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The η-Invariant and Wall Non-Additivity

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Let M^{4k-1} be a closed $4k-1$ dimensional Riemannian manifold. In [1] was studied the following invariant of M^{4k-1},

$$\eta(M^{4k-1}) = \int_{X^{4k}} \mathscr{L}_{4k} - \sigma(X^{4k}), \tag{1}$$

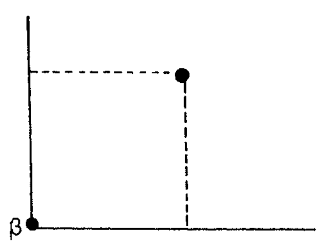
where X^{4k} is a $4k$-dimensional Riemannian manifold with $\partial X = M^{4k-1}$ and a neighborhood of ∂X metrically of the form $M \times [0, 1)$, $\mathscr{L}_{4k}$ denotes the $4k$ degree term in the total $\mathscr{L}$-polynomial built from the curvature forms on X, and $\sigma(X^{4k})$ denotes the signature (or index) of X^{4k}. That (1) defines an invariant of M follows essentially from the additive nature of signature. That is, if X^{4k}, Y^{4k} are two manifolds-with-boundary, $\partial X \cong \partial Y$, then the signature of the manifold obtained by gluing X to Y is determined by the signature of X and Y as follows:

$$\sigma(X \cup_\partial - Y) = \sigma(X) - \sigma(Y). \tag{2}$$

We now wish to define an invariant analogous to the η-invariant above for the pairs (M^{4k-1}, β^{4k-2}), $\beta \subset M$ a separating closed submanifold which possesses a neighborhood metrically of the form $\beta \times (-1, 1)$. We define

$$\eta(M^{4k-1}, \beta^{4k-2}) = \int_{X^{4k}} \mathscr{L}_{4k} - \sigma(X^{4k}) \tag{3}$$

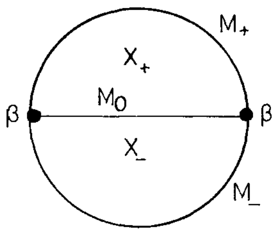
where X^{4k} is a $4k$-dimensional Riemannian manifold *with a corner at β*. More precisely $\partial X^{4k} = M^{4k-1}$ with a neighborhood of $M - \beta$ of the form $(M - \beta) \times [0, 1)$ and a neighborhood of β of the form $\beta \times ([0, 1) \times [0, 1))$ metrically.



We will write $\tilde{\partial} X = (M, \beta)$ and (M, β) as a manifold with a bend.

Proposition 1. *$\eta(M^{4k-1}, \beta^{4k-2})$ is a well-defined invariant of the pair (M, β). That is, if X_1^{4k}, X_2^{4k} are two manifolds as above with $\tilde{\partial} X_i = (M, \beta)$, then $\int\limits_{X_1} \mathscr{L} - \sigma(X_1) = \int\limits_{X_2} \mathscr{L} - \sigma(X_2)$.*

This proposition will follow from study of what happens to signature when manifolds with boundary are glued together only along portions of their boundary. The setup goes as follows. Let $X_\pm^{4k}$ denote two $4k$-dimensional Riemannian manifolds with $\partial X_\pm^{4k} = M_\pm^{4k-1} \cup M_0^{4k-1}$ where $M_\pm$ and M_0 are $4k-1$ dimensional manifolds-with-boundary and $\partial M_\pm = \partial M_0 = M_\pm \cap M_0 \subset \partial X_\pm^{4k}$. Let $\beta = \partial M_\pm = \partial M_0$.



Wall, in [3], defines an invariant $\delta(\beta;\, M_-,\, M_0,\, M_+)$, which we will refer to as Wall's non-additivity invariant and which fits into the following formula relating the signatures of $X_+ \cup_{M_0} X_-$ and $X_\pm$:

$$\sigma(X_+ \cup_{M_0} - X_-) = \sigma(X_-) - \sigma(X_+) + \delta(\beta; M_-, M_0, M_+)\,. \tag{4}$$

We will discuss δ more fully in Sect. 1 when we give the proofs of our results.

The second observation we wish to discuss relates the η-invariant of closed manifolds to η-invariants for pairs as discussed above. Let M^{4k-1} be a closed manifold. Assume that there exists a Riemannian manifold X^{4k} such that

a) $\partial X = M$ with a neighborhood metrically of the form $M \times [0, 1)$,

b) there exists $\beta^{4k-2} \subset M$ a closed separating submanifold with a metric neighborhood in X of the form $\beta \times ([0, 1) \times (-1, 1))$, and

c) there exists $M_0^{4k-1} \subset X$ a submanifold, $\partial M_0 = \beta = \partial X \cap M_0$, separating X into 2 components X_+, X_- such that there are neighborhoods of β in $X_\pm$ metrically of the form $\beta \times [0, 1) \times [0, 1)$ and neighborhoods of M_0 in $X_\pm$ of the form $M_0 \times [0, 1)$ metrically.

Then if we let $M_\pm$ be the component of $M - \beta$ lying in $X_\pm$, we see that $(M_\pm \cup_\beta M_0 = N_\pm, \beta)$ defines a pair of manifolds as above for each of $\pm$. Our second observation is

Proposition 2. $\eta(M^{4k-1}) = \eta(N_+^{4k-1}, \beta^{4k-2}) - \eta(N_-^{4k-1}, \beta) + \delta(\beta; M_-, M_0, M_+)$.

Notice that this proposition decomposes the calculation of the a priori global invariant $\eta(M)$ into similar sorts of pieces each of which have at least a chance of being more simple than $\eta(M)$ itself, together with a global homological invariant of the configuration. So on the one hand Proposition 2 chops up the η-invariant of a closed manifold into the sum of two η-invariants of manifolds with corners and a homological term. While on the other hand Proposition 2 gives an analytical calculation of Wall's non-additivity invariant. The obvious question now though concerns whether $\eta(N,\beta)$ is a spectral invariant of the pair (N,β). To answer the question along the lines of the study done in [1] one would need a careful analysis of a parametrix for the signature operator in a neighborhood of the corner β in X. As of now such an analysis by this author is not complete.

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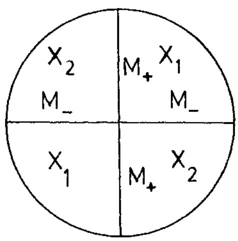
For the proofs of Propositions 1 and 2 we need but two facts concerning $\delta(\beta; M_-, M_0, M_+)$. First that δ is defined in the following situation. If M_1, M_2, M_3 are $4k-1$ dimensional manifolds with boundary, and $\beta=\partial M_1=\partial M_2=\partial M_3$, then $\delta(\beta; M_1, M_2, M_3)$ is defined as an invariant of the kernels of the inclusion induced maps of $H_{2k-1}(\beta;\mathbb{R})$ into $H_{2k-1}(M_i;\mathbb{R})$. Second that δ is an alternating function of its final three arguments.

To prove Proposition 1 let (M,β) and X_1, X_2 be given and let $M-\beta=M_+\cup M_-$. Then $X_\pm=X_1\cup_{M_\pm}X_2$ for $\pm$ are smooth Riemannian manifolds with smooth boundaries

$$\partial X_\pm = M_\mp \cup_\beta M_\mp = D(M_\mp) \text{ the double of } M_\mp .$$

Now construct $X=X_+\cup_{D(M)}X_+$. Schematically,

X is  , $\partial X=\phi$.

The Hirzebruch signature formula [2] gives

$$\int_X \mathscr{L}=\sigma(X).$$

But our assumptions on the metrics and the orientations necessary to glue manifolds compatibly gives

$$\int_X \mathscr{L}=2\left\{\int_{X_1}\mathscr{L}-\int_{X_2}\mathscr{L}\right\}=\sigma(X)=2\sigma(X_+)$$

by Novikov additivity. Wall non-additivity then provides

$$\sigma(X_+)=\sigma(X_1)-\sigma(X_2)+\delta(\beta; M_-, M_+, M_-).$$

Finally, the antisymmetric nature of δ's dependence on its arguments means that $\delta(\beta; M_-, M_+, M_-)=0$ and so

$$\int_{X_1}\mathscr{L}-\int_{X_2}\mathscr{L}=\sigma(X_1)-\sigma(X_2).$$

Then $\eta(M,\beta)$ is independent of X with $\tilde{\partial}X=(M,\beta)$. The proof of Proposition 2 is just as simple. We are given M^{4k-1} such that $M=\partial X$, $X=X_+\cup_{M_0}X_-$, and

$$\tilde{\partial}X_\pm=(N_\pm,\beta).$$

Now

$$\eta(M)=\int_X \mathscr{L}-\sigma(X)=\int_{X_+}\mathscr{L}-\int_{X_-}\mathscr{L}-\{\sigma(X_+)-\sigma(X_-)+\delta(\beta;M_+,M_0,M_-)\}$$

where $\partial X_\pm = M_\pm \cup_\beta M_0$. By Proposition 1

$$\eta(N_\pm,\beta)=\int_{X_\pm}\mathscr{L}-\sigma(X_\pm),$$

and so

$$\eta(M)=\eta(N_+,\beta)-\eta(N_-,\beta)+\delta(\beta;M_-,M_0,M_+).$$

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