The structure set of an arbitrary space, the algebraic surgery exact sequence and the total surgery obstruction

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Abstract

The algebraic theory of surgery gives a necessary and sufficient chain level condition for a space with n-dimensional Poincaré duality to be homotopy equivalent to an n-dimensional topological manifold. A relative version gives a necessary and sufficient chain level condition for a simple homotopy equivalence of n-dimensional topological manifolds to be homotopic to a homeomorphism. The chain level obstructions come from a chain level interpretation of the fibre of the assembly map in surgery.

The assembly map $A : H_n(X;\mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi_1(X)])$ is a natural transformation from the generalized homology groups of a space X with coefficients in the 1-connective simply-connected surgery spectrum $\mathbb{L}_\bullet$ to the non-simply-connected surgery obstruction groups $L_*(\mathbb{Z}[\pi_1(X)])$. The $(\mathbb{Z},X)$-category has objects based f.g. free $\mathbb{Z}$-modules with an X-local structure. The assembly maps A are induced by a functor from the $(\mathbb{Z},X)$-category to the category of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules. The generalized homology groups $H_*(X;\mathbb{L}_\bullet)$ are the cobordism groups of quadratic Poincaré complexes over $(\mathbb{Z},X)$. The relative groups $\mathbb{S}_*(X)$ in the algebraic surgery exact sequence of X

$$\dots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

are the cobordism groups of quadratic Poincaré complexes over $(\mathbb{Z},X)$ which assemble to contractible quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X)]$.

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional simple Poincaré complex X is the cobordism class of a quadratic Poincaré complex over $(\mathbb{Z},X)$ with contractible assembly over $\mathbb{Z}[\pi_1(X)]$, which measures the homotopy invariant part of the failure of the link of each simplex in X to be a homology sphere. The total surgery obstruction is $s(X) = 0$ if (and for $n \geqslant 5$ only if) X is simple homotopy equivalent to an n-dimensional topological manifold.

The Browder-Novikov-Sullivan-Wall surgery exact sequence for an n-dimensional topological manifold M with $n \geqslant 5$

$$\dots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)])$$

is identified with the corresponding portion of the algebraic surgery exact sequence

$$\dots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}_{n+1}(M) \to H_n(M;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)]) \ .$$

The structure invariant $s(h) \in \mathbb{S}^{TOP}(M) = \mathbb{S}_{n+1}(M)$ of a simple homotopy equivalence of n-dimensional topological manifolds $h : N \to M$ is the cobordism class of an n-dimensional quadratic Poincaré complex in $(\mathbb{Z},M)$ with contractible assembly over $\mathbb{Z}[\pi_1(M)]$, which measures the homotopy invariant part of the failure of the point inverses $h^{-1}(x)$ $(x \in M)$ to be acyclic. The structure invariant is $s(h) = 0$ if (and for $n \geqslant 5$ only if) h is homotopic to a homeomorphism.

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1 Introduction

The structure set of a differentiable n-dimensional manifold M is the set $\mathbb{S}^O(M)$ of equivalence classes of pairs (N,h) with N a differentiable manifold and $h : N \to M$ a simple homotopy equivalence, subject to $(N,h) \sim (N',h')$ if there exist a diffeomorphism $f : N \to N'$ and a homotopy $f \simeq h'f : N \to M$. The differentiable structure set was first computed for $N = S^n$ ($n \geqslant 5$), with $\mathbb{S}^O(S^n) = \Theta^n$ the Kervaire-Milnor group of exotic spheres. In this case the structure set is an abelian group, since the connected sum of homotopy equivalences $h_1 : N_1 \to S^n$, $h_2 : N_2 \to S^n$ is a homotopy equivalence

$$h_1 \# h_2 \ : \ N_1 \# N_2 \to S^n \# S^n \ = \ S^n \ .$$

The Browder-Novikov-Sullivan-Wall theory for the classification of manifold structures within the simple homotopy type of an n-dimensional differentiable manifold M with $n \geqslant 5$ fits $\mathbb{S}^O(M)$ into an exact sequence of pointed sets

$$\dots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^O(M) \to [M, G/O] \to L_n(\mathbb{Z}[\pi_1(M)])$$

corresponding to the two stages of the obstruction theory for deciding if a simple homotopy equivalence $h : N \to M$ is homotopic to a diffeomorphism:

(i) The primary obstruction in $[M, G/O]$ to the extension of h to a normal bordism $(f,b) : (W; M, N) \to M \times ([0,1]; \{0\}, \{1\})$ with $f| = 1 : M \to M$. Here G/O is the classifying space for fibre homotopy trivialized vector bundles, and $[M, G/O]$ is identified with the bordism of normal maps $M' \to M$ by the Browder-Novikov transversality construction.

(ii) The secondary obstruction $\sigma_*(f,b) \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$ to performing surgery on (f,b) to make (f,b) a simple homotopy equivalence, which depends on the choice of solution in (i). Here, it is necessary to use the version of the L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ in which modules are based and isomorphisms are simple, in order to take advantage of the s-cobordism theorem.

The Whitney sum of vector bundles makes G/O an H-space (in fact an infinite loop space), so that $[M, G/O]$ is an abelian group. However, the surgery obstruction function $[M, G/O] \to L_n(\mathbb{Z}[\pi_1(M)])$ is not a morphism of groups, and in general the differentiable structure set $\mathbb{S}^O(M)$ does not have a group structure (or at least is not known to have), abelian or otherwise.

The 1960's development of surgery theory culminated in the work of Kirby and Siebenmann [4] on high-dimensional topological manifolds, which revealed both a striking similarity and a striking difference between the differentiable and topological catgeories. Define the structure set of a topological n-dimensional manifold M exactly as before, to be the set $\mathbb{S}^{TOP}(M)$ of equivalence classes of pairs (N,h) with N a topological manifold and $h : N \to M$ a simple homotopy equivalence, subject to $(N,h) \sim (N',h')$ if there exist a homeomorphism $f : N \to N'$. The similarity is that again there is a surgery exact sequence for $n \geqslant 5$

$$\dots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)]) \qquad (*)$$

corresponding to a two-stage obstruction theory for deciding if a simple homotopy equivalence is homotopic to a homeomorphism, with G/TOP the classifying space for fibre homotopy trivialized topological block bundles. The difference is that the topological structure set $\mathbb{S}^{TOP}(M)$ has an abelian group structure and G/TOP has an infinite loop space structure with respect to which $(*)$ is an exact sequence of abelian groups. Another difference is given by the computation $\mathbb{S}^{TOP}(S^n) = 0$, which is just a restatement of the generalized Poincaré conjecture in the topological category : for $n \geqslant 5$ every homotopy equivalence $h : M^n \to S^n$ is homotopic to a homeomorphism.

Originally, the abelian group structure on $(*)$ was suggested by the characteristic variety theorem of Sullivan [15] on the homotopy type of G/TOP, including the computation $\pi_*(G/TOP) = L_*(\mathbb{Z})$. Next, Quinn [6] proposed that the surgery obstruction function should be factored as the composite

$$[M, G/TOP] \;=\; H^0(M; \underline{G/TOP}) \;\cong\; H_n(M; \underline{G/TOP}) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])$$

with $\underline{G/TOP}$ the simply-connected surgery spectrum with 0th space G/TOP, identifying the topological structure sequence with the homotopy exact sequence of a geometrically defined spectrum-level assembly map. This was all done in Ranicki [8], [9], but with algebra instead of geometry.

The algebraic theory of surgery was used in [9] to define the algebraic surgery exact sequence of abelian groups for any space X

$$\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \dots \,. \qquad (**)$$

The expression of the L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ as the cobordism groups of quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X)]$ (recalled in the notes on the foundations of algebraic surgery) was extended to $H_*(X; \mathbb{L}_\bullet)$ and $\mathbb{S}_*(X)$, using quadratic Poincaré complexes in categories containing much more of the topology of X than just the fundamental group $\pi_1(X)$. The topological surgery exact sequence of an n-dimensional manifold M with $n \geqslant 5$ was shown to be in bijective correspondence with the corresponding portion of the algebraic surgery sequence, including an explicit bijection

$$s \;:\; \mathbb{S}^{TOP}(M) \to \mathbb{S}_{n+1}(M) \;;\; h \mapsto s(h) \,.$$

The structure invariant $s(h) \in \mathbb{S}_{n+1}(M)$ of a simple homotopy equivalance $h : N \to M$ measures the chain level cobordism failure of the point inverses $h^{-1}(x) \subset N$ $(x \in M)$ to be points.

The Browder-Novikov-Sulivan-Wall surgery theory deals both with the existence and uniqueness of manifolds in the simple homotopy type of a geometric simple n-dimensional Poincaré complex X with $n \geq 5$. Again, this was first done for differentiable manifolds, and then extended to topological manifolds, with a two-stage obstruction :

(i) The primary obstruction in $[X, B(G/TOP)]$ to the existence of a normal map $(f, b) : M \to X$, with $b : \nu_M \to \widetilde{\nu}_X$ a bundle map from the stable normal bundle ν_M of M to a topological reduction $\widetilde{\nu}_X : X \to BTOP$ of the Spivak normal fibration $\nu_X : X \to BG$.

(ii) The secondary obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(M)])$ to performing surgery to make (f,b) a simple homotopy equivalence, which depends on the choice of solution in (i).

For $n \geqslant 5$ X is simple homotopy equivalent to a topological manifold if and only if there exists a topological reduction $\widetilde{\nu}_X : X \to BTOP$ for which the corresponding normal map $(f,b) : M \to X$ has surgery obstruction $\sigma_*(f,b) = 0$. The algebraic surgery exact sequence $(**)$ unites the two stages into a single invariant, the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$, which measures the chain level cobordism failure of the points $x \in X$ to have Euclidean neighbourhoods. For $\pi_1(X) = \{1\}$, $n = 4k$ the condition $s(X) = 0 \in \mathbb{S}_{4k}(X)$ is precisely the Browder condition that there exist a topological reduction $\widetilde{\nu}_X$ for which the signature of X is given by the Hirzebruch formula

$$\text{signature}(X) \; = \; \langle \mathcal{L}(-\widetilde{\nu}_X), [X] \rangle \in \mathbb{Z} \; .$$

2 Geometric Poincaré assembly

This section describes the assembly for geometric Poincaré bordism, setting the scene for the use of quadratic Poincaré bordism in the assembly map in algebraic L-theory. In both cases assembly is the passage from objects with local Poincaré duality to objects with global Poincaré duality.

Given a space X let $\Omega^P_n(X)$ be the bordism group of maps $f : Q \to X$ from n-dimensional geometric Poincaré complexes Q. The functor $X \mapsto \Omega^P_*(X)$ is homotopy invariant. If $X = X_1 \cup_Y X_2$ it is not in general possible to make $f : Q \to X$ Poincaré transverse at $Y \subset X$, i.e. $f^{-1}(Y) \subset Q$ will not be an $(n-1)$-dimensional geometric Poincaré complex. Thus $X \mapsto \Omega^P_*(X)$ does not have Mayer-Vietoris sequences, and is not a generalized homology theory. The general theory of Weiss and Williams [19] provides a generalized homology theory $X \mapsto H_*(X; \Omega^P_\bullet)$ with an assembly map $A : H_*(X; \Omega^P_\bullet) \to \Omega^P_*(X)$. However, it is possible to obtain A by a direct geometric construction : $H_n(X; \Omega^P_\bullet)$ is the bordism group of Poincaré transverse maps $f : Q \to X$ from n-dimensional Poincaré complexes Q, and A forgets the transversality. The coefficient spectrum $\Omega^P_\bullet$ is such that

$$\pi_*(\Omega^P_\bullet) \; = \; \Omega^P_*(\{\text{pt.}\}) \; ,$$

and may be constructed using geometric Poincaré n-ads.

In order to give a precise geometric description of $H_n(X; \Omega^P_\bullet)$ it is convenient to assume that X is the polyhedron of a finite simplicial complex (also denoted X). The *dual cell* of a simplex $\sigma \in X$ is the subcomplex of the barycentric subdivision X'

$$D(\sigma, X) \; = \; \{\widehat{\sigma}_0\widehat{\sigma}_1 \ldots \widehat{\sigma}_n \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \cdots < \sigma_n\} \subset X' \; ,$$

with boundary the subcomplex

$$\partial D(\sigma, X) \; = \; \bigcup_{\tau > \sigma} D(\tau, X) \; = \; \{\widehat{\tau}_0\widehat{\tau}_1 \ldots \widehat{\tau}_n \,|\, \sigma < \tau_0 < \tau_1 < \cdots < \tau_n\} \subset D(\sigma, X) \; .$$

Every map $f : M \to X$ from an n-manifold M can be made transverse across the dual cells, meaning that for each $\sigma \in X$

$$(M(\sigma), \partial M(\sigma)) \ = \ f^{-1}(D(\sigma, X), \partial D(\sigma, X))$$

is an $(n - |\sigma|)$-dimensional manifold with boundary. Better still, for an n-dimensional PL manifold M every simplicial map $f : M \to X'$ is already transverse in this sense, by a result of Marshall Cohen.

A map $f : Q \to X$ is *n-dimensional Poincaré transverse* if for each $\sigma \in X$

$$(Q(\sigma), \partial Q(\sigma)) \ = \ f^{-1}(D(\sigma, X), \partial D(\sigma, X))$$

is an $(n - |\sigma|)$-dimensional geometric Poincaré pair.

Proposition. $H_n(X; \Omega^P_\bullet)$ is the bordism group of Poincaré transverse maps $Q \to X$ from n-dimensional geometric Poincaré complexes. □

It is worth noting that

(i) The identity $1 : X \to X$ is n-dimensional Poincaré transverse if and only if X is an n-dimensional homology manifold.

(ii) If a map $f : Q \to X$ is n-dimensional Poincaré transverse then Q is an n-dimensional geometric Poincaré complex. The global Poincaré duality of Q is assembled from the local Poincaré dualities of $(Q(\sigma), \partial Q(\sigma))$. For $f = 1 : Q = X \to X$ this is the essence of Poincaré's original proof of his duality for a homology manifold.

The *Poincaré structure group* $\mathbb{S}^P_n(X)$ is the relative group in the *geometric Poincaré surgery exact sequence*

$$\dots \to H_n(X; \Omega^P_\bullet) \xrightarrow{A} \Omega^P_n(X) \to \mathbb{S}^P_n(X) \to H_{n-1}(X; \Omega^P_\bullet) \to \dots \ ,$$

which is the cobordism group of maps $(f, \partial f) : (Q, \partial Q) \to X$ from n-dimensional Poincaré pairs $(Q, \partial Q)$ with $\partial f : \partial Q \to X$ Poincaré transverse. The *total Poincaré surgery obstruction* of an n-dimensional geometric Poincaré complex X is the image $s^P(X) \in \mathbb{S}^P_n(X)$ of $(1 : X \to X) \in \Omega^P_n(X)$, with $s^P(X) = 0$ if and only if there exists an $\Omega^P_\bullet$-coefficient fundamental class $[X]_P \in H_n(X; \Omega^P_\bullet)$ with $A([X]_P) = (1 : X \to X) \in \Omega^P_n(X)$.

In fact, it follows from the Levitt-Jones-Quinn-Hausmann-Vogel Poincaré bordism theory that $\mathbb{S}^P_n(X) = \mathbb{S}_n(X)$ for $n \geqslant 5$, and that $s^P(X) = 0$ if and only if X is homotopy equivalent to an n-dimensional topological manifold. The geometric Poincaré bordism approach to the structure sets and total surgery obstruction is intuitive, and has the virtue(?) of dispensing with the algebra altogether. Maybe it even applies in the low dimensions $n = 3, 4$. However, at present our understanding of the Poincaré bordism theory is not good enough to use it for foundational purposes. So back to the algebra!

3 The algebraic surgery exact sequence

This section constructs the quadratic L-theory assembly map A and the algebraic surgery exact sequence

$$\cdots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \ldots \qquad (**)$$

for a finite simplicial complex X. A '$(\mathbb{Z},X)$-module' is a based f.g. free $\mathbb{Z}$-module in which every basis element is associated to a simplex of X. The construction of $(**)$ makes use of a chain complex duality on the $(\mathbb{Z},X)$-module category $\mathbb{A}(\mathbb{Z},X)$.

The quadratic L-spectrum $\mathbb{L}_\bullet$ is 1-connective, with connected 0th space $\mathbb{L}_0 \simeq G/TOP$ and

$$\pi_n(\mathbb{L}_\bullet) \;=\; \pi_n(\mathbb{L}_0) \;=\; L_n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\text{mod } 4) \text{ (signature)/8} \\ \mathbb{Z}_2 & \text{if } n \equiv 2 (\text{mod } 4) \text{ (Arf invariant)} \\ 0 & \text{otherwise} \end{cases}$$

for $n \geqslant 1$. From the algebraic point of view it is easier to start with the 0-connective quadratic L-spectrum $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})$, such that

$$\pi_n(\overline{\mathbb{L}}_\bullet) \;=\; \begin{cases} L_n(\mathbb{Z}) & \text{if } n \geqslant 0 \\ 0 & \text{if } n \leqslant -1 \end{cases}$$

with disconnected 0th space $\overline{\mathbb{L}}_0 \simeq L_0(\mathbb{Z}) \times G/TOP$. The two spectra are related by a fibration sequence $\mathbb{L}_\bullet \to \overline{\mathbb{L}}_\bullet \to \mathbb{K}(L_0(\mathbb{Z}))$ with $\mathbb{K}(L_0(\mathbb{Z}))$ the Eilenberg-MacLane spectrum of $L_0(\mathbb{Z})$.

The algebraic surgery exact sequence was constructed in Ranicki [9] using the $(\mathbb{Z},X)$-module category of Ranicki and Weiss [12]. (This is a rudimentary version of controlled topology, cf. Ranicki [11]).

A $(\mathbb{Z},X)$*-module* is a direct sum of based f.g. free $\mathbb{Z}$-modules

$$B \;=\; \sum_{\sigma\in X} B(\sigma)\ .$$

A $(\mathbb{Z},X)$-module morphism $f : B \to C$ is a $\mathbb{Z}$-module morphism such that

$$f(B(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} C(\tau)\ ,$$

so that the matrix of f is upper triangular. A $(\mathbb{Z},X)$-module chain map $f : B \to C$ is a chain equivalence if and only if each $f(\sigma,\sigma) : B(\sigma) \to C(\sigma)$ $(\sigma \in X)$ is a $\mathbb{Z}$-module chain equivalence. The universal covering projection $p : \widetilde{X} \to X$ is used to define the $(\mathbb{Z},X)$-module assembly functor

$$A \;:\; \mathbb{A}(\mathbb{Z},X) \to \mathbb{A}(\mathbb{Z}[\pi_1(X)])\ ;\ B \mapsto \sum_{\widetilde{\sigma}\in\widetilde{X}} B(p\widetilde{\sigma})$$

with $\mathbb{A}(\mathbb{Z},X)$ the category of $(\mathbb{Z},X)$-modules and $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$ the category of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules. In the language of sheaf theory $A = q_!p^!$ (cf. Verdier [16]), with $q : \widetilde{X} \to \{\text{pt.}\}$.

The involution $g \mapsto \overline{g} = g^{-1}$ on $\mathbb{Z}[\pi_1(X)]$ extends in the usual way to a duality involution on $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$, sending a based f.g. free $\mathbb{Z}[\pi_1(X)]$-module F to the dual f.g. free $\mathbb{Z}[\pi_1(X)]$-module $F^* = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(F,\mathbb{Z}[\pi_1(X)])$. Unfortunately, it is not possible to define a duality involution on $\mathbb{A}(\mathbb{Z},X)$ (since the transpose of an upper triangular matrix is a lower triangular matrix). See Chapter 5 of Ranicki [9] for the construction of a 'chain duality' on $\mathbb{A}(\mathbb{Z},X)$ and of the L-groups $L_*(\mathbb{A}(\mathbb{Z},X))$. The chain duality associates to a chain complex C in $\mathbb{A}(\mathbb{Z},X)$ a chain complex TC in $\mathbb{A}(\mathbb{Z},X)$ with

$$TC(\sigma)_r \;=\; \sum_{\tau\geqslant\sigma} \mathrm{Hom}_{\mathbb{Z}}(C_{-|\sigma|-r}(\tau),\mathbb{Z}) \;.$$

Example. The simplicial chain complex $C(X')$ is a $(\mathbb{Z},X)$-module chain complex, with assembly $A(C(X'))$ $\mathbb{Z}[\pi_1(X)]$-module chain equivalent to $C(\widetilde{X})$. The chain dual $TC(X')$ is $(\mathbb{Z},X)$-module chain equivalent to the simplicial cochain complex $D = \mathrm{Hom}_{\mathbb{Z}}(C(X),\mathbb{Z})^{-*}$, with assembly $A(D)$ which is $\mathbb{Z}[\pi_1(X)]$-module chain equivalent to $C(\widetilde{X})^{-*}$. □

The quadratic L-group $L_n(\mathbb{A}(\mathbb{Z},X))$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C,ψ) in $\mathbb{A}(\mathbb{Z},X)$.

Proposition. ([9], 14.5) The functor $X \mapsto L_*(\mathbb{A}(\mathbb{Z},X))$ is the generalized homology theory with $\mathbb{L}_\bullet(\mathbb{Z})$-coefficients

$$L_*(A(\mathbb{Z},X)) \;=\; H_*(X;\mathbb{L}_\bullet(\mathbb{Z})) \;. \qquad \square$$

The coefficient spectrum $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})$ is the special case $R = \mathbb{Z}$ of a general construction. For any ring with involution R there is a 0-connective spectrum $\mathbb{L}_\bullet(R)$ such that

$$\pi_*(\overline{\mathbb{L}}_\bullet(R)) \;=\; L_*(R) \;,$$

which may be constructed using quadratic Poincaré n-ads over R.

The assembly functor $A : \mathbb{A}(\mathbb{Z},X) \to \mathbb{A}(\mathbb{Z}[\pi_1(X)])$ induces assembly maps in the quadratic L-groups, which fit into the *4-periodic algebraic surgery exact sequence*

$$\dots \to H_n(X;\overline{\mathbb{L}}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \overline{\mathbb{S}}_n(X) \to H_{n-1}(X;\overline{\mathbb{L}}_\bullet) \to \dots$$

with the *4-periodic algebraic structure set* $\overline{\mathbb{S}}_n(X)$ the cobordism group of $(n-1)$-dimensional quadratic Poincaré complexes (C,ψ) in $\mathbb{A}(\mathbb{Z},X)$ such that the assembly $A(C)$ is a simple contractible based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex. (See section 4.5 for the geometric interpretation). A priori, an element of the relative group $\overline{\mathbb{S}}_n(X) = \pi_n(A)$ is an n-dimensional quadratic $\mathbb{Z}[\pi_1(X)]$-Poincaré pair $(C \to D,(\delta\psi,\psi))$ in $\mathbb{A}(\mathbb{Z},X)$. Using this as data for algebraic surgery results in an $(n-1)$-dimensional quadratic Poincaré complex (C',ψ') in $\mathbb{A}(\mathbb{Z},X)$ such that the assembly $A(C')$ is a simple contractible based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex.

Killing $\pi_0(\overline{\mathbb{L}}_\bullet) = L_0(\mathbb{Z})$ in $\overline{\mathbb{L}}_\bullet$ results in the 1-connective spectrum $\mathbb{L}_\bullet$, and the *algebraic surgery exact sequence*

$$\dots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots \qquad (**)$$

with $\mathbb{S}_n(X)$ the *algebraic structure set*. The two types of structure set are related by an exact sequence

$$\cdots \to H_n(X; L_0(\mathbb{Z})) \to \mathbb{S}_n(X) \to \overline{\mathbb{S}}_n(X) \to H_{n-1}(X; L_0(\mathbb{Z})) \to \dots \ .$$

4 The structure set and the total surgery obstruction

This chapter states the results in Chapters 16,17,18 of Ranicki [9] on the L-theory orientation of topology, the total surgery obstruction and the structure set.

The algebraic theory of surgery fits the homotopy category of topological manifolds of dimension $\geqslant 5$ into a pullback square

$$\begin{array}{ccc} \{\text{topological manifolds}\} & \longrightarrow & \{\text{local algebraic Poincaré complexes}\} \\ \big\downarrow & & \big\downarrow \\ \{\text{geometric Poincaré complexes}\} & \longrightarrow & \{\text{global algebraic Poincaré complexes}\} \end{array}$$

where local means $\mathbb{A}(\mathbb{Z}, X)$ and global means $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$. In words : the homotopy type of a topological manifold is the homotopy type of a geometric Poincaré complex with a local algebraic Poincaré structure.

4.1 The L-theory orientation of topological block bundles

The topological k-block bundles of Rourke and Sanderson [13] are topological analogues of vector bundles. By analogy with the classifying spaces $BO(k)$, BO for vector bundles there are classifying spaces $B\widetilde{TOP}(k)$ for topological block bundles, and a stable classifying space $BTOP$. It is known from the work of Sullivan [15] and Kirby-Siebenmann [4] that the classifying space for fibre homotopy trivialized topological block bundles

$$G/TOP \ = \ \text{homotopy fibre}(BTOP \to BG)$$

has homotopy groups $\pi_*(G/TOP) = L_*(\mathbb{Z})$. A map $S^n \to G/TOP$ classifies a topological block bundle $\eta : S^n \to B\widetilde{TOP}(k)$ with a fibre homotopy trivialization $J\eta \simeq \{*\} : S^n \to BG(k)$ $(k \geqslant 3)$. The isomorphism $\pi_n(G/TOP) \to L_n(\mathbb{Z})$ is defined by sending $S^n \to G/TOP$ to the surgery obstruction $\sigma_*(f, b)$ of the corresponding normal map $(f, b) : M \to S^n$ from a topological n-dimensional manifold M, with $b : \nu_M \to \nu_{S^n} \oplus \eta$. Sullivan [15] proved that G/TOP and BO have the same homotopy type localized away from 2

$$G/TOP[1/2] \ \simeq \ BO[1/2] \ .$$

(The localization $\mathbb{Z}[1/2]$ is the subring $\{\ell/2^m \,|\, \ell \in \mathbb{Z}, m \geqslant 0\} \subset \mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting 2. The localization $X[1/2]$ of a space X is a space such that

$$\pi_*(X[1/2]) \ = \ \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \ .$$

Thus $X \mapsto X[1/2]$ kills all the 2-primary torsion in $\pi_*(X)$.)

Let $\mathbb{L}^\bullet = \mathbb{L}(\mathbb{Z})^\bullet$ be the symmetric L-spectrum of $\mathbb{Z}$, with homotopy groups

$$\pi_n(\mathbb{L}^\bullet) \;=\; L^n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\text{mod } 4) \text{ (signature)} \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\text{mod } 4) \text{ (deRham invariant)} \\ 0 & \text{otherwise .} \end{cases}$$

The hyperquadratic $\mathbb{L}$-spectrum of $\mathbb{Z}$ is defined by

$$\widehat{\mathbb{L}}^\bullet \;=\; \text{cofibre}(1+T : \mathbb{L}_\bullet \to \mathbb{L}^\bullet) \; .$$

It is 0-connective, fits into a (co)fibration sequence of spectra

$$\cdots \to \mathbb{L}_\bullet \xrightarrow{1+T} \mathbb{L}^\bullet \to \widehat{\mathbb{L}}^\bullet \to \Sigma\mathbb{L}_\bullet \to \dots \; ,$$

and has homotopy groups

$$\pi_n(\widehat{\mathbb{L}}^\bullet) \;=\; \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_8 & \text{if } n \equiv 0 (\text{mod } 4) \text{ and } n > 0 \\ \mathbb{Z}_2 & \text{if } n \equiv 2,3 (\text{mod } 4) \\ 0 & \text{if } n \equiv 1 (\text{mod } 4) \; . \end{cases}$$

An *h-orientation* of a spherical fibration $\nu : X \to BG(k)$ with respect to a ring spectrum h is an h-coefficient Thom class in the reduced h-cohomology $U \in \dot{h}^k(T(\nu))$ of the Thom space $T(\nu)$, i.e. a $\dot{h}$-cohomology class which restricts to $1 \in \dot{h}^k(S^k) = \pi_0(h)$ over each $x \in X$.

Theorem ([9], 16.1) (i) The 0th space $\mathbb{L}_0$ of $\mathbb{L}_\bullet$ is homotopy equivalent to G/TOP

$$\mathbb{L}_0 \;\simeq\; G/TOP \; .$$

(ii) Every topological k-block bundle $\nu : X \to B\widetilde{TOP}(k)$ has a canonical $\mathbb{L}^\bullet$-orientation

$$U_\nu \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet) \; .$$

(iii) Every $(k-1)$-spherical fibration $\nu : X \to BG(k)$ has a canonical $\widehat{\mathbb{L}}^\bullet$-orientation

$$\widehat{U}_\nu \in \dot{H}^k(T(\nu);\widehat{\mathbb{L}}^\bullet) \; ,$$

with $\dot{H}$ denoting reduced cohomology. The *topological reducibility obstruction*

$$t(\nu) \;=\; \delta(\widehat{U}_\nu) \in \dot{H}^{k+1}(T(\nu);\mathbb{L}_\bullet)$$

is such that $t(\nu) = 0$ if and only if ν admits a topological block bundle reduction $\widetilde{\nu} : X \to B\widetilde{TOP}(k)$. Here, δ is the connecting map in the exact sequence

$$\cdots \to \dot{H}^k(T(\nu);\mathbb{L}_\bullet) \to \dot{H}^k(T(\nu);\mathbb{L}^\bullet) \to \dot{H}^k(T(\nu);\widehat{\mathbb{L}}^\bullet) \xrightarrow{\delta} \dot{H}^{k+1}(T(\nu);\mathbb{L}_\bullet) \to \dots \; .$$

The topological block bundle reductions of ν are in one-one correspondence with lifts of $\widehat{U}_\nu$ to a $\mathbb{L}^\bullet$-orientation $U_\nu \in H^k(T(\nu);\mathbb{L}^\bullet)$. □

Example. Rationally, the symmetric L-theory orientation of $\nu : X \to B\widetilde{TOP}(k)$ is the $\mathcal{L}$-genus

$$U_\nu \otimes \mathbb{Q} \;=\; \mathcal{L}(\nu) \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H^{4*}(X;\mathbb{Q})\ .$$ □

Example. Localized away from 2, the symmetric L-theory orientation of $\nu : X \to B\widetilde{TOP}(k)$ is the $KO[1/2]$-orientation of Sullivan [15]

$$U_\nu[1/2] \;=\; \Delta_\nu \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet)[1/2] \;=\; \widetilde{KO}^k(T(\nu))[1/2]\ .$$ □

4.2 The total surgery obstruction

The *total surgery obstruction* $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional geometric Poincaré complex X is the cobordism class of the $\mathbb{Z}[\pi_1(X)]$-contractible $(n-1)$-dimensional quadratic Poincaré complex (C,ψ) in $\mathbb{A}(\mathbb{Z},X)$ with $C = C([X]\cap - : C(X)^{n-*} \to C(X'))_{*+1}$, using the dual cells in the barycentric subdivision X' to regard the simplicial chain complex $C(X')$ as a chain complex in $\mathbb{A}(\mathbb{Z},X)$.

Theorem ([9], 17.4) The total surgery obstruction is such that $s(X) = 0 \in \mathbb{S}_n(X)$ if (and for $n \geqslant 5$ only if) X is homotopy equivalent to an n-dimensional topological manifold.
Proof A regular neighbourhood $(W,\partial W)$ of an embedding $X \subset S^{n+k}$ (k large) gives a Spivak normal fibration

$$S^{k-1} \to \partial W \to W \;\simeq\; X$$

with Thom space $T(\nu) = W/\partial W$ S-dual to $X_+ = X \cup \{\text{pt.}\}$. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ has image the topological reducibility obstruction

$$t(\nu) \in \dot{H}^{k+1}(T(\nu);\mathbb{L}_\bullet) \;\cong\; H_{n-1}(X;\mathbb{L}_\bullet)\ .$$

Thus $s(X)$ has image $t(\nu) = 0 \in H_{n-1}(X;\mathbb{L}_\bullet)$ if and only if ν admits a topological block bundle reduction $\widetilde{\nu} : X \to B\widetilde{TOP}(k)$, in which case the topological version of the Browder-Novikov transversality construction applied to the degree 1 map $\rho : S^{n+k} \to T(\nu)$ gives a normal map $(f,b) = \rho| : M = f^{-1}(X) \to X$. The surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ has image

$$[\sigma_*(f,b)] \;=\; s(X) \in \operatorname{im}(L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X)) \;=\; \ker(\mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet))$$

The total surgery obstruction is $s(X) = 0$ if and only if there exists a reduction $\widetilde{\nu}$ with $\sigma_*(f,b) = 0$. □

Example. For a simply-connected space X the assembly map $A : H_*(X;\mathbb{L}_\bullet) \to L_*(\mathbb{Z})$ is onto, so that

$$\mathbb{S}_n(X) \;=\; \ker(A : H_{n-1}(X;\mathbb{L}_\bullet) \to L_{n-1}(\mathbb{Z})) \;=\; \dot{H}_{n-1}(X;\mathbb{L}_\bullet)\ ,$$

with $\dot{H}$ denoting reduced homology. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of a simply-connected n-dimensional geometric Poincaré complex X is just the obstruction to the topological reducibility of the Spivak normal fibration $\nu_X : X \to BG$. □

There are also relative and rel ∂ versions of the total surgery obstruction.

For any pair of spaces $(X, Y \subseteq X)$ let $\mathbb{S}_n(X,Y)$ be the relative groups in the exact sequence

$$\cdots \to H_n(X,Y;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(Y)] \to \mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X,Y) \to H_{n-1}(X,Y;\mathbb{L}_\bullet) \to \dots .$$

The *relative total surgery obstruction* $s(X,Y) \in \mathbb{S}_n(X,Y)$ of an n-dimensional geometric Poincaré pair is such that $s(X,Y) = 0$ if (and for $n \geqslant 6$ only if) (X,Y) is homotopy equivalent to an n-dimensional topological manifold with boundary $(M, \partial M)$. In the special case $\pi_1(X) = \pi_1(Y)$

$$s(X,Y) \in \mathbb{S}_n(X,Y) = H_{n-1}(X,Y;\mathbb{L}_\bullet)$$

is just the obstruction to the topological reducibility of the Spivak normal fibration $\nu_X : X \to BG$, which is the π-π theorem of Chapter 4 of Wall [17].

The *rel ∂ total surgery obstruction* $s_\partial(X,Y) \in \mathbb{S}_n(X)$ of an n-dimensional geometric Poincaré pair (X,Y) with manifold boundary Y is such that $s_\partial(X,Y) = 0$ if (and for $n \geqslant 5$ only if) (X,Y) is homotopy equivalent rel ∂ to an n-dimensional manifold with boundary.

4.3 The L-theory orientation of topological manifolds

An n-dimensional geometric Poincaré complex X determines a symmetric $\mathbb{Z}[\pi_1(X)]$-Poincaré complex $(C(X'), \phi)$ in $\mathbb{A}(\mathbb{Z}, X)$, with assembly the usual symmetric Poinacaré complex $(C(\widetilde{X}), \phi(\widetilde{X}))$ representing the symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])$.

Example. For $n = 4k$ $\sigma^*(M) \in L^{4k}(\mathbb{Z}[\pi_1(M)])$ has image

$$\text{signature}(X) \;=\; \text{signature}(H^{2k}(X;\mathbb{Q}), \cup) \in L^{4k}(\mathbb{Z}) = \mathbb{Z} . \qquad \square$$

A triangulated n-dimensional manifold M determines a symmetric Poincaré complex $(C(M'), \phi)$ in $\mathbb{A}(\mathbb{Z}, M)$. The *symmetric L-theory orientation* of M is the $\mathbb{L}^\bullet$-coefficient class

$$[M]_{\mathbb{L}} \;=\; (C(M'), \phi) \in L^n(\mathbb{A}(\mathbb{Z}, M)) \;=\; H_n(M;\mathbb{L}^\bullet)$$

with assembly

$$A([M]_{\mathbb{L}}) \;=\; \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(M)]) .$$

Example. Rationally, the symmetric L-theory orientation is the Poincaré dual of the $\mathcal{L}$-genus

$$[M]_{\mathbb{L}} \;=\; \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \in H_n(M;\mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H_{n-4*}(M;\mathbb{Q}) \;=\; H^{4*}(M;\mathbb{Q}) .$$

Thus $A([M]_{\mathbb{L}}) = \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(M)])$ is a $\pi_1(M)$-equivariant generalization of the Hirzebruch signature theorem for a $4k$-dimensional manifold

$$\text{signature}(M) \;=\; \langle \mathcal{L}(-\widetilde{\nu}_M), [M] \rangle \in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z} . \qquad \square$$

Example. Localized away from 2, the symmetric L-theory orientation is the $KO[1/2]$-orientation $\Delta(M)$ of Sullivan [15]

$$[M]_{\mathbb{L}} \otimes \mathbb{Z}[1/2] \;=\; \Delta(M) \in H_n(M;\mathbb{L}^\bullet)[1/2] \;=\; KO_n(M)[1/2] . \qquad \square$$

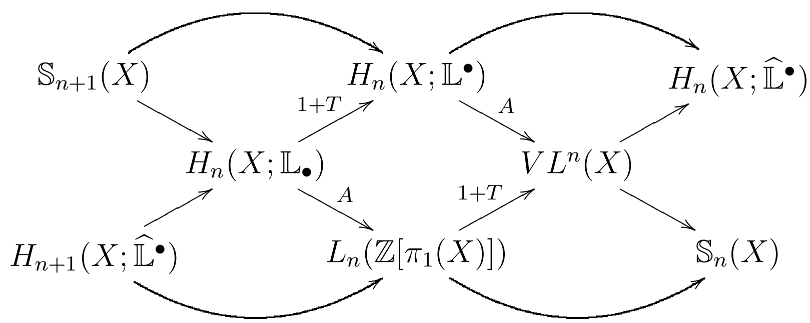
See Chapter 16 of [9] for the detailed definition of the *visible symmetric L-groups* $VL^*(X)$ of a space X, with the following properties :

(i) $VL^n(X)$ is the cobordism group of n-dimensional symmetric complexes (C, ϕ) in $\mathbb{A}(\mathbb{Z}, X)$ such that the assembly $A(C, \phi)$ is an n-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$, and such that each $(C(\sigma), \phi(\sigma))$ $(\sigma \in X^{(n)})$ is a 0-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z})$.

(ii) The (covariant) functor $X \mapsto VL^*(X)$ is homotopy invariant.

(iii) The visible symmetric L-groups $VL^*(K(\pi, 1))$ of an Eilenberg-MacLane space $K(\pi, 1)$ of a group π are the visible symmetric L-groups $VL^*(\mathbb{Z}[\pi])$ of Weiss [18].

(iv) The VL-groups fit into a commutative braid of exact sequences



(v) Every n-dimensional simple Poincaré complex X has a visible symmetric signature $\sigma^*(X) \in VL^n(X)$ with image the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$.

An *h-orientation* of an n-dimensional Poincaré complex X with respect to ring spectrum h is an h-homology class $[X]_h \in h_n(X)$ which corresponds under the S-duality isomorphism $h_n(X) \cong \dot{h}^{k+1}(T(\nu))$ to an h-coefficient Thom class $U_h \in \dot{h}^k(T(\nu))$ of the Spivak normal fibration $\nu : X \to BG(k)$ (k large, $X \subset S^{n+k}$).

Theorem ([9], 16.16) Every n-dimensional topological manifold M has a canonical $\mathbb{L}^\bullet$-orientation $[M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet)$ with assembly

$$A([M]_{\mathbb{L}}) \ = \ \sigma^*(M) \in VL^n(M) \ . \qquad \square$$

If M is triangulated by a simplicial complex K then

$$[M]_{\mathbb{L}} \ = \ (C, \phi) \in H_n(M; \mathbb{L}^\bullet) \ = \ L^n(\mathbb{Z}, K)$$

is the cobordism class of an n-dimensional symmetric Poincaré complex (C, ϕ) in $\mathbb{A}(\mathbb{Z}, K)$ with $C = C(K')$.

Example. The canonical $\mathbb{L}^\bullet$-homology class of an n-dimensional manifold M is given rationally by the Poincaré dual of the $\mathcal{L}(M)$-genus $\mathcal{L}(M) \in H^{4*}(M; \mathbb{Q})$

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} \ = \ \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \in H_n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} \ = \ H_{n-4*}(M; \mathbb{Q}) \ . \qquad \square$$

Theorem ([9], pp. 190–191) For $n \geqslant 5$ an n-dimensional simple Poincaré complex X is simple homotopy equivalent to an n-dimensional topological manifold if and only if there exists a symmetric L-theory fundamental class $[X]_{\mathbb{L}} \in H_n(X;\mathbb{L}^\bullet)$ with assembly

$$A([X]_{\mathbb{L}}) \;=\; \sigma^*(X) \in VL^n(X)\ . \qquad \square$$

In the simply-connected case $\pi_1(X) = \{1\}$ with $n = 4k$ this is just :

Example. For $k \geqslant 2$ a simply-connected $4k$-dimensional Poincaré complex X is homotopy equivalent to a $4k$-dimensional topological manifold if and only if the Spivak normal fibration $\nu_X : X \to BG$ admits a topological reduction $\widetilde{\nu}_X : X \to BTOP$ for which the Hirzebruch signature formula

$$\text{signature}(X) \;=\; \langle \mathcal{L}(-\widetilde{\nu}_X), [X]\rangle \in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}$$

holds. The if part is the topological version of the original result of Browder [1] on the converse of the Hirzebruch signature theorem for the homotopy types of differentiable manifolds.

$\square$

4.4 The structure set

The *structure invariant* of a homotopy equivalence $h : N \to M$ of n-dimensional topological manifolds is is the rel ∂ total surgery obstruction

$$s(h) \;=\; s_\partial(W, M \cup N) \in \mathbb{S}_{n+1}(W) \;=\; \mathbb{S}_{n+1}(M)$$

of the $(n+1)$-dimensional geometric Poincaré pair with manifold boundary $(W, M \cup N)$ defined by the mapping cylinder W of h.

Here is a more direct description of the structure invariant, in terms of the point inverses $h^{-1}(x) \subset N$ $(x \in M)$. Choose a simplicial complex K with a homotopy equivalence $g : M \to K$ such that g and $gh : N \to K$ are topologically transverse across the dual cells $D(\sigma, K) \subset K'$. (For triangulated M take $K = M$). Then $s(h)$ is the cobordism class

$$s(h) \;=\; (C, \psi) \in \mathbb{S}_{n+1}(K) \;=\; \mathbb{S}_{n+1}(M)$$

of a $\mathbb{Z}[\pi_1(M)]$-contractible n-dimensional quadratic Poincaré complex (C, ψ) in $\mathbb{A}(\mathbb{Z}, K)$ with

$$C \;=\; C(h : C(N) \to C(K'))_{*+1}\ .$$

Theorem ([9], 18.3, 18.5) (i) The structure invariant is such that $s(h) = 0 \in \mathbb{S}_{n+1}(M)$ if (and for $n \geqslant 5$ only if) h is homotopic to a homeomorphism.
(ii) The Sullivan-Wall surgery sequence of an n-dimensional topological manifold M with $n \geqslant 5$ is in one-one correspondence with a portion of the algebraic surgery exact sequence, by a bijection

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathbb{S}^{TOP}(M) & \longrightarrow & [M, G/TOP] & \longrightarrow & L_n(\mathbb{Z}[\pi_1(M)]) \\
& \| & & \cong \big\downarrow s & & \cong \big\downarrow t & & \| \\
\cdots \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathbb{S}_{n+1}(M) & \longrightarrow & H_n(M;\mathbb{L}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1(M)])
\end{array}$$

The higher structure groups are the rel ∂ structure sets

$$\mathbb{S}_{n+k+1}(M) \;=\; \mathbb{S}^{TOP}_{\partial}(M \times D^k, M \times S^{k-1}) \quad (k \geqslant 1)$$

of homotopy equivalences $(h, \partial h) : (N, \partial N) \to (M \times D^k, M \times S^{k-1})$ with $\partial h : \partial N \to M \times S^{k-1}$ a homeomorphism. □

Example. For a simply-connected space M the assembly maps $A : H_*(M; \mathbb{L}_\bullet) \to L_*(\mathbb{Z})$ are onto. Thus for a simply-connected n-dimensional manifold M

$$\begin{aligned}\mathbb{S}^{TOP}(M) \;&=\; \mathbb{S}_{n+1}(M) \\ &=\; \ker(A : H_n(M; \mathbb{L}_\bullet) \to L_n(\mathbb{Z})) \;=\; \dot{H}_n(M; \mathbb{L}_\bullet) \\ &=\; \ker(\sigma_* : [M, G/TOP] \to L_n(\mathbb{Z}))\end{aligned}$$

with σ_* the surgery obstruction map. The structure invariant $s(h) \in \mathbb{S}^{TOP}(M)$ of a homotopy equivalence $h : N \to M$ is given modulo 2-primary torsion by the difference of the canonical $\mathbb{L}^\bullet$-orientations

$$s(h)[1/2] \;=\; (h_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}}, 0) \in \dot{H}_n(M; \mathbb{L}^\bullet)[1/2] \;=\; \dot{H}_n(M; \mathbb{L}_\bullet)[1/2] \oplus H_n(M)[1/2] \; .$$

Rationally, this is just the difference of the Poincaré duals of the $\mathcal{L}$-genera

$$\begin{aligned}s(h) \otimes \mathbb{Q} \;&=\; h_*(\mathcal{L}(N) \cap [N]_{\mathbb{Q}}) - \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \\ &\in \mathbb{S}_n(M) \otimes \mathbb{Q} \;=\; \dot{H}_n(M; \mathbb{L}_\bullet) \otimes \mathbb{Q} \;=\; \sum_{4k \neq n} H_{n-4k}(M; \mathbb{Q}) \; .\end{aligned}$$

□

Example. Smale [14] proved the generalized Poincaré conjecture: if N is a differentiable n-dimensional manifold with a homotopy equivalence $h : N \to S^n$ and $n \geqslant 5$ then h is homotopic to a homeomorphism. Stallings and Newman then proved the topological version: if N is a topological n-dimensional manifold with a homotopy equivalence $h : N \to S^n$ and $n \geqslant 5$ then h is homotopic to a homeomorphism. This is the geometric content of the computation of the structure set of S^n

$$\mathbb{S}^{TOP}(S^n) \;=\; \mathbb{S}_{n+1}(S^n) \;=\; 0 \quad (n \geqslant 5) \; .$$

□

Here are three consequences of the Theorem in the non-simply-connected case, subject to the canonical restriction $n \geqslant 5$:

(i) For any finitely presented group π the image of the assembly map

$$A \;:\; H_n(K(\pi, 1); \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi])$$

is the subgroup consisting of the surgery obstructions $\sigma_*(f, b)$ of normal maps $(f, b) : N \to M$ of closed n-dimensional manifolds with $\pi_1(M) = \pi$.

(ii) The Novikov conjecture for a group π is that the higher signatures for any manifold M with $\pi_1(M) = \pi$

$$\sigma_x(M) \;=\; \langle x \cup \mathcal{L}(M), [M] \rangle \in \mathbb{Q} \quad (x \in H^*(K(\pi, 1); \mathbb{Q}))$$

are homotopy invariant. The conjecture holds for π if and only if the rational assembly maps

$$A \;:\; H_n(K(\pi,1);\mathbb{L}_\bullet)\otimes\mathbb{Q} \;=\; H_{n-4*}(K(\pi,1);\mathbb{Q}) \to L_n(\mathbb{Z}[\pi])\otimes\mathbb{Q}$$

are injective.

(iii) The topological Borel rigidity conjecture for an n-dimensional aspherical manifold $M = K(\pi,1)$ is that every simple homotopy equivalence of manifolds $h : N \to M$ is homotopic to a homeomeorphism, i.e. $\mathbb{S}^{TOP}(M) = \{*\}$, and more generally that

$$\mathbb{S}^{TOP}_\partial(M\times D^k, M\times S^{k-1}) \;=\; \{*\}\;\;(k \geqslant 1)\ .$$

The conjecture holds for π if and only if the assembly map

$$A \;:\; H_{n+k}(K(\pi,1);\mathbb{L}_\bullet) \to L_{n+k}(\mathbb{Z}[\pi])$$

is injective for $k = 0$ and an isomorphism for $k \geqslant 1$.

See Chapter 23 of Ranicki [9] and Chapter 8 of Ranicki [10] for the algebraic Poincaré transversality treatment of the splitting obstruction theory for homotopy equivalences of manifolds along codimension q submanifolds, involving natural morphisms $\mathbb{S}_*(X) \to LS_{*-q-1}$ to the LS-groups defined geometrically in Chapter 11 of Wall [17]. The case $q = 1$ is particularly important : a homotopy invariant functor is a homology theory if and only if it has excision, and excision is a codimension 1 transversality property.

4.5 Homology manifolds

An n-dimensional Poincaré complex X has a *4-periodic total surgery obstruction* $\overline{s}(X) \in \overline{\mathbb{S}}_n(X)$ such that $\overline{s}(X) = 0$ if (and for $n \geqslant 6$ only if) X is simple homotopy equivalent to a compact ANR homology manifold (Bryant, Ferry, Mio and Weinberger [2]). The $\mathbb{S}$- and $\overline{\mathbb{S}}$-groups are related by an exact sequence

$$0 \to \mathbb{S}_{n+1}(X) \to \overline{\mathbb{S}}_{n+1}(X) \to H_n(X;L_0(\mathbb{Z})) \to \mathbb{S}_n(X) \to \overline{\mathbb{S}}_n(X) \to 0\ .$$

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional homology manifold X is the image of the Quinn [7] resolution obstruction $i(X) \in H_n(X;L_0(\mathbb{Z}))$, such that $i(X) = 0$ if (and for $n \geqslant 6$ only if) there exists a map $M \to X$ from an n-dimensional topological manifold M with contractible point inverses. The homology manifold surgery sequence of X with $n \geqslant 6$ is in one-one correspondence with a portion of the 4-periodic algebraic surgery exact sequence, by a bijection

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(X)]) & \longrightarrow & \mathbb{S}^H(X) & \longrightarrow & [X, L_0(\mathbb{Z})\times G/TOP] & \longrightarrow & L_n(\mathbb{Z}[\pi_1(X)]) \\
 & \| & & \cong\big\downarrow \overline{s} & & \cong\big\downarrow \overline{t} & & \| \\
\cdots \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(X)]) & \longrightarrow & \overline{\mathbb{S}}_{n+1}(X) & \longrightarrow & H_n(X;\overline{\mathbb{L}}_\bullet) & \xrightarrow{\;A\;} & L_n(\mathbb{Z}[\pi_1(X)])
\end{array}$$

with $\mathbb{S}^H(X)$ the structure set of simple homotopy equivalences $h : Y \to X$ of n-dimensional homology manifolds, up to s-cobordism.

Example. The homology manifold structure set of S^n $(n \geqslant 6)$ is

$$\mathbb{S}^H(S^n) \ = \ \overline{\mathbb{S}}_{n+1}(S^n) \ = \ L_0(\mathbb{Z}) \ ,$$

detected by the resolution obstruction. □

See Chapter 25 of Ranicki [9] and Johnston and Ranicki [3] for more detailed accounts of the algebraic surgery classification of homology manifolds.

The homology manifold surgery exact sequence of [2] required the controlled algebraic surgery exact sequence

$$H_{n+1}(B;\overline{\mathbb{L}}_\bullet) \to \mathbb{S}_{\epsilon,\delta}(N,f) \to [N,\partial N;G/TOP,*] \to H_n(B;\overline{\mathbb{L}}_\bullet)$$

which has now been established by Pedersen, Quinn and Ranicki [5].

References

[1] W. Browder, *Homotopy type of differentiable manifolds*, in *Proc. Århus Topology Conference* (1962), reprinted in *Novikov Conjectures, Index Theorems and Rigidity*, Lond. Math. Soc. Lecture Notes 226, Cambridge Univ. Press 97–100 (1995)

[2] J. Bryant, S. Ferry, W. Mio, S. Weinberger, *Topology of homology manifolds*, Ann. of Maths. 143, 435–467 (1996)

[3] H. Johnston and A. Ranicki, *Homology manifold bordism*, Trans. A.M.S. 352, 5093–5137 (2000)

[4] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. Math. Stud. 88, Princeton University Press (1977)

[5] E. K. Pedersen, F. Quinn and A. Ranicki, *Controlled surgery with trivial local fundamental groups*, e-print http://arXiv.org/abs/math.GT/0119269 (2001)

[6] F. Quinn, $B_{(TOP_n)^{\sim}}$ *and the surgery obstruction*, Bull. A.M.S. 77, 596–600 (1971)

[7] ———, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. 34, 284–291 (1987)

[8] A. Ranicki, *The total surgery obstruction*, Proc. 1978 Arhus topology conference, Springer Lecture Notes 763, 275–316 (1979)

[9] ———, *Algebraic L-theory and Topological Manifolds*, Cambridge Tracts in Mathematics 102, Cambridge (1992)

[10] ———, *On the Novikov Conjecture*, Proc. 1993 Oberwolfach Conference on the Novikov Conjectures, Rigidity and Index Theorems, Vol. 1, L.M.S. Lecture Notes 226, 272–337, Cambridge (1995)

[11] ———, *Singularities, double points, controlled topology and chain duality*, Documenta Math. 4, 49–107 (1999)

[12] ——— and M. Weiss, *Chain complexes and assembly*, Math. Z. 204 , 157–186 (1990)

[13] C.P. Rourke and B. Sanderson, *On topological neighbourhoods*, Compositio Math. 22, 387–424 (1970)

[14] S. Smale, *Generalized Poincaré conjecture in dimensions greater than four*, Ann. of Maths. 74, 391–406 (1961)

[15] D. Sullivan, *Geometric periodicity and the invariants of manifolds*, Proc. 1970 Amsterdam Conf. on Manifolds, Lecture Notes in Mathematics 197, 44–75 (1971)

[16] J.L. Verdier, *On a theorem of Wilder*, Applications of Categorical Algebra, Proc. Symp. Pure Math. XVII, A.M.S., 184–191 (1970)

[17] C.T.C. Wall, *Surgery on compact manifolds*, 1st Edition, Academic Press (1970), 2nd Edition, Mathematical Surveys and Monographs 69, A.M.S. (1999)

[18] M. Weiss, *Visible L-theory*, Forum Math. 4, 465–498 (1992)

[19] ______ and B. Williams, *Assembly*, in *Novikov Conjectures, Index theorems and Rigidity*, Volume 2, L.M.S. Lecture Notes 227, 332–352, Cambridge (1995)