Problems Concerning Embeddings of Manifolds*

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Abstract

Very little is known in general about estimating the smallest integer l such that a manifold M^n embeds in $\boldsymbol{R}^{n+k+l}$ if it immerses in $\boldsymbol{R}^{n+k}$. Indeed there are relatively few examples where k and l can be estimated accurately. There are old examples [10] for which l is known to be arbitrarily large, for those examples l can grow like $\log n$ and there are recent examples [8] where l can grow linearly with n. The main difficulty in resolving questions of this kind is that the only general methods known for proving non-embedding and non-immersion results involve doing calculations with characteristic classes and the estimates that they give are very similar for the two problems. In this paper an account is given of various methods that can be used to study examples.

The first methods are the classical ones of algebraic topology—characteristic classes and the Wu-Haefliger-Hirsch obstructions. The next is that of studying the Thom complex of the normal bundle; this can give both negative and positive embedding results. This method leads naturally to the study of possible complements in a sphere, a method refined by G. Cooke [5]. The third method(§5) is a reformulation and extension of Whitney's method described by J.F.P. Hudson [11]. This can be used to show that one can have the maximal possible difference between the immersion and embedding dimensions when the target manifold is not necessarily Euclidean space. The immersion $S^n \to \boldsymbol{R}P^n$ is not homotopic to an embedding in $\boldsymbol{R}P^n \times \boldsymbol{R}^n$. Finally, some interesting special cases are studied and some open questions are highlighted.

§1 Characteristic Classes

Suppose M is a closed manifold and let

$$w(M) = 1 + w_1(M) + w_2(M) + \cdots$$

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be its total (tangential) Stiefel-Whitney class and

$$\bar{w}(M) = 1 + \bar{w}_1(M) + \bar{w}_2(M) + \cdots$$

be defined by $w(M)\bar{w}(M) = 1$.

Then if M^n immerses in $\boldsymbol{R}^{n+k}$ one knows that $\bar{w}_i(M) = 0$ for $i > k$, and, if M^n embeds in $\boldsymbol{R}^{n+k}$ then $\bar{w}_k(M)$ also vanishes.

Any attempts to improve directly on these methods lead to the study of secondary and higher order characteristic classes but they quickly become unmanageable. The next approach seems more tractable.

§ 2 Wu-Haefliger-Hirsch Method

First it is convenient to define certain useful ranges of dimensions. Suppose $f: M^n \to Q^q$ is a map between manifolds. The set $\Sigma_k(f)$ consisting of the k-tuple points of f is the subset of M of which at least k distinct points are identified under f. If f is in general position then, by linear algebra, one has that

$$\dim \Sigma_k(f) \leqslant kn - (k-1)q.$$

The various ranges of dimensions referred to above are those that imply, using this inequality, that various $\Sigma_k(f)$ are empty. So the stable range is where $2n - q < 0$ i.e. $q > 2n$, the metastable range is where $2q > 3n$ etc.

Let $S(\tau M)$ denote the sphere bundle of the tangent bundle of M and $M \times M \backslash \Delta$ be the deleted square (Δ is the diagonal). Both these spaces have free involutions, the first by taking the antipodal map in each fibre and the second by switching the factors. The normal bundle of Δ in $M \times M$ can be identified with τM and hence one can construct an equivariant inclusion

$$S(\tau M) \longrightarrow M \times M \backslash \Delta.$$

A theorem of Wu[24] states that, in the metastable range, there is an embedding of M in $\boldsymbol{R}^q$ if there is an equivariant map

$$M \times M \backslash \Delta \longrightarrow S^{q-1}.$$

Analogously, the theorem of Haefliger and Hirsch states that, in the same range, there is an immersion of M in $\boldsymbol{R}^q$ if there is an equivariant map

$$S(\tau M) \longrightarrow S^{q-1}.$$

So, from this viewpoint, since the inclusion $S(\tau M) \hookrightarrow M \times M \backslash \Delta$ seems complicated, there is no reason to expect to find embeddings when one knows that there is an immersion in a particular Euclidean space. In terms of the notation introduced at the beginning of this paper, one seeks a small l such that there is always an extension g:

$$\begin{array}{ccc} S(\tau M) & \xrightarrow{f} & S^{n+k-1} \\ \downarrow & & \downarrow \\ M \times M \backslash \Delta & \xrightarrow[g]{} & S^{n+k+l-1} \end{array}$$

Below the metastable range, it is known that there are examples of exotic spheres Σ^n that do not embed in very low codimension [9]. Being stably parallelizable, all exotic spheres immerse in Euclidean space with codimension one. The paper [9] reduces the problem of embedding a given homotopy sphere to a problem in the homotopy groups of spheres. Now that information about these homotopy groups is much more substantial, a more general pattern can be seen. I have been informed by M. Mahowald that he believes that using the homotopy elements described in [13] one can construct homotopy spheres of dimension 2^{j+1} that do not embed in codimension $2^j - 3$. This would essentially be a maximal possible difference between the embedding and immersion dimensions for an exotic sphere since they all embed at the bottom of the metastable range, because they embed topologically and hence smoothly by [2].

§ 3 The Thom Construction

If a compact M^n embeds in $\boldsymbol{R}^{n+k}$ with normal bundle ν then the Thom construction gives a map $S^{n+k} \longrightarrow T(\nu)$ of degree one onto the Thom space. This implies that $T(\nu)$ is homotopy equivalent to $S^{n+k} \vee X$ where X is homotopy equivalent to the Thom complex of ν restricted to $M \backslash point$. Under very favourable circumstances one might be able to show that this cannot happen. A list of all possible normal bundles ν would be made and then a check that each possible $T(\nu)$ does not decompose as above.

One case where this method works particularly well is when one is considering codimension one embeddings. If M is orientable, then the only possible normal bundle is the trivial line bundle, so if M^n embeds in $\boldsymbol{R}^{n+1}$, there would be a degree one map $S^{n+1} \longrightarrow \Sigma M$, the suspension of M. As examples, consider $\boldsymbol{RP}^3$, $\boldsymbol{RP}^7$ which are parallelizable and hence have immersions in codimension one. There is a nonzero cohomology operation—the Postnikov square with values in $H^{n+1}(\Sigma M; \boldsymbol{Z}/4)$ in both these cases and so ΣM does not decompose. The same argument can be used for some lens spaces. It was pointed out by Epstein [6] that, in this codimension one case, if $M \backslash point$ embeds in $\boldsymbol{R}^{n+1}$ then the fact that ΣM decomposes can already be deduced.

In a higher range of dimensions, codimension at least three for piecewise linear and the metastable range of smooth, the criterion that $T(\nu)$ is homotopy equivalent to $S^{n+k} \vee X$ is almost necessary and sufficient for the existence of an embedding (see Browder [2] for a survey). The existence of an embedding can be deduced only in one greater codimension. The codimension two case has been considered by Cappell and Shaneson [3]. In the case of $\boldsymbol{RP}^7$ this method was used in [15] to find embeddings.

§ 4 The Study of Complements

First, we give another method for proving the non-existence of codimension one embeddings. Suppose M^n embeds in S^{n+1}; it divides S^{n+1} into two connected pieces A and B which we can take to be such that $A \cap B = M$. Then, using the Mayer-Vietoris sequence, we see that

$$H_r(M) \cong H_r(A) \oplus H_r(B) \quad \text{for } 1 \leqslant r < n,$$

indeed, there is a map $\Sigma M \longrightarrow \Sigma A \vee \Sigma B$ inducing this isomorphism.

By duality, one has

$$\tilde{H}_r(A) \cong \tilde{H}^{n-r}(B) \quad \text{for all } r.$$

In suitable cases, this information already leads to a contradiction. For example, suppose $\boldsymbol{R}P^3$ embeds in S^4. Then $\boldsymbol{Z}/2 \cong H_1(\boldsymbol{R}P^3) \cong H_1(A) \oplus H_1(B)$. Suppose $H_1(A) = \boldsymbol{Z}/2$ and $H_1(B) = 0$. Then $H_2(A; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$ and so $H_2(B; \boldsymbol{Z}/2) = 0$. This contradicts the duality isomorphism. The same argument applies to three dimensional lens spaces and, using Steenrod squares, to $\boldsymbol{R}P^7$. An interesting example to show what happens when one has a codimension one embedding is the following, see Massey [14]. Let $Q^3 = S^3/\pi$ where π is the subgroup of unit quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$, then there is an embedding $Q^3 \subset S^4$ whose complements A, B both deformation rectract onto $\boldsymbol{R}P^2$. The suspension of $Q \backslash point$ is the suspension of $\boldsymbol{R}P^2 \vee \boldsymbol{R}P^2$, reflecting the fact that the abelianisation of π is $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$.

In higher codimensions there have been a number of studies of complements, culminating in the paper by G. Cooke [5] which, unfortunately, contains a small error in the details—as pointed out by Connolly and Williams [4]. Cooke's results(when corrected) are based on the following result.

Suppose that K is a two cell complex embedded in $S^{m+r+l+2}$ with complement homotopy equivalent to L. If the attaching map in K is $\alpha \in \pi_{r+m} S^m$ and that in L is $\beta \in \pi_{r+l} S^l$ then $\Sigma^l \alpha = (-1)^{ml} \Sigma^m \beta \in \pi_{r+m+l} S^{m+l}$. In particular, if one knows that $\Sigma^l \alpha$ does not desuspend $l+1$ times then one knows that $m \leqslant l$. In his paper [1] Adams proved that the Whitehead product $[\iota_{n-1}, \iota_{n-1}] : S^{2n-3} \longrightarrow S^{n-1}$ does not desuspend $\rho(n)$ times where $\rho(n)$ is the Radon-Hurwitz number. Based on this fact, Hsiang and Sczcarba gave the first examples of manifolds that require arbitrary more dimensions before it is possible to deform an immersion to an embedding; Over S^{n-1} there is an $n-\rho(n)$ dimensional vector bundle ξ such that $\xi \oplus trivial \cong \tau(S^{n-1})$. Let M be the $n-\rho(n)$ dimensional sphere bundle over S^{n-1}: $M = S(\xi \oplus \varepsilon^1)$. Then M is homotopy equivalent to

$$(S^{n-\rho} \vee S^{n-1}) \cup_{w + J\xi} e^{2n-\rho-1}$$

where $J: \pi_{n-2} SO(r) \longrightarrow \pi_{n+r-2} S^r$ is the J-homomorphism and w is the Whitehead product of the two identity maps. By the definition of ξ, $\Sigma^{\rho-1} J\xi = [\iota_{n-1}, \iota_{n-1}]$.

Now, suppose this manifold M embeds in $S^{2n-\rho-1-k}$. Let $i: S^{n-1} \longrightarrow M$ be the inclusion (induced by the obvious non-vanishing section of $\xi \oplus \varepsilon$), then by adding a cone on i one has an embedding of a complex that is homotopy equivalent to $S^{n-\rho} \cup_{J\xi} e^{2n-\rho-1}$ in $S^{2n-\rho+k}$ and whose complement is homotopy equivalent to $S^k \cup_\beta e^{n+k-1}$. By Cooke's result, $\Sigma^{n-\rho}\beta = \pm \Sigma^k J\xi$; hence either $\Sigma^k J\xi \simeq 0$ so $k \geqslant \rho(n)$, or $\Sigma^{n-k-1}\beta = \pm \Sigma^{\rho-1} J\xi$ so $n-k-1 \leqslant \rho(n)$. If $n > 16$ one has $\rho(n) \leqslant n - \rho(n) + 1$ so $k \geqslant \rho(n)$ in all such cases. As $\rho(n)$ is an unbounded function of n, this gives the required result. In the case $n = 16$ ($\rho(16) = 9$), it seems still not known whether M^{22} can be embedded in $\boldsymbol{R}^{29}$.

The method outlined here is very useful for relatively simple manifolds such as sphere bundles over spheres. It gives results which are non-embedding results up to homotopy. This will imply piecewise linear non-embedding results in codimension three or more by the Casson-Sullivan Theorem(see [22]) ,and smooth nonembedding results in the metastable range [2].

§ 5 Hudson's Method

In 1944 H. Whitney [23] gave a method for proving that manifolds M^n embed in $\boldsymbol{R}^{2n}$. His idea was to consider the double points of an immersion and to show that they occurred in pairs that could be cancelled. In his book [11], J.F.P. Hudson recast and generalised this method. He worked in the piecewise linear category.

Let $F: M^n \longrightarrow Q^{n+k}$ be a PL mapping in general position and such that there are no triple points (this is automatic in the metastable range $3n < 2k$). Let S_2F be the double point set of the map F:

$$S_2(F) = \{x \in M \mid \text{there is } y \neq x \text{ such that } F(x) = F(y)\}.$$

We assume that M has been triangulated so that $S_2(F)$ is a subcomplex. If σ_1, σ_2 are $(2n-q)$-dimensional simplices in $S_2(F)$ whose images coincide, let S_1, S_2 be their links in M and let Σ be the link of $F(\sigma_1)$ in Q. The map F embeds S_1 and S_2 in Σ, let $\varphi(F, \sigma_1)$ be their linking number $mod\ 2$. Regard

$$c(F) = \sum_{\sigma_1 \in S_2(F)} \varphi(F, \sigma_1)\sigma_1$$

as a $(2n-q)$-dimensional $mod\ 2$ chain on M. Hudson checks that this gives a well defined homology class

$$a(F) \in H_{2n-q}(M; \boldsymbol{Z}/2).$$

which is independent of the triangulations chosen and depends only on the homotopy class of F. Clearly, $a(F)$ vanishes if F is an embedding and hence if F is homotopic to an embedding. In our case we will have $q = 2n$ and so $a(F)$ can be regarded as an integer $mod\ 2$.

Let $f: S^n \longrightarrow P^n$ be the double covering map onto real projective space; f is an immersion denoted by

$$f(x_0,x_1,\cdots,x_n)=[x_0:x_1:\cdots:x_n].$$

Consider the map $F:S^n\longrightarrow P^n\times \boldsymbol{R}^n$ given by

$$F(x_0,x_1,\cdots,x_n)=([x_0:x_1:\cdots:x_n],(x_1,x_2,\cdots,x_n)).$$

This map F has a single double point, namely

$$F(1,0,\cdots,0)=F(-1,0,\cdots,0).$$

We can assume that the spaces are triangulated suitably and we need to calculate the linking number of the images of the boundaries of the neighbourhoods of these two points. This calculation can be done using differential topology. Let $\underline{x}=(x_1,x_2,\cdots,x_n)$, then near $(1,0,\cdots,0)$, F is approximated by

$$F(x_0,\underline{x})=(\underline{x},\underline{x})$$

using coordinates on P^n near $[1:0:\cdots:0]$ to be $(x_1/x_0,x_2/x_0,\cdots,x_n/x_0)$. Near $(-1,0,\cdots,0)$, F is approximated by

$$F(x_0,\underline{x})=(-\underline{x},\underline{x}).$$

The images of these maps are both n-dimensional vector subspaces of $\boldsymbol{R}^{2n}$ and are linearly independent. Hence their unit spheres have linking number equal to one in S^{2n-1}. This shows that the linking number we wish to calculate is also one. Since there is only one double point, Hudson's result gives that F is not homotopic to an embedding.

§ 6 Some Examples and Problems

In terms of embedding and immersion questions, the manifolds that have been most studied are those (projective spaces, Dold manifold, etc.) whose characteristic classes are relatively easy to calculate. At the other extreme are the parallelizable manifolds such as Lie groups. The following general method can be used to find low codimension embeddings of some compact Lie groups. Suppose H and K are two subgroups of G such that $H\cap K=\{1\}$ then the product of the quotient maps gives an embedding

$$G\longrightarrow G/H\times G/K.$$

One can often find good embeddings of suitable quotients G/H and G/K thus giving one of G. A good start can be obtained by considering a maximal torus T in G. The adjoint action of G on its Lie algebra $\mathfrak{G}$ has principal orbit G/T, and so G/T embeds in $\mathfrak{G}$ with trivial normal bundle. One can then make good use of the second part of the following

Proposition Let H and K be the subgroups of the compact Lie group G with $H\cap K=\{1\}$. Suppose G/H embeds in $\boldsymbol{R}^N$ and G/K embeds in $\boldsymbol{R}^M$, then G embeds in $\mathbb{R}^{N+M}$. If the embedding of G/K has trivial normal bundle then G embeds in $\boldsymbol{R}^{N+M-d}$ where d is the codimension of G/K in $\boldsymbol{R}^N$.

Corollary Let G be a compact Lie group, T a maximal torus and H a subgroup of

G not meeting T. If G/H embeds in $\boldsymbol{R}^N$ then G embeds in $\boldsymbol{R}^{N+d-l}$ were $d=\dim G$ and $l=\operatorname{rank} G=\dim T$.

Proof G embeds in $G/H\times G/T$ and G/T embeds in $\mathfrak{G}$ with trivial normal bundle, so one has $G/T\times \boldsymbol{R}^l\subset \boldsymbol{R}^d$. Hence

$$G/H\times G/T\hookrightarrow \boldsymbol{R}^N\times G/T=\boldsymbol{R}^{N-l}\times G/T\times \boldsymbol{R}^l\subset \boldsymbol{R}^{N-l}\times \boldsymbol{R}^d.$$

This is essentially the method used in [17] and (using $G/(S^3\times\cdots S^3)$) in [21] to give good embedding results. It has also been used in [20] for homogeneous spaces. It can also be used to give the results of [17] for Spin(n) although a slightly different argument was given there. This simplification was suggested to me by a remark made by Derek Hacon. Let T denote the involution on Spin(n) whose quotient is SO(n). The Lemma of [17, page 156] constructs an equivariant map f: $(\mathrm{Spin}(n),T)\longrightarrow(S^L,-I)$ where $L=k^2-k+1$ $(n=2k)$ and $=k^2+k+1$ $(n=2k+1)$. The product map $f\times\pi:\mathrm{Spin}(n)\longrightarrow S^L\times \mathrm{SO}(n)$ is an embedding and SO(n) embeds in $\boldsymbol{R}^N$ where $N=2k^2$ $(n=2k)$ and $=2k^2+2k+1$ $(n=2k+1)$. Hence Spin(n) embeds in $\boldsymbol{R}^{N+L}$ where $N+L=3k^2-k+1$ $(n=2k)$ and $3k^2+3k+2$ $(n=2k+1)$ i. e. codimension k^2+1 and $(k+1)^2+1$ respectively.

An important question is to find methods that might indicate to what extent these embeddings are best possible i.e. to find methods for proving good non-embedding results for parallelizable manifolds. A good test problem might be:

Show that there is no fixed l such that SO(n) embeds in codimension l for every n. Even to try to show that, for large n, SO(n) (or Spin(n)) does not embed in codimension three seems to be a challenging problem.

The general problem of the embedding question for finite coverings also does not seem to be very tractable. For example, if $\widetilde{M}\longrightarrow M$ is a covering, what invariants will estimate a reasonable l so that if $M\subset \boldsymbol{R}^{n+k}$ then $\widetilde{M}\subset \boldsymbol{R}^{n+k+l}$? One might at least hope that for odd coverings this question might have a reasonably neat answer. The method outlined above for Spin(n) would of course work well for coverings that can be trivialised over a small number of sets.

An interesting example is Q^3, the quotient of S^3 by the unit quaternions whose double cover is a Lens space L^3 with fundamental group cyclic of order 4. Q^3 embeds in $\boldsymbol{R}^4$ but L^3 does not.

There are still many open questions concerning the embedding of simplicial complexes in Euclidean space. From Kuratowski's theorem on planar graphs one knows that if the Betti number of a graph Γ is less than 4 then Γ is planar. A higher dimensional analogue of this result is still missing, although it has been proved [16] that if $b_n(K^n)\leqslant 1$ then K^n embeds in $\boldsymbol{R}^{2n}$. A seemingly reasonable conjecture would be that if $b_n(K^n)<2^{n+1}$ then K^n embeds in $\boldsymbol{R}^{2n}$. If K^n denotes the iterated join of $(n+1)$ copies of 3 points then $b_nK^n=2^{n+1}$ and it is known, by the Flores-van Kampen theorem [7], [12], that K^n does not embed in $\boldsymbol{R}^{2n}$. Recent results by K. Sarkaria[18,19] suggest that further progress in this direction might soon be made.

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