Big\downarrow \text{excision} \\ H^t_{*+n}(E,\partial E;\mathbb{Z}) & \xrightarrow[\text{inclusion}]{} & H^t_{*+n}(E, \pi^{-1}(N-U) \cup \partial E; \mathbb{Z}), \end{array}$$

and v is the preimage of the image of the fundamental class in $H^t_m(E, \partial E; \mathbb{Z})$. We want to show v is a fundamental class for F.

Next assume N is a smooth manifold (by taking a thickening and taking the induced thickening of N. The conclusions for F will be the same). Take a handlebody decomposition of $(N, \partial N)$. Denote the cellular chains with respect to this handlebody by $C^c_*(N, \partial N)$, and the cellular chains of the dual handlebody by $C^c_*(N)$. Intersection gives an isomorphism $I: C^c_*(N, \partial N) \xrightarrow{\simeq} \hom_\Gamma(C^c_*(N), \Gamma)$ where $\Gamma = \mathbb{Z}(\pi_1 N)$. Use these filtrations of N to construct the homology and cohomology spectral sequences of the fibration $(E, \partial_1 E) \longrightarrow N$. These have E' terms

$$E^1_{ij}(\mathcal{H}) = H^t_j(F, \partial F; \Lambda) \otimes_\Gamma C^c_i(N, \partial N)$$

$$E_1^{ij}(\mathcal{H}^*) = H^j(F; \Lambda) \otimes_\Gamma C^i_c(N) .$$

Rotate the cohomology sequence so that it is a homology spectral sequence with $E^1_{ij} = E_1^{n-i, m-m-j}(\mathcal{H}^*)$, then the map $(\cap V) \otimes_\Gamma I$ is a map of spectral sequences. This map resolves $\cap[E, \partial E]$ in the homology and cohomology of E, by construction of v. Since these spectral sequences are bounded, I is an isomorphism, $C^c_j(N, \partial N)$ are finitely generated free, and $(\cap v) \otimes_\Gamma I$ abuts to an isomorphism, Moore's comparison theorem [11] applies to show that an isomorphism is induced on the E^2 terms. In particular the $E^2_{*,0}$ maps are $\cap v\colon H^*(F; \Lambda) \xrightarrow{\simeq} H^t_{m-n-*}(F, \partial F; \Lambda)$. Thus F is an E-Poincaré space. It can also be seen to be an F-Poincaré space since if $\overline{F} \longrightarrow \tilde{E}$ is the induced cover $\pi_1 \overline{F}$ acts trivially on $\tilde{F}$.

If X is a regular value of f, then for some disc U about x, $f^{-1}(U)$ is a disc bundle over $f^{-1}(x)$. Thus naturality of the Thom isomorphism shows $f^{-1}(x) \longrightarrow \pi^{-1}(x)$ is degree 1. The normal bundle of $f^{-1}(x)$ is the normal bundle of M restricted to $f^{-1}(x)$ minus the normal bundle of x in M. Since the classifying map of the normal bundle of M factors through $M \longrightarrow E$, $f^{-1}(x) \longrightarrow \pi^{-1}(x)$ can be made a normal map in a natural way.

The analysis for a submanifold instead of a point is similar; only the notation changes a little. □

Now if N is a PL manifold (and $m-n-k+j \geq 5$ if $\mathcal{C}$ = TOP), then we can make f transversal to a triangulation of N, and obtain a surgery map $f^{-1}(\sigma) \to \pi^{-1}(\sigma)$ over each simplex of N. If σ is a p-simplex this map is a p-simplex of $\mathbb{L}^h_{m-n}(\pi^{-1}\sigma)$, so all of these maps fit together to give a section of the bundle $\mathbb{L}^h_{m-n}(\pi) \to N$. If $\partial_0 M \to \partial_0 N$ and $\partial_{h-1} M \to N$ are already fibrations, then holding them fixed gives a section in $\Gamma_{(N,\partial_0 N)} \mathbb{L}^h_{m-n}(\underline{\delta}_0 \pi)$. By $\underline{\delta}_0\pi$ of a bundle $\pi: E \to B$ over a k-ad we mean the bundle $\delta_{k-1} E \to B$.

The homotopy class of this section depends only on the homotopy class (rel the fixed fibered boundaries) of f, and is zero if f is homotopic to a block fibration (with manifold fibers). Thus this section is an obstruction to the solution of our problem. Note that when assembled into $\mathbb{L}_m(E)$ this section is trivial, since it becomes the obstruction to the structure on E as a surgery map to be cobordant to a homotopy equivalence. This is essentially the solution of the problem, as a formal globalization will show.

If $\pi : E \to B$ is a fibration with k-ad base and Poincaré (j-k+1)-ad fiber F of dimension p, and $\partial_0 E \to \partial_0 B$, $\partial_{k-1} E \to B$ are manifold block fibrations, then there is a sequence of bundle maps over N, $S(\pi, \underline{\partial}_0 \pi) \to NM(\pi, \underline{\partial}_0 \pi) \to \mathbb{L}_p(\underline{\delta}_0 \pi)$. Here S and NM of a bundle are defined as $\mathbb{L}\,\pi$ in 3.2, by $S(\pi, \partial_0\pi) = \cup_{\sigma^q \in B} S(\pi^{-1}\sigma^8, \partial_{k-1}\pi^{-1}\sigma^q)$, etc. If $p+k-j \geq 4$ this sequence is a homotopy fibration over N in the sense that the maps of fibers over each simplex of N are a homotopy fibration. Thus the induced maps of section spaces is a homotopy fibration.

The sections of the middle space are, by transversality, $\Gamma_B NM_{\mathcal{L}}(\pi, \partial_0\pi) \xrightarrow{\simeq} NM_{\mathcal{L}}(E, \partial_{k-1}E) \xrightarrow{\simeq} \Delta(E, \partial_{k-1}E, G/\mathcal{L})$. There is a natural assembly map from this sequence of section spaces to the structure sequence of E, which is the main point of this section.

<u>4.2.2. Diagram</u>: Suppose N is a PL manifold k-ad, E Poincaré j-ad, $\pi: E \longrightarrow N$ a fibration with finite complex fibers which is a $\mathcal{L}$-block fibration $\partial_0 E \longrightarrow \partial_0 N$ and $\partial_{k-1}E \longrightarrow B$. If the normal bundle of E reduces to $B_{\mathcal{L}}$ and $p+k-j \geq 4$ then there is a natural square of fibrations

$$\begin{array}{ccccc}
\Omega X & \longrightarrow & * & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_{(N,\partial_0 N)} S_{\mathcal{L}}(\pi, \underline{\partial}_0\pi) & \longrightarrow & \Delta(E, \partial_0 E \cup \partial_{k-1}E; G/\mathcal{L}) & \longrightarrow & \Gamma_{(N,\partial_0 N)} \mathbb{L}_p(\underline{\delta}_0\pi) \\
\downarrow A_\pi & & \downarrow 1 & & \downarrow A_\pi \\
S_{\mathcal{L}}(E, \partial_0 E \cup \partial_{k-1}E) & \longrightarrow & \Delta(E, \partial_0 E \cup \partial_{k-1}E; G/\mathcal{L}) & \longrightarrow & \mathbb{L}_{n+p}(\delta_0\delta_{k-1}E)
\end{array}$$

The construction of a section in the problem above is just the construction of the analogue of the boundary homomorphism in homology, and gives a map $S_{\mathcal{L}}(E, \partial_0 E \cup \partial_{k-1}E) \longrightarrow X$ whose fiber is now seen to be $\Gamma S_{\mathcal{L}}(\pi, \underline{\partial}_0\pi)$.

This is a formal solution to the problem. Sections of $\Gamma S_{\mathcal{E}}(\pi, \partial_0\pi)$ assemble to give block fibrations if $S^s_{\mathcal{E}}$ is used, by the s-cobordism theorem, and give maps with $f^{-1}(\sigma) \longrightarrow \pi^{-1}(\sigma)$ a homotopy equivalence for S^h. The latter sort of map is easily seen to be h-cobordant to a block fibration, with torsion in a nice subquotient of the Whitehead group. To reduce the corresponding block fibration to some finer sort of fibration becomes a problem of maps of bundles with fibers different classifying spaces. This study of these classifying spaces is very rudimentary as yet and has little to do with the techniques of this thesis. For example, the problem of reducing a block fibration to a fibration involves the space $\tilde{\mathcal{E}}N/\mathcal{E}(N)$, essentially "pseudoisotopy modulo isotopy." The best result known is Rourke's theorem that if N is 1-connected of dimension ≥ 5, then $\tilde{\mathcal{E}}N/\mathcal{E}N$ is also 1-connected.

The solution to the block fibration problem, however, is seen to be again a question about the fiber of the assembly map of $\mathbb{L}$ spaces. Conversely other attacks on the solution of this problem will yield information about the assembly map and hence $\mathbb{L}$ spaces. In particular embedding (splitting) theorems apply directly to this in special cases, and it seems likely that techniques providing a geometric proof of the periodicity theorem would yield useful information here also. At the present we cannot improve much on what can be deduced formally from what we already know about the assembly map in special cases, so an independent investigation will not be made.

Some results can be obtained immediately from the naturality of the diagram 4.2.2. For example, if we change categories, independence of $\mathbb{L}$ on the category shows the obstructions to fibering remain the same. This gives a fibration version of Sullivan's hauptvermutung:

<u>4.2.3. Corollary</u>: If $f: M \longrightarrow N$ is a TOP block fibration over a PL manifold with fibers a k-ad F of dimension p, $p - k \geq 4$, and M and N are ζ (= PL or DIFF) manifolds, then f is homotopic to a ζ block fibration.

Clearly there is a more complicated relative form as in 4.2.2.

More detailed results also follow from restrictions on the fibration. For example if $F \longrightarrow E \longrightarrow B$ is π_1-split ($\pi_1 F \longrightarrow \pi_1 E \longrightarrow \pi_1 B$ is a split short exact sequence) the bundle $\mathbb{L}_j(\underset{\sim}{\delta}_0 \pi)$ becomes a product, and the section space a mapping space. In this form we can apply the results of 3.3. Note that although the base of the fibration must be PL to get block fibrations, the category $\mathcal{S}$ is still arbitrary.

4.2.4. Corollary: Suppose $f: M \longrightarrow N$ is a π_1-split map from a Poincaré j-ad to a PL h-ad, which is a block fibration on $\partial_0 M \longrightarrow \partial_0 N$ and $\partial_{k-1} M \longrightarrow N$, and has fiber a finite complex (j-k+1)-ad. If $m-n-j+k \geq 4$, then

1) if $\pi_1 \delta_0 N \in$ Wald, then the columns of diagram 4.2.2 are projections. Thus each $\mathcal{S}$-manifold structure on M rel $\partial_0 M \cup \partial_{k-1} M$ is homotopic to a $\mathcal{S}$-block fibration in a natural (canonical) way.

2) if $\delta_0 N$ is a Waldhausen $K(G,1)$, then any $\mathcal{S}$ structure on M rel $\partial_0 M \cup \partial_{k-1} M$ is homotopic to a unique $\mathcal{S}$ block fibration.

Proof: 3.3.6 applied to the lower right vertical map in 4.2.2.

On the other hand if $\pi_1 F \in$ Wald we can apply the results of 3.3 to the structure sequence which appears in the fibers of the bundle maps whose sections form the middle row of 4.2.2 $\square$

Slightly more generally, say that a TOP manifold p-ad K satisfies the Poincaré conjecture if $S^s_{TOP}(K, \partial_0 K)$ is contractible. According to 3.3.6, Waldhausen $K(G,1)$'s satisfy the Poincaré conjecture.

Suppose the homotopy fiber of $f: M \longrightarrow N$ is a manifold (j-k+1)-ad of dimension $m-n \geq 4+j-k$ which satisfies the Poincaré conjecture. Then the fibration $S^s_{TOP}(K, \partial_0 K) \longrightarrow S^s_{TOP}(f, \partial_0 f) \longrightarrow N$ has trivial fibers, and the space of sections is contractible. In this situation 4.2.2 becomes

$$\begin{array}{ccccc}
\Omega X & \longrightarrow & * & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Delta(M, \partial_0 M \cup \partial_{k-1} M; G/TOP) & \xrightarrow{\simeq} & \Gamma_{(N, \partial_0 N)} \mathbb{L}^s_{m-n}(f, \partial_0 f) \\
\downarrow & & \downarrow 1 & & \downarrow \\
X & \longrightarrow & \Delta(M, \partial_0 M \cup \partial_{k-1} M; G/TOP) & \longrightarrow & \mathbb{L}^s_m(\partial_0 \partial_{k-1} M) \\
\| & & & & \\
S^s_{TOP}(M, \partial_0 M \cup \partial_{k-1} M) & & & &
\end{array}$$

and we have found another expression for the fiber of an assembly map.

4.2.5. Corollary: If $(K, \partial_0 K)$ is a TOP p-ad of dimension k which satisfies the Poincaré conjecture and $k-p \geq 4$, then any two block fibrations with fiber $(K, \partial_0 K)$ which agree when restricted to ∂_0 and have homotopy equivalent total spaces are isomorphic as fibrations.

Proof: These are sections of $S^s_{TOP}(f, \underset{\sim}{\partial}_0 f) \longrightarrow B$ which has trivial fiber, so they are homotopic. □

Lastly we apply all this to sphere fibrations. Let $\xi : E \longrightarrow B$ be a homotopy S^n fibration, $n \geq 5$, and M_ξ the mapping cylinder, the associated homotopy disc fibration. If the normal bundle of M_ξ has a reduction to B_{TOP}, then the boundary map and the assembly give a diagram of fibration:

$$
\begin{array}{ccccc}
\Gamma_{(B,\partial_0 B)}S_{TOP}(D,\xi) & \longrightarrow & \Delta(M_\xi, E\cup f^{-1}(\partial_0 B); G/TOP) & \longrightarrow & \Delta(B, \Omega^{n+1}G/TOP) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Delta(M_\xi; G/TOP) & \xrightarrow{\simeq} & \Delta(B, \Delta(D^n, G/TOP)) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_{(B,\partial_0 B)}S_{TOP}(\xi) & \longrightarrow & \Delta(E; G/TOP) & \longrightarrow & \Delta(B, \Omega^n G/TOP)
\end{array}
$$

The fibers on the left are contractible if they are not empty (recall these are fibrations over the identity component of the spaces on the right).

If ξ is equivalent to a $\widetilde{TOP}(S^n)$ fibration, then the top map is a homotopy equivalence. An investigation of the multiplicative structure of $\mathbb{L}$ spaces (involving yet another characterization of $\mathbb{L}$ as a cobordism space) shows that the top map is the inverse of a Thom isomorphism. This would recover Sullivan's theorem that a $\widetilde{TOP}(S^n)$-bundle has a Thom class in $K_{G/TOP}$ theory. A good deal of his remarkable converse to this theorem can also be obtained from this theory.

Bibliography

1. H. Bass, Algebraic K-theory, W. A. Benjamin, 1968.

2. W. Browder, Surgery on simply-connected manifolds (to appear).

3. __________, Actions of finite groups, Proc. Georgia Conference, 1969 (to appear).

4. S. Cappell (to appear).

5. A. J. Casson, Fibrations over spheres, Topology 6 (1967), 489-499.

6. J. Cerf, Topologie de certains espaces de plongements, Bull. Soc. Math., France 89 (1961), 227-380.

7. D. M. Kan, Abstract homotopy III, Proc. Nat. Acad. USA 42 (1956) 419-421.

8. R. C. Kirby and L. C. Siebenmann, On the triangulation of manifolds and the hauptvermutung, Bull. AMS 75 (1969), 742-749.

9. J. A. Lees, Immersions and surgeries of topological manifolds, Bull. AMS 75 (1969), 529-534.

10. J. P. May, Simplicial objects in algebraic topology, Van Nostrand, Princeton, 1967.

11. J. C. Moore in Séminaire Henri Cartan 1954-1955, Paris, Ecole Normale Superieure, 1956.

12. F. S. Quinn, A geometric formulation of surgery, Proc. Georgia Conference, 1969 (to appear).

13. C. P. Rourke, The hauptvermutung according to Sullivan, Notes IAS, 1968.

14. C. P. Rourke and B. J. Sanderson, Block bundles I, II, III, Ann. Math. 87 (1968), 1-28, 256-287, 431-483.

15. __________, On the homotopy theory of Δ-sets (to appear).

16. J. L. Shaneson, Wall's surgery obstruction groups for $\mathbb{Z} \times G$, Bull. AMS 74 (1968), 467-471.

17. J. L. Shaneson, Wall's surgery obstruction groups for $\mathbb{Z} \times G$, mimeograph notes, Yale University, 1968.

18. M. Spivak, Spaces satisfying Poincaré duality, Topology 6 (1967), 77-102.

19. R. E. Stong, Notes on cobordism theory, Princeton University Press, 1968.

20. D. Sullivan, Triangulating homotopy equivalences, thesis, Princeton University, 1965.

21. F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. Math. 87 (1968), 56-88.

22. __________, On Whitehead groups of free products with amalgamations (to appear).

23. C. T. C. Wall, Classification problems in differential topology, IV, Topology 5 (1966), 73-94.

24. __________, Finiteness conditions for CW complexes, I Ann. Math. 81 (1965), 56-69; II Proc. Royal Soc. A, 295 (1966), 163-189.

25. __________, Poincaré complexes I, Ann. Math 86 (1967), 211-245.

26. __________, Topological space-forms, Proc. Georgia Conference 1969 (to appear).

27. __________, Surgery of compact manifolds (to appear).

Abstract

In this thesis we present a theory of spaces whose homotopy groups are the various sets of equivalence classes arising in nonsimply connected surgery. Long exact sequences appear naturally as the long exact homotopy sequences of fibrations of spaces. The treatment is also geometric in that algebra is shunned for geometric techniques whenever possible.

After the definition and elementary theory of the spaces is developed, two maps, the pullback map and the assembly map, are defined (Section 3.2). The rest of the thesis is essentially devoted to an investigation of the assembly map. Using it and a new theorem of Cappell, some calculations of surgery spaces generalizing those of Shaneson and Wall are given. In 4.1 the structure sequence for manifold structures on a Poincaré space is investigated and found to be a special case of the assembly map. Finally in 4.2 the assembly is used to give a formal solution to the problem: If a map $f: M \longrightarrow N$ of closed manifolds has fiber the homotopy type of a finite complex, what are the obstructions to it being homotopic to a block fibration with manifold fibers.