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GAUSS' LINKING NUMBER REVISITED

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ABSTRACT

In this paper we provide a mathematical reconstruction of what might have been Gauss' own derivation of the linking number of 1833, providing also an alternative, explicit proof of its modern interpretation in terms of degree, signed crossings and intersection number. The reconstruction presented here is entirely based on an accurate study of Gauss' own work on terrestrial magnetism. A brief discussion of a possibly independent derivation made by Maxwell in 1867 completes this reconstruction. Since the linking number interpretations in terms of degree, signed crossings and intersection index play such an important role in modern mathematical physics, we offer a direct proof of their equivalence. Explicit examples of its interpretation in terms of oriented area are also provided.

Keywords: Linking number; potential; degree; signed crossings; intersection number; oriented area.

Mathematics Subject Classification 2010: 57M25, 57M27, 78A25

1. Introduction

The concept of linking number was introduced by Gauss in a brief note on his diary in 1833 (see Sec. 2 below), but no proof was given, neither of its derivation, nor of its topological meaning. Its derivation remained indeed a mystery. Nevertheless this concept was seminal, and proved to be fundamental in the subsequent development of knot theory, general topology and modern topological field theory (see, for example, [23]). In this paper we provide a plausible mathematical proof of what Gauss himself might have done, bridging this possible derivation with a modern

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proof of the alternative, equivalent definitions of linking number in terms of degree, crossing signs and intersection number.

Reconstructions of possible derivations made by Gauss have been offered by historians, in particular Epple [8–10], and physicists [15]. These reconstructions rely on ideas that can certainly be traced back to Gauss, even though direct derivation of the linking number from Gauss's own work is left to speculation. Here we offer a plausible reconstruction entirely based on Gauss' own work on terrestrial magnetism; this is done in Sec. 3 by providing a mathematical reconstruction that relies step by step on specific, published results of Gauss. An alternative derivation of the linking number, made independently by Maxwell in 1867 (discussed previously in [10]), is also briefly re-examined in Sec. 4, as a little complement to the mathematical reconstruction made in Sec. 3.

It is well known that the Gauss linking number formula admits several alternative interpretations. Rolfsen [21] gives eight of them, proving their equivalence, even though at times by very concise statements. This is the case, for instance, for the interpretation of the linking number in terms of degree, where his brief reference to Spivak's *Calculus on Manifolds* (p. 135) seems all too demanding for non-experts. Since this equivalence is still source of inspiration for modern mathematics (see [1, 2, 11, 18]), we believe that a proof of the interpretation in terms of degree deserves a discussion somewhat more accessible to non-specialists. This is done in Sec. 5. Similar considerations apply to its interpretation in terms of signed crossings and intersection number, two different ways of computing the linking number, that have proven to be very useful in many physical contexts (see [3, 4, 19, 20, 26]). This discussion is also presented in Sec. 5. Finally in Sec. 6 direct inspection of the linking number by the Gauss map is presented for three different links.

2. Gauss' Note on the Linking Number

It is well known that Gauss had a great interest in topology — then known as "geometry of position" (*geometria situs*) — even though he never published anything on it. From his personal correspondence with Olbers it is known that he became deeply interested in the subject from the early years of 1800 (see [7, p. 221]): a few notes on knots are from that period. One of the oldest is dated 1794, bearing the heading *A Collection of Knots*, and it contains 13 sketches of knots with English names written beside them. Two other sheets of paper with sketches of knots are one dated 1819, and the other bearing the notation "Riedl, Beiträge zur Theorie des Sehnenwinkels, Wien, 1827" ([7, p. 222]). Few other remarks referring to the knotting of closed curves are reproduced in his *Werke* (VIII, pp. 271–285).

The concept of linking number was introduced by Carl Friedrich Gauss in a brief note on a page of his personal diary — a kind of logbook of his most important discoveries ([5, p. 19]) — dated January 22, 1833 (see Fig. 1(a) and the English translation in Fig. 1(b)). This note remained unknown for 34 years, until it was

[4.]

Von der *Geometria Situs*, die LEIBNITZ ahnte und in die nur einem Paar Geometern (EULER und VANDERMONDE) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z'-\mathrm{d}z\,\mathrm{d}y')+(y'-y)(\mathrm{d}z\,\mathrm{d}x'-\mathrm{d}x\,\mathrm{d}z')+(z-z')(\mathrm{d}x\,\mathrm{d}y'-\mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4m\pi$$

und m die Anzahl der Umschlingungen.

Der Werth ist gegenseitig, d. i. er bleibt derselbe, wenn beide Linien gegen einander umgetauscht werden. 1833. Jan. 22.

(a)

[4.]

Of the *geometria situs,* which was forseen by LEIBNITZ, and into which only a pair of geometers (EULER and VANDERMONDE) were granted a bare glimpse, we know and have, after a century and a half, little more than nothing.

A principal problem at the *interface* of *geometria situs* and *geometria magnitudinis* will be to count the intertwinings of two closed or endless curves.

Let x, y, z be the coordinates of an undetermined point on the first curve; x', y', z' those of a point on the second and let

$$\iint \frac{(x'-x)(\mathrm{d}y\mathrm{d}z'-\mathrm{d}z\mathrm{d}y')+(y'-y)(\mathrm{d}z\mathrm{d}x'-\mathrm{d}x\mathrm{d}z')+(z'-z)(\mathrm{d}x\mathrm{d}y'-\mathrm{d}y\mathrm{d}x')}{\left[(x'-x)^2+(y'-y)^2+(z'-z)^2\right]^{\frac{3}{2}}} = V$$

then this integral taken along both curves is

$$= 4m\pi$$

m being the number of intertwinings.

The value is reciprocal, i.e. it remains the same if the curves are interchanged. 1833. Jan. 22.

(b)

Fig. 1. (a) Excerpt from the *Nachlass zur Electrodynamik*, published in the V volume of Gauss' *Werke* (1867). In this note Carl Friedrich Gauss introduces the concept of linking number. (b) English translation.

discovered and published in the *Nachlass zur Elektrodynamik*, as part of Gauss' posthumous *Werke* [14].

In this note the notion of linking number is merely introduced to count the number of times a closed (or endless) curve encircles a second closed (or endless) curve in space. There is no explicit reference neither to a physical system, electrical or magnetic, nor to an application. Thus, inclusion of this note in the *Nachlass zur Electrodynamik* (supported by Schäfer, one of Gauss scientific biographers, [22]) generated some controversy. The mathematical reconstruction presented in the next section, however, advocates in favor of Schäfer's choice.

3. Derivation of the Linking Number Based on Gauss' Work

Gauss developed an interest in the Earth magnetism and its mysterious origin as early as 1806, but any publication on the subject was postponed until the 1830s, when firm observational data were readily available. The first paper on the subject, entitled *Intensitas vis magneticae terrestris ad mensuram absoluta revocata* [12], was read before the Royal Society of Göttingen on 15 December 1832. In this paper Gauss presents the first systematic use of absolute units to measure magnetic quantities. Stimulated by Faraday's discovery of induced currents, made in 1831, Gauss and Weber started to collect electromagnetic measurements from 22 October 1832. During this work, in 1833, they were led to anticipate the discovery of Kirchhoff's laws of branched circuits [17]. Apparently, all the fundamental concepts for a mathematical theory of terrestrial magnetism were conceived as early as 1806, and were readily formulated in 1822 (see [17, p. 306]). Gauss' theory on terrestrial magnetism was therefore ready for a test in 1832 (see [7, p. 158]), but work on its publication was postponed till the winter of 1838, when experimental observations were finally confirmed. Gauss' paper on this, entitled *Allgemeine Theorie des Erdmagnetismus*, finally appeared in 1839 [13].

In this paper the magnetic potential induced at any point on the Earth's surface is worked out in terms of an infinite series of spherical functions. In considering the magnetic effects produced by some "magnetized fluid", supposedly present in the Earth interior, Gauss examines the potential $V = V(P')$ induced at an exterior point $P' = P'(x', y', z')$, placed at a distance $r = |P - P'|$ from a magnetic source at $P = P(x, y, z)$. The theory of magnetic potential, derived there from first principles, is applied then to calculate the value of the magnetic effects at various geographical locations. By acknowledging Ampère's result on the interpretation of the magnetic effects in terms of electric currents (see [13, Secs. 2, 36, 37; pp. 188, 229, 230]), Gauss states the following theorem (see [13, Sec. 38, p. 230]):

Theorem 3.1 (Gauss, 1839). *The potential of a magnetic shell Σ at any point is equal to the solid angle which it subtends at that point multiplied by its magnetic strength.*

"Magnetic shell" means an infinitely thin magnetized layer given by an orientable surface, whose opposite sides have opposite polarity. Theorem 3.1 is derived from geometric considerations of certain elementary expressions for the magnetic potential (Fig. 2). Of central importance here is indeed the result of Ampère,

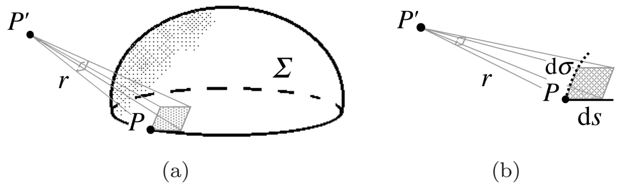


Fig. 2. (a) The potential $V = V(P')$ induced by a magnetic shell of unit strength and surface Σ at an exterior point P', placed at a distance r from a source point P, can be calculated by integrating the elementary contribution $d\Sigma$ over Σ, via the solid angle subtended by $d\Sigma$. (b) The volume of the elementary solid angle subtended by $d\Sigma$ can be worked out in terms of the volume of the pyramid of base $d\Sigma$ and side r.

mentioned by Gauss. This can be stated (cf., for example, [25]) in terms of the following theorem.

Theorem 3.2 (Ampère, 1831). *A closed galvanic circuit C produces the same effect as a magnetic shell Σ of any form having the circuit C for its edge.*

A "galvanic circuit" is indeed an electrically charged wire carrying electric current. By using this theorem, the value of the potential at P', induced by a magnetic surface Σ, bounded by $C = \partial\Sigma$, is equivalent to the potential at P' induced by an electric current flowing in the circuit identified by C. According to Ampère's theorem, the magnetic effects are evidently independent of the shape of Σ, so that Σ (given, for example, by a spherical cap, as in Fig. 3(a)) can be replaced by any other (topologically equivalent) surface bounded by C: the laminar disk on the right-hand side of Fig. 3(a), for instance, is an example. Hence, we have (see [25, Corollary 2, p. 272]) the following corollary.

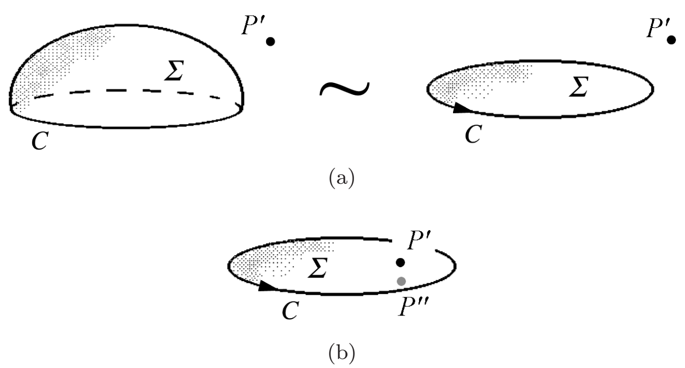


Fig. 3. (a) By Ampère's theorem the potential induced by a magnetized shell Σ bounded by C at an exterior point P' is equivalent to that induced at the same point by an electric current flowing in the circuit identified by C. (b) The potential of a point P', infinitely close to the magnetized shell Σ, and that of its antipodal P'', placed on the opposite side of Σ, differ by 4π.

Corollary 3.3 (Thomson, 1850). *The potential of a magnetic shell at any point is independent of the form of the shell itself, and depends solely on the bounding line or edge.*

Thinking of the Earth surface as a magnetized sphere threaded by a south–north oriented magnetization, Gauss gives the following example (see [25, Sec. 38, p. 231]): consider two magnetic shells (representing the Earth hemispheres) Σ_1 and Σ_2, having common boundary (that is the equator) and unit magnetic strength. The value of the potential at an external point P' depends on the position of this point with respect to the two shells. By considerations on the solid angle and the respective orientation of Σ_1, whose normal vector is positively oriented toward the exterior, and Σ_2, negatively oriented toward the exterior, Gauss states that the potential due to Σ_1, at any point in the region interior to the two shells, exceeds the potential due to Σ_2 by 4π (the solid angle of the unit sphere). When P' moves from a position very near to the positive side of the shell to its antipodal P'', infinitely close to the start point on the negative side of the shell, through a path around the edge [see Figs. 3(b) and 4(a)], the solid angle thus measured must continuously increase by 4π. We know that indeed the potential is a multi-valued function of the position. Following Gauss, Lord Kelvin (then W. Thomson) states this result as follows (see [25, Corollary 3, p. 272]):

Corollary 3.4 (Thomson, 1850). *Of two points infinitely near one another on the two sides of a magnetic shell, but not infinitely near its edge, the potential at that one which is on the north polar side exceeds the potential at the other by the constant* 4π.

Gauss' note on the linking number can now be proven in full. Let $C = \partial\Sigma$ and C' a smooth, simple, closed curve in $\mathbb{R}^3$, encircling C m times. Let $P = P(x, y, z) \in C$, $P' = P'(x', y', z') \in C'$, and $V = V(P')$ be the potential induced by Σ at P'. We have the following proposition.

Proposition 3.5 (Gauss, 1832).

(i) $$\iint \frac{l(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}} = V; \quad (3.1)$$

(ii) $$V = 4m\pi, \quad (3.2)$$

where $m = m(C, C')$ *is the linking number of* C *and* C';

(iii) $$m(C, C') = m(C', C). \quad (3.3)$$

Proof. The potential induced by Σ at the point P' is given by Gauss' Theorem 3.1. The volume of the elementary solid angle $d\omega$ at P' (see Fig. 2(a)) can be worked out in terms of the volume of the pyramid of solid angle $d\omega$ and side r. The base of the pyramid has elementary area $d\Sigma$. We now use Ampère's theorem: since the induction due to Σ is equivalent to that due to the electric current flowing in C (see Fig. 3), the elementary solid angle can be evaluated in terms of direction cosines (see the discussion by Maxwell [16, Secs. 419–420 and Eqs. (1) and (6)], partly reproduced in Sec. 4 below) by

$$\frac{1}{3}r^3 d\omega = \frac{1}{3}r^3 \Pi ds d\sigma, \tag{3.4}$$

where

$$r = [(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{1/2} \tag{3.5}$$

is the distance between P' and P, and Π is given by

$$\Pi = \frac{1}{r^3} \det \begin{pmatrix} x'-x & y'-y & z'-z \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{dx'}{d\sigma} & \dfrac{dy'}{d\sigma} & \dfrac{dz'}{d\sigma} \end{pmatrix}. \tag{3.6}$$

The potential is thus given by

$$\begin{aligned} V &= \int_\Omega d\omega = \int_C \int_{C'} \Pi ds d\sigma \\ &= \int_C \int_{C'} \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}. \end{aligned} \tag{3.7}$$

where the double integral is extended to C and C'. Statement (i) is thus proven. Since C' encircles C m times (m integer; see Fig. 4(b)), at each turn around C the potential $V = V(p')$ increases by 4π (cf. Corollary 3.4; see also [16, Sec. 421]); hence, after m times, we have $V = m4\pi$; thus Eq. (3.2) of (ii) is proven. Since m is an algebraic number, it does not depend on the shape of the two curves in space: it is therefore an invariant of the topology of the embedded curves. By direct

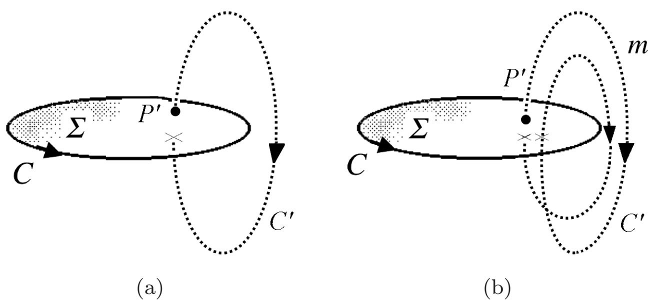


Fig. 4. (a) The potential $V(P')$ jumps by 4π as P' passes through the surface Σ. (b) If the point P' passes through Σ m times, then $V(P')$ increases by $m4\pi$.

inspection of the integrand of (3.7) we can see that $m(C, C') = m(C', C)$. Thus (iii) is proven, together with Proposition 3.5. □

4. Maxwell's Rediscovery of the Linking Number

Maxwell developed an interest in knots as early as November 1867, prompted by Thomson's theory of vortex atoms and Tait's engagement in classification work. Indeed, several concepts introduced and discussed by Tait in his first paper on knots (1877, [24]) were actually suggested by Maxwell, as their private correspondence demonstrate. Maxwell's primary interests lay then squarely in foundational aspects of electricity and magnetism, an interest that kept him well acquainted with Gauss' work on terrestrial magnetism and his solid angle interpretation of the magnetic potential. As we shall see from the material presented below, it seems almost certain that he could be unaware of Gauss' note by December, 1867. In any case, the introductory material presented in Chapter III, Vol. II of his *Treatise* [16] is entirely devoted to Gauss' work on magnetic potential. After introducing Gauss' theorem for magnetic shells (see [16, Sec. 409]), Maxwell proceeds to apply the solid angle interpretation to calculate the potential. Starting from the very definition of solid angle ([16, Secs. 417]), Maxwell puts forward three alternative methods ([16, Secs. 417–420]) to compute this quantity, by relying on geometric and physical arguments.

The physical reasoning of [16, Sec. 419], for instance, is quite illuminating: by interpreting the potential at a point P as the work done by a unit magnetic pole at P against the force exerted by a magnetic shell, while it moves from an infinite distance to the point P, he notes that the potential (i.e. the solid angle) must be the result of a double line integration: one performed along the boundary curve C of the shell (by making use of Theorem 3.2), and the other one performed along the path followed by the pole at P, as it approaches C. If we denote by s and σ the respective arc-lengths of these two paths, the elementary contribution $d\omega$ to this solid angle can thus be expressed as

$$d\omega = \Pi \, ds \, d\sigma, \tag{4.1}$$

where Π remains to be determined.

Now, consider a small displacement $d\sigma$ of P towards C (see Fig. 5). As the pole approaches C from infinity, the circuit C will be seen (from P) to move apparently to C' of the same distance $d\sigma$, in the opposite direction. The elementary arc ds of C will thus be seen to span an elementary surface of area $ds\, d\sigma$. The volume of the pyramid with base this surface, vertex in P and solid angle $d\omega$ is indeed given by Eq. (3.4), where, with reference to Fig. 5, $r = \overline{PQ}$ denotes the distance between the point $P = P(\xi, \eta, \zeta)$ and a point $Q = Q(x, y, z)$ on C. This volume may equally be expressed in terms of the direction cosines of ds, $d\sigma$ and r with respect to the

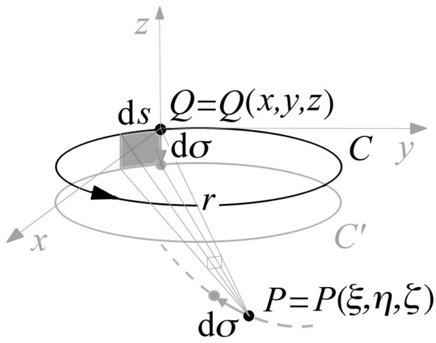


Fig. 5. Maxwell's interpretation of the solid angle in terms of apparent displacement of the boundary curve C.

triad (x, y, z), so that

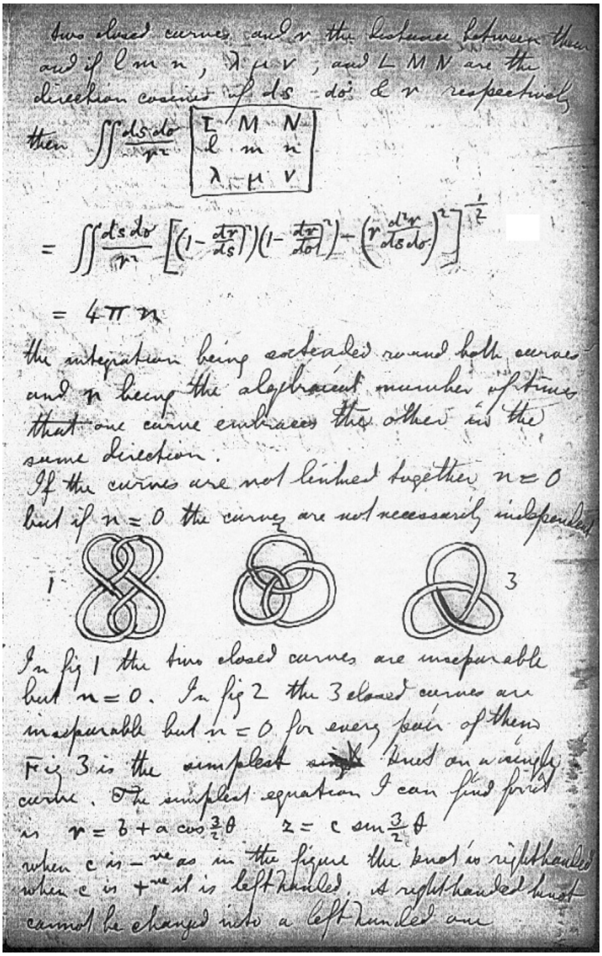
$$d\omega = \frac{ds\,d\sigma}{r^2}\begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = \frac{ds\,d\sigma}{r^2}\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{d\xi}{d\sigma} & \dfrac{d\eta}{d\sigma} & \dfrac{d\zeta}{d\sigma} \\ \dfrac{\xi - x}{r} & \dfrac{\eta - y}{r} & \dfrac{\zeta - z}{r} \end{vmatrix}$$

$$= \frac{ds\,d\sigma}{r^2}\left[\left(1 - \left(\frac{dr}{ds}\right)^2\right)\left(1 - \left(\frac{dr}{d\sigma}\right)^2\right) - \left(r\frac{d^2 r}{ds d\sigma}\right)^2\right]^{1/2} . \qquad (4.2)$$

Note that the determinant on the r.h.s. of the equation above can be reduced to that of Eq. (3.6). These expressions are essentially those given by Eqs. (5) and (6) of [16, Sec. 420].

These calculations can be dated precisely, since they are communicated in a letter to Tait of December 4, 1867 (see Fig. 6). This letter is quite remarkable: not only Maxwell derives here the linking number, but he quickly proceeds to point out its shortcomings, by considering two cases (given by the first two diagrams shown) of zero linking number for two essential links. These considerations will be reproduced by Tait in his first paper on knots [24].

A careful reading of this (and other) correspondence indicates that Maxwell must have derived this result independently, presumably giving Gauss due credit only afterwards, on the occasion of the publication of the *Treatise*. Further confirmation of this can be found in a postcard to Tait, dated January 24, 1877 (see Fig. 7), where explicit reference to Gauss' original note is made.



two closed curves and r the distance between them
and if $l\ m\ n$, $\lambda\ \mu\ \nu$, and $L\ M\ N$ are the
direction cosines of ds $d\sigma$ & r respectively
then $\iint \frac{ds\,d\sigma}{r^2} \begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$

$$= \iint \frac{ds\,d\sigma}{r^2}\left[\left(1-\frac{dr}{ds}^2\right)\left(1-\frac{dr}{d\sigma}^2\right)-\left(r\frac{d^2r}{ds\,d\sigma}\right)^2\right]^{\frac{1}{2}}$$

$= 4\pi n$

the integration being extended round both curves
and n being the algebraical number of times
that one curve embraces the other in the
same direction.
If the curves are not linked together $n = 0$
but if $n = 0$ the curves are not necessarily independent

1 3

In fig 1 the two closed curves are inseparable
but $n = 0$. In fig 2 the 3 closed curves are
inseparable but $n = 0$ for every pair of them
Fig 3 is the simplest ~~single~~ knot on a single
curve. The simplest equation I can find for it
is $r = b + a\cos\frac{3}{2}\theta$ $z = c\sin\frac{3}{2}\theta$
when c is $-$ve as in the figure the knot is righthanded
when c is $+$ve it is lefthanded. A righthanded knot
cannot be changed into a left handed one

Fig. 6. Third sheet of a letter by J. C. Maxwell to P. G. Tait, dated December 4, 1867. Calculation of the linking number "[...] n being the algebraical number of turns that one curve embraces the other in the same direction." The first two diagrams show examples of essential links with $n = 0$. The third diagram shows a trefoil knot, described by the parametric equation given at the bottom of the page. Cambridge University Library, papers of James Clerk Maxwell, Ms. Add. 7655, Box **1**(Ib), 7 (unpublished). Courtesy of the Syndics of Cambridge University Library.

5. Interpretation of the Linking Number in Terms of Degree, Signed Crossings and Intersection Number

In this section we concentrate on the interpretation of the linking number in terms of degree, signed crossings and intersection number, by providing a more direct, explicit and, to some extent, elementary proof of their equivalence.

It is useful to recall some basic definitions. Let γ_1, γ_2 be two smooth, disjoint, closed, oriented curves in S^3, and $\mathbf{r}_1(t_1)$, $\mathbf{r}_2(t_2)$ their parametrizations, with $\{t_1, t_2\} \in [0, 2\pi]$. To each pair $(Q_1, Q_2) \in \gamma_1 \times \gamma_2$ there corresponds a point (t_1, t_2)

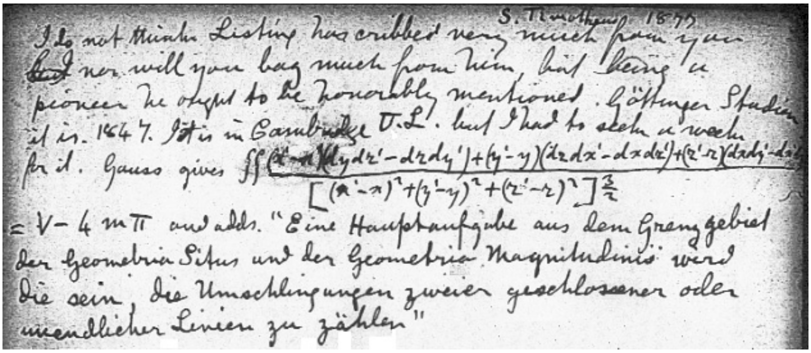

I do not think Listing has cribbed very much from you
but nor will you bag much from him, but being a
pioneer he ought to be honourably mentioned. Göttinger Studien
it is. 1847. It is in Cambridge U.L. but I had to seek a week
for it. Gauss gives $\iint \frac{(x'-x)(dy\,dz'-dz\,dy')+(y'-y)(dz\,dx'-dx\,dz')+(z'-z)(dx\,dy'-dy\,dx')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}$
$= V - 4m\pi$ and adds. "Eine Hauptaufgabe aus dem Grenzgebiet
der Geometria Situs und der Geometria Magnitudinis wird
die sein, die Umschlingungen zweier geschlossener oder
unendlicher Linien zu zählen"

Fig. 7. Extract from a postcard to P.G. Tait dated January 24, 1877. After a few comments on Listing's work, Maxwell communicates the Gauss linking number formula, quoting Gauss' annotation (see Figure 1). The term $V - 4m\pi$ should be read $V = 4m\pi$. Cambridge University Library, papers of James Clerk Maxwell, Ms. Add. 7655, Box **1**(Ib), 86: # 497 (unpublished). Courtesy of the Syndics of Cambridge University Library.

on the torus $\mathbb{T}$. The *Gauss map* $\boldsymbol{\psi} : \mathbb{T} \to S^2$ associates to each point (t_1, t_2) the unit vector

$$\mathbf{n}(t_1, t_2) = \frac{\mathbf{r}_1(t_1) - \mathbf{r}_2(t_2)}{|\mathbf{r}_1(t_1) - \mathbf{r}_2(t_2)|}. \tag{5.1}$$

Now, let M and N be two compact, unbounded, oriented manifolds of same dimensions, and $\mathbf{f} : M \to N$ a continuous function. We have the following definition.

Definition 5.1. The *degree* of $\mathbf{f}$ is defined by

$$\deg(\mathbf{f}) := \sum_{x \in Y} \operatorname{sign}\left[\det\left(\frac{\partial \mathbf{f}}{\partial x}\right)_x\right], \tag{5.2}$$

where Y denotes the set of regular points for which $\det(\partial\mathbf{f}/\partial x) \neq 0$.

The original formula of Gauss gives the following definition.

Definition 5.2. The *linking number* $Lk^{(1)} = Lk^{(1)}(\gamma_1, \gamma_2)$ of γ_1 and γ_2 is defined by

$$Lk^{(1)}(\gamma_1, \gamma_2) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2')}{|\mathbf{r}_2 - \mathbf{r}_1|^3} dt_1 dt_2. \tag{5.3}$$

Here primes denote derivatives with respect to the argument. Gauss' original notation is recovered by taking $\mathbf{r}_1 = \mathbf{r}_1(s) = (x(s), y(s), z(s))$ for P and $\mathbf{r}_2 = \mathbf{r}_2(\sigma) = (\xi(\sigma), \eta(\sigma), \zeta(\sigma))$ for P'; hence (cf. Eq. (3.6)) $r = |\mathbf{r}_2(\sigma) - \mathbf{r}_1(s)|$ and

$$\frac{(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2')}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = \Pi. \tag{5.4}$$

An alternative definition can be given in terms of degree.

Definition 5.3. The *linking number* $Lk^{(2)} = Lk^{(2)}(\gamma_1, \gamma_2)$ of γ_1 and γ_2 is defined by

$$Lk^{(2)}(\gamma_1, \gamma_2) := \deg(\boldsymbol{\psi}). \tag{5.5}$$

Consider now the indented, oriented diagram $D_{\boldsymbol{\nu}}(\mathcal{L})$ of the (tame) link $\mathcal{L} = \gamma_1 \sqcup \gamma_2$, obtained by projecting $\mathcal{L}$ along $\boldsymbol{\nu}$ onto the plane, allowing under- and over-crossings. Let $D_{\boldsymbol{\nu}}(\mathcal{L})$ be a good projection of $\mathcal{L}$, that is one for which the standard projection has nodal points of multiplicity at most two. We assign to each apparent crossing k of $\gamma_1 \sqcap \gamma_2$ the number $\epsilon(k) = \pm 1$ according to standard convention. We have the following definition.

Definition 5.4. The *linking number* $Lk^{(3)} = Lk^{(3)}(\gamma_1, \gamma_2)$ of γ_1 and γ_2 is defined by

$$Lk^{(3)}(\gamma_1, \gamma_2) := \frac{1}{2} \sum_{k \in \gamma_1 \sqcap \gamma_2} \epsilon(k). \tag{5.6}$$

Finally, let us choose a Seifert surface M of γ_1, such that $M \cap \gamma_2$ is made of a finite number of transversal intersections. According to their relative sign, the algebraic sum of these is given by $I(M, \gamma_2)$. We have the following definition.

Definition 5.5. The *linking number* $Lk^{(4)} = Lk^{(4)}(\gamma_1, \gamma_2)$ of γ_1 and γ_2 is defined by

$$Lk^{(4)}(\gamma_1, \gamma_2) := I(M, \gamma_2). \tag{5.7}$$

In all these cases, we have $Lk^{(i)}(\gamma_1, \gamma_2) = Lk^{(i)}(\gamma_2, \gamma_1)$ $(i = 1, \ldots, 4)$.

5.1. *Equivalence of definitions*

Proposition 5.6. *The following holds true*:

$$Lk^{(1)} = Lk^{(2)} = Lk^{(3)} = Lk^{(4)}. \tag{5.8}$$

Proof. (i) $Lk^{(2)} = Lk^{(1)}$. Consider the set of preimages

$$Y = \{\mathbf{T}_1 = (t_{1_1}, t_{1_2}), \ldots, \mathbf{T}_n = (t_{n_1}, t_{n_2})\}$$

of any regular point of $\boldsymbol{\psi}$ on S^2. By Eq. (5.2) any point of Y contributes ± 1 to the value of $\deg(\boldsymbol{\psi})$, depending on the orientation of the surface $\boldsymbol{\psi}(\mathbb{T})$. At any point $\boldsymbol{\psi}(\mathbf{T}_i)$ $(i = 1, \ldots, n)$ this orientation is given by $\boldsymbol{\nu}(\mathbf{T}_i) = (\frac{\partial \mathbf{n}}{\partial t_1} \times \frac{\partial \mathbf{n}}{\partial t_2})_{\mathbf{T}_i}$ normal to the surface at $\mathbf{T}_i$, thus contributing $+1$ or -1 depending on the orientation of $\boldsymbol{\nu}(\mathbf{T}_i)$ (outwards or inwards) with respect to the sphere. The sign is given by

$$\mathbf{n}(\mathbf{T}_i) \cdot \boldsymbol{\nu}(\mathbf{T}_i) = \left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\mathbf{T}_i}; \tag{5.9}$$

hence

$$\deg(\boldsymbol{\psi}) = \sum_{\mathbf{T}_i \in Y} \operatorname{sgn}\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\mathbf{T}_i}. \tag{5.10}$$

Since the latter does not depend on any particular regular value $\mathbf{y}$ of $\boldsymbol{\psi}$, and by Sard's theorem the critical values of $\boldsymbol{\psi}$ form a null set, we have

$$\deg(\boldsymbol{\psi}) = \frac{1}{4\pi}\int_{S^2} \sum_{\mathbf{T}_i \in Y} \operatorname{sgn}\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\mathbf{T}_i} d^2\mathbf{y}, \tag{5.11}$$

where the right-hand side is an average over S^2. If $\mathbf{y} \notin \boldsymbol{\psi}(\mathbb{T})$, then the set $\boldsymbol{\psi}^{-1}(\mathbf{y})$ is empty; thus, the integration domain can be reduced to $\boldsymbol{\psi}(\mathbb{T})$. By exploiting the continuity of $\boldsymbol{\psi}$ we can sub-divide $\boldsymbol{\psi}(\mathbb{T})$ into m regions $\mathcal{R}_k$, each having n_k number of preimages (constant). Since each $\mathcal{R}_k$ is the image of the n_k regions $\mathcal{S}_{k,i}$ ($i = 1, \ldots, n_k$), the orientation of each $\mathcal{R}_{k,i} = \boldsymbol{\psi}(\mathcal{S}_{k,i})$ is constant for any $i = 1, \ldots, n_k$. Moreover, since $\mathbb{T} = \bigcup_{k=1}^{m}(\bigcup_{i=1}^{n_k} \mathcal{S}_{k,i})$, then Eq. (5.11) can be written as

$$\deg(\boldsymbol{\psi}) = \frac{1}{4\pi}\sum_{k=1}^{m}\sum_{i=1}^{n_k}\int_{\mathcal{R}_k} \operatorname{sgn}\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\boldsymbol{\psi}^{-1}(\mathbf{y})\cap\mathcal{S}_{k,i}} d^2\mathbf{y}. \tag{5.12}$$

We now make a change of variable by taking $\boldsymbol{\tau} = \boldsymbol{\psi}^{-1}(\mathbf{y})$, so that

$$\deg(\boldsymbol{\psi}) = \frac{1}{4\pi}\sum_{k=1}^{m}\sum_{i=1}^{n_k}\int_{\mathcal{S}_{k,i}} \operatorname{sgn}\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\boldsymbol{\tau}} \left|\frac{\partial \mathbf{n}}{\partial t_1} \times \frac{\partial \mathbf{n}}{\partial t_2}\right|_{\boldsymbol{\tau}} d^2\boldsymbol{\tau}. \tag{5.13}$$

Thus

$$\deg(\boldsymbol{\psi}) = \frac{1}{4\pi}\int_{\mathbb{T}} \left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\boldsymbol{\tau}} d^2\boldsymbol{\tau}. \tag{5.14}$$

By some tedious, but straightforward algebra we can see that

$$\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right) = \frac{(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2')}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{5.15}$$

Equation (5.15) is thus proven and Eq. (5.14) becomes

$$\deg(\boldsymbol{\psi}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2')}{|\mathbf{r}_2 - \mathbf{r}_1|^3} dt_1 dt_2. \tag{5.16}$$

Definitions 5.2 and 5.3 are thus equivalent.

(ii) $Lk^{(2)} = Lk^{(3)}$. We have $\mathbf{T}_0 = (t_{1_0}, t_{2_0}) \in \boldsymbol{\psi}^{-1}(\mathbf{v})$ if and only if

$$\frac{\mathbf{r}_1(t_{1_0}) - \mathbf{r}_2(t_{2_0})}{|\mathbf{r}_1(t_{1_0}) - \mathbf{r}_2(t_{2_0})|} = \mathbf{v}, \tag{5.17}$$

i.e. if and only if $\pi_{\mathbf{v}}(\mathbf{r}_1(t_{1_0})) = \pi_{\mathbf{v}}(\mathbf{r}_2(t_{2_0}))$, i.e. if and only if $\pi_{\mathbf{v}}(\mathbf{T}_0)$ is a crossing in $D_{\boldsymbol{\nu}}(\mathcal{L})$, where γ_1 goes over γ_2. Let $\mathbf{P}_k(\gamma_i)$ be the unit vector tangent to $\pi_{\mathbf{v}}(\gamma_i)$ at k. Without loss of generality, we assume $\pi_{\mathbf{v}}(\mathbf{r}'_i(t_{i_0})) \neq \mathbf{0}$. By standard sign convention, we have

$$\epsilon(k) = -(\mathbf{v}, \mathbf{P}_k(\gamma_1), \mathbf{P}_k(\gamma_2)). \tag{5.18}$$

Now let $\mathbf{N}$ denote a vector normal to Π; then, we have

$$\mathrm{sgn}(\mathbf{N}, \mathbf{r}'_1, \mathbf{r}'_2)_{\mathbf{T}_0} = \mathrm{sgn}(\mathbf{N}, \pi_{\mathbf{v}}(\mathbf{r}'_1), \pi_{\mathbf{v}}(\mathbf{r}'_2))_{\mathbf{T}_0}. \tag{5.19}$$

Moreover

$$\frac{(\pi_{\mathbf{v}}(\mathbf{r}'_1) \times \pi_{\mathbf{v}}(\mathbf{r}'_2))_{\mathbf{T}_0}}{|\pi_{\mathbf{v}}(\mathbf{r}'_1) \times \pi_{\mathbf{v}}(\mathbf{r}'_2)|_{\mathbf{T}_0}} = (\mathbf{P}_k(\gamma_1) \times \mathbf{P}_k(\gamma_2))_{\mathbf{T}_0}. \tag{5.20}$$

Hence, by Eq. (5.18) and by using Eqs. (5.17) and (5.20), we have

$$\epsilon(k) = \mathrm{sgn}(\mathbf{r}_2 - \mathbf{r}_1, \pi_{\mathbf{v}}(\mathbf{r}'_1), \pi_{\mathbf{v}}(\mathbf{r}'_2))_{\mathbf{T}_0}; \tag{5.21}$$

since $\mathbf{r}_2(t_{2_0}) - \mathbf{r}_1(t_{1_0})$ is orthogonal to Π, by (5.19) we have

$$\epsilon(k) = \mathrm{sgn}(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}'_2)_{\mathbf{T}_0}. \tag{5.22}$$

By assuming that $\mathbf{v} \in S^2$ is a regular value of $\boldsymbol{\psi}$, by Eqs. (5.2) and (5.15), we have

$$\deg(\boldsymbol{\psi}) = \sum_{\mathbf{T}_i \in \boldsymbol{\psi}^{-1}(\mathbf{y})} \mathrm{sgn}\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right)_{\mathbf{T}_i} = \sum_{k \in K} \epsilon(k), \tag{5.23}$$

where K is the number of over-crossings (γ_1 over γ_2). By considering the total number of crossings (over *and* under) we have that indeed $Lk^{(2)} = Lk^{(3)}$. Hence, Definitions 5.3 and 5.4 are equivalent.

(iii) $Lk^{(2)} = Lk^{(4)}$. We recall that the degree of a (continuous) function restricted to the boundary of the manifold where it is defined is zero. Let us construct a continuous function $\tilde{\boldsymbol{\psi}}$ from a manifold N to S^2 such that $\gamma_1 \times \gamma_2$ is a component of ∂N and $\tilde{\boldsymbol{\psi}}|_{\gamma_1 \times \gamma_2} = \boldsymbol{\psi}$. Let $\tilde{\boldsymbol{\psi}}(x, y) = (y - x)/|y - x|$ and $N = (M \times \gamma_2) \setminus \bigcup_{m \in M \cap \gamma_2} \mathcal{B}(m, \epsilon)$, where $\mathcal{B}_\epsilon(m)$ is a ball of small radius ϵ, centered on m: $\tilde{\boldsymbol{\psi}}$ is well-defined. One component of ∂N is $\gamma_1 \times \gamma_2$; the union of the other components of ∂N is given by $\bigcup_{m \in M \cap \gamma_2} S^2_\epsilon(m)$, the orientation of each sphere being opposite to that inherited from M. Now, from $\deg(\tilde{\boldsymbol{\psi}}|_{\partial N}) = 0$, we have

$$\deg(\tilde{\boldsymbol{\psi}}|_{\gamma_1 \times \gamma_2}) + \deg(\tilde{\boldsymbol{\psi}}|_{\bigcup_{m \in M \cap \gamma_2} S^2_\epsilon(m)}) = 0. \tag{5.24}$$

Hence,

$$Lk^{(2)}(\gamma_1, \gamma_2) = \deg(\boldsymbol{\psi}) = -\deg(\tilde{\boldsymbol{\psi}}|_{\bigcup_{m \in M \cap \gamma_2} S^2_\epsilon(m)}). \tag{5.25}$$

Moreover, we have

$$\deg(\tilde{\boldsymbol{\psi}}|_{\bigcup_{m\in M\cap\gamma_2} S^2_\epsilon(m)}) = -I(M,\gamma_2), \tag{5.26}$$

and therefore $Lk^{(2)} = Lk^{(4)}$. This completes the demonstration of Proposition 5.6. □

6. Gauss' Integral in Terms of Oriented Area

As we saw from the discussion in Sec. 3, Gauss' integral can be interpreted in terms of oriented area contributions. Its evaluation by the Gauss map is indeed useful. We provide three elementary examples of direct inspection of $\boldsymbol{\psi}(\mathbb{T})$ through the *oriented area* $\mathbf{A}$. Let us sub-divide $\mathbb{T}$ into regions $\mathbb{T}_j$, in which $\frac{\partial \mathbf{n}}{\partial t_1} \times \frac{\partial \mathbf{n}}{\partial t_2}$ has constant sign. The oriented area of each $\mathbb{T}_j$ is given by

$$\pm \iint \left| \frac{\partial \mathbf{n}}{\partial t_1} \times \frac{\partial \mathbf{n}}{\partial t_2} \right| dt_1 dt_2,$$

where the sign depends on the orientation of the surface. $\mathbf{A}$ is given by summing up (algebraically) the contributions from the positive and negative regions $\mathbb{T}_j$. Since $(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}) = \pm |\frac{\partial \mathbf{n}}{\partial t_1} \times \frac{\partial \mathbf{n}}{\partial t_2}|$ has the sign of the oriented surface $\boldsymbol{\psi}(\mathbb{T})$, the oriented area is given by

$$\mathbf{A} = \int_0^{2\pi} \int_0^{2\pi} \left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2} \right) dt_1 dt_2. \tag{6.1}$$

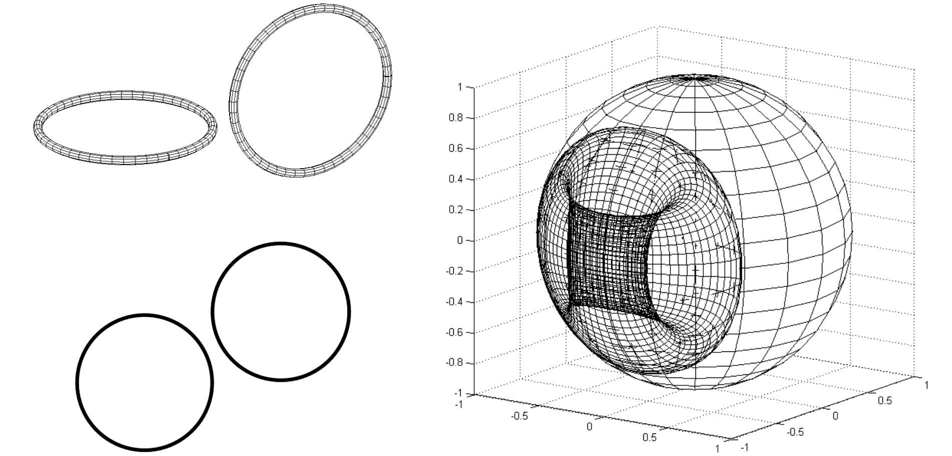
Thus, by dividing by the area of the unit sphere S^2, we have $\deg(\boldsymbol{\psi}) = \mathbf{A}/4\pi$. Since $\boldsymbol{\psi}(\mathbb{T})$ is a continuous mapping of a closed surface and $\boldsymbol{\psi}(\mathbb{T}) \subseteq S^2$, then the oriented area of $\boldsymbol{\psi}(\mathbb{T})$ will be a multiple integer of 4π and $\deg(\boldsymbol{\psi})$ an integer. Since positive and negative contributions cancel out, the degree is simply given by counting the number of times $\boldsymbol{\psi}(\mathbb{T})$ covers S^2 *essentially* (see also, for example, [6]).

6.1. *Un-link, Hopf link and link* $\mathbf{4^2_1}$

The linking number can be estimated by direct inspection of the oriented area of $\boldsymbol{\psi}(\mathbb{T})$. To illustrate this, let us consider the following three examples.

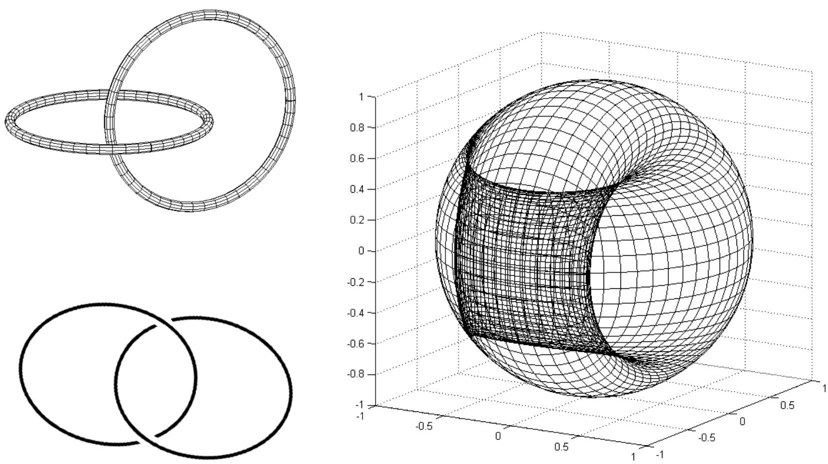
First, let us consider the Gauss map associated with the trivial link 0^2_1 (the "un-link") (see Fig. 8). Any point on the unit sphere (Fig. 8(b)) has same degree, that is $\deg(\boldsymbol{\psi}) = 0$. Different grid levels identify regions that contribute differently to the oriented area. The lighter region is not covered by $\boldsymbol{\psi}(\mathbb{T})$. The intermediate region is doubly covered by $\boldsymbol{\psi}(\mathbb{T})$, with opposite orientations and contributions to the degree (hence $\deg(\boldsymbol{\psi}) = 0$). The darker region is covered four times: two inherited from the double covering of the intermediate region and two, of opposite orientations, of its own. Thus, the overall contribution to the oriented area is zero, thus $Lk(0^2_1) = 0$.

A second example is provided by the Hopf link 2^2_1 (see Fig. 9). Any point on the sphere (Fig. 9(b)) has $\deg(\boldsymbol{\psi}) = \pm 1$. Two distinct regions can be identified: the



(a) (b)

Fig. 8. (a) The un-link 0_1^2 with $Lk(0_1^2) = 0$ viewed (top) in space and (bottom) in its minimal diagram representation. (b) $\boldsymbol{\psi}(\mathbb{T})$ resulting from the Gauss map of the un-link 0_1^2.

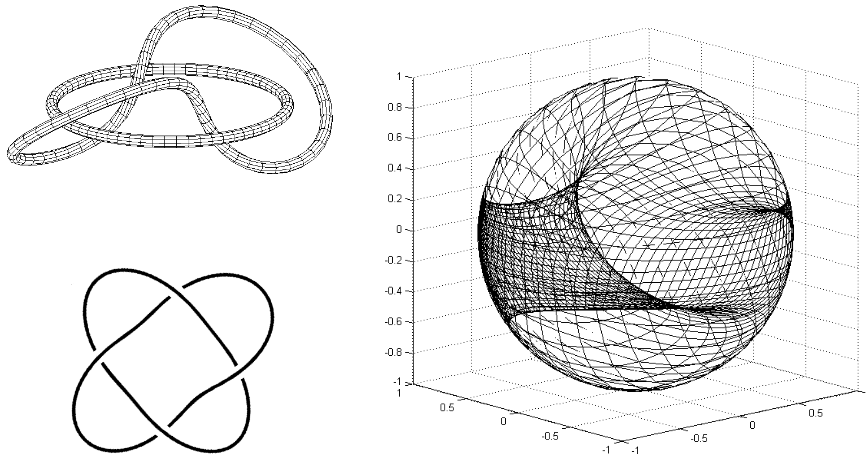


(a) (b)

Fig. 9. (a) The Hopf link 2_1^2 with $Lk(2_1^2) = \pm 1$ viewed (top) in space and (bottom) in its minimal diagram representation. (b) $\boldsymbol{\psi}(\mathbb{T})$ resulting from the Gauss map of the Hopf link 2_1^2.

light region is covered once, depending on the surface orientation. The dark region is covered twice, by two opposite orientations; one covering is inherited from the light region. The oriented area is thus given by $\pm 4\pi$, hence $Lk(2_1^2) = \pm 1$.

Finally we consider the link 4_1^2 of Fig. 10. From Fig. 10(b) we see that $\deg(\boldsymbol{\psi}) = \pm 2$, with two distinct regions: the light region is covered twice, with $\deg(\boldsymbol{\psi}) = \pm 2$, whereas the dark region is covered four times. The oriented area is thus given by $\pm 8\pi$, hence $Lk(4_1^2) = \pm 2$.



(a) (b)

Fig. 10. (a) The link 4_1^2 with $Lk(4_1^2) = \pm 2$ viewed (top) in space and (bottom) in its minimal diagram representation. (b) $\boldsymbol{\psi}(\mathbb{T})$ resulting from the Gauss map of the link 4_1^2.

7. Conclusion

A mathematical reconstruction of a possible derivation made by Gauss of the linking number has been presented, together with explicit proofs of modern, equivalent interpretations of the linking number in terms of degree, signed crossings and intersection number. The mathematical reconstruction offered here is based entirely on first principles of potential theory due to Gauss. This proves two important points: (i) that its derivation is entirely consistent with the contemporary work of Gauss on magnetic potential, and that indeed it can be worked out by using explicit results of Gauss on terrestrial magnetism; (ii) that the concept of linking number is intimately related to the foundations of potential theory and modern mathematical physics, and that indeed contributes to the topological foundations of physical theory. Further support to this comes from the subsequent derivation made by Maxwell, which, as demonstrated by the documents provided here, is most probably authentically independent.

Since the interpretations of the linking number in terms of degree, signed crossings and intersection number find numerous applications in modern mathematical physics, the proofs of their equivalence presented here bridge an existing gap between the original definition and the alternative meanings present in the literature, which prove sometimes hard to reconcile for non-experts. Moreover, direct estimate of the linking integral by the Gauss map of oriented areas is applied to three examples, the un-link, the Hopf link and the 4_1^2 link, to show how Gauss' original concepts and ideas in potential theory are still so fruitful in modern science.

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