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# An Invariant of Knot Cobordism*

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## Introduction

In this paper we study a knot cobordism invariant. This is an integer modulo 2 assigned to each knot cobordism class. The definition and proofs of some of the known properties of this invariant are presented in Section 1.

By using the Seifert surface $F$ of a knot $k$ we are able to calculate $\varphi(k)$, the cobordism invariant of $k$, directly from the integral matrix $V$ associated with a canonical set of curves on $F$. Using the matrix $V$ again we define a quadratic form mod 2 over the vector space $H_1(F, \mathbf{Z}_2)$. The Arf invariant of this quadratic form turns out to be $\varphi(k)$.

In Section 5 we show that $\varphi(k)$ can be computed directly from the Alexander polynomial $\Delta(t)$ of $k$. In fact, if $\Delta(t) = c_0 + c_1 t + \cdots + c_0 t^{2n}$, then $\varphi(k) \equiv c_{n-1} + c_{n-3} + \cdots + c_r \bmod 2$, $r = 0$ or $1$.

Finally in Section 6 we construct from a Seifert surface $F$ of a knot $k$ an orthonormally framed 2-manifold $(M, W)$ in $R^5$ and show that $\varphi(k) = \delta(M, W)$, the $\delta$-invariant of Pontryagin given in [6]. This enables us to define a homomorphism $\theta: C_1 \to \pi_5(S^3)$, where $C_1$ is the knot cobordism group.

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## 1. Definition of the Knot Cobordism Invariant

Let $f: S^2 \to M^4$ be a combinatorial embedding of the 2-sphere $S^2$ into a closed, oriented, simply connected, differentiable 4-manifold $M^4$. Let $f$ be differentiable and regular except at one point $x_0 \in S^2$, and suppose there exists a differentiably embedded 4-disk $D^4$ such that $D^4 \subset M^4$, $f(x_0)$ (the singularity of $f$) is at the center of $D^4$, and $f(S^2) \cap D^4$ is a knot in $S^3 = bD^4$. Now $f$ represents a certain homology class $\xi$ contained in $H_2(M^4, \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers. Let us assume that $\xi$ is dual to $w_2(M^4)$, the 2-nd Stiefel-Whitney class of $M^4$. We shall describe the above assumptions on the map $f$ by saying $f: S^2 \to M^4$ is *admissible* for the knot $k$.

For any such map $f:S^2 \to M^4$ admissible for $k$, we define a residue class modulo 2 by

$$\varphi(k) \equiv \frac{\xi \cdot \xi - \sigma(M^4)}{8} \mod 2,$$

where $\xi \cdot \xi$ is intersection number and $\sigma(M^4)$ is the signature of the manifold as defined in [4]. It is known that $\xi \cdot \xi \equiv \sigma(M^4) \mod 8$ (see [2]) so that the above residue class is well defined.

We shall show that the number mod 2 depends only on the cobordism class of $k$ (see [3]):

LEMMA 1.1. *If $k$ and $k'$ are cobordant knot types, then*

$$\varphi(k) \equiv \varphi(k') \mod 2,$$

*and thus $\varphi(k)$ is independent of the admissible map and manifold used to compute it.*

Proof: Let

$$f_1:S^2 \to M_1^4 \quad \text{and} \quad f_2:S^2 \to M_2^4$$

be admissible for $k$ and $k'$, respectively. Let $D_1^4$ and $D_2^4$ be 4-discs in $M_1^4$ and $M_2^4$, respectively, containing the singularities of $f_1$ and $f_2$ at their respective centers. Further, let $M_1^4 \cap D_1^4 = k$ and $M_2^4 \cap D_2^4 = k'$. Take in $M_2^4$ a 4-disc $\Delta$ contained in and concentric to $D_2^4$. Between $b\Delta$ and $bD_2^4$ we can differentiably embed in $D_2^4$ an oriented 2-manifold $N^2$ of genus zero such that $N^2$ meets $bD_2^4$ and $b\Delta$ orthogonally, $N^2 \cap bD_2^4 = k'$, and $N^2 \cap b\Delta$ has the knot type of $k$ oppositely oriented. We can form the connected sum $\bar{M}_1^4 \# M_2^4$, using the discs $D_1^4$ and $\Delta$ in such a way that $k$ in $D_1^4$ is identified with $k \subset b\Delta$. Here $\bar{M}_1^4$ denotes $M_1^4$ with the opposite orientation. Using the image of $S^2$ under $f_1$ and $f_2$, respectively, occurring outside of $D_1^4$ and $D_2^4$ and connecting them in $\bar{M}_1^4 \# M_2^4$ by means of $N^2$, one obtains a differentiable embedding $f:S^2 \to \bar{M}_1^4 \# M_2^4$.

If $f$ represents the class $\xi \in H_2(\bar{M}_1^4 \# M_2^4, \mathbf{Z})$ and $f_i$ represents the class $H_2(M_i^4, \mathbf{Z})$ for $i = 1$ and 2, then using the natural isomorphism

$$H_2(\bar{M}_1^4 \# M_2^4) \simeq H_2(\bar{M}_1^4) + H_2(M_2^4),$$

we have $\xi = \bar{\xi}_1 + \xi_2$, where $\bar{\xi}_1$ corresponds through the change of orientation to $\xi_1$.

Since $\xi_i$ is dual to $w_2(M_i)$, for $i = 1$ and 2, it follows that $\xi$ is dual to

$$w_2(\bar{M}_1^4 \# M_2^4) = w_2(\bar{M}_1^4) + w_2(M_2^4).$$

It follows from Theorem 1 in [4] that, since $f$ is differentiable,

$$\frac{\xi \cdot \xi - \sigma(\bar{M}_1^4 \# M_2^4)}{8} \equiv 0 \mod 2.$$



Therefore

$$\frac{\xi_1 \cdot \xi_1 - \sigma(M_1^4)}{8} \equiv \frac{\xi_2 \cdot \xi_2 - \sigma(M_2^4)}{8} \mod 2,$$

and the lemma follows.

LEMMA 1.2. *$\varphi(k)$ is defined for all knot types $k$.*

Proof: To prove this we need to construct an admissible map for arbitrary $k$. Let $D^4$ be a 4-disc with boundary $S^3$. We assume $k$ is differentiably embedded in $S^3$. Let $U$ be a tubular neighborhood of $k$ in $S^3$, then there is a diffeomorphism $\varphi:S^1 \times D^2 \to S^3$ such that $\varphi(S^1 \times D^2) = U$ and $\varphi(S^1 \times o) = k$; $o$ is the center of the standard 2-disc $D^2$. Considering $S^1 \times D^2 = (bD^2) \times D^2$, let $M_0^4$ be the differentiable manifold obtained from the disjoint union $D^4 \cup (D^2 \times D^2)$ by identifying $U$ and $S^1 \times D^2$ by means of $\varphi$ and then "rounding" the "corners". Let $f:S^2 \to M_0^4$ be the embedding such that

$$f(S^2) = Ck \cup (o \times D^2) \subset M_0^4.$$

Now there exists a parallelizable, simply connected 4-manifold $V$ such that $bV$ is diffeomorphic to $bM_0^4$. This follows by observing that $bM_0^4$ embedded in $R^{3+k}$ has an orthonormal $k$-frame for $k$ sufficiently large. By the subjectivity of the Whitehead $J$-homomorphism $J:\pi_3(SO(n)) \to \pi_{n+3}(S^n)$; and by picking the frame properly on $bM_0^4$ one obtains a parallelizable 4-manifold with $bM_0^4$ as boundary. The method of framed spherical modifications (see [5]) gives the simply connected manifold $V$.

Forming the union of $V$ and $M_0^4$ and identifying their boundaries, we obtain a simply connected, closed, oriented, 4-manifold $M^4$. Furthermore, the map $f$ gives an embedding $f':S^2 \to M^4$. Let $\xi \in H_2(M^4, \mathbf{Z})$ be represented by $f'$. By picking the map properly one can make $\xi \cdot \xi$ take on the value $+1$, in which case $bM_0^4$ will be a homology sphere. We assume $\varphi$ was so chosen.

Now by Wu's formula, $\xi$ will be dual to $w_2(M^4)$ if and only if $\xi' \cdot x = x \cdot x$ for all classes $x \in H_2(M^4, \mathbf{Z}_2)$. $\mathbf{Z}_2$ represents the integers modulo 2 and $\xi'$ is obtained from $\xi$ by reducing coefficients modulo 2. Since $bM_0^4$ is a homology sphere, there is a natural isomorphism

$$H_2(M^4, \mathbf{Z}_2) \simeq H_2(M_0^4, \mathbf{Z}_2) + H_2(V, \mathbf{Z}_2).$$

Since $V$ is parallelizable, $y \cdot y = 0$ for $y$ in $H_2(V, \mathbf{Z}_2)$ and hence $\xi$ is dual to $w_2(M^4)$ if and only if $\xi' \cdot x = x \cdot x$ for all classes $x$ in $H_2(M_0^4, \mathbf{Z}_2)$. Since the class $\xi$ represented by $f'$ generates $H_2(M_0^4, \mathbf{Z})$, it follows that $\xi$ is dual to $w_2(M^4)$. Therefore $f'$ is admissible for $k$, and the lemma follows.

Let $k_1 \# k_2$ denote the sum of two types $k_1$ and $k_2$ as defined in [3], then we have

LEMMA 1.3. $\varphi(k_1 \# k_2) = \varphi(k_1) + \varphi(k_2)$.

Proof: Let $f_i:S^2 \to M_i$ be admissible for $k_i$ for $i = 1, 2$. Let $D_1^4$ and $D_2^4$ be the special 4-discs containing the singularities of $f_1$ and $f_2$, respectively, and such that $k_i \subset bD_i$, $i = 1, 2$. We form the connected sum $M_1^4 \# M_2^4$ in such a way that $D_1^4 \cup D_2^4$ becomes a 4-disc $D^4$ in $M_1^4 \# M_2^4$, $f_1(S^2) \# f_2(S^2)$ is embedded in $M_1^4 \# M_2^4$ and $f_1(S^2) \# f_2(S^2)$ intersects $bD^4$ orthogonally and in the knot type $k_1 \# k_2$. We obtain an embedded 2-sphere by embedding the cone over $k_1 \# k_2$ in $D^4$ with vertex at the center of $D^4$ and taking the union of it with the part of $f_1(S^2) \# f_2(S^2)$ outside $D^4$. This gives an embedding $f: S^2 \to M^4$ which has one singularity at the vertex of the cone over $k_1 \# k_2$. If $f$ represents $\xi \in H_2(M_1^4 \# M_2^4, \mathbf{Z})$ and $f_i$ represents $\xi_i \in H_2(M_i, \mathbf{Z})$, it follows easily that $f$ is admissible for $k_1 \# k_2$ and $\xi = \xi_1 + \xi_2$. Since $\sigma(M_1^4 \# M_2^4) = \sigma(M_1^4) + \sigma(M_2^4)$, the lemma follows.

LEMMA 1.4. *If $k$ is the trivial knot, then $\varphi(k) = 0$.*

Proof: Compute $\varphi(k)$ from the trivial embedding $f: S^2 \to S^4$.

From the three lemmas and the definition of the group of knot cobordism classes we have

THEOREM 1. *The knot cobordism invariant $\varphi$ defines a homomorphism*

$$\varphi: C_1 \to \mathbf{Z}_2 ,$$

*where $C_1$ is the group of knot cobordism classes.*

LEMMA 1.5. *Let $3_1$ denote the right-hand trefoil knot* (see knot table in [7]), *then* $\varphi(3_1) = 1$.

Proof: Let $PC(2)$ be the complex projective plane and $\gamma$ the generator of $H_2(PC(2), \mathbf{Z})$, then in the proof of Theorem 2 in [2] it is shown that $3 \cdot \gamma$ is represented by an embedding $f: S^2 \to PC(2)$. This embedding is admissible for $3_1$. Thus

$$\varphi(3_1) \equiv \frac{9 - 1}{8} \mod 2 .$$

LEMMA 1.6. *If $k^*$ is the reflected inverse of $k$* (see [3], page 142), *then* $\varphi(k) = \varphi(k^*)$.

Proof: The cobordism class of $k^*$ is the inverse of the cobordism class of $k$ (see [3], page 142). The lemma then follows from Theorem 1.

## 2. Proper Links

Let $L = k_1 \cup k_2 \cup \cdots \cup k_n$ be a link in $S^3$, where $k_i$ are the component knots and are oriented. We say that $L$ is a *proper link* if the sum of the linking numbers between $k_i$ and the rest of the component knots is an even integer for each choice of $i = 1, \cdots, n$.

Let $D_1^4$ and $D_2^4$ be two concentric 4-discs in $R^4$, where $D_1^4 \subset D_2^4$. Consider a knot $k \subset bD_1^4$ and a link $L \subset bD_2^4$. Suppose there exists an oriented 2-manifold $N^2$ of genus zero differentiably and regularly embedded between $bD_1^4$ and $bD_2^4$ such that $N^2 \cap bD_1^4 = k$ and $L = N^2 \cap bD_2^4$. Assume also that $N^2$ meets $bD_1^4$ and $bD_2^4$ orthogonally. In this case we shall say that $k$ is *related* to the link $L$. We then have

THEOREM 2. *If $k$ and $k'$ are two knots related to the same link $L$, and if further $L$ is a proper link, then $\varphi(k) = \varphi(k')$.*

Proof: Let $L = k_1 \cup k_2 \cup \cdots \cup k_n$ be differentiably embedded in the boundary $S^3$ of the standard 4-disc $D^4$. For $i = 1, \cdots, n$, let $U_i$ be a tubular neighborhood of $k_i \subset S^3$ such that $U_i \cap U_j = \varnothing$ for $i \neq j$. There exist diffeomorphisms

$$\varphi_i: S^1 \times D^2 \to S^3$$

such that $\varphi_i(S^1 \times D^2) = U_i$ and $\varphi_i(S^1 \times o) = k_i$, where $i = 1, \cdots, n$ and $o$ is the center of the standard 2-disc $D^2$. Let $H_i$, $i = 1, \cdots, n$, be $n$ copies of $D^2 \times D^2$. We embed $S^1 \times D^2$ in $H_i$ by considering $S^1 \times D^2 = (bD^2) \times D^2$. We form the simply connected differentiable manifold $M_0^4$ by taking the disjoint union $D^4 \cup H_1 \cup \cdots \cup H_n$ and identifying $U_i$ in $D^4$ with $S^1 \times D^2$ in $H_i$ by means of the map $\varphi_i$, this being done for each $i = 1, \cdots, n$.

Again there exists a simply connected, parallelizable 4-manifold $V$ with $bV$ diffeomorphic to $bM_0^4$. Taking the union of $V$ and $M_0^4$ and identifying boundaries, we obtain a simply connected, closed, oriented 4-manifold $M^4$.

Considering the union of the cone over $k_i \subset S^3$ with vertex at the center of $D^4$ and $o \times D^2 \subset H_i$ for each $i = 1, \cdots, n$, we get combinatorial embeddings

$$f_i: S^2 \to M^4 \tag{2.1}$$

such that each $f_i$ is regular except for one singularity due to the vertex of the cones at the center of $D^4$. Let $f_i$ represent the homology class $\xi_i$ in $H_2(M^4, \mathbf{Z})$, and let $\xi = \xi_1 + \cdots + \xi_n$. Now for $i \neq j$

$$\xi_i \cdot \xi_j = L(k_i, k_j) , \tag{2.2}$$

where $L(k_i, k_j)$ = linking number in $S^3$. By choosing the maps $\varphi_i$ properly, the self-intersection numbers $\xi_i \cdot \xi_i$, $i = 1, \cdots, n$, can be made to take on independently any integer values. It follows that the maps $\varphi_i$ can be chosen to make the matrix of intersection numbers $\|\xi_i \cdot \xi_j\|$ a unimodular matrix. This in turn implies, by using Poincaré duality, that the manifold $bM_0^4$ will be a homology 3-sphere.

Assuming that the $\varphi_i$ are so chosen, we have then a direct sum decomposition

$$H_2(M^4) \simeq H_2(M_0^4) + H_2(V) , \tag{2.3}$$

the coefficients being $\mathbf{Z}$ or $\mathbf{Z}_2$.

We claim now that the class $\xi$ is dual to $w_2(M^4)$. To show this let $\xi_i'$ and $\xi'$ be the classes in $H_2(M^4, \mathbf{Z}_2)$ corresponding to $\xi_i$ and $\xi$, respectively, for $i = 1, \cdots, n$. $\xi$ will be dual to $w_2(M^4)$ if

$$\xi' \cdot x = x \cdot x \tag{2.4}$$

$H_2(M_0^4, \mathbf{Z}_2)$, and that $V$ is parallelizable, (2.4) will follow if

(2.5) $$\xi' \cdot \xi_i' = \xi_i' \cdot \xi_i'$$

for $i = 1, \cdots, n$. But by (2.2),

(2.6) $$\xi' \cdot \xi_i' \equiv \xi_i' \cdot \xi_i' + L(k_i, L - k_i) \mod 2 ,$$

for $i = 1, \cdots, n$. And since $L$ is a proper link,

$$L(k_i, L - k_i) \equiv 0 \mod 2 ,$$

for $i = 1, \cdots, n$. Hence $\xi$ is dual to $w_2(M^4)$.

The theorem follows if we can construct two maps $f: S^2 \to M^4$ and $f': S^2 \to M^4$ which are admissible for $k$ and $k'$, respectively, and both represent the class $\xi$. This is done as follows:

Let $D_1^4$ be a 4-disc contained in and concentric to $D_2^4$. Since $k$ is related to $L$, there is a surface $N^2$ with properties listed at the beginning of this paragraph.

Let $A$ be the union of all the $o \times D_2 \subset H_i$, $i = 1, \cdots, n$, and $Ck$ the cone over $k \subset bD_1^4$ with vertex at the center of $D_1^4$, then $Ck \cup N^2 \cup A$ gives a combinatorially embedded 2-sphere in $M^4$. This yields the map $f: S^2 \to M^4$ which clearly represents $\xi$ and is admissible for $k$. Similarly we get the map $f': S^2 \to M^4$.

## 3. Calculation of $\varphi(k)$ from Seifert Surfaces

In this section we follow closely the notation of [3] on Seifert surfaces.

Let $F$ be a Seifert surface of genus $h$ of a knot $k$ in $R^3$, and let $a_1, \cdots, a_{2h}$ be a canonical set of closed, oriented curves on $F$. This defines a projection of $F$ on a plane in $R^3$ as in [3], page 151, where $F$ is considered as a 2-disc with "bands" $B_1, \cdots, B_{2h}$ attached. The band $B_i$ has the curve $a_i$ running along it for $i = 1, \cdots, 2h$. We say the band $B_i$ has a given over-crossing (under-crossing) with $B_j$ if, in the projection of $F$, $a_i$ goes over (under) $a_j$.

Suppose $F$ and $F'$ are Seifert surfaces of $k$ and $k'$, respectively. Let $F'$ be obtained from $F$ by taking the standard projection of $F$ and replacing a given over-crossing of $B_i$ with $B_j$ by an under-crossing of $B_i$ with $B_j$. See for example Figure 1.

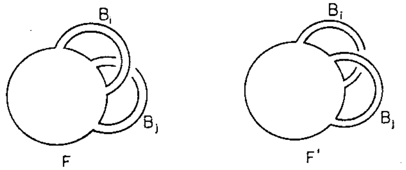


Figure 1

LEMMA 3.1. *If $F$ and $F'$ are related as above, then*

$$\varphi(k) = \varphi(k') .$$

Proof: The proof consists of finding a proper link $L$ to which $k$ and $k'$ are both related. To this end attach to the band $B_j$ of $F$ a 2-disc $e$ on each side of the over-crossing of $B_i$ with $B_j$ as in Figure 2. The boundary of the resulting

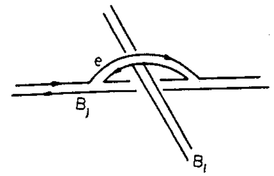


Figure 2

surface is an oriented link $L$ of multiplicity 2. $L$ is clearly a proper link. It follows also from the discussion in [3], page 134, that $k$ and $k'$ are both related to $L$. The lemma now follows by Theorem 2.

To a Seifert surface $F$ of genus $2h$ and a canonical set of curves $a_i$ of $F$ there is associated a $2h \times 2h$ integral matrix $V = (v_{ij})$ as defined in [3], page 152.

THEOREM 3. *Let $F$ be a Seifert surface of genus $2h$ with canonical set of curves $a_i$ and with boundary knot $k$. Let $V = (v_{ij})$ be the associated integral matrix, then*

$$\varphi(k) \equiv \sum_{i=1}^{h} v_{2i-1,2i-1} v_{2i,2i} \mod 2 .$$

Proof: By successively changing over-crossing bands to under-crossing bands as described before Lemma 3.1, we obtain a Seifert surface $F'$ and a canonical set of curves $a_i'$ such that if $V' = (v_{ij}')$ is the associated integral matrix, then

(3.1) $$v_{ii}' = \begin{cases} 1 & \text{for } v_{ii} \text{ odd} , \\ 0 & \text{for } v_{ii} \text{ even} , \end{cases}$$

for $i = 1, \cdots, 2h$,

(3.2) $$v_{2j-1,2j}' = 1$$

for $j = 1, \ldots, h$, and $v_{ij}' = 0$ in all other cases. Furthermore, we may assume that if $k'$ is the boundary of $F'$, then $k' = k_1 \# k_2 \# \cdots \# k_h$, where $k_i$ is a trivial knot if $v_{2i-1,2i-1}' \cdot v_{2i,2i}' = 0$, and $k_i$ is a trefoil knot if $v_{2i-1,2i-1}' \cdot v_{2i,2i}' = 1$, for $i = 1, \cdots, h$. Therefore by Lemma 1.5 and Theorem 1,

$$\varphi(k') \equiv \sum_{i=1}^{h} v_{2i-1,2i-1}' \cdot v_{2i,2i}' \mod 2 ,$$

and hence by (3.1),

$$\varphi(k') \equiv \sum_{i=1}^{h} v_{2i-1,2i-1} \cdot v_{2i,2i} \mod 2 .$$

But by Lemma 3.1, $\varphi(k) = \varphi(k')$ and hence the theorem follows.

## 4. A Quadratic Form Modulo 2 of a Seifert Surface

Let $F$ be a Seifert surface of genus $h$ of the knot $k$ in $S^3$ and let $a_1, \cdots, a_{2h}$ be a canonical set of curves on $F$. Let $V = (v_{ij})$ be the integral matrix associated with $a_i$ and $F$.

We define a quadratic form mod 2 over the vector space $H_1(F, \mathbf{Z}_2)$ as follows:

Let $x \in H_1(F, \mathbf{Z}_2)$, then $x = \Sigma x_i a_i$, where $x_i \in \mathbf{Z}_2$ and $a_i$ is considered as a homology class mod 2. Define

(4.1) $$Q(x) \equiv \sum_{i=1}^{2h} \sum_{j=1}^{2h} x_i v_{ij} x_j \mod 2 .$$

Theorem 4. *$Q$ is a quadratic form modulo 2, that is*

(4.2) $$Q(x+y) = Q(x) + Q(y) + x \cdot y$$

*for all $x, y$ in $H_1(F, \mathbf{Z}_2)$, where $x \cdot y$ means intersection number.*

Proof: Let $x = \Sigma x_i a_i$ and $y = \Sigma y_i a_i$, then by (4.1),

(4.3) $$Q(x+y) \equiv Q(x) + Q(y) + \sum_{i,j} x_i(v_{ij} + v_{ji})y_i \mod 2 .$$

But for $i = 1, \cdots, 2h$ and $j = 1, \cdots, 2h$,

(4.4) $$v_{ij} + v_{ji} \equiv a_i \cdot a_j \mod 2$$

(see [3], page 152). Substituting (4.4) into (4.3) gives (4.2).

There is a connection between the quadratic form $Q$ of $F$ and $\varphi(k)$, where $k$ is the boundary knot of $F$, which is given as follows:

Theorem 5. *Let $Q$ be the quadratic form associated with the Seifert surface $F$ and canonical basis $a_i$, then*

$$\alpha(Q) = \varphi(k) ,$$

*where $\alpha(Q)$ is the Arf invariant of $Q$* (see [1]), *and $k$ is the boundary of $F$.*

Proof: From (4.1) and the definition of $\alpha(Q)$ it follows that

$$\alpha(Q) \equiv \sum v_{2i-1,2i-1} v_{2i,2i} \mod 2 .$$

The theorem then follows from Theorem 3.

## 5. Computation of $\varphi(k)$ from the Alexander Polynomial

Let $\Delta(t)$ denote the Alexander polynomial of the knot $k$. We assume that $\Delta(t)$ is normalized, that is $\Delta(t)$ has only positive powers of $t$. Then $\Delta(t)$ is in $\mathbf{Z}[t]$ and is determined up to a multiple $\pm t^n$, where $n$ is a positive integer. Considering $(2, (1+t)^4)$ as an ideal in $\mathbf{Z}[t]$ we have

Theorem 6. *If $\Delta(t)$ is the normalized Alexander polynomial of the knot $k$, then*

(a) $\varphi(k) = 0$ *if and only if* $\Delta(t) \equiv t^n \mod (2, (1+t)^4)$,

*and*

(b) $\varphi(k) = 1$ *if and only if* $\Delta(t) \equiv t^n(1 + t + t^2) \mod (2, (1+t)^4)$,

*where $n$ is some non-negative integer.*

Proof: Let $F$ be a Seifert surface for the knot $k$, and $V = (v_{ij})$ the matrix of crossing numbers associated with any canonical basis for $F$. It is known that

(5.1) $$\Delta(t) = \pm t^n \det A(t) ,$$

where

(5.2) $$A(t) = I + (1-t)V ,$$

and

$$I = \underbrace{\begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & \cdot & 0 & & \\ & 0 & \cdot & & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{bmatrix}}_{2h} \Bigg\} 2h ,$$

$\Delta(t)$ being the normalized Alexander polynomial of $k$ (see [3], page 154).

We pick now a canonical basis $a_i$ for $F$ such that, if $V$ is the resulting matrix of crossing numbers, then

(5.3) $$v_{2i-1,2i} \equiv 1 \mod 2 , \qquad v_{2i,2i-1} \equiv 0 \mod 2$$

for $i = 1, \cdots, h$. This can be done since in any event $v_{2i-1,2i} - v_{2i,2i-1} = 1$ for $i = 1, \cdots, h$. Let $A(t) = (a_{ij}(t))$, then from (5.2)

(5.4) $$a_{2i,2i-1}(t) \equiv 1 \mod 2 ,$$

(5.5) $$a_{2i-1,2i}(t) \equiv t \mod 2$$

for $i = 1, \cdots, h$, and

(5.6) $$a_{ij}(t) \equiv (1+t)v_{ij} \mod 2$$

in all other cases.

We want now to evaluate det $A(t)$, calculating modulo 2. To this end let $S_{2h}$ be the group of permutations of $1, \cdots, 2h$. For $\alpha = 0, \cdots, h$ and $\beta = 0, \cdots, h$ let $S_{\alpha,\beta}$ be the subset of $S_{2h}$ consisting of permutations $\phi$ such that

(a) $\phi(2i) = 2i - 1$ has exactly $\alpha$ solutions for $i = 1, \cdots, h$,

and

(b) $\phi(2i - 1) = 2i$ has exactly $\beta$ solutions for $i = 1, \cdots, h$.

From the definition of $S_{\alpha,\beta}$ it follows that $S_{h,h-1}$ and $S_{h-1,h}$ are empty. Also if $\phi \in S_{\alpha,\beta}$, then $\phi^{-1} \in S_{\beta,\alpha}$ for $\alpha = 0, \cdots, h$ and $\beta = 0, \cdots, h$.

Now calculating modulo 2,

(5.7) $$\det A(t) \equiv \sum_{\alpha,\beta=0}^{h} \sum_{\phi \in S_{\alpha,\beta}} a_{1,\phi(1)} \cdots a_{2h,\phi(2h)} \mod 2,$$

where $a_{ij}$ means $a_{ij}(t)$. We define a matrix $B = (b_{ij})$ by

(5.8) $$b_{2i,2i-1} = 1, \qquad b_{2i-1,2i} = 1, \qquad i = 1, \cdots, h,$$

and by

(5.9) $$b_{ij} = v_{ij} = v_{ji} = b_{ji}$$

in all other cases. Let

(5.10) $$\epsilon_{\alpha,\beta} \equiv \sum_{\phi \in S_{\alpha,\beta}} b_{1,\phi(1)} \cdots b_{2h,\phi(2h)} \mod 2$$

for $\alpha = 0, \cdots, h$ and $\beta = 0, \cdots, h$, then it follows from (5.4) through (5.10) that

(5.11) $$\det A(t) \equiv \sum_{\alpha,\beta=0}^{h} \epsilon_{\alpha,\beta}(1 + t)^{2h-\alpha-\beta} \cdot t^{\beta} \mod 2.$$

Since $\phi \in S_{\alpha,\beta}$ implies $\phi^{-1} \in S_{\beta,\alpha}$, and since $B$ is a symmetric matrix, it follows that

(5.12) $$\epsilon_{\alpha,\beta} \equiv \epsilon_{\beta,\alpha} \mod 2.$$

Let $\phi_{ij}$ be the permutation

(5.13)
$$(1, 2)(3, 4) \cdots \widehat{(2i - 1, 2i)} \cdots \widehat{(2j - 1, 2j)} \cdots (2h - 1, 2h)(2i, 2i - 1, 2j, 2j - 1),$$

where $i = 1, \cdots, h, j = 1, \cdots, h, i < j$, and $\widehat{\ }$ means the given cycle is missing. Let

$$\phi_{ii} = (1, 2)(3, 4) \cdots \widehat{(2i - 1, 2i)} \cdots (2h - 1, 2h), \quad i = 1, \cdots, h,$$

then $S_{h-1,h-1}$ consists of the permutations $\phi_{ij}$, $\phi_{ij}^{-1}$, and $\phi_{11}, \cdots, \phi_{hh}$, where $i = 1, \cdots, h, j = 1, \cdots, h$ and $i < j$. By the symmetry of $B$, it follows that

(5.14) $$\sum_{i<j} b_{1,\phi_{ij}(1)} \cdots b_{2h,\phi_{ij}(2h)} \equiv \sum_{i<j} b_{1,\phi_{ij}^{-1}(1)} \cdots b_{2h,\phi_{ij}^{-1}(2h)} \mod 2;$$

(5.15) $$\epsilon_{h-1,h-1} \equiv \sum_{i=1}^{h} v_{2i,2i} \cdot v_{2i-1,2i-1} \mod 2.$$

It is clear that $\epsilon_{h,h} \equiv 1 \mod 2$ and $\epsilon_{h-1,h} = \epsilon_{h,h-1} \equiv 0$; thus by (5.7), (5.11), and (5.12),

(5.16) $$\det A(t) \equiv t^h + \epsilon_{h-1,h-1}(1 + t)^2 t^{h-1} + \epsilon_{h,h-2}(1 + t)^2(1 + t^2)t^{h-2} + (1 + t)^4 P(t) \mod 2,$$

where $P(t)$ is some polynomial in $t$. Hence by Theorem 3 and (5.15),

(5.17) $$\det A(t) \equiv t^h + \varphi(k) \cdot (1 + t)^2 t^{h-1} \mod (2, (1 + t)^4).$$

Thus if $\varphi(k) = 0$, we have det $A(t) \equiv t^h \mod (2, (1 + t)^4)$, and if $\varphi(k) = 1$, then det $A(t) \equiv t^{h-1}(1 + t + t^2) \mod (2, (1 + t)^4)$. Since it is impossible that

$$t^a \equiv t^b(1 + t + t^2) \mod (2, (1 + t)^4)$$

for any choice of non-negative integers $a$ and $b$, it follows from (5.1) that $\Delta(t)$ must be congruent to either $t^a$ or $t^b(1 + t + t^2) \mod (2, (1 + t)^4)$ but not both. This proves the theorem.

THEOREM 7. *Let $\Delta(t) = c_0 + c_1 t + \cdots + c_n t^n + \cdots + c_0 t^{2n}$ be the Alexander polynomial of the knot $k$, then*

$$\varphi(k) \equiv c_{n-1} + c_{n-3} + \cdots + c_r \mod 2,$$

*where $r = 0$ for $n$ odd, $r = 1$ for $n$ even.*

Proof: The proof consists in calculating modulo the ideal $(2, (1 + t)^4)$ using the fact that $t^4 \equiv 1 \mod (2, (1 + t)^4)$. Since $\Delta(1) = 1$, it follows that $c_n \equiv 1 \mod 2$ and thus

(5.18) $$\Delta(t) \equiv t^n + \sum_{i=0}^{n-1} c_i(t^i + t^{2n-i}) \mod (2, (1 + t)^4).$$

But

(5.19) $$t^i + t^{2n-i} \equiv 0 \mod (2, (1 + t)^4)$$

for $n$ even and $i$ even, or for $n$ odd and $i$ odd. On the other hand,

(5.20) $$t^i + t^{2n-i} \equiv t + t^3 \mod (2, (1 + t)^4),$$

for $n$ even and $i = 1, 3, \cdots, n - 1$. Also

(5.21) $$t^i + t^{2n-i} \equiv 1 + t^2 \mod (2, (1 + t)^4),$$

for $n$ odd and $i = 2, 4, \cdots, n - 1$. Thus by (5.18) through (5.21),

(5.22) $$\Delta(t) \equiv \begin{cases} t^n + c(t + t^3) & \text{for } n \text{ even}, \\ t^n + c'(1 + t^2) & \text{for } n \text{ odd}, \end{cases}$$

modulo the ideal $(2, (1+t)^4)$, where $c = c_{n-1} + c_{n-3} + \cdots + c_1$ and $c' = c_{n-1} + c_{n-3} + \cdots + c_0$. It follows easily from (5.22) and Theorem 6 that $\varphi(k)$ is equal to $c$ or $c'$ reduced modulo 2.

## 6. The δ-Invariant of Pontryagin

In this section we follow closely the notation of [6], §15. Consider the Seifert surface $F \subset R^3 \times o \subset R^3 \times R^2$, where $o$ is the origin of $R^2$. $F$, being oriented, has a natural orthonormal 1-frame of vectors in $R^3 \times o$. The suspension of this frame gives an orthonormal 3-frame $U$ on $F$ in $R^5$. Let $k$ be the boundary knot. There exists a differentiably embedded 2-disc $\Delta \subset R^5$ such that $b\Delta = k$ and $\Delta$ meets $R^3 \times o$ tangentially along $k$. Let $M^2 = F \cup \Delta$, then $M^2$ is a closed 2-manifold embedded in $R^5$ of genus equal to that of $F$. The frame $U$ on $F$ can be extended to an orthonormal 3-frame $W$ on $M^2$ in $R^5$ since the obstruction to extending $U$ is $w_2(M^2) = 0$. If we apply the Thom-Pontryagin construction to $(M^2, W)$, we obtain a homotopy class $\theta(F) \in \pi_5(S^3)$. We shall prove that $\theta$ defines a homomorphism $\theta: C_1 \to \pi_5(S^3)$ given by $\theta(\alpha) = \theta(F)$, where $F$ is a Seifert surface of any knot $k$ in the cobordism class $\alpha$. To this end we have

LEMMA 6.1. *$\delta(M^2, W) = \varphi(k)$, where $(M^2, W)$ is obtained from a Seifert surface $F$ as above, and $\delta(M^2, W)$ is the Pontryagin invariant defined in* [6], §15.

Proof: The computation of $\delta(M^2, W)$ requires a canonical basis for $H_1(M^2, \mathbf{Z}_2)$. Clearly a canonical set of curves $a_i$, $i = 1, \cdots, 2h$, on $F$ gives then a canonical basis $a_i$ for $H_1(M^2, \mathbf{Z}_2)$. Let $V = (v_{ij})$ be the matrix of crossing numbers corresponding to the curves $a_i$. It follows from the definition of $\delta(a_i)$ and $v_{ii}$ that

$$\delta(a_i) \equiv v_{ii} \mod 2$$

for $i = 1, \cdots, 2h$. But

$$\delta(M^2, W) \equiv \sum_{j=1}^{h} \delta(a_{2j-1}) \cdot \delta(a_{2j}) \mod 2\,,$$

and thus by Theorem 3

$$\delta(M^2, W) = \varphi(k)\,.$$

Let $\psi: \pi_5(S^3) \to \mathbf{Z}_2$ be the isomorphism obtained from the Thom-Pontryagin construction and $\delta(M^2, W)$ (see [6]), then

THEOREM 8. *The map $\theta: C_1 \to \pi_5(S^3)$ is a homomorphism and*

$$\begin{array}{ccc} C_1 & \xrightarrow{\theta} & \pi_5(S^3) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathbf{Z}_2 & \end{array}$$

*is commutative.*

Proof: We must show that $\theta$ is well defined. Let $k_1$ and $k_2$ be cobordant knots, and $(M_1, W_1)$ and $(M_2, W_2)$ be framed 2-manifolds constructed from any Seifert surfaces of $k_1$ and $k_2$, respectively. By Lemma 6.1, $\delta(M_1, W_1) = \delta(M_2, W_2)$ since $\varphi(k_1) = \varphi(k_2)$; therefore $(M_1, W_1)$ and $(M_2, W_2)$ are homologous (see [6], Theorem 24). It follows that the resulting homotopy class in $\pi_5(S^3)$ is the same for $k_1$ and $k_2$ independent of the choices of the Seifert surfaces.

To see that $\theta$ is a homomorphism let $k_1$ and $k_2$ have Seifert surfaces $F_1$ and $F_2$, respectively. We then have a Seifert surface for $k_1 \# k_2$ by attaching $F_1$ to $F_2$ in the obvious manner. Let $(M_i, W_i)$ be constructed from $F_i$, $i = 1, 2$, and $(M, W)$ from $F$, where we may take $M_1$ and $M_2$ as disjoint in $R^5$. Clearly $(M, W)$ is homologous to $(M_1 \cup M_2, V)$, where the frame $V$ is obtained from $W_1$ and $W_2$. It follows from this that $\theta$ is a homomorphism.

The commutativity of the diagram follows from the definition of $\psi$ and Lemma 6.1.

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