Computing Cocycles on Simplicial Complexes[*]

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Abstract

In this note, working in the context of simplicial sets [17], we give a detailed study of the complexity for computing chain level Steenrod squares [20, 21], in terms of the number of face operators required. This analysis is based on the combinatorial formulation given in [5]. As an application, we give here an algorithm for computing cup–i products over integers on a simplicial complex at chain level.

1 Introduction

Cohomology operations are tools for calculating n-cocycles on the cohomology of spaces (see, for example, [16, 19]). Unfortunately, up to the present, no symbolic computational system includes *general* methods for finding representative n–cocycles on the cohomology of spaces, algebras, groups, etc. Recently, several methods for finding 2–cocycles representing 2–dimensional cohomology classes of finite groups have been designed (see [4, 12, 14]). The method established in [14] is based on the general theory presented in [13] and it seems that can be generalized to higher dimension without effort.

In this paper, we describe a different procedure based on a combinatorial formulation given in [5] for an important class of chain level cohomology operations called *Steenrod squares*. The formula we obtain in [5] is essentially an explicit simplicial description of the original formula given by Steenrod [20] for the cup–i product on simplicial complexes. We note that a mod–2 explicit formulation of the Steenrod coproduct on the chain of a simplicial set has also been given in (6.2) of Hess [11], using a different method.

We work with simplicial sets [17] which are combinatorial analogs of topological spaces. First, our concern here is to study the "complexity" (in terms of number of face operators involved) of an algorithm for computing (at chain level) the integer cup–i products, using the formulation given in [5]. Finally, as an application, we give an algorithm for computing chain level Steenrod squares on simplicial complexes.

We integrate here tools of Combinatorics and Computer Algebra in a work of Algebraic Topology, opening a door to a computational development in the search of cocycles in any degree (see [1] and [6]). A treatment of some of our methods has already been presented in [7].

In the literature, there is plenty of information about cup–i products and Steenrod squares (see [23] and [3] for a non–exhaustive account of results). We think that the algorithmic technique explained here could be substantially refined if it is suitably combined with relevant and well–known results on these cohomology operations and with techniques of homological perturbation for manipulating explicit homotopy equivalences (see [2, 8, 9, 10]).

We are grateful to Prof. Julio Rubio for his helpful suggestions for improving the algorithms showed here.

2 Topological and Algebraic Preliminaries

The aim of this section is to give some simplicial and algebraic preliminaries in order to put into context the problem of computing n–cocycles (via cup–i products and Steenrod squares). Most of the material given in this section can be found in [15], [17] and [19].

A *simplicial set* X is a sequence of sets $X_0, X_1, \ldots$, together with *face operators* $\partial_i : X_n \to X_{n-1}$ and *degeneracy operators* $s_i : X_n \to X_{n+1}$ $(i = 0, 1, \ldots, n)$, which satisfy the following simplicial identities:

$$
\begin{array}{lll}
\textbf{(s1)} & \partial_i \partial_j = \partial_{j-1} \partial_i & \text{if } i < j \ ; \\
\textbf{(s2)} & s_i s_j = s_{j+1} s_i & \text{if } i \leq j \ ; \\
\textbf{(s3)} & \partial_i s_j = s_{j-1} \partial_i & \text{if } i < j \ , \\
\textbf{(s4)} & \partial_i s_j = s_j \partial_{i-1} & \text{if } i > j+1 \ , \\
\textbf{(s5)} & \partial_j s_j = 1_X = \partial_{j+1} s_j \ .
\end{array}
$$

The elements of X_n are called n–*simplices*. A simplex x is *degenerate* if $x = s_i(y)$ for some simplex y and degeneracy operator s_i; otherwise, x is *non degenerate*.

Let R be a ring which is commutative with unit. Given a simplicial set X, let us denote $C_*(X)$ by the chain complex $\{C_n(X), d_n\}$, in which $C_n(X)$ is the free R–module

generated by X_n and $d_n : C_n(X) \to C_{n-1}(X)$ is a R–module map of degree -1 called *differential*, defined by $d_n = \sum_{i=0}^n (-1)^i \partial_i$.

Let $s(C_*(X))$ be the graded R–module generated by all the degenerate simplices of X. In $C_*(X)$, we have that $d_n(s(C_{n-1}(X))) \subset s(C_{n-2}(X))$, then $C_*^N(X) = \{C_n(X)/s(C_{n-1}(X)), d_n\}$ is a chain complex called *the normalized chain complex associated to X*.

Since $d_n\, d_{n+1} = 0$, we can define the *homology* of X, $H_*(X)$, that is the family of modules $H_n(X) = \mathrm{Ker}\, d_n/\mathrm{Im}\, d_{n+1}$.

Now, the *cochain complex* associated to $C_*^N(X)$, denoted by $C^*(X; R)$, is the free R–module generated by all the R–module maps from $C_*^N(X)$ into R, together with a map called *codifferential* defined by $(\delta^n c)(x) = c(d_{n+1}(x))$ if $x \in C_{n+1}^N(X)$ and $c \in C^n(X; R)$. We will say that $c \in C^n(X; R)$ is a n–*cocycle* if $\delta(c) = 0$, and c is a n–*coboundary* if there exists another cochain $c' \in C^*(X; R)$ such that $c = \delta(c')$.

In this way, we define the *cohomology* of X with coefficients in R by $H^*(X) = \mathrm{Ker}\, \delta^n/\mathrm{Im}\, \delta^{n-1}$. Notice that a cocycle c represents a class of cohomology.

3   Complexity for Computing Steenrod Squares

First of all, let us show the explicit formula of the cup–n product $\smile_n$ on $C^*(X; R)$ given in [5]. The chain level Steenrod squares $Sq^i : C^j(X; \mathbf{Z}_2) \to C^{j+i}(X; \mathbf{Z}_2)$ are defined from this operation in a very easy way,

$$Sq^i(c) = c \smile_n c, \ \text{ where } n = j - i \ .$$

They satisfy that if c is a j–cocycle, then $Sq^i(c)$ is a $(i+j)$–cocycle.

Theorem 3.1 [5] *Let R be the ground ring and X a simplicial set. Let $c \in C^p(X; R)$, $c' \in C^q(X; R)$ and $x \in C_{p+q-n}^N(X)$; if n is even, then*

$$c \smile_n c'(x) = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})}$$

$$c(\partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \cdot \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m\, x)$$
$$\bullet c'(\partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1}\, x)$$

and if n is odd, then

$$c \smile_n c'(x) = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})}$$

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$$c(\partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1} \, x)$$
$$\bullet c'(\partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \cdot \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m \, x)$$

where $m = p + q - n$, the symbol $\bullet$ is the product in R,

$$A(n) \;=\; \begin{cases} 1 & \text{if } n \equiv 3,4,5,6 \ mod \ 8 \ , \\ 0 & \text{otherwise,} \end{cases}$$

$$B(n,m,\bar{\imath}) \;=\; \begin{cases} \displaystyle\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j} & \text{if } n \equiv 1,2 \ mod \ 4 \ , \\[2em] \displaystyle\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm & \text{if } n \equiv 0,3 \ mod \ 4 \ , \end{cases}$$

$$C(n,\bar{\imath}) \;=\; \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor}(i_{2j} + i_{2j-1})(i_{2j-1} + \cdots + i_0)$$

and

$$D(n,m,\bar{\imath}) \;=\; \begin{cases} (m + i_n)(i_n + \cdots + i_0) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

being $\bar{\imath} = (i_0, i_1, \ldots, i_n)$.

As we can see, the general organization of face operators in these formulae is simple in the sense that we distinguish in some way $n+1$ face operators $\partial_{i_0}, \partial_{i_1}, \ldots, \partial_{i_n}$; but the signs involved follow a complicated formula. Working over $\mathbf{Z}_2$, this problem is eliminated.

The aim of this section is to give an idea of the complexity of the algorithm for computing n–cocycles based on the previous formulation.

First of all, let us begin by giving a different description of the cup–n formula. Let us consider an alphabet with only two letters: 0 and 1. So, words in this alphabet are sequences of letters 0 and 1. We count the letters of a word from the left to the right and we will suppose that the first letter on the left is in zero position.

Let m and n be two nonnegative integers such that $n \leq m$. And let $i_0, i_1, \ldots, i_n \in \mathbf{Z}$ so that $0 \leq i_0 < i_1 \cdots < i_n \leq m$, then the notation $(i_0, i_1 \ldots, i_n)_m$ represents the word with $m + 1$ letters such that there are zeros in the positions $i_0, i_1, \ldots, i_n$ and ones in the rest, that is,

$$\overset{i_0}{\;}\quad \overset{i_1}{\;}\quad \overset{i_2}{\;}\quad \overset{i_3}{\;}\qquad \overset{i_n}{\;}$$
$$1 \cdots 1 \;\; 0 \;\; 1 \cdots 1 \;\; 0 \;\; 1 \cdots 1 \;\; 0 \;\; 1 \cdots 1 \;\; 0 \;\; \cdots \cdots \;\; 0 \;\; 1 \cdots 1 \; .$$

In the word above, by j–*block* $(1 \leq j \leq n)$ we mean the block of ones in $i_{j-1} + 1$ until $i_j - 1$ positions and zero in i_j position. The 0–block has ones in 0 until $i_0 - 1$ positions and zero in i_0 position; and the $(n+1)$–block has ones in $i_n + 1$ until m positions. That is,

$$
\underbrace{\overset{}{}}_{\text{0–block}} \quad \underbrace{}_{\text{1–block}} \quad \underbrace{}_{\text{2–block}} \quad \cdots \quad \underbrace{}_{\text{n–block}} \quad \underbrace{}_{\text{$(n+1)$–block}}
$$

$$
1 \cdots 1 \; \overset{i_0}{0} \quad 1 \cdots 1 \; \overset{i_1}{0} \quad 1 \cdots 1 \; \overset{i_2}{0} \quad \cdots \cdots \quad 1 \cdots 1 \; \overset{i_n}{0} \quad 1 \cdots 1
$$

Eventually, the $(n+1)$–block can be the empty word.

Now, given a word $(i_0, i_1, \ldots, i_n)_m$ we can make a pair of words denoted by $((i_0, i_1, \ldots, i_n)_m^+, (i_0, i_1 \ldots, i_n)_m^-)$, in the following way. If n is even, then

- the first word of the pair, denoted by $(i_0, i_1, \ldots, i_n)_m^+$, can be obtained from the word $(i_0, i_1, \ldots, i_n)_m$ preserving the j–blocks with j odd, that is,

$$
\underbrace{1 \cdots 1 \, 0}_{\text{1–bl.}} \quad \underbrace{1 \cdots 1 \, 0}_{\text{3–bl.}} \quad \underbrace{1 \cdots 1 \, 0}_{\text{5–bl.}} \quad \cdots \cdots \quad \underbrace{1 \cdots 1 \, 0}_{(n-1)\text{–bl.}} \quad \underbrace{1 \cdots 1}_{(n+1)\text{–bl.}} \quad ;
$$

- the second word of the pair, denoted by $(i_0, i_1, \ldots, i_n)_m^-$ can be obtained from the word $(i_0, i_1, \ldots, i_n)_m$ preserving the j–blocks with j even, that is,

$$
\underbrace{1 \cdots 1 \, 0}_{\text{0–bl.}} \quad \underbrace{1 \cdots 1 \, 0}_{\text{2–bl.}} \quad \underbrace{1 \cdots 1 \, 0}_{\text{4–bl.}} \quad \cdots \cdots \quad \underbrace{1 \cdots 1 \, 0}_{(n-2)\text{–bl.}} \quad \underbrace{1 \cdots 1 \, 0}_{n\text{–bl.}} \quad .
$$

If n is odd, then the procedure is analogous.

Some examples are:

- the word 1101101 represented by $(2, 5)_6$ is associated with the pair of words:

$$
((2,5)_6^+, (2,5)_6^-) = (110, 1101) \; ;
$$

- the word 00110 represented by $(0, 1, 4)_4$ is associated with the pair of words:

$$
((0,1,4)_4^+, (0,1,4)_4^-) = (0, 0110) \; .
$$

It is easy to see that we can recover the original word $(i_0, i_1, \ldots, i_n)_m$ from the pair $((i_0, i_1, \ldots, i_n)_m^+, (i_0, i_1, \ldots, i_n)_m^-)$ suitably combining the j–blocks of both words.

For example, if we have the pair

$$
(111101011, 011100) \; ,
$$

we first count the number of letters (in this case, $m = 14$), we determine the j–blocks in each word of the pair

$$\left(\; \overbrace{11110}^{0–\text{bl.}} \; \overbrace{10}^{1–\text{bl.}} \; \overbrace{11}^{2–\text{bl.}} \; , \; \overbrace{0}^{0–\text{bl.}} \; \overbrace{1110}^{1–\text{bl.}} \; \overbrace{0}^{2–\text{bl.}} \; \right)$$

and finally, we reconstruct the original word alternating the blocks of both words

$$0\ 11110\ 1110\ 10\ 0\ 11 = (0, 5, 9, 11, 12)_{14} \; .$$

Identifying the letter 1 in the position k with ∂_k and 0 with the identity, the general formula for the cup–n product admits the following representation:

$$c \smile_n c'(x)$$

$$= \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})}$$
$$c((i_0, i_1, \ldots, i_n)^+_m x) \bullet c'((i_0, i_1, \ldots, i_n)^-_m x) \; .$$

And the problem of counting the number of summands in the formula of the cup–n product is equivalent to that of finding all the possible ways to put $n + 1$ zeros in $m + 1$ possible places, that is,

$$\binom{m+1}{n+1} \; .$$

But, taking into account that c is a p–cochain and c' is a q–cochain, then we only have to consider the summands of the formulae having $q - n$ face operators in the first factor and $p - n$ in the second one. Hence, in an analogous way that in [5], a new combinatorial definition of cup–n product is given in the following theorem.

Theorem 3.2 *Let R be the ground ring and X a simplicial set. If $c \in C^p(X; R)$, $c' \in C^q(X; R)$ and $x \in C^N_{p+q-n}(X)$, then*

$$c \smile_n c'(x)$$

$$= \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})} \qquad (1)$$
$$c((i_0, i_1, \ldots, i_n)^+_m x) \bullet c'((i_0, i_1, \ldots, i_n)^-_m x)$$

where $m = p + q - n$, $\bullet$ is the product in R,

$$S(k) = i_{k+1} - i_{k-2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left(\lambda(n) - \left\lfloor \frac{n}{2} \right\rfloor \right) + \left\lfloor \frac{k}{2} \right\rfloor$$

being $\lambda(n) = p$ if n even and $\lambda(n) = q$ otherwise; and $i_0 = S(0)$.

Proof.

Let us start with $c \in C^p(X; R)$ and $c' \in C^q(X; R)$. If $n < p$ or $n < q$ then $c \smile_n c'$ is zero because there is not any summand in the formula with $q - n$ face operators in the first factor and $p - n$ face operators in the second one. So, let us suppose that $n \leq p$ and $n \leq q$.

If $n = 0$, then $p + q - i_0 = q$ and $i_0 = p$, so $i_0 = p$.

If $n = 1$, then $i_1 - 1 - i_0 = q - 1$ and $p + q - 1 - i_1 + i_0 = p - 1$. So, $i_1 - i_0 - q = 0 = q - i_1 + i_0$ and hence, $i_0 = i_1 - q$ and $i_1 \geq q$.

Let us suppose that n is even (if n is odd, the proof is analogous), then the number of face operators in the first factor of the summands is

$$p + q - n - i_n + \cdots + i_{2k+1} - 1 - i_{2k} + \cdots + i_1 - 1 - i_0 \ , \tag{2}$$

and in the second one

$$i_n - 1 - i_{n-1} + \cdots + i_{2k} - 1 - i_{2k-1} + \cdots + i_2 - 1 - i_1 + i_0 \ . \tag{3}$$

Since we only have to consider in the formula for $c \smile_n c'$, the summands that the number of face operators in the first factor is $q - n$ and $p - n$ in the second one, that is, (2) is $q - n$ and (3) is $p - n$, then

$$p + q - n - i_n + \cdots + i_{2k+1} - 1 - i_{2k} + \cdots + i_1 - 1 - i_0 - p + n$$
$$= i_n - 1 - i_{n-1} + \cdots + i_{2k} - 1 - i_{2k-1} + \cdots + i_2 - 1 - i_1 + i_0 - q + n$$

and hence,

$$i_0 = i_1 - i_2 + i_3 - \cdots - i_n + p - \frac{n}{2} \ . \tag{4}$$

Taking into account in (4) that $i_0 \geq 0$, we get

$$i_1 \geq i_2 - i_3 + \cdots + i_n - p + \frac{n}{2} \ .$$

Using $i_0 \leq i_1 - 1$ in (4), we have

$$i_2 \geq i_3 - i_4 + \cdots + i_{n-1} - i_n + p - \frac{n}{2} + 1 \ .$$

In general, let us suppose that

$$i_k \geq i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left(p - \frac{n}{2}\right) + \left\lfloor \frac{k}{2} \right\rfloor \ ,$$

for all $1 \leq k \leq \ell$, and let us prove that this expression is true in $\ell + 1$ with ℓ odd (if ℓ even, the proof is similar). In the case $k = \ell - 1$, since $i_\ell - 1 \geq i_{\ell-1}$, we have

$$i_\ell - 1 \geq i_\ell - i_{\ell+1} + \cdots + (-1)^{\ell+n-2} i_n + (-1)^{\ell+n-1} \left(p - \frac{n}{2}\right) + \frac{\ell - 1}{2}$$

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and simplifying, we conclude

$$i_{\ell+1} \geq i_{\ell+2} - i_{\ell+3} + \cdots + (-1)^{\ell+n}i_n + (-1)^{\ell+n+1}\left(p - \frac{n}{2}\right) + \frac{\ell+1}{2} \ .$$

□ Now, let us study the number of summands in the formula above. Given a p–cochain c, a q–cochain c' and a nonnegative integer n, the problem of counting all the summands in the formula of $c \smile_n c'$ is equivalent to that of finding all the pairs of words $((i_0, i_1, \ldots, i_n)_m^+, (i_0, i_1, \ldots, i_n)_m^-)$ such that the first word has $q - n$ letters 1 and the second word has $p - n$ letters 1. We obtain the following result.

Theorem 3.3 *Let R be the ground ring. Let X be a simplicial set and n a nonnegative integer. If $c \in C^p(X; R)$ and $c' \in C^q(X; R)$, then the number of summands taking part in the formula (1) for $c \smile_n c'$ is*

$$\binom{q - \left\lfloor \frac{n+1}{2} \right\rfloor}{\left\lfloor \frac{n}{2} \right\rfloor} \binom{p - \left\lfloor \frac{n}{2} \right\rfloor}{\left\lfloor \frac{n+1}{2} \right\rfloor} .$$

Proof.

First, let us suppose that n is even. Our proof starts with the observation that the first factor of a summand of the formula (1) has $q - n$ face operator if and only if the word $(i_0, i_1, \ldots, i_n)_m^+$ associated to it has $q - n$ letters 1 and $\frac{n}{2}$ letters 0. Then the number of words $(i_0, i_1, \ldots, i_n)_m^+$ having exactly $q - n$ letters 1 is the number of all the possible ways to put $\frac{n}{2}$ zeros in $q - n + \frac{n}{2}$ places,

$$\binom{q - \frac{n}{2}}{\frac{n}{2}} \ .$$

Analogously, the word $(i_0, i_1, \ldots, i_n)_m^-$ associated to the second factor has $p - n$ letters 1 and $\frac{n}{2} + 1$ letters 0. Then the number of words $(i_0, i_1, \ldots, i_n)_m^-$ having $p - n$ letters 1 is the number of all the possible ways to put $\frac{n}{2}$ zeros (the last zero can not be changed) in $p - n + \frac{n}{2}$ places, that is,

$$\binom{p - \frac{n}{2}}{\frac{n}{2}} \ .$$

And the same reasoning applied to the case n odd gives us with the result that there are

$$\binom{q - \frac{n+1}{2}}{\frac{n-1}{2}}$$

possible words $(i_0, i_1, \ldots, i_n)_m^+$ with $q - n$ letters 1, and

$$\binom{p - \frac{n-1}{2}}{\frac{n+1}{2}}$$

Table 1: Number of summands

	in the formula of Theorem 3.1	in the formula of Theorem 3.2
$c_3 \smile_2 c_4$	20	6
$c_6 \smile_5 c_6$	28	12
$c_{12} \smile_4 c_{10}$	11,628	1,260
$c_{25} \smile_5 c_{30}$	18,009,460	621,621
$c_{60} \smile_5 c_{70}$	4,925,156,775	68,222,616
$c_6 \smile_5 c_{700}$	162,699,437,009,655	970,224
$c_{60} \smile_{50} c_{60}$	225,368,761,961,739,396	33,701,394,635,724,816
$c_6 \smile_5 c_{7000}$	163,331,343,055,757,216,550	97,902,024

words $(i_0, i_1, \ldots, i_n)_m^-$ with $p - n$ letters one.

□ Let us see with several examples, the improvement of the last formulae of the cup–n product given in Theorem 3.2 respect to the first formulae given in Theorem 3.1. Let us note c_p if $c \in C^p(X; R)$.

Taking into account that Steenrod squares are defined using cup–n products, the following corollary holds.

Corollary 3.4 *Let* $\mathbf{Z}_2$ *be the ground ring. Let* i *be a nonnegative integer and* $c \in C^j(X; \mathbf{Z}_2)$, *then the number of summands taking part in the formula of* $Sq^i(c)$ *is*

$$\binom{\left\lfloor \frac{m}{2} \right\rfloor}{\left\lfloor \frac{n}{2} \right\rfloor} \binom{\left\lfloor \frac{m+1}{2} \right\rfloor}{\left\lfloor \frac{n+1}{2} \right\rfloor},$$

where $m = i + j$ *and* $n = j - i$.

4   Simplicial Complexes

Now, let us study a particular simplicial set. A (combinatorial) *simplicial complex* [18, 22] is a collection P of nonempty finite subsets of some vertex set V such that if $\tau \subset \sigma \subset V$ and $\sigma \in P$, then $\tau \in P$. If the vertex set is ordered, we call P an *ordered* simplicial complex. To every such ordered simplicial complex we associate a simplicial set $SS(P)$ as follows. The set $SS_n(P)$ consists of all ordered $(n + 1)$–tuples $\langle v_0, v_1, \ldots, v_n \rangle$ of vertices (called *n–simplices*), possibly including repetition, such that the underlying set

$\{v_0, v_1, \ldots, v_n\}$ is in P (note that $v_0 \leq v_1 \leq \cdots \leq v_n$). This set is endowed with face and degeneracy operators defined by:

$$\partial_i \langle v_0, \ldots, v_n \rangle = \langle v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \rangle$$

and

$$s_i \langle v_0, \ldots, v_n \rangle = \langle v_0, \ldots, v_i, v_i, \ldots, v_n \rangle .$$

Notice that a simplex is degenerate if it has repeated vertices; otherwise, the simplex is non degenerate.

Summing up, a simplicial complex P can be considered as a combinatorial version of a triangulated polyhedron. The strong combinatorial structure in the first one (more precisely, in $SS(P)$) is due to considering the degeneracy operators.

From now on, due to the fact that we will work only with ordered simplicial complexes, we will call them simplicial complexes, and in order to simplify the explanation, we will identify the ordered simplicial complex P with the associated simplicial set $SS(P)$. Then if $v \in P_q$, we will say that the *dimension* of v is q. By abuse of notation, we will say that a simplex belongs to P if it belongs to P_ℓ for some ℓ.

Let x and y be two simplices of P. We will note $x \leq y$ if x is a projection of y. It is clear that a simplicial set can be given by the set of all the simplices with maximal dimension; and a simplex belongs to P if it is a projection of some maximal simplex of P.

Let x and y be two simplices of a simplicial complex P. Let us define two operations between simplices. Let $\{z \in P : x \leq z \text{ and } y \leq z\}$, then we define $x \cup y$ as the simplex of this set with smallest dimension (it is easy to see that $x \cup y$ is unique). And let $\{z \in P : z \leq x \text{ and } z \leq y\}$, then $x \cap y$ is the simplex of this set with highest dimension (observe that $x \cap y$ is unique, too). On the other hand, the formulation of cup–n products given in Theorem 3.2 on a simplicial complex is the following.

Proposition 4.1 *Let R be the ground ring and P a simplicial complex with a finite number of vertices. If $c \in C^p(P)$ and $c' \in C^q(P)$, then for all nonnegative integer n, $c \smile_n c' \in C^{p+q-n}(P)$ is defined by the following formulae. Let $m = p + q - n$ and $x = \langle v_0, \ldots, v_m \rangle \in C_m(P)$, then if n is even,*

$$c \smile_n c'(x)$$
$$= \sum_{i_n = S(n)}^{m} \sum_{i_{n-1} = S(n-1)}^{i_n - 1} \cdots \sum_{i_1 = S(1)}^{i_2 - 1} (-1)^{A(n) + B(n,m,\bar{i}) + C(n,\bar{i}) + D(n,m,\bar{i})}$$

$$c(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots v_{i_2}, v_{i_3}, \ldots, v_{i_{n-2}}, v_{i_{n-1}}, \ldots, v_{i_n}\rangle)$$
$$\bullet c'(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots v_{i_3}, v_{i_4}, \ldots, v_{i_{n-1}}, v_{i_n}, \ldots v_m\rangle) \; ;$$

and if n is odd, the formula is analogous.

In these formulas, $\bullet$ is the product in R,

$$S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$$

for all $0 \leq k \leq n$, and $i_0 = S(0)$.

Proof. Using the formula from Theorem 3.2, we only have to notice that

$$\partial_0 \cdots \partial_\ell \langle v_0, \ldots, v_m\rangle = \langle v_{\ell+1}, \ldots, v_m\rangle \; ,$$
$$\partial_\ell \cdots \partial_s \langle v_0, \ldots, v_m\rangle = \langle v_0, \ldots, v_{\ell-1}, v_{s+1}, \ldots, v_m\rangle \; ,$$
$$\partial_s \cdots \partial_m \langle v_0, \ldots, v_m\rangle = \langle v_0, \ldots, v_{s-1}\rangle \; .$$

$\square$ For example, the formula

$$c \smile_1 c'(\langle v_0, v_1, \ldots v_m\rangle)$$

$$= \sum_{j=0}^{p-1} (-1)^{j+(p-1+j)q} c(\langle v_0, \ldots, v_j, v_{j+q}, \ldots, v_m\rangle) \bullet c'(\langle v_j, \ldots, v_{j+q}\rangle)$$

coincides with that of Steenrod given on p. 293 of [20], up to the sign $(-1)^{p+q}$.

5   Algorithms

We are interested in designing algorithms for computing cocycles using cup–n products. In order to do this, we need the following notation.

Given a simplicial complex P, two nonnegative integers n and m, and three simplices x, y, z such that $z = x \cup y = \langle v_0, \ldots, v_m\rangle$ is a m–simplex and $x \cap y = \langle v_{i_0}, \ldots, v_{i_n}\rangle$ is a n–simplex, let us define the simplices

$$z^0 = \langle v_0, \ldots, v_{i_0}\rangle \; ,$$
$$z^j = \langle v_{i_{j-1}}, \ldots, v_{i_j}\rangle \quad \text{for } 1 \leq j \leq n \; ,$$
$$z^{n+1} = \langle v_{i_n}, \ldots, v_m\rangle \; .$$

We have the following result.

Proposition 5.1 *Let R be the ground ring. Let P be a simplicial complex, n a nonnegative integer, $c \in C^p(P)$ and $c' \in C^q(P)$. Let C (resp. C') be the set of non degenerate simplices of P such that $c(x) \neq 0$ if and only if $x \in C$ (resp. $c'(x) \neq 0$ if and only if $x \in C'$). Let $m = p + q - n$ and let $z = \langle v_0, \ldots, v_m \rangle$ be a simplex of P. Define the set*

$$D_z = \{ \quad (x_r, y_s) : x_r \in C, \, y_s \in C', \, x_r \cup y_s = z,$$
$$x_r \cap y_s = \langle v_{i_0}, \ldots, v_{i_n} \rangle \text{ is a } n\text{--simplex with } i_0 = S(0),$$
$$x_r = \bigcup_{j \text{even}} z^j \} \ .$$

Then,

$$c \smile_n c'(z) = \sum_{(x, y) \in D_z} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})} c(x) \bullet c'(y)$$

where $\bullet$ is the product in R.

Proof.

Using the formula in Proposition 4.1 for $c \smile_n c'$, it is not difficult to see that a summand of the formula is not zero if the first factor is a simplex of C and the second factor is a simplex of C'. Hence, the simplices $x_r \in C$ and $y_s \in C'$, are both factors of a summand if and only if $x_r \cup y_s = z$, $x_r \cap y_s = \langle v_{i_0}, \ldots, v_{i_n} \rangle$ is a n--simplex with $i_0 = S(0)$ (therefore, the rest of the inequalities $S(k) \leq i_k \leq i_{k+1}$, $1 \leq k \leq n - 1$, and $S(n) \leq i_n \leq m$ are verified) and $x_r = \bigcup_{j \text{ even}} z^j$.

□ Translating this result to a more algorithmic language, we obtain the following method in which the output is expressed as a formal sum of simplices.

Procedure 5.2 *Algorithm for computing cup--n products.*

Input: *the ground ring R,*
        a simplicial complex P,
        a p--cochain c and a q--cochain c'.
Construct the set C of p--simplices so that $x \in C$ if and only if $c(x) \neq 0$.
Construct the set C' so that $y \in C'$ if and only if $c'(y) \neq 0$.
Initially, $D := \{\ \}$.
for *each $x \in C$ and $y \in C'$,* **do**
    $z := x \cup y = \langle v_0, \ldots, v_m \rangle$,
    if *$x \cap y = \langle v_{i_0}, \ldots, v_{i_n} \rangle$ is a n--simplex with $n = p + q - m$,*
      $i_0 = S(0)$ and $x = \bigcup_{j \text{even}} z^j$ **then**
      $D := D \cup \{(x, y)\}$.
    endif;
endfor;
Let $cup := 0$.
for *each $(x, y) \in D_z$* **do**

$cup := cup + (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})} c(x) \bullet c(y) \, z.$

endfor;

Output: *a formal sum, $cup = \sum \lambda_j z_j$, such that*

if λz is a summand of cup, being $\lambda \in R$ and z a m–simplex, then

$c \smile_n c'(z) = \lambda$, *where $n = p + q - m$,*

and $c \smile_n c'(z) = 0$ otherwise.

Now, in order to compute cocycles, for example, working in $\mathbf{Z}_2$, we need the following formula given in [20]:

$$\delta(c \smile_n c')$$
$$= u \smile_{n-1} v + v \smile_{n-1} u + \delta u \smile_n v + u \smile_n \delta v \ .$$

It is clear that if both c and c' are cocycles, then the "commutativity" of the cup–$(n-1)$ product will determine the obtention of cocycles via cup–n products. And, in the particular case $c = c'$, the chain level Steenrod squares appear in a natural way. We will develop machinery which takes advantage of this fact in a future work.

The following result is a simple consequence of Proposition 5.1.

Corollary 5.3 *Let $\mathbf{Z}_2$ be the ground ring. Let P be a simplicial set and c a j–cocycle. Let C be the set of non degenerate j–simplices of P such that $c(x) = 1$ if and only if $x \in C$. Let i be a positive integer and $z = \langle v_0, \ldots, v_m \rangle$, a $(i+j)$–simplex of P. Define the set*

$$D_z = \{ \quad (x_r, x_s) : x_r, x_s \in C, \ r < s, \ x_r \cup x_s = z,$$
$$x_r \cap x_s = \langle v_{i_0}, \ldots, v_{i_n} \rangle \text{ is a } n\text{–simplex with } n\text{=}j\text{-}i,$$
$$i_0 = S(0) \text{ and } x_r = \bigcup_{j \text{ even}} z^j \text{ or } x_r = \bigcup_{j \text{ odd}} z^j \ \}.$$

If cardinal of D_z is even, then $Sq^i(c)(z) = 0$. Otherwise, $Sq^i(c)(z) = 1$.

In this corollary, we consider only $i > 0$ because it is well–known that $Sq^0(c) = c$. And we can observe that for knowing the cocycle $Sq^i(c)$, it is sufficient to evaluate it over the simplices z such that D_z is nonempty.

Using that Steenrod squares $Sq^i(c_j)$ are cup–$(j-i)$ products, we can adapt the last algorithm to these operations. Due to the fact that they are cohomology operations, the first step is to determine if the cochain c is a cocycle.

Procedure 5.4 *Algorithm for computing chain level Steenrod squares.*

Input: *a simplicial complex P and a j–cochain c.*

Construct the set $C = \{x_1, x_2, \ldots, x_k\}$ of j–simplices so that
$x \in C$ if and only if $c(x) = 1$.
Let $O := \{\ \}$.
for *each x_r, $x_s \in C$, $r < s$, and $x_r \cup x_s$ is a $(j+1)$–simplex* **do**
  $O := O \cup \{(x_r, x_s, x_r \cup x_s)\}$.
endfor;
if *O is empty* **then**
 c is not a cocycle
else *let $co := \{\ \}$.*
   for *each $(x_r, x_s, x_r \cup x_s) \in O$* **do**
     if *there exists a pair (y, z) in co such that $x_r \cup x_s = z$* **then**
      if *x_r is not a summand of y,* **then**
       $co := (co \setminus \{(y, z)\}) \cup \{(y + x_r, z)\}$.
      endif;
      if *x_s is not a summand of y,* **then**
       $co := (co \setminus \{(y, z)\}) \cup \{(y + x_s, z)\}$.
      endif;
     else $co := co \cup \{(x_r + x_s, x_r \cup x_s)\}$.
     endif;
   endfor;
endif;
if *there exists some pair in co such that*
 the number of summands in the first element is odd **then**
 c is not a cocycle,
else *let $S := 0$.*
   for *each $x_r, x_s \in C$, $r < s$,* **do**
   $z = x_r \cup x_s = \langle v_0, \ldots, v_m \rangle$,
    if *$x_r \cap x_s = \langle v_{i_0}, \ldots, v_{i_n} \rangle$ where $n = 2j - m$,*
     $i_0 = S(0)$
     and $x_r = \bigcup_{t\ \text{even}} z^t$ or $x_r = \bigcup_{t\ \text{odd}} z^t$ **then**
     $S := S + z$.
    endif;
   endfor;
endif.
Output: *a formal sum of simplices S such that*
    if the m–simplex z is a summand of S then
    $Sq^i(c)(z) = 1$ where $i = m - j$,
    and $Sq^i(c)(z) = 0$ otherwise.

Note that the resulting cocycles $Sq^i(c)$ can be coboundaries or not. This means that the last algorithm does not determine Steenrod squares at cohomology level.

  The previous procedures can be easily implemented using any Computer Algebra package or functional programming language.

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