NONCOMMUTATIVE RATIONAL FUNCTIONS AND FARBER'S INVARIANTS OF BOUNDARY LINKS

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To D. B. Fuchs on the occasion of his sixtieth birthday

Abstract

M. Farber in [2] constructed invariants of μ-component boundary links with values in the algebra of noncommutative rational functions. In this paper we simplify his algebraic constructions and express them by using noncommutative generalizations of determinants introduced by Gelfand and Retakh. In particular, for every finite-dimensional module N over the algebra of noncommutative polynomials $k\langle x_1, \ldots, x_\mu\rangle$ we construct a characteristic rational power series χ_N. If k is an algebraically closed field (of arbitrary characteristic) and N is semi-simple, the series χ_N determines N.

1 Introduction

Alexander polynomials of knots and links are constructed from the Alexander module over $\mathbf{Z}[t_1, \ldots, t_\mu]$ (where μ is the number of components of the link) which corresponds to the maximal abelian covering of the link complement. They contain essentially all *commutative* information about the link (or rather its fundamental group).

Farber in [2] constructed invariants of μ-component boundary links with values in the algebra of noncommutative rational functions. When $\mu = 1$ (i.e. when link is a knot) this invariant is equivalent to the Alexander polynomial.

Farber associated to any n-dimensional boundary link a sequence of rational *noncommutative* power series ϕ_i, $i = 1, \ldots, n$ providing rather strong link invariants. To compute ϕ_i one has to calculate a finite number of integers (the traces of certain linear maps acting on the homology of a Seifert manifold).

The core of Farber's construction is a characteristic rational power series ϕ_M associated to any finite-dimensional module M over a certain k-algebra P_μ with $\mu + 1$ generators having a subalgebra isomorphic to the algebra $k\langle x_1, \ldots, x_\mu\rangle$ of noncommutative polynomials. When k is an algebraically closed field of zero characteristic and the P_μ-module M is semisimple, the series ϕ_M determines M up to an isomorphism.

In this paper we construct a characteristic rational power series χ_N for any finite-dimensional $k\langle x_1, \ldots, x_\mu\rangle$-module N. When k is an algebraically closed field of *arbitrary* characteristic and N is semi-simple, the series χ_N determines N up to an isomorphism.

If N is also a P_μ-module and the actions of P_μ and $k\langle x_1, \ldots, x_\mu\rangle$ are compatible, the series ϕ_N and χ_N determine each other.

Note that our construction of χ_N is simpler than the construction of ϕ_N and this gives us a hope that this invariant can be applied to a wider class of links.

Alexander polynomial can be computed in terms of minors of the Alexander matrix and it would be very interesting to have a similar interpretation of its noncommutative generalization. In this paper we make the first step in the direction of applying noncommutative determinants to obtain invariants of links. We show that characteristic series χ and ϕ can be expressed by means of quasideterminants, a noncommutative generalization of determinants introduced by Gelfand and Retakh (see, for example, [6]).

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2 Characteristic functions of finite-dimensional $k\langle X\rangle$-modules

Let k be a field and $X = \{x_1, \ldots, x_\mu\}$ be a set of noncommuting variables. Denote by $k\langle X\rangle$ (resp. $k\langle\langle X\rangle\rangle$) the k-algebra of noncommutative polynomials (resp. formal series) in $x_1, \ldots, x_\mu$. Denote by X^* the free monoid generated by X. Then X^* is a k-basis of $k < X >$. The elements of X^* will be called *words*.

Definition 2.1 The ring $\mathcal{R}$ of *noncommutative rational series* is defined as the smallest k-subalgebra of $k\langle\langle X\rangle\rangle$ satisfying the following properties.

(i) $k\langle X\rangle \subset \mathcal{R}$;

(ii) if $g \in \mathcal{R}$ and g is invertible in $k\langle\langle X\rangle\rangle$ then $g^{-1} \in \mathcal{R}$.

General theory of rational series may be found in [1]. Here we remind just basic facts.

Rational series are called sometimes *noncommutative rational functions.* They can be characterized in terms of *Fox derivatives*.

Definition 2.2 The *Fox derivative* ∂_i on $k\langle\langle X\rangle\rangle$ is a k-endomorphism defined by

$$\partial_i(x_j w) = \delta_{ij} w, \quad \partial_i 1 = 0$$

for any word $w \in X^*$.

It satisfies the Leibniz rule: for formal series f and g

$$\partial_i(fg) = (\partial_i f)g + \epsilon(f)\partial_i g,$$

where $\epsilon(f)$ means the constant term of f.

Fox derivatives turn $k\langle\langle X\rangle\rangle$ into a right $k\langle X\rangle$-module. This structure may also be described by the following construction: the action of polynomial P on series S is given by $(S \circ P, w) = (S, Pw)$, for $w \in X^*$, where $(\ , \ )$ denotes the canonical pairing between $k\langle\langle X\rangle\rangle$ and $k\langle X\rangle$, defined by

$$(S, P) = \sum_{w \in X^*} s_w p_w, \quad \text{if } S = \sum_{w \in X^*} s_w w, \quad P = \sum_{w \in X^*} p_w w.$$

In this notations $w \circ x_i = \partial_i w$.

Remark 2.3 Let $\Lambda = k[F_\mu]$ be the group algebra of the free group F_μ with μ generators $g_1, \ldots, g_\mu$. The *Magnus embedding*

$$\Lambda \hookrightarrow k\langle\langle X\rangle\rangle, \quad g_i \mapsto 1 + x_i, \quad g_i^{-1} \mapsto \sum_{n \geq 0} (-x_i)^n$$

maps Λ into $\mathcal{R}$.

By Leibniz rule the image of Λ under the Magnus embedding is a $k\langle X\rangle$-submodule of $\mathcal{R}$.

The structure of $k\langle X\rangle$-module on $k\langle\langle X\rangle\rangle$ that we consider here is different from the one obtained by the canonical embedding $k\langle X\rangle \hookrightarrow k\langle\langle X\rangle\rangle$.

Rational series can be characterized in terms of the Fox derivatives. The following proposition belongs to Schützenberger [8] and Fliess [4].

Proposition 2.4 *A series f is rational if and only if the k-vector space spanned by all Fox derivatives $\partial_{i_k} \ldots \partial_{i_2}\partial_{i_1} f$ of f is finite-dimensional.*

Therefore, to each rational function $\chi \in \mathcal{R}$ there corresponds a finite-dimentional vector space M_χ spanned by the Fox derivatives of χ of all orders. The space M_χ has a natural $k\langle X\rangle$-module structure given by $x_i \mapsto \partial_i, \quad i = 1, \dots, \mu$.

Vice versa, to each finite-dimensional $k\langle X\rangle$-module M there corresponds a rational function χ_M.

Definition 2.5 Let M be a $k\langle X\rangle$-module, finite dimensional over k, and $u_i \in \mathrm{End}_k(M)$ the endomorphism of M corresponding to the action of $x_i \in k\langle X\rangle$.

Define a k-algebra homomorphism

$$\alpha : k\langle X\rangle \to \mathrm{End}_k M, \quad x_i \mapsto u_i.$$

The *characteristic function* $\chi_M \in k\langle\langle X\rangle\rangle$ of M is defined by

$$\chi_M = \sum_{w \in X^*} Tr(\alpha(w))w.$$

Example 2.6 Let k be an algebraically closed field and $\mu = 1$, i.e. M is a $k[x]$-module. Let $d = \dim(M)$ and $\lambda_i, i = 1, \dots, d$ be eigenvalues of $\alpha(x)$. Then

$$\chi_M = \sum_{1 \le i \le d} \frac{1}{1 - \lambda_i x}.$$

It is easy to prove the following fact.

Proposition 2.7 *The characteristic function χ_M is a rational series which is additive for short exact sequences of $k\langle X\rangle$-modules.*

The following theorem shows that a simple $k\langle X\rangle$-module can be recovered from its characteristic function.

Theorem 2.8 *Let k be an algebraically closed field and M be a simple finite-dimensional $k\langle X\rangle$-module. Then M_{χ_M} is isomorphic (as a $k\langle X\rangle$-module) with the direct sum of $d = \dim(M)$ copies of M.*

We will deduce this result from a theorem of Fliess which we state now.

Let M be a finite dimensional right $k\langle X\rangle$-module. Then the dual space M' is a left $k\langle X\rangle$-module. Let $m \in M$ (resp. $\phi \in M'$) generate M (resp. M') under the $k\langle X\rangle$-action.

Define

$$S = \sum_{w \in X^*} (m(\alpha(w)))\phi w,$$

where α is defined as before (x_i acts on the right on M', $i = 1, \dots, \mu$ and similarly ϕ acts on the right on M).

Theorem 2.9 (Fliess [4]) *Module M is isomorphic to M_S as right $k\langle X\rangle$-modules and, under this isomorphism m (resp. ϕ) corresponds to S (resp. to the constant term map $\epsilon : k\langle\langle X\rangle\rangle \to k$).*

The proof is mechanical: we define the isomorphism by $m' \to \sum_{w\in X^*}(m'(\alpha w))\phi w$. Then a straightforward verification shows that it is well-defined, injective and surjective. Note that k needs not to be algebraically closed here.

Proof of Theorem 2.8. Let $\{m_1,\dots,m_d\}$ be a basis of M, and $\{\phi_1,\dots,\phi_d\}$ be the dual basis of M'. Define $S_i = \sum_{w\in X^*}(m_i(\alpha w))\phi_i$. Then $\chi_M = \sum_{i=1}^d S_i$.

Let N be a direct sum of d copies of M, $m = (m_1,\dots,m_d) \in N$, $\phi = (\phi_1,\dots,\phi_d) \in N'$. Then χ_M corresponds to the triple N, m, ϕ as is described in the Theorem 2.8, so that M_{χ_M} is isomorphic to N, if we can verify the hypothesis of this theorem.

Since k is algebraically closed, and M is simple $k\langle X\rangle$-module, the subalgebra of $\mathrm{End}_k(M)$ is, by Burnside's theorem, all of $\mathrm{End}_k(M)$. Note that this algebra coincides with $\{\alpha P | P \in k\langle X\rangle\}$. Hence, we may find $P \in k\langle X\rangle$ such that $m_i(\alpha P) = m_i$ and $m_j(\alpha P) = 0$ for $j \neq i$. Thus $m(\alpha P) = (0,\dots,0,m_i,0,\dots,0)$ which implies that m generates all of N under this action, M being simple. For the other hypothesis of Theorem 2.8 it is similar: one replaces M by its dual M' on which the u_j's act simply on the left.

Remark 2.10 Note that the result is not true if k is not algebraically closed. The smallest example is $k = \mathrm{R}$, $M = ke_1 \oplus ke_2$, and action of u given by imitating multiplication by complex number i, i.e.

$$e_1 u = -e_2, \quad e_2 u = e_1.$$

Then $\chi = \chi_M = 2 - 2x^2 + 2x^4 - 2x^6 + \dots$, and M_χ is isomorphic to M and not to $M \oplus M$.

There is certainly a version of Theorem 2.8 when k is not algebraically closed, where the "arithmetic" of k plays some role. Also, for effective computational purposes, $k = \mathbf{Q}$ seems to be the best field.

If M, instead of being simple, is only semi-simple, there is a variant of Theorem 2.8.

Theorem 2.11 *Let M be a semi-simple module over $k\langle X\rangle$. If $M = \bigoplus_{r=1}^{q} M_r$, where M_r, $r = 1,\dots,q$ is simple, then M_{χ_M} is isomorphic to $\bigoplus_{r=1}^{q} \bigoplus_{r=1}^{\dim(M_r)} M_r$.*

As an application of Theorem 2.8 we obtain the following important result.

Theorem 2.12 *Let k be algebraically closed. Two finite-dimensional semi-simple $k\langle X\rangle$-modules are isomorphic if and only if their characteristic functions coincide.*

3 P_μ-modules

In his study of boundary links and links modules M. Farber [2] introduced an algebra P_μ with $\mu+1$ generators and defined characteristic functions of finite-dimensional P_μ-modules.

Let P_μ be a k-algebra defined by $\mu+1$ generators $z, \pi_i,\ i = 1, \ldots, \mu$, and relations

$$\pi_i \pi_j = \delta_{ij}\pi_i, \quad \pi_1 + \ldots + \pi_\mu = 1.$$

Modules over P_μ are automatically $k\langle X\rangle$-modules via the homomorphism $\delta : k\langle X\rangle \to P_\mu$ defined by $x_i \mapsto -z\pi_i$. Denote $\delta(x_i)$ by ∂_i, $i = 1, \ldots, \mu$.

Proposition 3.1 *The homomorphism δ is an embedding.*

Proof. Indeed, this follows from the two simple facts that, first, the image of δ in P_μ coincides with the subalgebra $zP_\mu \subset P_\mu$, and that, second, this subalgebra is freely generated by μ elements $z, z\pi_1, \ldots, z\pi_{\mu-1}$. Using the identity $\sum_i \pi_i = 1$, every word of type $z^{a_1+1}\pi_{i_1} z^{a_2+1}\pi_{i_2} \ldots z^{a_k}, \quad a_j \geq 0, \quad 1 \leq i_j \leq \mu - 1$ can be written as a polynomial in $\partial_i = \delta(x_i) = -z\pi_i$ in a unique way.

Recall that X^* is the free monoid generated by the alphabet $X = \{x_1, \ldots, x_\mu\}$. Every word $w \in X^*$ can be uniquely written as $w = x_j w'$ for some j and $w' \in X^*$. Define action of the generators of P_μ on X^* as

$$\pi_i(x_j w) = \delta_{ij} x_j w,\ i, j = 1, \ldots, \mu, \quad z(x_k w) = -w.$$

This gives a P_μ-module structure on the augmentation ideal $k\langle\langle X\rangle\rangle_+$ of $k\langle\langle X\rangle\rangle$ (i.e. the formal series without constant terms).

Denote by Λ_0 the image of the Magnus embedding of $\Lambda = k[F_\mu] \to k\langle\langle X\rangle\rangle$. The action of P_μ on X^* defines a P_μ-module structure on $k\langle\langle X\rangle\rangle/\Lambda_0$.

Proposition 3.2 *Let $\mathcal{R}$ be the ring of noncommutative rational series. Then $\mathcal{R}/\Lambda_0$ is invariant under P_μ-action on $k\langle\langle X\rangle\rangle/\Lambda_0$.*

Farber [2] introduced the following notion of characteristic function for finite-dimensional P_μ-modules.

Definition 3.3 *Let A be a finite-dimensional P_μ-module. Its characteristic function is the series*

$$\phi_A = \sum_{k=1}^{\mu} \sum_{\alpha} Tr(\pi_k \partial_\alpha) x^\alpha x_k,$$

where $\alpha = (i_1, \ldots, i_p)$, $\partial_\alpha = \partial_{i_p} \ldots \partial_{i_1}$, $x^\alpha = x_{i_1} \ldots x_{i_p}$.

Proposition 3.4 *The characteristic function is a rational series which is additive for short exact sequences of P_μ-modules.*

Corollary 3.5 *Let M be finite-dimensional P_μ-module and let C_i, be its distinct composition factors appearing with multiplicities $m_i \geq 1$, $i = 1, \ldots, n$. Then*

$$\phi_M = \sum_{i=1}^{n} m_i \phi_{C_i}.$$

Example 3.6 If $\mu = 1$ then

$$\phi_A = \sum_{j=1}^{n} \frac{x}{1 + \lambda_j x},$$

where $n = \dim_k A$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the operator z.

In general, function ϕ_A is an invariant capable of capturing only *semi-simple* information about P_μ-modules. To study non-semisimple modules we will need more subtle invariants.

Farber [2] proved the following result similar to our Theorem 2.12.

Theorem 3.7 *Suppose k is an algebraically closed field of characteristic zero. Let A and B be finite-dimensional semi-simple P_μ-modules. Then $\phi_A = \phi_B$ if and only if A is isomorphic to B.*

This result follows from the version of our Theorem 2.8 for P_μ-modules. It shows that the P_μ-submodule of $\mathcal{R}/\Lambda_0$ generated by χ_A is closely related to A.

Definition 3.8 A P_μ-module M is called *primitive* if one of the generators $\pi_1, \ldots, \pi_\mu, z$ of P_μ acts as the identity on M and the other generators act trivially.

A P_μ-module A is called *primitive-free* if it has no primitive composition factors.

Theorem 3.9 ([2]) *If k is algebraically closed and A is simple non-primitive P-module of $\dim A = d$, then $P_\mu \phi_A \subset \mathcal{R}/\Lambda_0$ is P_μ-isomorphic to dA, the direct sum of d copies of A.*

Remarks 3.10

(1) If one takes an alphabet with $\mu + 1$ letters, then the characteristic series χ_M on this alphabet for a P_μ-module is equivalent to Farber's series; i.e. the knowledge of one is equivalent to the knowledge of the other.

(2) It it easy to construct example of simple P_μ-modules which are not $k\langle x_1, \ldots, x_\mu \rangle$-simple under the homomorphism $x_i \mapsto -z\pi_i$. Let M be a three-dimensional vector space with a basis

e_i, $i = 1, 2, 3$. Define an action of algebra P_3 on M by setting $\pi_i e_j = \delta_{ij} e_j$, $\quad i, j = 1, 2, 3$.

The action of z in this basis is given by a matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

One can see that M is a simple P_3-module under this action, and the two-dimensional image of z is invariant under action of ∂_i for $i = 1, 2, 3$.

Therefore, it is not clear whether the main results about semi-simple P_μ-modules of this section follow directly from the results obtained in Section 2.

4 Quasideterminants and characteristic functions

In this section we express characteristic functions constructed in Sections 2 and 3 via *quasideterminants.*

Quasideterminants were introduced by Gelfand and Retakh (see, for example, [6]) and are defined as follows. Let A be an $m \times m$-matrix over an algebra R. For any $1 \leq i, j \leq m$, let $r_i(A)$, $c_j(A)$ be the i-th row and the j-th column of A. Let A^{ij} be the submatrix of A obtained by removing the i-th row and the j-th column from A. For a row vector r let $r^{(j)}$ be r without the j-th entry. For a column vector c let $c^{(i)}$ be c without the i-th entry. Assume that A^{ij} is invertible. Then the quasideterminant $|A|_{ij} \in R$ is defined by the formula

$$|A|_{ij} = a_{ij} - r_i(A)^{(j)}(A^{ij})^{-1}c_j(A)^{(i)},$$

where a_{ij} is the ij-th entry of A. Let $B = (b_{ij})$ be a matrix of order n with formal entries and E_n a unit n-matrix. The following proposition was proved in [5].

Proposition 4.1 *In the ring $\mathbf{Z}\langle\langle b_{ij}\rangle\rangle$ of formal series with integer coefficients generated by noncommuting variables b_{ij} we have*

$$|E_n - B|_{ii}^{-1} = 1 + \sum_{k_1,\ldots,k_p} b_{ik_1} b_{k_1k_2} \ldots b_{k_pi},$$

where the sum is over all $1 \leq k_1, \ldots, k_p \leq n$, $p = 1, 2, \ldots$.

Let k be a field, $X = \{x_1, \ldots, x_\mu\}$ set of noncommuting variables, and M be an n-dimensional $k\langle\langle X\rangle\rangle$-module. Choose a basis of M and let A_i, $i = 1, \ldots, \mu$, be the matrix of the operator x_i in this basis.

The following proposition gives an expression of the characteristic function χ_M of M via quasideterminants.

Proposition 4.2 *The sum*

$$\sum_{i=1}^{n} |E_n - x_1 A_1 - \ldots - x_\mu A_\mu|_{ii}^{-1}$$

does not depend on the choice of the basis and is equal to the characteristic function χ_M of the $k\langle X\rangle$-module M.

A similar result holds for Farber's characteristic functions.

Let M be a finite-dimensional P_μ-module. Fix a basis of M and denote by B_i, $i = 1, \ldots, \mu$ the matrix of the the operator $(1-z)\pi_i$ in this basis.

Proposition 4.3 *The sum*

$$\sum_{i=1}^{n} |E_n - x_1 B_1 - \ldots - x_\mu A_\mu|_{ii}^{-1}$$

does not depend on the choice of the basis and is equal to the characteristic function ϕ_M of the P_μ-module M.

5 Link modules

Let F_μ be a free group with μ generators t_i and $\Lambda = k[F_\mu]$ its group ring. A finitely-generated left Λ-module M is called *link module* if $\mathrm{Tor}_q^\Lambda(k, M) = 0$ for all $q \geq 0$, where k is regarded as a right Λ-module via the augmentation map.

The following characterization of link modules allows to consider them as P_μ-modules.

Theorem 5.1 (Sato [7]) *M is a link module if and only if every element m of M has a unique representation*

$$m = \sum_{i=1}^{\mu} (t_i - 1)m_i, \ m_i \in M.$$

Corollary 5.2 *In the notations of the theorem, formulas $\pi_i m = (t_i - 1)m_i$, $zm = \sum m_i$, make every link module a P_μ-module. In addition, $\partial_i m = m_i$.*

Note that every P_μ-module is also a $k\langle x_1, \ldots, x_\mu\rangle$-module under the canonical embedding $\delta : k\langle x_1, \ldots, x_\mu\rangle \to P_\mu$.

Definition 5.3 ([3]) A finite-dimensional P_μ-submodule A of a link module M is called *lattice* if A generates M over Λ.

It is easy to prove the following fact.

Theorem 5.4 ([3]) *Every link module contains a lattice. Intersection of all lattices in M is a lattice which is called the minimal lattice of M.*

Definition 5.5 *If M is a link module, we set $\phi_M = \phi_A$ and $\chi_M = \chi_A$, where A is the minimal lattice in M.*

Theorem 5.6 (Farber [2]) *Let M and N be semi-simple link modules with $\phi_M = \phi_N$. If the field k is algebraically closed and* $\mathrm{char}(k) = 0$, *then M and N are isomorphic as P_μ-modules.*

The proof of the theorem is based on Theorem 3.7.

6 Boundary links

An n-dimensional μ-component link is an oriented smooth submanifold Σ of S^{n+2}, where $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ is an ordered disjoint union of μ submanifolds of S^{n+2}, each diffeomorphic to S^n. It is called a *boundary link* if there exists an oriented submanifold V of dimension $n+1$ of S^{n+2}, such that $V = V_1 \cup \ldots \cup V_\mu$ and $\partial V_i = \Sigma_i$, $i = 1, \ldots, \mu$. If, in addition, each V_i is connected, we say that V is a *Seifert manifold* for Σ.

The homology groups of the Seifert manifold $H_*(V; k)$ have several natural operations. The first μ operations are the projections $\pi_i : H_*(V; k) \to H_*(V_i; k)$. The last operation z is defined as follows. For $Y = S^{n+2} \setminus V$ let $I_\pm : V \to Y$ be small shifts in the direction of positive and negative normals to V, respectively. The map

$$i_{+*} - i_{-*} : H_k(V) \to H_k(Y)$$

is an isomorphism for any $k = 0, 1, 2, \ldots$ (see [2]) and we define z by $z = (i_{+*} - i_{-*})^{-1} i_{+*}$.

Let Σ be a link in S^{n+2} and $X = S^{n+2} \setminus T(\Sigma)$ the complement of its tubular neighborhood.

Fix a base point $* \in X$. Then connecting the meridians of Σ (small loops around each component Σ_i) with $*$ we obtain elements $m_1, \ldots, m_\mu \in \pi_1(X, *)$ defined up to conjugation. If $\Sigma = \partial V$ is a boundary link then there is an epimorphism (defined up to conjugation) $\sigma : \pi_1(X, *) \to F_\mu$ defined as follows. If α is a loop in X intersecting V transversally, first in component V_{i_1}, then V_{i_2}, etc, we set $\sigma[\alpha] = t_{i_k}^{\varepsilon_k} \ldots t_{i_2}^{\varepsilon_2} t_{i_1}^{\varepsilon_1}$, where $\varepsilon_i = \pm 1$ is the local intersection index between α and V at i-th intersection point.

Consider the covering $\widetilde{X} \to X$ corresponding to the kernel of σ. The group of deck transformations of this covering space is F_μ and therefore the homology $H(\widetilde{X}; k)$ is a $\Lambda = k[F_\mu]$-module. For $0 \le i \le n$ this module is a link module.

There is a canonical lifting map $S^{n+2} \setminus \Sigma \to \widetilde{X}$ giving a homomorphism of P_μ-modules $f : H_i(V; k) \to H_i(\widetilde{X}; k)$. The image of f is a lattice. If $H_i(V; k)$ has no primitive P_μ-submodules, then f is a monomorphism and the image of $H_i(V; k)$ is a minimal sublattice of $H_i(\widetilde{X}; k)$ (cf. [2, 3]).

Applying the constructions of Sections 2 and 3 to the minimal lattices in homology groups $H_i(\tilde{X}, k)$ we obtain sequences of rational series χ_i and ϕ_i, $i = 1, \ldots, \mu$. Thus we have a sequence of *noncommutative invariants* associated to every boundary link.

These invariants are stronger than the well-known commutative invariants. In particular, Farber [2] constructed an example of a link one of whose Alexander modules vanishes, but the corresponding characteristic function is non-trivial.

Example 6.1 For $\mu = 1$ the invariants ϕ, χ, and the Alexander polynomial determine each other. Let $\Delta_i(t)$ be the Alexander polynomial of $H_i(\tilde{X}; \mathbf{Q})$ where $\tilde{X}$ is an infinite cyclic cover of the complement of the knot. Then if

$$\Delta_i(t) = \prod_{j=1}^{n} (t - \nu_j), \quad \nu_i \in \mathbf{C},$$

the characteristic functions are

$$\chi(x) = \sum_{j=1}^{n} 1/(1-\lambda_j x)$$

and

$$\phi(x) = \sum_{j=1}^{n} x/(1+\lambda_j x),$$

where

$$\lambda_j = 1/(1-\nu_j), \quad j = 1, \ldots, n.$$

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