Computation of Cohomology Operations on Finite Simplicial Complexes *

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Abstract

We propose a method for calculating cohomology operations for finite simplicial complexes.

Of course, there exist well-known methods for computing (co)homology groups, for example, the "reduction algorithm" consisting in reducing the matrices corresponding to the differential in each dimension to the Smith normal form, from which one can read off (co)homology groups of the complex [Mun84], or the "incremental algorithm" for computing Betti numbers [DE93]. However, there is a gap in the literature concerning general methods for computing cohomology operations.

For a given finite simplicial complex K, we sketch a procedure including the computation of some primary and secondary cohomology operations and the A_{∞} -algebra structure on the cohomology of K. This method is based on the transcription of the reduction algorithm mentioned above, in terms of a special type of algebraic homotopy equivalences, called a contraction, of the (co)chain complex of K to a "minimal" (co)chain complex M(K). For instance, whenever the ground ring is a field or the (co)homology of K is free, then M(K) is isomorphic to the (co)homology of K. Combining this contraction with the combinatorial formulae for Steenrod reduced pth powers at cochain level developed in [GR99] and [Gon00], these operations at cohomology level can be computed. Finally, a method for calculating Adem secondary cohomology operations Φ_q : $Ker(Sq^2H^q(K)) \rightarrow H^{q+3}(K)/Sq^2H^q(K)$ is showed.

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1 Introduction

Particular important topological invariants are the (co)homology groups. In a certain way, these groups measure the degree of connectedness of the space. Although there are plenty of programs for calculating (co)homology groups of finite simplicial complexes, we have not found any general software for computing cohomology operations.

Our main motivation is the design of a program for computing all sort of cohomology invariants on finite simplicial complexes: (co)homology groups, cup product, Bockstein cohomology operation, cohomology operations determined by homomorphisms of coefficient groups, Steenrod squares and reduced *p*th powers, Pontrjagin squares and *p*th powers, the A_{∞} -algebra structure of cohomology, higher cohomology operations, etc. In this paper, we give a solution to the problem of computing Steenrod squares and reduced *p*th powers [Ste47, ES62] and Adem secondary cohomology operations [Ade52, Ade58]. Our approach is based on the translation of the well-known "reduction" algorithm for computing (co)homology groups [Mun84] in terms of homotopy equivalences. In that way, we have a description of the generators of the (co)homology groups in terms of cochains. In fact, this is sufficient to enable us to determine the effect of the induced maps between cohomology groups corresponding to cochain maps. Using the same approach we think that the rest of primary cohomology operations could be attacked.

2 Background

We give a brief summary of concepts and notation used in the following sections. Our terminology follows Munkres [Mun84].

For $0 \leq q \leq n$, a *q*-simplex σ in \mathbb{R}^n is the convex hull of a set T of q + 1 affinely independent points $(v_0, ..., v_q)$. The dimension of σ is $|\sigma| = q$. For every non-empty $U \subset T$, the simplex τ defined by U is a face of σ . A simplicial complex K is a collection of simplices satisfying the following properties:

- If τ is a face of σ and $\sigma \in K$ then $\tau \in K$.
- If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is either empty or a face of both.

The set of all the q-simplices of K is denoted by $K^{(q)}$. The largest dimension of any simplex in K is the dimension of K. A simplex σ in K is maximal if it is not face of any simplex in K. Therefore, K can be given by the set of its maximal simplices. A subset $L \subset K$ is a subcomplex of K if it is a simplicial complex itself. All simplices in this paper have finite dimension and all simplicial complexes are finite collections. From now on, K denotes a finite simplicial complex. The oriented q-simplex $\sigma = [v_0, ..., v_q]$ is the equivalence class of the particular ordering $(v_0, ..., v_q)$. Two orderings are equivalent if they differ from one another by an even permutation.

Let Λ denote an abelian group. A formal sum, $\lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n$, where $\lambda_i \in \Lambda$ and σ_i are oriented q-simplices, is called a q-chain. The chain complex canonically associated to K, denoted by $C_*(K)$, is the family of groups such that in each dimension q, $C_q(K)$ is the group of q-chains in K. The boundary of a q-simplex $\sigma = [v_0, v_1, ..., v_q]$ is the (q-1)-chain

$$\partial_q \sigma = \sum_{i=0}^q (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_q],$$

where the hat means that v_i is omitted. By linearity, the boundary operator ∂_q can be extended to q-chains, where it is a homomorphism. It is clear that for each q-simplex σ_j there exist unique integers λ_{ij} such that

$$\partial_q(\sigma_j) = \sum_{\tau_i \in K^{(q-1)}} \lambda_{ij} \tau_i$$

The matrix $A_q = (\lambda_{ij})$ is the *matrix of* ∂_q relative to the bases $K^{(q)}$ and $K^{(q-1)}$. The group of q-cycles, $Z_q(K)$, is the kernel of ∂_q , and define $Z_0(K) = C_0(K)$. The group of q-boundaries, $B_q(K)$, is the image of ∂_{q+1} , that is, the subgroup of q-chains $b \in C_q(K)$ for which there exists a (q+1)-chain a with $b = \partial_{q+1}a$. It can be shown that $\partial_q \partial_{q+1}$ is null so $B_q(K)$ is a subgroup of $Z_q(K)$. Then, the qth homology group

$$H_q(K) = Z_q(K) / B_q(K)$$

can be defined for each integer q. Let K and L be two simplicial complex. A chain map $f: C_*(K) \to C_*(L)$ is a family of homomorphisms

$$\{f_q: C_q(K) \to C_q(L)\}_{q \ge 0}$$

such that $\partial_q f_q = f_{q-1}\partial_q$, for all q.

Dual concepts to the previous ones can be defined. The *cochain complex* canonically associated to K, $C^*(K)$, is the family

$$C^{*}(K) = \{C^{q}(K), \delta_{q}\}_{q \ge 0},\$$

where

$$C^q(K) = \operatorname{Hom}(C_q(K); \Lambda) = \{c : C_q(K) \to \Lambda, \quad c \text{ is a homomorphism} \}$$

and $\delta^q: C^q(K) \to C^{q+1}(K)$ called the *coboundary operator* is given by

$$\delta^q(c)(a) = c(\partial_{q+1}a),$$

where $c \in C^q(K)$ and $a \in C_{q+1}(K)$. Observe that a *q*-cochain can be defined only on $K^{(q)}$ and extended to $C_q(K)$ by linearity. Moreover, if Λ is a ring, then a basis of $C^q(K)$ is the set of homomorphisms

$$\sigma^*: C_q(K) \to \Lambda \,,$$

such that if $\tau \in K^{(q)}$, then $\sigma^*(\tau) = 1$ if $\tau = \sigma$, and $\sigma^*(\tau) = 0$ otherwise. $Z^q(K)$ and $B^q(K)$ are the kernel of δ^q and the image of δ^{q-1} , respectively. The elements in $Z^q(K)$ are called *q*-cocycles and those in $B^q(K)$ are called *q*-coboundaries. It is also satisfied that $\delta^q \delta^{q-1} = 0$ so the *q*th cohomology group

$$H^q(K) = Z^q(K)/B^q(K)$$

can also be defined for each integer q. If Λ is a ring, the cohomology of K is also a ring with the *cup product*

$$\smile: H^p(K) \otimes H^q(K) \to H^{p+q}(K)$$

defined at cocycle level by

$$c \smile c'(v_0, v_1, \ldots, v_{p+q}) = \mu(c(v_0, \ldots, v_p) \otimes c'(v_p, \ldots, v_{p+q}))$$

where $v_0 < v_1 < \cdots < v_{p+q}$, c is an p-cocycle, c' is a q-cocycle and μ is the product on Λ .

We use in this paper a special type of homotopy equivalences. A contraction r of a chain complex N_* to another chain complex M_* is a set of three homomorphisms (f, g, ϕ) where $f: N_n \to M_n$ (projection) and $g: M_n \to N_n$ (inclusion) are chain maps and satisfy that $fg = 1_M$, and $\phi: N_n \to N_{n+1}$ (homotopy operator) satisfies that

$$1_N - gf = \phi \partial_N + \partial_N \phi \,.$$

Moreover, $\phi g = 0$, $f\phi = 0$, $\phi\phi = 0$. A contraction up to dimension n of N_* to M_* consists in a set of three homomorphisms (f, g, ϕ) such that

$$f_k: N_k \to M_k$$
, $g_k: M_k \to N_k$ and $\phi_{k-1}: N_{k-1} \to N_k$

are defined for all $k \leq n$, $\phi_n = 0$, and the conditions of being a contraction are satisfied up to dimension n. Starting from a contraction $r = (f, g, \phi)$ of N_* to M_* , it is possible to give another contraction $r^* = (f^*, g^*, \phi^*)$ of $\text{Hom}(N; \Lambda)$ to $\text{Hom}(M; \Lambda)$ as follows:

$$f^* : \operatorname{Hom}(N_n; \Lambda) \to \operatorname{Hom}(M_n; \Lambda), \qquad g^* : \operatorname{Hom}(M_n; \Lambda) \to \operatorname{Hom}(N_n; \Lambda),$$

$$\phi^* : \operatorname{Hom}(N_n; \Lambda) \to \operatorname{Hom}(N_{n-1}; \Lambda),$$

are such that

 $f^*(c) = cg$, $g^*(c') = cf$ and $\phi^*(c) = c\phi$, where $c \in \operatorname{Hom}(N_n; \Lambda)$ and $c' \in \operatorname{Hom}(M_n; \Lambda)$.

3 "Minimal" Chain Complexes

It is possible to translate the results of the "reduction algorithm", discussed at length in [Mun84], in terms of homotopy equivalences. Combining this translation with modern homological perturbation techniques, algorithms for computing algebraic invariants, such as the A_{∞} -algebra structure on the cohomology of K and primary and secondary cohomology operations can be designed in an easy way.

First of all, it is necessary to recall the reduction algorithm for computing homology groups of a finite simplicial complex K. This method consists in reducing the matrix A of the boundary operator in each dimension q, relative to given bases of $C_q(K)$ and $C_{q-1}(K)$, to its Smith normal form A' (a matrix of integers satisfying that all its elements are zero except for $\lambda'_{11} \geq 1$ and $\lambda'_{11}/\lambda'_{22}/\cdots/\lambda'_{\ell\ell}$ for some integer ℓ). This reduction is done in each dimension q modifying the given base of $C_{q-1}(K)$, using the following "elementary row operations" on the matrix A:

- (1) Exchange row i by row k.
- (2) Multiply row i by -1.
- (3) Replace row i by row i + n (row k), where n is an integer and $k \neq i$.

Of course, there are similar "column operations" on A corresponding to changes of basis of $C_q(K)$. With this operations, the Smith normal form A' of A can be obtained, relative to some bases $\{a_1, \ldots, a_r\}$ of $C_q(K)$ and $\{e_1, \ldots, e_s\}$ of $C_{q-1}(K)$. Then,

- (1) $\{a_{\ell+1}, ..., a_r\}$ is a basis of $Z_q(K)$,
- (2) $\{\lambda'_{11}e_1, \ldots, \lambda'_{\ell\ell}e_\ell\}$ is a basis of $B_{q-1}(K)$.

Obviously, a dual treatment for $C^*(K)$ and, consequently, for the cohomology $H^*(K)$, can be done.

A chain complex $M_*(K)$ is called *minimal* if in each dimension q, $M_q(K)$ is a finitely generated free abelian group and the Smith normal form A' of the differential of $M_q(K)$ has the first element λ'_{11} different from 1. An *algebraic minimal model* of K is a minimal chain complex $M_*(K)$ together with a contraction of $C_*(K)$ to $M_*(K)$. Indeed, there is an algebraic minimal model for any finite simplicial complex K and any two algebraic minimal models of K are isomorphic.

Now, let us construct inductively an algebraic minimal model of a given finite simplicial complex K. Suppose that an algebraic minimal model up to dimension q-1is already constructed. That is, we have a minimal chain complex $M'_*(K)$ such that $M'_i(K) = 0, i \ge q$, and a contraction up to dimension q-1, (f', g', ϕ') , of $C_*(K)$ to $M'_*(K)$. Reduce the matrix of $\partial_q : C_q(K) \to C_{q-1}(K)$ to its Smith normal form A'. If the elements $\lambda'_{11} = \cdots = \lambda'_{tt} = 1$, for $t \le \ell$ (that is, $\partial(a_i) = e_i$ for $1 \le i \le t$), then define $M_*(K)$ as follows:

$$M_i(K) = M'_i(K), \quad \text{for } i \neq q - 1, q$$
$$M_{q-1}(K) = M'_{q-1}(K) - \Lambda[e_1, \dots, e_t]$$
$$M_q(K) = C_q(K) - \Lambda[a_1, \dots, a_t]$$

where $\Lambda[a_1, \ldots, a_t]$ and $\Lambda[e_1, \ldots, e_t]$ are the free abelian groups generated by $\{a_1, \ldots, a_t\}$ and $\{e_1, \ldots, e_t\}$, respectively. The formulae for the component morphisms of the contraction up to dimension q, (f, g, ϕ) , of $C_*(K)$ to $M_*(K)$ are:

$$\begin{split} f(x) &= \begin{cases} f'(x) & \text{if } x \in \Lambda[e_{t+1}, \dots, e_s] \text{ or } x \in C_i(K), \ i < q, \\ 0 & \text{if } x \in \Lambda[e_1, \dots, e_t] \text{ or } x \in \Lambda[a_1, \dots, a_t], \\ x & \text{if } x \in \Lambda[a_{t+1}, \dots, a_r], \end{cases} \\ g(y) &= \begin{cases} g'(y) & \text{if } y \in M_i(K), \ i < q, \\ y & \text{if } y \in M_n(K), \end{cases} \\ \phi(x) &= \phi'(x) & \text{if } x \in C_i(K), \ i < q - 1, \\ \phi(e_i) &= a_i & \text{if } 1 \le i \le t, \\ \phi(e_i) &= 0 & \text{if } t + 1 \le i \le s. \end{cases} \end{split}$$

In this way, we can determine an algebraic minimal model for a finite simplicial complex K. Observe that whenever Λ is a field or the homology of K is free, then $M_*(K)$ is isomorphic to $H_*(K)$ and, therefore, we can obtain a contraction of $C_*(K)$ to its homology.

Passing to cohomology does not represent a problem and a dual process can be done without effort.

The fact of dealing with contractions is highly important in obtaining topology invariants such as the A_{∞} -algebra structure of the cohomology of K [GS86]. In particular, if $\Lambda = \mathbf{Q}$, then from the previous contraction connecting $C^*(K)$ with $H^*(K)$, it is possible to design an algorithm computing the commutative A_{∞} -algebra structure of $H^*(K)$ reflecting the complete rational homotopy type of K [Kad98]. We will see in the next section that the homotopy equivalence data structure is also essential in computing cohomology operations.

4 Steenrod Cohomology Operations

Let us suppose $\Lambda = \mathbf{Z}_p$ (*p* being a prime), then it is possible to construct an algebraic minimal model for any finite simplicial complex *K*, in which the associated contraction (f^*, g^*, ϕ^*) connects $C^*(K)$ with its cohomology. From this data and the combinatorial formulae for Steenrod squares and reduced *p*th powers [Ste47, ES62] at cochain level in terms of face operators established in [GR99, Gon00], Steenrod cohomology operations can effectively be computed.

For instance, the formula for the Steenrod reduced power

$$\mathcal{P}_1: H^*(X) \to H^{*p-1}(X)$$

at cochain level [Gon00] is:

$$P_{1}(c)(\sigma) = \sum_{j=1}^{p-1} \sum_{i=jq}^{(j+1)q-1} (-1)^{(i+1)(q+1)+1} \\ \mu(c(v_{0}, \dots, v_{q})) \\ \otimes (v_{q}, \dots, v_{2q}) \\ \vdots \\ \otimes c(v_{(j-2)q}, \dots, v_{(j-1)q}) \\ \otimes c(v_{(j-1)q}, \dots, v_{i-q}, v_{i}, \dots, v_{(j+1)q-1})$$

$$\otimes c(v_{(j+1)q-1}, \dots, v_{(j+2)q-1})$$

$$\vdots$$

$$\otimes c(v_{(p-2)q-1}, \dots, v_{(p-1)q-1})$$

$$\otimes c(v_{(p-1)q-1}, \dots, v_{pq-1})$$

$$\otimes c(v_{i-q}, \dots, v_i))$$

where c is a q-cocycle, $\sigma = (v_0, v_1, \ldots, v_{pq-1})$ is a (pq-1)-simplex such that $v_0 < v_1 < \cdots < v_{pq-1}$ and μ is the product on \mathbf{Z}_p . Therefore, for calculating the cohomology class $\mathcal{P}_1(\alpha)$ with $\alpha \in H^q(K)$, we only have to compute $f^*P_1g^*(\alpha)$.

In the particular case of Steenrod squares,

$$Sq^i: H^*(K; \mathbf{Z}) \to H^{*+i}(K; \mathbf{Z}_2),$$

we can express them in a matrix form due to the fact that these cohomology operations are homomorphisms. Moreover, the process of diagonalization of such matrices can give us detailed information about the kernel and image of these cohomology operations.

5 Adem Secondary Cohomology Operations

For attacking the computation of secondary cohomology operations, we will see in this section that the homotopy operator ϕ^* of the contraction associated to an algebraic minimal model of a simplicial complex K is essential.

First of all, we shall indicate how Adem secondary cohomology operations

$$\Psi_q: N^q(K) \to H^{q+3}(K; \mathbf{Z}_2) / Sq^2 H^{q+1}(K; \mathbf{Z})$$

can be constructed (see [Ade52]). $N^q(K)$ denotes the kernel of $Sq^2 : H^q(K; \mathbb{Z}) \to H^{q+2}(K; \mathbb{Z}_2)$ These operations appear using the known relation:

$$Sq^2Sq^2\alpha + Sq^3Sq^1\alpha = 0$$

for any $\alpha \in H^*(K; \mathbb{Z})$. For this particular relation there exist cochain mappings

$$E_j: C^*(K \times K \times K \times K) \to C^{*-j}(K)$$

such that mod 2

$$(c \smile_{q-2} c) \smile_q (c \smile_{q-2} c) + (c \smile_{q-1} c) \smile_q (c \smile_{q-1} c) = \delta E_{3q-3}(c^4),$$

where \smile_k is the cup-k product [Ste47] and c is a q-cochain. Recall that, at cochain level, $Sq^i(c) = c \smile_{j-i} c \mod 2$, where c is a j-cocycle. Then Ψ_q is defined at cochain level by

$$\psi_q(c) = b \smile_{i+1} b + b \smile_{i+2} \delta b + E_{3i+3}(c) + \eta(c) \smile_{i-1} \eta(c) + \eta(c) \smile_i \delta \eta(c),$$

where c is a q-cocycle representative of a cohomology class of $N^q(K)$, b is a (q+1)-cochain such that $c \smile_{q-2} c = \delta b$ and $\eta(c) = \frac{1}{2}(c \smile_q c + c)$.

If \mathbf{Z}_2 is the ground ring, formulae for computing cup-*i* products are well-known [Ste47]. A method for obtaining "economical" formulae for E_{3i+3} in terms of face operations is given in [Gon00]. For example,

$$E_{3}(c^{4})(\sigma) = \mu(c(v_{0}, v_{2}, v_{3}) \otimes c(v_{0}, v_{1}, v_{2}) \otimes c(v_{3}, v_{4}, v_{5}) \otimes c(v_{2}, v_{3}, v_{5}) + c(v_{0}, v_{4}, v_{5}) \otimes c(v_{3}, v_{4}, v_{5}) \otimes c(v_{0}, v_{1}, v_{2}) \otimes c(v_{0}, v_{1}, v_{2}) + c(v_{0}, v_{1}, v_{5}) \otimes c(v_{3}, v_{4}, v_{5}) \otimes c(v_{1}, v_{2}, v_{3}) \otimes c(v_{1}, v_{2}, v_{3}) + c(v_{0}, v_{1}, v_{2}) \otimes c(v_{2}, v_{4}, v_{5}) \otimes c(v_{2}, v_{3}, v_{4}) \otimes c(v_{2}, v_{3}, v_{4}) + c(v_{0}, v_{1}, v_{2}) \otimes c(v_{2}, v_{3}, v_{5}) \otimes c(v_{3}, v_{4}, v_{5}) \otimes c(v_{3}, v_{4}, v_{5})),$$

where c is a 2-cochain, $\sigma = (v_0, v_1, ..., v_5)$ is a 5-simplex such that $v_0 < v_1 < \cdots < v_5$ and μ is the product on \mathbb{Z}_2 . Therefore, the steps for computing Ψ_q are the following:

- 1. Take $\alpha \in N^q(K)$ making use of the diagonalization of the matrix of Sq^2 in dimension q.
- 2. Compute $b = \phi^* S q^2 g^*(\alpha)$.
- 3. Compute $f^*\psi g^*(\alpha)$.

Note that it is very easy to prove that

$$g^*(\alpha) \smile_{q+2} g^*(\alpha) = \delta \phi^* S q^2 g^*(\alpha) \,,$$

using the relation $1 - g^* f^* = \phi^* \delta + \delta \phi^*$.

6 Some Comments

All these results can be given in a more general framework working with not necessarily finite simplicial complexes. Nevertheless, a contraction of the chain complex associated to the simplicial complex to its (co)homology must exist in order to develop the method.

Concerning the complexity, obtaining a contraction of a finite simplicial complex K to its (co)homology can be done using Delfinado–Edelsbrunner incremental algorithm [ELZ00] which runs in time as most cubic in the number of simplices of the complex if the group of coefficients is a field. On the other hand, another datum to take into account is the number of summands of the formulae for computing cohomology operations at cocycle level. For example the number of summands of P_1 over a q-cocycle c and a (pq-1)-simplex σ is (p-1)q.

Finally, in order to obtain the image of any cohomology operations at cochain level over a representative cocycle using our formulae, we have to compute them over a basis of $C_*(K)$ in the desired dimension. A way of decreasing the complexity of this is to do a "topological" thinning of the simplicial complex K in order to obtain a thinned simplicial subcomplex $M_{\text{top}}(K)$ of K, such that there exists a contraction of $C_*(K)$ to $C_*(M_{\text{top}}(K))$ For example, one way to construct it is using simplicial collapses [For99]. Then we can apply our machinery to compute cohomology operations in the thinned simplicial complex $M_{\text{top}}(K)$ and the results can be easily interpreted in the "big" simplicial complex K.

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