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HIGH-DIMENSIONAL KNOTS WITH $\pi_1 \cong \mathbb{Z}$ ARE DETERMINED BY THEIR COMPLEMENTS IN ONE MORE DIMENSION THAN FARBER'S RANGE

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ABSTRACT. The surgery theory of Browder, Lashof, and Shaneson reduces the study of high-dimensional smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ with $\pi_1 \cong \mathbb{Z}$ to homotopy theory. We apply Williams's Poincaré embedding theorem to a highly connected Seifert surface. Then such knots are determined by their complements if the $\mathbb{Z}$-cover of the complement is $[(n+2)/3]$-connected; we improve Farber's work by one dimension.

0. INTRODUCTION

A high-dimensional knot will mean a smooth, oriented, codimension two embedding $\Sigma^n \hookrightarrow S^{n+2}$ of an exotic sphere, with $n \geq 5$. See the survey of Kervaire and Weber [KW] for more details. For our purposes, two knots $\Sigma_i^n \hookrightarrow S^{n+2}$ are said to be *equivalent* if there is diffeomorphism $\phi: S^{n+2} \xrightarrow{\cong} S^{n+2}$ such that $\phi(\Sigma_1^n) = \Sigma_2^n$. A knot $\Sigma^n \hookrightarrow S^{n+2}$ has a *complement* $X = \overline{S^{n+2} - \Sigma^n \times D^2}$, and is *determined by its complement* if it is equivalent to any other knot with diffeomorphic complement. The orientation of Σ^n and the trivialization of the normal bundle neighborhood give a preferred diffeomorphism $\beta: \Sigma^n \times S^1 \xrightarrow{\cong} \partial X$. We will call the composite

$$\alpha: S^n \times S^1 \xrightarrow{\varpi \times \mathrm{id}} \Sigma^n \times S^1 \xrightarrow{\beta} \partial X \xrightarrow{\imath} X \qquad (\varpi : S^n \xrightarrow{\simeq} \Sigma^n \text{ the degree one map})$$

the *attaching map* of the knot. Let $\tau: S^n \times S^1 \xrightarrow{\simeq} S^n \times S^1$ be the homotopy equivalence (actually a diffeomorphism) given by the generator of $\pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}/2$.

A knot $\Sigma^n \hookrightarrow S^{n+2}$ is called *r-simple* [Ke2, Fa1] if the $\mathbb{Z}$-cover $\widetilde{X}$ of the complement is r-connected. Levine [Le1], Kearton [Ke1], and Kojima [Ko] showed that $[(n-1)/2]$-simple knots were determined by their complements, using ambient surgery on the Seifert surface. Using Wall's thickening theory [Wa3], Farber [Fa2] proved this for $([n/3]+1)$-simple knots. We extend Farber's result by one dimension.

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Theorem A. *For $n+3 \leq 3q$ and $n \geq 5$, $(q-1)$-simple smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ are determined by their complements.*

Theorem B. *Let $\Sigma^n \hookrightarrow S^{n+2}$ be a knot with complement X and attaching map $\alpha \colon S^n \times S^1 \to X$. There exists a homotopy equivalence $\zeta \colon X \xrightarrow{\simeq} X$ so that the diagram*

$$\begin{array}{ccc} S^n \times S^1 & \xrightarrow{\alpha} & X \\ \tau\downarrow & & \downarrow\zeta \\ S^n \times S^1 & \xrightarrow{\alpha} & X \end{array} \tag{1}$$

commutes up to homotopy, if the knot is $(q-1)$-simple, $n+3 \leq 3q$, and $n \geq 5$.

We prove Theorem B via Williams's Theorem 1.7; in our range the Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ of a Seifert surface is determined by its "unstable normal invariant" $\rho \in \pi_{n+2}(\Sigma(M/\partial M))$. We construct another Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ suggested by $\alpha \cdot \tau$ (Lemma 1.4), with the same unstable normal invariant ρ (Theorem 1.6). Theorem 1.7 implies the two Poincaré embeddings are "concordant". We use this concordance to construct the homotopy automorphism ζ.

That Theorem B implies Theorem A is well known for PL knots, by the work of Browder [Br1], and Lashof and Shaneson [LS], which we extend to smooth knots in §2.

Our proof requires Levine's result [Le3], that there exists highly connected Seifert surfaces. Together with Barratt [BR], circa '82, we have a purely homotopy proof which uses $\mathbb{Z}$-equivariant Hopf invariants and Ranicki's [Ra] equivariant S-duality.

We conjecture that high-dimensional knots with $\pi_1 \cong \mathbb{Z}$ are determined by their complements. If $\pi_1 \not\cong \mathbb{Z}$, there exist counterexamples due to Cappell and Shaneson, Gordon, and Suciu [CS, Go, Su]. Theorem 1.6, which is true outside our range $n+3 \leq 3q$, and the appendix provide some evidence for the conjecture.

We have a homotopy theoretic proof [Ri] of Theorem 1.7, completing Williams's program [Wi2], proving the result using Browder-Quinn Poincaré surgery [Br4, Qu].

Given a subspace $A \subset X$ and a map $f \colon A \to Y$, we will write the identification space $X \cup_f Y$ as $\varinjlim (X \xleftarrow{\iota} A \xrightarrow{f} Y)$, the *colimit* or *pushout* of the diagram.

1. Poincaré embeddings and the proof of Theorem B

In Theorem B we can eliminate the condition that ζ be a homotopy equivalence.

Lemma 1.1. *Any self-map ζ of the knot complement X making diagram* (1) *commute up to homotopy is a homotopy equivalence.*

Proof. Diagram (1) implies that ζ is a self-map of the Poincaré pair $(X, \partial X)$, and $\zeta_*[X] = [X] \in H_{n+2}(X, \partial X) \xrightarrow{\cong} H_{n+1}(\partial X)$. Furthermore, ζ induces the identity on $\pi_1(X) \cong \mathbb{Z}$. By naturality of $\mathbb{Z}$-equivariant Poincaré duality [Le2, Wa2], the composite

$$H_{\Lambda}^{n+2-*}(\widetilde{X}, \partial\widetilde{X}; \Lambda) \xrightarrow{\zeta^*} H_{\Lambda}^{n+2-*}(\widetilde{X}, \partial\widetilde{X}; \Lambda) \xrightarrow{[X]\cap\cdot} H_*(\widetilde{X}) \xrightarrow{\zeta_*} H_*(\widetilde{X})$$

is the cap product isomorphism $[X] \cap \cdot$, where $\Lambda = \mathbb{Z}[\mathbb{Z}]$. Hence ζ_* is surjective. Since the group ring Λ is Noetherian and X is a finite complex, $\zeta_*: H_*(\widetilde{X}) \to H_*(\widetilde{X})$ must be an isomorphism; hence, ζ is a homotopy equivalence by the Whitehead theorem. □

For a smooth knot $\Sigma^n \hookrightarrow S^{n+2}$ with complement $(X, \partial X)^{n+2}$, Alexander duality and relative transversality give a map $h: X \to S^1$ which is transverse to the point $1 \in S^1$, with inverse image $h^{-1}(1) = (M, \partial M)$, and $\partial M = \Sigma^n$. M^{n+1} is called a *Seifert surface* for the knot. By the relative tubular neighborhood theorem, there is a codimension zero embedding $M \times I \subset X$ extending the embedding $\Sigma^n \times I \subset \partial X = \Sigma^n \times S^1 = \Sigma^n \times I \cup \Sigma^n \times I$, where I is the interval $[-1, 1]$. Let $A = \partial(M \times I)$.

Let $W = \overline{X - M \times I}$ be the *Seifert surface complement.* The knot complement X is then obtained by gluing together $M \times [-1, 1]$ and W along their common boundary $M \times \{-1, 1\}$. By writing X as the union of the Seifert surface and its complement, we obtain the decomposition $S^{n+2} = M \times I \cup W \cup \Sigma^n \times D^2$. Let $\widehat{W}$ be the manifold with corners $\widehat{W} = W \cup \Sigma^n \times D^2$, so that $A = \partial\widehat{W}$. There is a deformation retraction $\varepsilon: \widehat{W} \to W$ which maps $\Sigma^n \times D^2$ onto $\Sigma^n \times I$. Let $f: A \to W$ be the composite of the inclusion $A = \partial(M \times I) = \partial\widehat{W} \subset \widehat{W}$ and the retraction $\varepsilon: \widehat{W} \to W$. Note that f is a cofibration, since ε restricts to a homeomorphism $\varepsilon: \partial\widehat{W} \to \partial W$.

Williams [Wi1] studies codimension zero *Poincaré embeddings* [Br1, Br2, Br3, Wa1] of an m-dimensional oriented finite Poincaré pair $(Y, \partial Y)$ in the sphere S^m, which consist of a *complement* Z along with an *attaching map* $f: \partial Y \to Z$, such that the pushout $Y \cup_f Z$ is homotopy equivalent to S^m. Williams [Wi1] defines two Poincaré embeddings with attaching maps $f_1: \partial Y \to Z_1$ and $f_2: \partial Y \to Z_2$ to be *concordant* if there exists a homotopy equivalence $\xi: Z_1 \to Z_2$ so that $f_2 \simeq \xi \cdot f_1: \partial Y \to Z_2$.

For the above Seifert surface embedding, $(M, \partial M)$ is an $(n+1)$-dimensional oriented Poincaré pair, with an oriented homeomorphism $\varpi: S^n \to \partial M$, given by the orientation of the knot. $(M \times I, A)$ is an $(n+2)$-dimensional Poincaré pair, with $A = M \times \{\pm 1\} \cup S^n \times I$, and the attaching map $f: A \to W$ gives a Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, by the homotopy equivalence $S^{n+2} = M \times I \cup \widehat{W} \xrightarrow{1\cup\varepsilon} M \times I \cup_f W$. Let $\omega: S^n \to M$ be the composite of ϖ and the inclusion $\imath: \partial M \to M$. Let $v_{\pm}: M \to W$ be the restriction of f to $M \times \pm 1 \subset A$. Let $\imath_{\pm}: M \to A$ be the inclusions $M \times \{\pm 1\} \subset A$. For any map $g: A \to W$ we denote by $g_{\pm}$ the restrictions $g_+: M \times \{\pm 1\} \subset A \xrightarrow{f} W$ and $g_-: S^n \times I \subset A \xrightarrow{f} W$.

Definition 1.2. Define $\delta: S^n \times I \xrightarrow{\cong} S^n \times I$ by $\delta(x, t) = (e^{i\pi(t+1)} \cdot x, t)$, using the standard action of $S^1 = \mathrm{SO}(2) \subset \mathrm{SO}(n+1)$ on S^n. We define the diffeomorphism τ of $S^n \times S^1 = \varinjlim (S^n \times I \xleftarrow{\imath} S^n \times \{\pm 1\} \xrightarrow{\imath} S^n \times I)$ to be the identity on the left $S^n \times I$ and δ on the right $S^n \times I$. We define the self-map

γ of $A = \varinjlim (M \times \{\pm 1\} \stackrel{\omega \times \mathrm{id}}{\longleftarrow} S^n \times \{\pm 1\} \stackrel{\iota}{\to} S^n \times I)$ to be the identity on $M \times \{\pm 1\}$ and δ on $S^n \times I$.

Lemma 1.3. (1) *The self-map* τ *of* $S^n \times S^1$ *is homotopic to the composite*

$$\tau' : S^n \times S^1 \xrightarrow{\text{coaction}} S^n \times S^1 \vee S^{n+1} \xrightarrow{\mathrm{id} \vee \eta} S^n \times S^1 \vee S^n \xrightarrow{\mathrm{id} \vee \iota_1} S^n \times S^1.$$

(2) *The self-map* γ *of* A *is homotopic to the composite*

$$A \xrightarrow{\text{coaction}} A \vee S^{n+1} \xrightarrow{\mathrm{id} \vee \eta} A \vee S^n \xrightarrow{\mathrm{id} \vee \omega} A \vee M \xrightarrow{\mathrm{id} \vee \iota_+ \bullet} A.$$

Proof. By the Barratt-Puppe sequence of the CW-complex $S^n \times S^1 = S^n \vee S^1 \cup e^{n+1}$, the self-map τ' is characterized up to homotopy by the property that: τ' induces the identity in homology, and the Hopf construction of the composite $S^n \times S^1 \stackrel{\tau'}{\to} S^n \times S^1 \stackrel{\pi_1}{\to} S^n$ is the generator $\eta \in \pi_{n+2}(S^{n+1}) \cong \mathbb{Z}/2$. But τ clearly satisfies both of these properties; hence (1). Now consider the rel boundary coaction map

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1}$$

of $S^n \times I$. We can define the coaction map of $A = M \times I \cup S^n \times I$ onto its top cell by glueing the identity map on the left half $M \times I$ to the above rel boundary coaction map. Furthermore, the self-map δ of $S^n \times I$ is the identity on $S^n \times \{\pm 1\} \cup N \times I$ for some point N. Therefore, δ is homotopic, rel boundary, to the composite

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1} \xrightarrow{\mathrm{id} \vee g} S^n \times I \vee S^n \xrightarrow{\mathrm{id} \vee \iota_1} S^n \times I$$

for some map $g \in \pi_{n+1}(S^n)$. By part (1), we see that $g = \eta \in \pi_{n+1}(S^n)$. By glueing in this rel boundary homotopy, the second assertion follows. □

Lemma 1.4. *Let* $f: A \to W$ *be the attaching map of a Poincaré embedding* $(M \times I, A) \hookrightarrow S^{n+2}$. *Then the composite* $A \stackrel{\gamma}{\to} A \stackrel{f}{\to} W$ *is also the attaching map of a Poincaré embedding* $(M \times I, A) \hookrightarrow S^{n+2}$. *Furthermore,* $(f \cdot \gamma)_+ = f_+ : M \times \{\pm 1\} \to W$ *and* $(f \cdot \gamma)_- = f_- \cdot \delta : S^n \times I \to W$.

Consider a Poincaré embedding $(M, A) \hookrightarrow S^m$ of an oriented, finite m-dimensional Poincaré pair (M, A), with complement W and attaching map $f: A \to W$. Let $\nu: S^m \stackrel{\simeq}{\to} M \cup_f W$ be the homotopy equivalence for which the composite $\rho: S^m \stackrel{\nu}{\to} M \cup_f W \to M/A$ is orientation preserving. Williams [Wi1] calls $\rho \in \pi_m(M/A)$ the *unstable normal invariant.* Williams [Wi2] shows that Browder's cofibration [Br3] $S^m \stackrel{\rho}{\to} M/A \stackrel{\Sigma f \cdot \partial}{\longrightarrow} \Sigma W$ is split by the degree one map $M/A \stackrel{\text{pinch}}{\longrightarrow} S^m$, that the composite

$$M/A \stackrel{\partial}{\to} \Sigma A \xrightarrow{\Sigma(f) \cdot \partial \vee \text{pinch}} \Sigma W \vee S^m \tag{2}$$

is a homotopy equivalence. From this we deduce

Lemma 1.5. *Let* $\rho, \rho' \in \pi_m(M/A)$ *be the unstable normal invariants of the Poincaré embeddings* $(M, A) \hookrightarrow S^m$ *with attaching maps* $f: A \to W$ *and* $f': A \to W'$ *Then*:

(1) $\rho = \rho'$ *if and only if the composite* $S^m \stackrel{\rho'}{\to} M/A \stackrel{\Sigma f \cdot \partial}{\longrightarrow} \Sigma W$ *is nullhomotopic.*

(2) *If $W' = W$ and the suspensions of the attaching maps $f, f': A \to W$ are homotopic, then the unstable normal invariants are equal, $\rho = \rho' \in \pi_m(M/A)$.*

Now consider our Seifert surface Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f: A \to W$, and unstable normal invariant $\rho \in \pi_{n+2}((M \times I)/A)$.

Theorem 1.6. *The Poincaré embeddings $(M \times I, A) \hookrightarrow S^{n+2}$ with attaching maps $f \cdot \gamma, f: A \to W$ have equal unstable normal invariant.*

Proof. This follows from the codimension one framed embedding of the Seifert surface, which implies the vanishing of $\Sigma\omega \in \pi_{n+1}(\Sigma M)$. That is, the cofibration sequence

$$S^{n+1} = \Sigma S^n \overset{\Sigma\omega}{\to} \Sigma M \overset{\Sigma\iota}{\to} \Sigma(M/S^n) \overset{\Sigma\partial}{\to} \Sigma^2 S^n = S^{n+2}$$

splits; we have a homotopy equivalence $\Sigma M \vee S^{n+2} \overset{\Sigma\iota\vee\rho}{\longrightarrow} \Sigma(M/S^n)$. Hence $\Sigma\iota: \Sigma M \to \Sigma(M/S^n)$ has a left homotopy inverse, which implies that $S^{n+1} \overset{\Sigma\omega}{\to} \Sigma M$ is nullhomotopic. By Lemma 1.3(2), $\Sigma\gamma \simeq \mathrm{id}: \Sigma A \to \Sigma A$. Hence, the maps $\Sigma f, \Sigma(f \cdot \gamma): \Sigma A \to \Sigma W$ are homotopic. The result follows from Lemma 1.5. □

We recall the uniqueness part of Williams's [Wi1] Poincaré embedding theorem.

Theorem 1.7. *Let (M, A) be an oriented, finite, m-dimensional Poincaré pair, with $\pi_1(A) = \pi_1(M) = 0$ and $m \geq 6$. Suppose M is n-dimensional as a CW-complex, and let $q = m - n - 1$. If $m < 3q$, then any two Poincaré embeddings of (M, A) in S^m whose unstable normal invariants are equal are concordant.*

Proof of Theorem B. Let $\Sigma^n \hookrightarrow S^{n+2}$ be a $(q-1)$-simple knot, $n + 3 \leq 3q$, with knot complement X^{n+2}, and attaching map $\alpha: S^n \times S^1 \to X$. Let M^{n+1} be a Seifert surface with resulting Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f: A \to W$. By a theorem of Levine [Le3] we can assume that M^{n+1} is $(q-1)$-connected. By Poincaré duality of the pair $(M, \partial M)^{n+1}$, M is then $(n+1-q)$-dimensional. Theorems 1.6 and 1.7 imply the Poincaré embeddings with attaching maps $f \cdot \gamma, f: A \to W$ are concordant. Let $\xi: W \overset{\simeq}{\to} W$ be a concordance, so $\xi \cdot f \simeq f \cdot \gamma: A \to W$. Since the geometric map f is a cofibration, we may assume that $\xi \cdot f = f \cdot \gamma: A \to W$. By Lemma 1.4 we have $\xi \cdot f_+ = f_+$ and $\xi \cdot f_- = f_- \cdot \delta$. The knot complement X is the pushout $X = \varinjlim (M \times I \overset{\iota}{\leftarrow} M \times \{-1, 1\} \overset{f_\pm}{\to} W)$, so we can define a self-map $\zeta: X \to X$ to be the identity on $M \times I$ and ξ on W. But the attaching map $\alpha: S^n \times S^1 \to X$ and the composite $\alpha \cdot \tau: S^n \times S^1 \to X$ are the induced maps of colimits of the following strictly commutative diagrams:

$$\begin{array}{ccccc}
M \times I & \xleftarrow{\iota} & M \times \{\pm 1\} & \xrightarrow{f_\pm} & W \\
\uparrow{\scriptstyle \omega\times\mathrm{id}} & & \uparrow{\scriptstyle w\times\mathrm{id}} & & \uparrow{\scriptstyle f_-} \\
S^n \times I & \xleftarrow{\iota} & S^n \times \{\pm 1\} & \xrightarrow{\iota} & S^n \times I
\end{array}$$

$$\begin{array}{ccccc} M\times I & \stackrel{\iota}{\longleftarrow} & M\times\{\pm1\} & \stackrel{f_+}{\longrightarrow} & W \\ \uparrow{\scriptstyle \omega\times\mathrm{id}} & & \uparrow{\scriptstyle \omega\times\mathrm{id}} & & \uparrow{\scriptstyle f_-\cdot\delta} \\ S^n\times I & \stackrel{\iota}{\longleftarrow} & S^n\times\{\pm1\} & \stackrel{\iota}{\longrightarrow} & S^n\times I \end{array}$$

Using Definition 1.2, we have $\zeta\cdot\alpha=\alpha\cdot\tau\colon S^n\times S^1\to X$, and thus diagram (1) commutes. By Lemma 1.1 $\zeta\colon X\to X$ is a homotopy equivalence. □

2. Proof of Theorem A; smooth knots and surgery

Lashof and Shaneson [LS, Theorem 2.1] show that any self-homotopy equivalence of a knot complement pair $(X,\partial X)^{n+2}$ is homotopic to a diffeomorphism, if $n\ge 5$ and $\pi_1(X)\cong\mathbb{Z}$. This follows from the Sullivan-Wall surgery exact sequence [Wa2, §10]

$$0=L_{n+3}(\mathbb{Z}[\mathbb{Z}]\stackrel{\mathrm{id}}{\to}\mathbb{Z}[\mathbb{Z}])\to\mathscr{S}^O(X_1)\to[X,G/O]=0.$$

Let $\phi\colon X_1\stackrel{\cong}{\to}X_2$ be a diffeomorphism between the knot complements of two smooth $(q-1)$-simple knots $\Sigma_i^n\hookrightarrow S^{n+2}$ with $n+3\le 3q$, $n\ge 5$, for $i=1,2$. The homotopy equivalence $\zeta\colon X_1\stackrel{\simeq}{\to}X_1$ of Theorem B is thus homotopic to a diffeomorphism $\theta\colon X_1\stackrel{\cong}{\to}X_1$. Following Browder [Br1, Corollary 2], we have an exact sequence

$$\text{(3)}\qquad \Gamma^{n+1}\oplus\Gamma^{n+2}\stackrel{\iota}{\to}\mathrm{Diff}(\Sigma_1^n\times S^1)\stackrel{\mathscr{F}}{\to}\mathscr{E}(S^n\times S^1)\cong\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2,$$

involving the pseudoisotopy and homotopy automorphism groups. The first two $\mathbb{Z}/2$ summands are given by the degree -1 maps of S^n and S^1, and the third summand is given by the self-map τ, which is detected by the Hopf construction (cf. Lemma 1.3). We have to modify Browder's argument slightly, since $\mathscr{F}$ will not be surjective if the exotic sphere Σ_1^n does not possess an orientation-preserving diffeomorphism (cf. [KM]).

Let $\beta_i\colon\Sigma_i^n\times S^1\stackrel{\cong}{\to}\partial X_i$ be the preferred diffeomorphisms, for $i=1,2$. The restriction of ϕ to the boundary gives a diffeomorphism $\partial\phi\colon\Sigma_1^n\times S^1\stackrel{\cong}{\to}\Sigma_2^n\times S^1$. If the Hopf construction of the composite $\pi_2\cdot\partial\phi\colon\Sigma_1^n\times S^1\to\Sigma_2^n$ is $\eta\in\pi_{n+2}(S^{n+1})\cong\mathbb{Z}/2$, then replace the diffeomorphism ϕ by the composite $\phi\cdot\theta$. By Browder's application of the Browder-Levine fibering theorem [Br1, Lemma 2], we can assume that $\partial\phi$ restricts to a diffeomorphism $\phi_0\colon\Sigma_1^n\stackrel{\cong}{\to}\Sigma_2^n$. Let $\varepsilon=\pm1$ be the degree of $\partial\phi$ on the S^1 factor. Consider the diffeomorphism $\psi=(\phi_0\times\varepsilon)^{-1}\cdot\partial\phi\in\mathrm{Diff}(\Sigma_1^n\times S^1)$, which induces the identity in homology. Since its Hopf construction is zero, $\psi\simeq\mathrm{id}$, so $\mathscr{F}(\psi)=\mathrm{id}$, and (3) shows that $\partial\phi=(\phi_0\times\varepsilon)\cdot\psi$ is pseudoisotopic to the composite

$$\Sigma_1^n\times S^1\stackrel{e}{\to}\Sigma_1^n\times S^1\stackrel{d\times 1}{\longrightarrow}\Sigma_1^n\times S^1\stackrel{\phi_0\times\varepsilon}{\longrightarrow}\Sigma_2^n\times S^1,$$

where $d\in\Gamma^{n+1}$ is a diffeomorphism of Σ_1^n, and $e\in\Gamma^{n+2}$ is obtained from the identity map of $\Sigma_1^n\times S^1$ by "connecting sum" with a diffeomorphism of an $(n+1)$-disk. We claim that $\partial\phi$ extends to a diffeomorphism of $\widetilde{\partial\phi}\colon\Sigma_1^n\times D^2\stackrel{\cong}{\to}\Sigma_2^n\times D^2$. Certainly $(\phi_0\cdot d)\times\varepsilon$ extends. But e must be pseudoisotopic to the identity, otherwise we could glue in the tubular neighborhoods to get a diffeomorphism between the standard sphere S^{n+2} and the exotic sphere represented by $e \in \Gamma^{n+2}$. By gluing together ϕ with the extension $\widetilde{\partial\phi}$, we have an equivalence of the two knots.

3. Appendix

Farber [Fa2] shows that PL $(q-1)$-simple knots with $n+3 < 3q$ are classified up to isotopy by a stable homotopy Seifert pairing, which amounts to the homotopy class $v_+ \in [M, W]$, which lies in a stable group by the Freudenthal suspension theorem. Using our S-duality map $\Delta\colon S^{n+3} \to \Sigma W \wedge \Sigma M$ [Ri] we have a bijection $[M, W] \overset{\Delta\cap\cdot}{\to} \pi^s_{n+3}((\Sigma W)^{[2]})$ in Farber's range. We note that Farber uses a dual S-duality map $M \wedge W \to S^{n+1}$. We show that Farber's stable homotopy invariant is essentially the second Hopf invariant $\lambda_2(\rho)$ of our unstable normal invariant.

Theorem 3.1. *The second Hopf invariant $\lambda_2(\rho)$ of the unstable normal invariant $\rho\colon S^{n+2} \to \Sigma(M/\partial M)$ is the S-dual of the map $v_+\colon M \to W$:*

$$\lambda_2(\rho) = [\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}(\mathrm{id} \wedge v_+) \cdot \Sigma\Delta \in \pi^s_{n+3}((\Sigma(M/\partial M))^{[2]}),$$

using the isomorphisms

$$[M, W] \xrightarrow[\cong]{\Delta\cap\cdot} \pi^s_{n+3}((\Sigma W)^{[2]}) \xrightarrow[\cong]{[\Sigma\iota\cdot(\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}} \pi^s_{n+3}((\Sigma(M/\partial M))^{[2]}).$$

Consider the general case of a Poincaré embedding $(M, A) \hookrightarrow S^m$, with complement W and attaching map $f\colon A \to W$ as in [Ri]. The *boundary map* $\partial\colon M/A \to \Sigma A$ is defined to be the homotopy class making the diagram

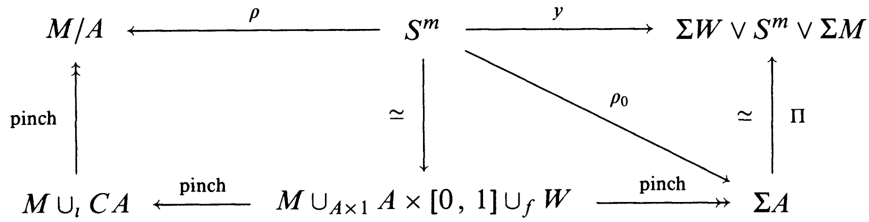
$$\begin{array}{ccc} M \cup_\iota CA & \xrightarrow{\text{pinch}} & \Sigma A \\ {\scriptstyle\text{pinch}}\downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle\partial} & \\ M/A & & \end{array} \qquad \left(\begin{aligned} CA &= A \times [0, 1]/A \times 0, \\ \Sigma A &= CA/A = A \wedge ([0, 1]/\{0, 1\}) \end{aligned}\right)$$

commute up to homotopy. Let $\rho_0 = \partial \cdot \rho\colon S^m \to \Sigma A$ be the composite. Extending the splitting (2), from Williams's work [Wi2] we have a homotopy equivalence

$$\Pi = x \cdot \Sigma f + y \cdot \text{pinch} + z \cdot \Sigma\iota\colon \Sigma A \xrightarrow{\simeq} \Sigma W \vee S^m \vee \Sigma M,$$

where x, y, and z are the inclusions of the three factors, and pinch: $\Sigma A \to S^m$ is the unique degree one homotopy class. The two maps Π, $y\colon \Sigma A \to \Sigma W \vee S^m \vee \Sigma M$ are equalized up to homotopy by the collapse map $M \cup_{A\times 1} A \times [0, 1] \cup_f W \to \Sigma A$: the first and third maps $x \cdot \Sigma f$ and $z \cdot \Sigma\iota$ can be nullhomotoped when restricted to $M \cup_{A\times 1} A \times [0, 1] \cup_f W$ since the ends M and W are "free" (as in the proof of the equivalence of Whitehead products and Samelson products

[Wh]). Thus the diagram

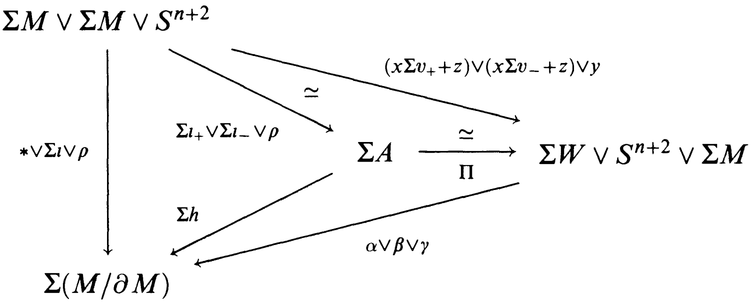


is homotopy commutative. Now apply Boardman and Steer's Cartan formula and composition formula [BS] to the equation $\Pi \cdot \rho_0 \simeq y\colon S^m \to \Sigma W \vee S^m \vee \Sigma M$. We obtain $\Pi \wedge \Pi \cdot \lambda_2(\rho_0) + \lambda_2(\Pi) \cdot \Sigma\rho_0 = 0$ and $\lambda_2(\Pi) = x\Sigma f \smile z\Sigma\iota$, which implies

$$\Pi \wedge \Pi \cdot \lambda_2(\rho_0) = -(x\Sigma f \smile z\Sigma\iota) \cdot \Sigma\rho_0 \in \pi_{m+1}((\Sigma W \vee S^m \vee \Sigma M)^{[2]}).$$

Proof of Theorem 3.1. Now consider a Seifert surface $(M \times I, A) \hookrightarrow S^{n+2}$, with complement W and attaching map $f\colon A \to W$. Let $h\colon A \to M/\partial M$ be defined by collapsing the subspace $M \times 1 \cup \partial M \times I$. Then we see that Σh is a homotopy retraction of $\partial\colon \Sigma(M/\partial M) \to \Sigma A$. Thus $\rho \simeq \Sigma h \cdot \rho_0\colon S^m \to \Sigma(M, \partial M)$, and $\lambda_2\rho = (\Sigma h)^{[2]} \cdot \lambda_2\rho_0$.

Let us factor $\Sigma h\colon \Sigma A \to \Sigma(M/\partial M)$ through the homotopy equivalence Π, by a map $\alpha \vee \beta \vee \gamma\colon \Sigma W \vee S^{n+2} \vee \Sigma M \to \Sigma(M/\partial M)$. The homotopy commutative diagram



yields $\alpha = \Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \in [\Sigma W, \Sigma(M/\partial M)]$, $\gamma = -\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+ \in [\Sigma M, \Sigma(M/\partial M)]$, and $\beta = \rho$. Recall that $\Pi^{[2]} \cdot \lambda_2(\rho_0) = -(x\Sigma f \smile z\Sigma\iota) \cdot \Sigma\rho_0$, which is the negative of the suspension of the S-duality map $\Delta\colon S^{n+2} \to \Sigma W \wedge M$. Thus,

$$\Sigma h = [\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \vee \rho \vee -\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+] \cdot \Pi,$$

$$(\Sigma h)^{[2]} \cdot \lambda_2(\rho_0) = [\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}(\mathrm{id} \wedge v_+) \cdot \Sigma\Delta,$$

and we are done. □

Remark. This provides further evidence for our conjecture: the two Seifert surface Poincaré embeddings that we consider are equivalent under a stronger equivalence relation than Farber's relation of isotopy.

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