SOME REMARKS ON THE KIRBY-SIEBENMANN CLASS

R. J. Milgram

In this note we study the relations that hold between the Kirby-Siebenmann class $\{KS\} \in H^4(B_{STOP};\ \mathbf{Z}/2)$ and the first Pontrajagin class.

The first result is that that the natural map $p_0 : B_{STOP} \longrightarrow B_{SG}$ does not detect $\{KS\}$ no matter what coefficients might be used. However, the homology dual of $\{KS\}$ is in the image of the Hurewicz map

$$\pi_4(B_{STOP}) \longrightarrow H_4(B_{STOP};\ \mathbf{Z}/2).$$

In fact there is a unique non-zero element $[KS] \in \pi_4(B_{STOP})$ of order 2, and $p_0([KS]) \neq 0 \in \pi_4(B_{SG})$. In particular this implies that $w_4 + \{KS\}$ is a mod(24) fiber-homotopy invariant of SPIN-TOP bundles. However, it is interesting to ask what happens when w_2 is non-zero. To understand this we introduce an intermediate classifying space, B_{TSG} for which we have a factorization

$$p_0 = p \cdot f, \quad B_{STOP} \xrightarrow{f} B_{TSG} \xrightarrow{p} B_{SG}.$$

B_{TSG} is univeral for the vanishing of transversality obstructions through dimension 5. Additionally, B_{TSG} is built out of finite groups ($\mathbf{Z}/2$-extensions of the symmetric groups S_n) in the same way that B_{SG} is constructed from the S_n. As a result, explicit construction of homotopy classes of maps into B_{TSG} is often possible.

We show that $H_4(B_{TSG};\ \mathbf{Z}/2) = \mathbf{Z}/2 \oplus \mathbf{Z}/48$ and that the homology dual of the Kirby-Siebenmann class maps to 24 times the second generator. Thus, this transversality theory does detect $\{KS\}$. But note also the $\mathbf{Z}/48$. Our main question is the extent to which it gives rise to a fiber homotopy invariant of topological $\mathbf{R}^n$-bundles. The general result is

Theorem I: *Let ξ, ψ be two stable $\mathbf{R}^n$-bundles over X, and suppose they are fiber homotopy equivalent. Then there is $\alpha \in H^2(X;\ \mathbf{Z}/2)$ and*

$$24\alpha^2 + P_1(\xi) + 24\{KS(\xi)\} \ = \ P_1(\psi) + 24\{KS(\psi)\}$$

in $H^4(X;\ \mathbf{Z}/48)$ where $P_1(\xi)$ is the $\mathbf{Z}/48$ reduction of the first Pontrajagin class.

In other words, there is an element $A \in H^4(B_{TSG};\ \mathbf{Z}/48)$ with $f^*(A) = P_1 + 24\{KS\}$, and (I) gives the effect of different liftings of a map $p_0 \cdot g : X \longrightarrow B_{STOP} \longrightarrow B_{SG}$ on A.

$H^2(B_{STOP};\ \mathbf{Z}/2) = \mathbf{Z}/2$ with generator w_2, so the possible factorizations of p_0 through B_{TSG} differ in their effect on A only by $24w_2^2$. In particular this gives

Corollary: *If M^4 is a compact closed topological manifold with even index, and ν is its stable normal bundle, then $w_2^2 = 0 \in H^2(M;\ \mathbf{Z}/2)$ and*

$$\nu^* f^*(A) = P_1(\nu) + 24\{KS(\nu)\}$$

is independent of the choice of f factoring p_0.

This note came about in answer to a question of Frank Quinn. He pointed out that in [M-M] the exact structure of B_{STOP}, and the various surgery maps in dimension 4 were never worked out. But currently it appears very useful to understand them. Of course, we do not attempt to work out explicit geometric methods for evaluating the new invariants. But knowing what they are and how they fit together should make that fairly direct.

The homotopy types of B_{SO}, B_{SG} in dimension ≤ 7

A Postnikov system for B_{SO} through dimension 7 is given by

$$(1) \qquad B_{SO} \longrightarrow K(\mathbf{Z}/2,\ 2) \longrightarrow K(\mathbf{Z},\ 5)$$

with K-invariant $2\{Sq^2Sq^1(\iota_2) + \iota_2 \cdot Sq^1(\iota_2)\}$. (Note that $H^5(K(\mathbf{Z}/2,\ 2);\ \mathbf{Z}) = \mathbf{Z}/4$ with generator having mod(1) reduction γ and

$$(2) \qquad \gamma = Sq^2Sq^1(\iota_2) + \iota_2 \cdot Sq^1(\iota_2).$$

Moreover, $\beta_4(\iota_2^2) = \gamma$.)

The stable homotopy of spheres is given in the first 6 dimensions by

$$(3) \qquad \pi_i^s(S^0) = \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}/2 & i = 1, \text{ generator } \eta \\ \mathbf{Z}/2 & i = 2, \text{ generator } \kappa_1 \\ \mathbf{Z}/24 & i = 3, \text{ generator } \nu \\ 0 & i = 4,\ 5 \\ \mathbf{Z}/2 & i = 6, \text{ generator } \kappa_2 = \nu^2 \end{cases}$$

and we will use the same names for the corresponding elements in $\pi_{i+1}(B_{SG}) \cong \pi_i^s(S^0)$. One relation that should be kept in mind is $\eta\kappa_1 = 12\nu$, since it also holds in $\pi_*(B_{SG})$, though the relation $\eta^2 = \kappa_1$ which holds stably does not hold in $\pi_*(B_{SG})$.

Lemma (4): *A Postnikov system for B_{SG} through 7 is given by*

$$K(\mathbf{Z}/2,\ 2) \times K(\mathbf{Z}/2,\ 3) \times K(\mathbf{Z}/2,\ 7) \longrightarrow K(\mathbf{Z}/24,\ 5)$$

where the K-invariant is $2\{Sq^2Sq^1(\iota_2) + \iota_2 \cdot Sq^1(\iota_2)\} + 4\{Sq^2(\iota_3)\}$.

Proof: With $\mathbf{Z}/24$-coefficients the K-invariant for B_{SG} maps back to the image of the corrisponding K-invariant for B_{SO}. Hence, the class in (2) must appear in the K-invariant. Also, the kernel of the map $H^5(K(\mathbf{Z}/2,\ 2,3);\ \mathbf{Z}/24) \longrightarrow H^5(K(\mathbf{Z}/2,\ 3);\ \mathbf{Z}/24)$ is generated by $4Sq^2(\iota_3)$. It follows that $4Sq^2(\iota_3)$ is the only term which can be added to the K-invariant. But, in fact, this term must be involved in the K-invariant because there is the homotopy relation which we have already noted $\eta\kappa_1 = 12\nu$, since η is detected by Sq^2.

In order to understand the integral homology of B_{SG}, B_{STOP}, and the intermediate space B_{TSG} which we will introduce shortly, we need a method for obtaining Bochstein information from K-invariants. The following result will suffice.

Lemma (5): *Let* $K(\mathbf{Z}/2^i,\ j) \times K(\mathbf{Z}/2,\ j+1) \xrightarrow{\kappa} K(\mathbf{Z}/2^s,\ j+1)$ *be given with*

$$\kappa = 2^w\beta(\iota_j) + 2^{s-1}(\iota_{j+1}),$$

then the fiber E of the map κ is $K(\mathbf{Z}/2^{i+s-w-1} \times \mathbf{Z}/2^w)$.

Proof: The homotopy exact sequence of the fibration in dimensions $j,\ j+1$ is

$$(6) \qquad 0 \longrightarrow \pi_{j+1}(E) \longrightarrow \mathbf{Z}/2 \xrightarrow{\kappa_*} \mathbf{Z}/2^s \xrightarrow{\partial} \pi_j(E) \longrightarrow \mathbf{Z}/2^j \longrightarrow 0$$

But the term $2^{s-1}\iota_{j+1}$ in $\kappa^*(\iota_{j+1})$ implies that κ_* is injective in (6). Thus E is a $K(\pi,\ j)$ and π is given as an extension in the sequence

$$0 \longrightarrow \mathbf{Z}/2^{s-1} \longrightarrow \pi_j(E) \longrightarrow \mathbf{Z}/2^i \longrightarrow 0.$$

The type of this extension is determined by the term $2^w(\beta(\iota_j))$ in $\kappa^*(\iota_{j+1})$. From this (5) follows.

(4) and (5) imply that there is a mod(8) Bochstein

$$\beta_8(\iota_2^2) = \{Sq^2(\iota_3)\} \text{ in } H^*(B_{SG};\ \mathbf{Z}/2).$$

Additionally, the Hurewicz image of ν is $\{w_4^*\} + 2\{\iota_2^{2,*}\}$ since this is already true in B_{SO}, where it is well known. As a consequence $H_4(B_{SG};\ \mathbf{Z}) = \mathbf{Z}/2 \oplus \mathbf{Z}/24$ with generators $\{w_4^*\}$, $\{w_2^{2,*}\}$ respectively, and 12ν is in the kernel of the Hurewicz map.

The structure of B_{STOP} through dimension 7

From the fiberings

$$(7) \qquad \begin{array}{ccccc} G/O & \longrightarrow & B_{SO} & \longrightarrow & B_{SG} \\ \downarrow & & \downarrow & & \downarrow \\ G/TOP & \longrightarrow & B_{STOP} & \longrightarrow & B_{SG} \end{array}$$

and the well known result of Kirby-Siebenmann that $\pi_4(G/TOP) = \pi_4(G/O) = \mathbf{Z}$, but that the map between them is multiplication by 2, we get the diagram of extensions in π_4,

$$(8) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\cdot 24} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/24 & \longrightarrow & 0 \\ & & \downarrow{\cdot 2} & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \pi_4(B_{STOP}) & \longrightarrow & \mathbf{Z}/24 & \longrightarrow & 0 \end{array}$$

The only way this diagram can commute is if $\pi_4(B_{STOP}) = \mathbf{Z}/2 \oplus \mathbf{Z}$ with the element of order 2 mapping to $12 \cdot \nu$, and the generator of the $\mathbf{Z}$-summand mapping to ν.

Lemma (9): $\pi_i(B_{STOP}) = \begin{cases} \mathbf{Z}/2 & i = 2 \\ \mathbf{Z} \oplus \mathbf{Z}/2 & i = 4 \\ 0 & 4 < i < 8. \end{cases}$ *Moreover, a Postnikov system for* B_{STOP} *through this range is given by*

$$(10) \qquad K(\mathbf{Z}/2,\ 2) \times K(\mathbf{Z}/2,\ 4) \longrightarrow K(\mathbf{Z},\ 5)$$

with K*-invariant* $2\{Sq^2Sq^1(\iota_2) + \iota_2 \cdot Sq^1(\iota_2)\}$.

(This is clear.)

In particular, the class $\{KS^*\} \in H_4(B_{STOP};\ \mathbf{Z})$ which is in the Hurewicz image of the element of order 2, must go to zero in $H_4(B_{SG};\ \mathbf{Z})$, since, in homotopy, it goes to 12ν. This shows that $\{KS^*\}$ has no homology (or cohomology) relations implied by the map into B_{SG}. However, in homotopy, the fact that it maps to 12ν should have some consequeences.

The space B_{TSG}

The failure to detect the Kirby-Siebenmann class in $H_*(B_{SG};\ \mathbf{Z})$ is the influence of the first exotic class ι_3. In fact, the term $4Sq^2(\iota_3)$ in the 5-dimensional K-invariant (4) is exactly the difficulty. (For example, if we kill w_2 but leave ι_3 in $H^*(B_{SG};\ \mathbf{Z}/2)$ the resulting space has only $\mathbf{Z}/4$-torsion in $H_4(\ \ ;\ \mathbf{Z})$.) Hence it is natural to consider the classifying space B_{TSG} obtained from B_{SG} by killing the exotic class ι_3. For definiteness, recall that ι_3 is detected with 0-indeterminacy in the Thom-complex MSG by applying the twisted secondary operation associated to the relation $(w_2 + Sq^2)(w_2 + Sq^2)$ to the Thom class, and using the Thom isomorphism to bring the class back to B_{SG}. For details see [R].

We have the fibration sequence

$$(11).\qquad K(\mathbf{Z}/2,\ 2) \longrightarrow B_{TSG} \xrightarrow{p} B_{SG} \xrightarrow{\iota_3} K(\mathbf{Z}/2,\ 3)$$

with K-invariant ι_3. This is the universal space for fiber homotopy transversality to hold in the Thom space, at least through dimension 5 (Compare [B-M]). Indeed, a fiber homotopy sphere bundle $\xi \longrightarrow X$ and reduction to B_{TSG} is equivalent to the condition $\iota_3(\xi) = 0 \in H^3(X;\ \mathbf{Z}/2)$, together with a specific choice of 2-dimensional cochain c so

$$\delta c \ = \ f^{\#}(\iota_3)$$

where $f\ :\ X \longrightarrow B_{SG}$ classifies ξ. This situation is very close, but certainly not the same as the situation studied in [F-K]. Also, there is a factorization of the canonical map $B_{STOP} \longrightarrow B_{SG}$ as

$$B_{STOP} \longrightarrow B_{TSG} \longrightarrow B_{SG}.$$

Precisely, there are exactly two such factorizations differing by a map

$$B_{STOP} \longrightarrow K(\mathbf{Z}/2,\ 2).$$

Now, we look at the 6-skeleton of B_{TSG}. This is the 6-skeleton of the 2-stage Postnikov system

$$K(\mathbf{Z}/2,\ 2) \times K(\mathbf{Z}/3,\ 4) \longrightarrow K(\mathbf{Z}/8,\ 5)$$

with K-invariant $2\{Sq^2Sq^1(\iota_2) + \iota_2 \cdot Sq^1(\iota_2)\}$. From (5) the resulting space has 4^{th} integral homology group given as

$$H_4(B_{TSG};\ \mathbf{Z}) \ = \ \mathbf{Z}/2 \oplus \mathbf{Z}/48$$

with generators $(w_4)^*$, $(w_2^2)^*$ respectively. Here, w_2 can be identified with ι_2. Note that this implies that the Kirby-Siebenmann class maps *non-trivially* to $24((w_2^2)^*)$.

The proof of theorem (I)

Lemma (12): *Let* $X \xrightarrow{f} B_{TSG}$ *be given and suppose* f' *is the composite*

$$X \xrightarrow{(\alpha, f)} K(\mathbf{Z}/2,\ 2) \times B_{TSG} \xrightarrow{\mu} B_{TSG}$$

where μ is the principal bundle map $K(\mathbf{Z}/2,\ 2) \times B_{TSG} \longrightarrow B_{TSG}$, then

$$f'^{,*}\{w_2^2\} \ = \ f^*\{w_2^2\} + 24\alpha^2 \in H^4(X;\ \mathbf{Z}/48).$$

Proof: $H^4(K(\mathbf{Z},\ 2) \times B_{TSG};\ \mathbf{Z}/16) = (\mathbf{Z}/2)^2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/16$ with generators

$$8(\iota_2 \otimes w_2),\ 8(1 \otimes w_4) \text{ of order } 2,\ (4\iota_2^2 \otimes 1) \text{ of order } 4, \text{ and } (1 \otimes w_2^2) \text{ of order } 16.$$

We will show that $\mu^*(w_2^2) \ = \ 8(\iota_2^2 \otimes 1) + 1 \otimes w_2^2$. We first note, by naturality and the primitivity of w_2^2 in $H^4(B_{SO};\ \mathbf{Z})$ that $8(\iota_2 \otimes w_2)$ is not in this image. Next, we look at the cohomology Serre spectral sequence of the fibering

$$K(\mathbf{Z}/2,\ 2) \longrightarrow B_{TSG} \longrightarrow B_{SG}$$

with $\mathbf{Z}/16$-coefficients. $E_2^{0\ 4} \ = \ H^4(K(\mathbf{Z}/2,_2;\ \mathbf{Z}/16) \ = \ \mathbf{Z}/4$, with generator $4\iota_2^2$. Also, $E_2^{4\ 0} \ = \ H^4(B_{SG};\ \mathbf{Z}/16) \ = \ \mathbf{Z}/2 \oplus \mathbf{Z}/8$ with generators $8w_4$, $2(w_2^2)$, and

$$E_2^{5\ 0} \ = \ H^5(B_{SG};\ \mathbf{Z}/16) \ = \ (\mathbf{Z}/2)^3 + \mathbf{Z}/8.$$

Here, only the $\mathbf{Z}/8$ is of interest. It has generator $Sq^2(\iota_3)$, so $d_5(4\iota_2^2) \ = \ 4Sq^2(\iota_3)$, and at $E_\infty^{i,j}$, $i + j = 4$, only $E^{0\ 4} = \mathbf{Z}/2$, $E^{4\ 0} \ = \mathbf{Z}/8 \oplus \mathbf{Z}/2$ are non-zero. Thus there is a non-trivial extension for $H^4(B_{TSG};\ \mathbf{Z}/16)$

$$0 \longrightarrow \mathbf{Z}/8 \text{ (generator } 2w_2^2) \longrightarrow \mathbf{Z}/16 \longrightarrow \mathbf{Z}/2 \text{ (generator } 8\iota_2^2) \longrightarrow 0.$$

But this forces the result.

Theorem (I) is direct from (12). The corollary follows, also, since the assumption of even index implies that $w_2(M^4)^2 = 0 \pmod 2$. Hence, either lifting gives the same map in cohomology with $\mathbf{Z}/48$-coefficients.

Concluding remarks

From Quillen's work we know that $B_{SG} \otimes \hat{\mathbf{Z}}_2$ can be identified with $B(B^+(SO(\mathbf{F}_3)))$ in dimensions ≤ 6, and as $B(B^+(\mathcal{S}_\infty))$ in all dimensions. Here, $\mathcal{S}_\infty$ is the infinite symmetric group. Similarly we can describe B_{TSG} as $B(B^+(\tilde{S}O(\mathbf{F}_3)))$ in this same range. Moreover, B_{TSG} can be given as $B(B^+(\tilde{\mathcal{S}}_\infty))$ in all dimensions. Here, these new groups are described by central extensions

$$\mathbf{Z}/2 \longrightarrow \tilde{S}O(\mathbf{F}_3) \longrightarrow SO(\mathbf{F}_3) \longrightarrow 0$$

$$\mathbf{Z}/2 \longrightarrow \tilde{\mathcal{S}}_\infty \longrightarrow \mathcal{S}_\infty \longrightarrow 0$$

where, for $\mathcal{S}_\infty$ the extension is the (unique) non-trivial one for which the transposition $(1,\ 2)$ continues to have order 2. This might be very useful in understanding Casson's recent results on the Rochlin invariant.

It seems direct to use the description above of B_{TSG} by finite models to calculate the order of the classes which carry the remaining Pontrajagin classes. I hope to return to this later.

Also, there is a second factorizing space for the map $B_{STOP} \longrightarrow B_{SG}$, namely the space where we kill *all* the exotic classes $\sigma(e_{2^i-1,2^i-1})$. The precise structure of these classes is not entirely known, but there is considerable information in [R]. So it should be possible to understand the higher torsion in the cohomology and homology of this intermediate classifying space. Moreover, it is likely that it is the universal space for the vanishing of transversality obstructions.

Bibliography

[B-M] G. Brumfiel-J.Morgan, *Homotopy theoretic consequences of N. Levitts obstruction theory to transversality for spherical fibrations*, Pac. J. Math (1976) 1-100

[F-K] M. Freedman-R. Kirby, *A geometric proof of Rochlin's theorem*, Algebraic and Geometric Topology, A.M.S. Proceedings of Symposia in Pure Mathematics, Vol. XXXII(1) (1978) 85-98

[M-M] Ib Madsen-R.J.Milgram, Classifying Spaces for Surgery and Cobordism of Manifolds, Ann. of Math Studies #92, Princeton U. Press (1979)

[R] Doug Ravenal, Thesis, Brandeis University (1970)

November, 1987
Sonderforschungsbereich 170
Göttingen Universität