

COBORDISM AND THE EULER NUMBER

BRUCE L. REINHART^{†‡}

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OUR OBJECTIVE in this paper is to obtain the Euler number as a cobordism invariant by making a more stringent definition of cobordism. In fact, we require the existence of a nonsingular vector field interior normal on one of a pair of cobording manifolds and exterior normal on the other. The new cobordism groups admit natural homomorphisms into the usual ones having as kernels cyclic groups generated by spheres. In even dimensions, these kernels are free cyclic and give the Euler number as an additional invariant. In odd dimensions, the kernels are zero except in oriented cobordism of dimension $4k + 1$, where the kernel is cyclic of order 2. In this case, two manifolds are cobordant if and only if they have the same Stiefel–Whitney numbers and bound a manifold of even Euler number. In the oriented case, the kernel is a direct summand, while in the nonoriented case, the even dimensional real projective spaces become of infinite order. Finally, we give some applications to the cobordism theory of bundles and to general relativity.

Throughout the paper, manifold will mean infinitely differentiable compact manifold, possibly with boundary.

§1. THE NONORIENTED CASE

We say that two closed manifolds M_1 and M_2 are cobordant if there exists a manifold with boundary N and a vector field F on N such that

- (i) $\partial N = M_1 \cup M_2$;
- (ii) F is interior normal on M_1 and exterior normal on M_2 ;
- (iii) F is nonsingular.

This is an equivalence relation, and the cobordism classes of manifolds of dimension n form a commutative semigroup under the operation of disjoint union. The class of the empty manifold is the zero of this semigroup. We shall denote the semigroup by \mathfrak{M}_0^n , reserving the notation \mathfrak{M}^n for the Thom cobordism group [7].

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[‡] RIAS and the University of Maryland, College Park, Maryland, U.S.A.

THEOREM (1). *Two manifolds are cobordant if and only if they have the same Stiefel–Whitney and Euler numbers. Moreover, \mathfrak{M}_0^n is a commutative group for $n > 0$.*

Let a surgery of type (p, q) be the replacement of a subset $D^{p+1} \times S^q$ by $S^p \times D^{q+1}$, where D^p is the p -disc, S^p is the p -sphere, and $p + q + 1 = n$. According to a theorem of Milnor [4, p. 40] two manifolds which are related by a surgery are cobordant in the Thom sense, hence have the same Stiefel–Whitney numbers [6].

LEMMA (1). *If $n = p + q + 1$ is even, then surgery of type (p, q) with p even increases the Euler number by 2, while that with p odd decreases the Euler number by 2.*

Proof. If p is even, then q is odd, and the Euler number of $D^{p+1} \times S^q$ is zero while that of $S^p \times D^{q+1}$ is $+2$.

LEMMA (2). *If n is odd, then the Euler number of M^n is zero. If M^n is a boundary, then it bounds a connected manifold of Euler number zero.*

Proof. The first statement is well known. Let $M = \partial N'$, and let the Euler number of N' be even. If $n > 1$, we make N' connected by surgeries of type $(n, 0)$ then use surgeries of type either $(n-1, 1)$ or $(n-2, 2)$ to make the Euler number zero. If $n = 1$, the circle bounds a cross cap, which has Euler number zero. If the Euler number of N' is odd, form the disjoint union of N' with real projective n -space and proceed as before.

LEMMA (3). *If n is even and $M^n = \partial N$, then the Euler number of N is half that of M .*

Proof. Let N' be the union of two copies of N , sewed together by the identity map on the boundary. Then N' has Euler number zero because of its dimension. On the other hand, $X(N') = 2X(N) - X(M)$.

Proof of Theorem (1). Suppose M_1 and M_2 have the same Stiefel–Whitney and Euler numbers. Then there is a manifold N such that $\partial N = M_1 \cup M_2$ [7, p.77]. Let N' be a closed tubular neighbourhood of M_2 in N , and let M'_2 be such that $\partial N' = M'_2 \cap M_2$. Let X be the common Euler number of M_1 and M_2 . By Lemmas (2) and (3), we may assume that X is also the Euler number of N . Since N' is diffeomorphic to $M_2 \times [0, 1]$ and $(N - \text{int}(N'))$ is diffeomorphic to N , each of these spaces has Euler number X . Consequently, each of them admits a vector field interior normal on the boundary and having at most one singular point, the index of this point being X . If we reverse the sense of the field at each point of N' , the index of the singularity changes to $-X$. (If $n + 1$ is even, this is true only because $X = 0$.) Hence if we use the unreversed field on $N - N'$ and the reversed field on N' , we get a vector field on N interior normal on M_1 , exterior normal on M_2 , and having the sum of the indices of the singular points equal to zero. It follows that there exists such a field which is nonsingular. Hence M_1 and M_2 are cobordant. The converse is proved by reversing the argument. Finally, we show the existence of inverses. Indeed, the inverse of M is a manifold which has the same Stiefel–Whitney numbers, but whose Euler number is the negative of the Euler number of M . By Lemma (1), such a manifold can be constructed by surgery, for positive even n . This completes the proof of the theorem.

There is a natural homomorphism of the group \mathfrak{M}_0^n onto \mathfrak{M}^n , defined by choosing

representatives and mapping each into its class in \mathfrak{M}^n . Let K^n be its kernel, so we have

$$0 \rightarrow K^n \xrightarrow{i} \mathfrak{M}_0^n \xrightarrow{j} \mathfrak{M}^n \rightarrow 0.$$

PROPOSITION (1). *Let x_j for $j \neq 2^k - 1$ be an indeterminate, and let $Z[x_2, x_4, x_5, \dots]$ be the polynomial algebra over the integers generated by these indeterminates. Then $\mathfrak{M}_0 = \sum_1^{00} \mathfrak{M}_0^n$ is isomorphic to the quotient of this polynomial algebra by elements of the form*

$$\begin{cases} 2x_{2m+1} \\ 2x_{2m} - 2x_{2j}x_{2k}, j+k=m. \end{cases}$$

Also K^n is zero for n odd and free cyclic for n even. The generator y_{2m} of K^{2m} is the class of the sphere S^{2m} , and $i(y_{2m}) = 2x_{2m}$.

Proof. We know [7, p.79] that \mathfrak{M}^n is isomorphic to the polynomial algebra over Z_2 with these generators. The classes x_{2m+1} correspond to manifolds of Euler number zero while the classes x_{2m} correspond to the even dimensional real projective spaces, which have Euler number 1. The proposition follows immediately from these facts.

§2. THE ORIENTED CASE

For this case we modify the definition of cobordism by requiring that M_1 , M_2 , and N be oriented, while the orientation induced on M_1 by N agrees with that of M_1 , and the orientation induced on M_2 is opposite to the given orientation of M_2 . (We shall write M'_2 instead of $-M_2$ for M_2 with orientation reversed, because M'_2 in fact is not the negative of M_2 in our theory.) Again we get a commutative semigroup Ω_0^n .

THEOREM (2). *Two manifolds M_1 and M_2 of dimension $n \neq 4k + 1$ are cobordant if and only if they have the same Stiefel-Whitney, Pontryagin and Euler numbers. If $n = 4k + 1$ and M_1 and M_2 have the same Stiefel-Whitney and Pontryagin numbers, let N be the manifold bounded by M_1 and M'_2 in the (usual) oriented sense. Then M_1 and M_2 are cobordant if and only if the Euler number of N is even. Ω_0^n is a commutative group for $n > 0$.*

LEMMA (4). *If M has dimension $4k + 1$ and $M = \partial N$, then the parity of the Euler number of N depends only upon M . Moreover, we may always choose N with Euler number 0 or 1. In dimension $4k - 1$, we may choose N with Euler number zero.*

Proof. Let $M = \partial N_1 = \partial N_2$ and let V be the manifold formed by identifying N_1 and N'_2 along the boundary. Then the Euler number of V is the sum of the Euler number of N_1 and N_2 , and is even because V is a closed oriented $4k + 2$ manifold. This shows that the parity of the Euler number of N depends only upon M . We may change it to zero or one by surgery. Since complex projective $2k$ space has Euler number $2k + 1$ and is orientable, we may use the technique of Lemma (2) to show that a $4k - 1$ manifold which bounds also bounds a manifold of Euler number zero.

Proof of Theorem (2). In the usual oriented cobordism, the invariants are the Stiefel-Whitney and Pontryagin numbers [8, p. 306]. Using this fact and Lemmas (1), (3) and (4), we can give a proof of Theorem (2) analogous to that for Theorem (1).

As before, we have a homomorphism of Ω_0^n onto Ω^n , and a group L^n defined by the sequence

$$(*) \quad 0 \rightarrow L^n \xrightarrow{i} \Omega_0^n \xrightarrow{j} \Omega^n \rightarrow 0.$$

PROPOSITION (2). L^{4k-1} is zero, L^{4k+1} is cyclic of order 2 and L^{2k} is free cyclic. In each case, the generator is the class of the sphere, and Ω_0^n is the direct sum of L^n and Ω^n .

Proof. If a manifold bounds in the usual sense, then the top Stiefel–Whitney class is zero, so the Euler number is even. Hence, in even dimensions a bounding manifold of Euler number 2 may be taken as generating L^n . This generator is of infinite order because the disjoint union of k such manifolds has Euler number $2k$, which is not zero. In dimension $4k+2$, the Euler number is always even. Hence, we may define a homomorphism $\phi: \Omega_0^n \rightarrow L^n$ by mapping each manifold onto the sphere with the same Euler number. Since ϕi is the identity, Ω_0^{4k+2} is a direct sum. In dimension $4k$, the Euler number and the index are congruent modulo 2, since each is congruent to the middle betti number. Hence, we define the map ϕ by mapping each manifold onto the sphere whose Euler number is equal to the difference of the Euler number and index of the given manifold. Since the sphere has index 0, ϕi is the identity, and the conclusion follows. In dimension $4k+1$, the kernel is generated by a manifold which bounds a manifold of odd Euler number. Since the sphere bounds the disc, which has Euler number 1, the sphere will do. In the terminology of Wall [8], let $g_\omega = \partial_3(X_{2a_1} \dots X_{2a_r})$. Then the classes g_ω generate the torsion part of Ω multiplicatively. Let $M_\omega \in g_\omega$. Then Ω_0^{4k+1} is generated additively by the classes of S^{4k+1} and products of M_ω . We show that $2M_\omega$ bounds a manifold of even Euler number, so the class of M_ω is of order two. Let $Q(m_i, n_i)$ be in the class X_{2a_i} . Then M_ω is embedded in $\Pi_i Q(m_i, n_i)$ and cutting along M_ω gives an oriented manifold bounded by two copies of M_ω and having Euler number equal to that of $\Pi_i Q(m_i, n_i)$. However $Q(m_i, n_i)$ has even Euler number, since its top Stiefel–Whitney class is zero. Hence the class of M_ω is of order 2. Since Ω_0^{4k+1} is generated by elements of order 2, every element is of order 2. Thus the sequence (*) splits. This concludes the proof of Proposition (2).

COROLLARY. In general, a two-sheeted covering of M is not corbordant to $2M$.

Proof. We have just shown that $2P(m, n)$, where $P(m, n)$ is the Dold manifold, bounds a manifold of even Euler number. On the other hand, it admits a twofold covering by $S^m \times P_n(C)$ which bounds a manifold of odd Euler number when n is even. This result may also be obtained by examining the construction given by Dold [3] for the manifold bounded by $2P(m, n)$ and $S^m \times P_n(C)$.

We owe the following remarks to Floyd and Conner, particularly the former. If we consider the bordism group [1, 2] $\Omega_n(BSO(n))$, we see that the classifying maps of the tangent bundles of two manifolds determine the same element of this group if and only if the manifolds have the same Euler, Pontryagin and Stiefel–Whitney numbers. From another point of view, they determine the same element if and only if they are cobordant in such a way that their tangent bundles can be extended to an oriented n -plane bundle on the big manifold. On the other hand, our cobordism makes the additional condition that the

oriented n -plane bundle on the big manifold be a subbundle of its tangent bundle. Proposition 2, combined with these remarks, then yields the following result.

PROPOSITION (3). *In dimensions not of the form $4k + 1$, the tangent bundles of two oriented n -manifolds are cobordant as oriented bundles if and only if they are cobordant in such a way that their tangent bundles can be extended to an oriented n -plane subbundle of the tangent bundle of the big manifold. In dimension $4k + 1$, this is false for the tangent bundle of the sphere.*

One further application may be noted. In relativity theory, one has no theory of statics. However, one may suppose that at time zero, the solutions of the Einstein equations are instantaneously static. Then the initial surface has zero scalar curvature. Recently, Misner [5] has shown that such surfaces exist with highly complicated topology. Suppose such surfaces are given at $t = 0$ and at $t = 1$. Then the Lorentz metric gives rise to a vector field interior normal at $t = 0$ and exterior normal at $t = 1$, where interior and exterior are with reference to the segment of space-time cut off by these surfaces. One may ask whether there is any relation between the topology of these surfaces. Since Ω_0^3 is zero, our results imply that no relation exists.

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*University of Maryland,
College Park,
Maryland, U.S.A.*