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Index Theory, Coarse Geometry, and Topology of Manifolds

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Preface

These are the lecture notes from the conference 'Index Theory, Coarse Geometry, and Topology of Manifolds' which was held in Boulder in August 1995. I have adhered fairly closely to the original scheme of the lectures, although the notes naturally contain rather more detail and in a few places (especially in lecture 7) describe constructions that were finalized only after the lectures were delivered.

It is of the nature of this subject to require a rather diverse background. I attempted to deal with this in the lectures by including a number of mini-surveys, and in this book most of lectures 1 and 6 and large parts of lectures 2 and 3 are devoted to such surveys. I hope that these will be useful.

The work reported on here has developed over a dozen years or so and during that time I have had the pleasure of learning from a great number of colleagues. Among them I would mention Alain Connes, Misha Gromov, Steve Hurder, Jonathan Rosenberg, Stephan Stolz, and Guoliang Yu. More recently I have benefited from the advice of Steve Ferry, Erik Pedersen, Andrew Ranicki, and Shmuel Weinberger, all of whom have (among other things) answered the numerous questions of a surgery neophyte with exemplary patience. Finally, a major debt of gratitude is owed to Nigel Higson, with whom I have collaborated on many of the projects that are summarized in this book.

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John Roe, October 1995

CHAPTER 1

Index Theory

The subject of this lecture series interrelates a number of different areas of mathematics, but in my mind it began with index theory, so that is where I will start.

Let M be a compact (smooth) manifold. Then we have the de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M),$$

in which d is the exterior derivative, and because $d^2 = 0$ we can form its cohomology, the *de Rham cohomology* of M. Notice that the vector spaces $\Omega^i(M)$ are infinite-dimensional, but we know that the de Rham cohomology is isomorphic to the ordinary cohomology, and is therefore finite-dimensional. When formulated in terms of de Rham cohomology this is a result about the solution spaces of certain partial differential equations, and we might look for a proof of it in terms of analysis.

Analysis of Dirac operators

Let M be any manifold (compact or not), and let $\Omega^i_c(M)$ denote the space of *compactly supported* i-forms on M. The first thing we need to do is to complete the spaces $\Omega^i_c(M)$ of *smooth* differential forms to Hilbert spaces, which are much more tractable from the perspective of functional analysis. Let us choose a Riemannian metric on M. This metric defines a positive measure μ on M, and it also gives rise to a Hermitian inner product on the cotangent bundle of M, and hence on all its associated exterior powers. We may therefore define an inner product $\langle \cdot, \cdot \rangle$, called the L^2 inner product, on each of the spaces $\Omega^i_c(M)$ by integrating the local inner products $(\cdot, \cdot)$ with respect to the measure μ:

$$\langle \alpha, \beta \rangle = \int_M (\alpha(x), \beta(x))\, d\mu(x).$$

By completing the spaces $\Omega^i_c(M)$ in this inner product, we obtain Hilbert spaces $\Omega^i_{L^2}(M)$ of *square integrable* forms.

The de Rham complex now becomes a complex of Hilbert spaces and unbounded operators. We recall that an *unbounded operator* T on a Hilbert space

H is a linear map from a dense subspace $\mathrm{Dom}(T) \subseteq H$, the *domain* of T, to H. In the case of the de Rham operator d we let $H = \Omega^*_{L^2}(M)$ and it is convenient to take the domain of d to be the space $\Omega^*_c(M)$ of smooth forms with compact support. One says that the unbounded operator T' is an *extension* of T (in symbols, $T \subseteq T'$) if $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T')$ and both operators agree on $\mathrm{Dom}(T)$.

Here are some important definitions from the theory of unbounded operators:

(i) The *adjoint* of T is the unbounded operator T^* defined by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ and with domain the largest for which this definition can make sense, that is, the domain of T^* is the set of all y for which $x \mapsto \langle Tx, y\rangle$ is a continuous linear map;

(ii) T is *symmetric* if $T \subseteq T^*$, and it is *self-adjoint* if $T = T^*$;

(iii) T is *essentially self-adjoint* if it is symmetric and has exactly one self-adjoint extension.

In the case of the de Rham complex, the operator d has a 'formal adjoint' d^*, which is the restriction of its Hilbert space adjoint (defined above) to $\Omega^*_c(M)$. The operator d^* is also a differential operator and it may be determined by integrating by parts: for instance, the formula

$$\int f\frac{dg}{dx}dx = -\int \frac{df}{dx}g dx$$

on $\mathbb{R}$ shows that the formal adjoint of d/dx, on $L^2(\mathbb{R})$, is $-d/dx$. We let $D = d + d^*$ be the symmetric operator obtained by adding d and d^*.

D is an example of a *generalized Dirac operator.* We do not need to know the details of this definition[1] ; suffice it to say that generalized Dirac operators are first order formally self-adjoint differential operators obtained from bundles of Clifford modules. We will however need some analytic information about such operators, beginning with

PROPOSITION 1.1: *A generalized Dirac operator D on a complete Riemannian manifold is essentially self-adjoint.*

See [**25**] or [**90**] for a proof. It is convenient to use the same letter D for the original operator and for its unique self-adjoint extension, and we will do this.

[1] But, for the sake of completeness, here is an outline. Let S be a vector bundle over a Riemannian manifold M. We say that S is a *Clifford bundle* if it is equipped with a linear map $\rho\colon TM \to \mathrm{End}(S)$ which obeys the *Clifford identity* $\rho(u)\cdot\rho(v) + \rho(v)\cdot\rho(u) = -\langle u, v\rangle 1$. There are natural notions of metric and of connection on S compatible with the Clifford action ρ. If S is equipped with such a metric and connection, a first order differential operator D on S is defined as the composite

$$C^\infty(S) \to C^\infty(T^*M \otimes S) \to C^\infty(TM \otimes S) \to C^\infty(S),$$

where the first arrow is given by the covariant derivative associated to the connection, the second arrow is given by the Riemannian metric, and the third arrow is given by the Clifford action. This operator D is the generalized Dirac operator associated to S.

Details may be found in [**90**] or in any of the other general references on index theory listed at the end of the chapter. In the case of the de Rham operator, S is the exterior algebra bundle of T^*M, and the Clifford action is the sum of interior and exterior multiplication.

Now we need a form of the spectral theorem[2].

THEOREM 1.2: (FUNCTIONAL CALCULUS) *Let T be an (unbounded) self-adjoint operator on a Hilbert space H. Then given any bounded Borel function f on $\mathbb{R}$ one can define a bounded operator $f(T)$ on H, such that the assignment $f \mapsto f(T)$ is a ring homomorphism, respects the involutions, and $f(T) = Tg(T)$ if $f(t) = tg(t)$. Moreover, $\|f(T)\| \leqslant \sup\{|f(t)| : t \in \mathbb{R}\}$.*

Recall that an operator $C\colon H \to H$ is called *compact* if it is a norm limit of finite rank operators. Such operators enjoy a particularly simple spectral theory; to any self-adjoint compact operator there is associated a decomposition of H into an infinite direct sum of finite-dimensional eigenspaces, with eigenvalues tending to zero.

Elliptic regularity theory says that a generalized Dirac operator has compact resolvent. More precisely

PROPOSITION 1.3: *Let D be a generalized Dirac operator on a compact manifold M. Let $f \in C_0(\mathbb{R})$, which means that f is a continuous function on $\mathbb{R}$ and tends to zero at infinity. Then $f(D)$ is compact.*

This implies that (on a compact manifold M) the operator D has discrete spectrum tending to infinity — one can see this by considering the spectral decomposition of the compact self-adjoint operator $(1+D^2)^{-1}$. If we let P_{λ_i} be the orthogonal projection onto the finite-dimensional eigenspace corresponding to the eigenvalue λ_i, then we can derive the convergent series expansion

$$f(D) = \sum_i f(\lambda_i) P_{\lambda_i} \tag{1.4}$$

for the operator $f(D)$ defined by functional calculus, $f \in C_0(\mathbb{R})$.

Proposition 1.3 does *not* remain true for non-compact complete M. Instead, what happens in that case is that the operator $f(D)$ is *locally compact.* To define this notion, notice that differential forms can be multiplied by functions; in fact, if φ is a bounded continuous function on M, then φ acts by multiplication as a bounded operator on the spaces of L^2 differential forms, and the norm of φ as an operator is dominated by the supremum of its absolute value. (A Hilbert space which is equipped with this sort of action of the continuous functions on M will be briefly referred to as an *M-module.*) By definition, an operator T is *locally compact* if φT and $T\varphi$ are compact whenever φ is compactly supported (or equivalently, by simple estimates, whenever $\varphi \in C_0(M)$).

EXAMPLE: Consider the self-adjoint operator $D = id/dx$ on $\mathbb{R}$ (which can in fact be regarded as a generalized Dirac operator). The functions $f(D)$ can in this case be described using Fourier analysis: $f(D)$ is just the operator which multiplies the Fourier transform $\hat{u}(\xi)$ of a function u by $f(\xi)$, in other words, it is

[2]The *spectrum* of a linear operator C on a Hilbert space H is the collection of $\lambda \in \mathbb{C}$ such that $C - \lambda 1$ fails to have a (bounded) inverse; it is a non-empty closed subset of $\mathbb{C}$. The term 'spectral theory' refers to the attempt to reconstitute C from its spectrum.

the operator of convolution by the inverse Fourier transform of f. In particular, suppose $f(\lambda) = e^{-\lambda^2/2}$. Then

$$f(D)u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2} u(y)\, dy,$$

in other words, $f(D)$ is a *smoothing operator* with Gaussian kernel. A smoothing operator with compactly supported kernel is a compact operator, so whenever $f(D)$ is cut down to a compact domain by multiplication by a compactly supported φ, we get a compact operator. That is, $f(D)$ is locally compact.

Let's confine our attention for the moment, however, to the case that M is a *compact* manifold. Then elliptic regularity has the following simple consequence: the complex $\Omega^*_{L^2}$ is the direct sum of two closed subcomplexes invariant under D, the complex $\mathcal{H}^* = \ker D$ of *harmonic* forms (on which the differential d is identically zero), and its orthogonal complement $\mathcal{H}^\perp$; moreover, the restriction of D to $\mathcal{H}^\perp$ is invertible, and this implies that $\mathcal{H}^\perp$ is an *acyclic* complex — it has trivial cohomology. Thus we get the *Hodge theorem* $H^i_{DR}(M) = \mathcal{H}^i(M)$. Since $\mathcal{H}^i(M)$, as an eigenspace of a generalized Dirac operator on a compact manifold, must be finite dimensional, this gives us the analytic explanation that we were seeking for the finite-dimensionality of de Rham cohomology.

Index theory

The notion of *index* is abstractly defined in functional analysis for all *Fredholm* operators: a Fredholm operator from one Hilbert space to another is a bounded operator which is invertible modulo the compact operators. Such an operator T has $\ker T$ and $\ker T^*$ finite-dimensional, and its index is, by definition, $\operatorname{Ind}(T) = \dim\ker T - \dim\ker T^*$. To relate this definition to the Dirac operators that we have been considering, let us introduce the notion of a *chopping function*.

DEFINITION 1.5: *A* chopping function *is a continuous odd function* $\chi\colon \mathbb{R} \to [-1,1]$ *such that* $\chi(t) \to \pm 1$ *as* $t \to \pm\infty$.

LEMMA 1.6: *Let D be a generalized Dirac operator on a compact manifold M. Then $\chi(D)$ is a Fredholm operator, and if χ_1 and χ_2 are two different chopping functions, then the corresponding Fredholm operators $\chi_1(D)$ and $\chi_2(D)$ differ by a compact operator.*

PROOF: Since $\chi^2 - 1$ and $\chi_1 - \chi_2$ are functions vanishing at infinity, the corresponding operators are compact by 1.3. □

Since $\chi(D)$ is self-adjoint, the index of $\chi(D)$ is zero. However, Dirac operators on even-dimensional manifolds frequently come equipped with a piece of extra structure called a *grading*: this is a self-adjoint involution[3] ε which anticommutes

[3] An *involution* is an operator whose square is 1.

with D, or, put differently, it is a decomposition $H = H_0 \oplus H_1$ of the domain, with respect to which one has matrix representations

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}, \qquad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The archetypal example of such a grading, for the de Rham operator $D = d + d^*$, is furnished by the decomposition $\Omega^* = \Omega^{\text{even}} \oplus \Omega^{\text{odd}}$ into forms of even degree and forms of odd degree. If such a grading is given, then $\chi(D)$ decomposes as

$$\chi(D) = \begin{bmatrix} 0 & \chi(D)_- \\ \chi(D)_+ & 0 \end{bmatrix},$$

and $\chi(D)_+$ is a Fredholm operator from H_0 to H_1, which can have a non-zero index, denoted $\operatorname{Ind}(D, \varepsilon)$ or simply $\operatorname{Ind}(D)$; it is independent of the choice of χ by the second part of lemma 1.6. In the example of the even/odd grading on differential forms, this index is simply the Euler characteristic. One therefore calls $D = d + d^*$ equipped with this grading the *Euler characteristic operator.*

More important for our purposes, however, will be two other examples: the *signature operator* and the (*spinor*) *Dirac operator.* These are graded operators defined on certain even-dimensional manifolds[4]. As an operator, the signature operator is simply the same $D = d + d^*$ that we have discussed before, but the grading is different. To define it, we need to assume that M is oriented; then the Riemannian metric and orientation on M give an identification of $\Omega^k(M)$ with $\Omega^{n-k}(M)$, $n = \dim M$ which can be expressed by a (bounded) operator $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ such that $*^2 = \pm 1$; $*$ is called the *Hodge star* operator. The Hodge star operator anticommutes with D; in particular, $*$ maps harmonic forms to harmonic forms, and (when we identify harmonic forms with cohomology) it can be thought of as an analytical implementation of Poincaré duality. It follows that by multiplying $*$ by appropriate powers of $\sqrt{-1}$ one can produce a new grading operator ε for D; equipped with this grading, D is called the signature operator. The index of the signature operator is equal to the *signature* of M as defined in differential topology [**76**], that is zero if the dimension is not congruent to 0 modulo 4, and, if the dimension is congruent to 0 modulo 4, the signature of the symmetric bilinear form defined by Poincaré duality on the middle-dimensional cohomology. This is a simple consequence of Hodge theory.

The spinor Dirac operator is defined only for spin manifolds M, that is manifolds equipped with a *spin structure.* It is helpful to think of such a structure as a 'refined orientation' of M: in fact, a spin structure is given by an orientation together with a principal $\operatorname{Spin}(n)$-bundle which double covers the $SO(n)$-bundle of oriented orthonormal frames. The group $\operatorname{Spin}(n)$ has a canonical faithful representation S, the spin-representation, and, if n is even,

[4] There are corresponding ungraded versions on odd-dimensional manifolds. We will discuss these in Chapter 4.

S decomposes as a direct sum $S_0 \oplus S_1$ of two "half-spin" representations. Associated to the given $\mathrm{Spin}(n)$-bundle on M these give two vector bundles S_0 and S_1 on M. The Dirac operator on sections of S is the composite of the covariant derivative operator associated to an appropriate connection, $\nabla\colon C^\infty(S) \to C^\infty(T^*M \otimes S)$, and the Clifford multiplication coming from the spin structure, $c\colon C^\infty(T^*M \otimes S) \to C^\infty(S)$. It is graded by the decomposition $S = S_0 \oplus S_1$.

The indices of our model operators have certain stability properties which will be important for us. The index of the *signature operator* is invariant under (oriented) homotopy equivalence, as follows immediately from the Hodge theorem: it is defined in purely cohomological terms. Although this is easy to see, we need to realize that it is rather surprising from the point of view of differential equations; a homotopy equivalence might completely destroy the local differential structure of a manifold. (Later, we will need to take a look at surgery theory, which investigates the relation between the differentiable and the homotopy classification of manifolds.)

The index of the *Dirac operator* is not an invariant of homotopy type. The 'stability property' for the index that is relevant here is the vanishing theorem of Lichnerowicz. This follows from the Weitzenböck formula[5]

$$D^2 = \nabla^*\nabla + \tfrac{1}{4}\kappa,$$

where κ denotes the scalar curvature. We note that $\nabla^*\nabla$ is a *positive*[6] operator, and it follows that if κ is strictly positive then D can have no kernel, so its index must be zero. In combination with the Atiyah-Singer index theorem this has proved to be a most powerful tool for investigating which manifolds can carry positive scalar curvature metrics; for the index of the Dirac operator is the $\widehat{\mathcal{A}}$-genus, a topological invariant (see below), and if this invariant is non-zero, then the manifold can carry no metric of positive scalar curvature.

Index theory and coarsening

It is time now to take a look at the index theorem, which we just mentioned. This provides a relationship between the indices of elliptic operators and some apparently rather different invariants of manifolds, the characteristic numbers associated to the tangent bundle. In general the index theorem takes the following form: it provides a local recipe to obtain, from an elliptic operator D together with grading ε, a characteristic class $\Im_D$ on M. Then it states that

$$\mathrm{Ind}(D,\varepsilon) = \langle \Im_D, [M] \rangle;$$

[5]There are similar formulae for all generalized Dirac operators, including the signature operator. They differ in the precise nature of the curvature term. It is the fact that the curvature term is the *scalar* curvature that makes the Dirac operator so useful.

[6]A symmetric operator T on a Hilbert space is *positive* if $\langle Tx, x\rangle \geqslant 0$ for all x in the domain of T.

the *global* invariant $\mathrm{Ind}(D, \varepsilon)$ is obtained by integrating the *local* invariant $\Im_D$ over M. The prototype of this is the Gauss-Bonnet theorem. In the case of our two model operators, the signature and Dirac operators, the relevant characteristic classes $\Im_D$ are called the $\mathcal{L}$ class and the $\widehat{\mathcal{A}}$ class respectively; they can be expressed in terms of the Pontrjagin classes of TM, as is done in the table below for a few low dimensions:

Dimension	$\mathcal{L}$ class	$\widehat{\mathcal{A}}$ class
4	$p_1/3$	$-p_1/24$
8	$(7p_2 - p_1^2)/45$	$(-4p_2 + 7p_1^2)/5760$
12	$(62p_3 - 13p_2p_1 + 2p_1^3)/945$	$(-16p_3 + 44p_1p_2 - 31p_1^3)/967680$

One proof of the index theorem (the so-called *heat equation proof*) proceeds as follows. Since D^+ and D^- are adjoint to one another, we find that $\ker D^+ = \ker D^-D^+$, and $\ker D^- = \ker D^+D^-$. Now let φ be a function on $\mathbb{R}^+$ with $\varphi(0) = 1$ and $\varphi(\lambda) = 0$ for all λ greater than the least nonzero eigenvalue of D^2; then the operator $\varphi(D^2)$ defined by the functional calculus is just the orthogonal projection onto the kernel $\ker D = \ker D^2$. Since the trace of a projection is just the dimension of its range, we get

$$\mathrm{Ind}(D, \varepsilon) = \mathrm{Tr}(\varepsilon\varphi(D^2)) = \mathrm{Tr}(\varphi(D^-D^+)) - \mathrm{Tr}(\varphi(D^+D^-)) \qquad (*)$$

Now elementary algebra tells us that given two operators A and B, the nonzero spectra of AB and BA coincide, and we may apply this with $A = D^-$ and $B = D^+$. Using formula 1.4 we see that if we allow φ to be *any* function with $\varphi(0) = 1$, provided only that it is fast-decaying enough for the right hand side to make sense, then $(*)$ will remain true: the additional contributions from the non-zero eigenspaces will simply cancel. In particular we get

$$\mathrm{Ind}(D, \varepsilon) = \mathrm{Tr}(\varepsilon e^{-tD^2}),$$

called the *heat equation formula* because the operator e^{-tD^2} governs the time evolution of the solution of the (spinor valued!) heat equation $\partial u/\partial t + D^2 u = 0$. Atiyah, Patodi, and Singer, following up an idea of Gilkey, showed that the right hand side becomes local in t as $t \downarrow 0$, and that it in fact admits an asymptotic expansion in which the leading term is the index form $\Im_D$. The index theorem follows.

We may say that the heat equation provides a process of interpolation between the local invariant $\Im_D$ (for small values of t) and the global invariant $\mathrm{Ind}(D, \varepsilon)$ (for large values). The interpolation process consists of successively switching off more and more of the high-energy modes of the operator D^2 as one passes from the local invariant $\Im_D$ to the index. Now it is well known to physicists that high energies correspond to short distances, and we may therefore envisage the operation of the heat equation process somewhat as follows — to obtain the local invariant $\Im_D$ we probe the structure of the manifold M with the highest energies at our command, but as t increases the energy used becomes less, and correspondingly our view of M becomes blurrier. Finally, when t is very large, our view of M is so blurry that all the local geometry is washed out. To our "coarsened" vision M is indistinguishable from a point, and so all that is left of D is the sole abstract topological invariant of a single Fredholm operator on a Hilbert space, namely the index. We have therefore arrived at the slogan, *index is coarsening.*

Notice that the *invariance* properties of the index, which we emphasized above, are intimately related to its 'coarse' aspect. For example consider the Lichnerowicz vanishing theorem. If $\kappa_0 > 0$ is a lower bound for the scalar curvature, then D^2 has no spectrum below $\kappa_0/4$; this means that the vanishing begins to take hold only at times t in the heat equation process for which $t\kappa_0 \gg 1$. We might ask whether the index is the sole relevant 'coarse' invariant: in the scalar curvature problem this would amount to asking whether a (spin) manifold with zero $\widehat{\mathcal{A}}$-genus *must* admit positive scalar curvature. It turns out that this is far from the case; for example, a notable result of Schoen-Yau and Gromov-Lawson is that no torus can admit a metric of positive scalar curvature, even though, being parallelizable, it has no nonzero characteristic numbers at all. If we hope then that index or 'coarsening' theory is going to tell us the whole story about this problem, we need to ask the following question: *Are there more refined ways of coarsening?*

The answer is yes. Such 'refined coarsenings' are usually known as *higher indices.* Our objective is to construct and interpret higher indices, particularly for *non-compact* manifolds M. For such manifolds the definition of the ordinary index does not make sense (the kernels are not finite-dimensional), and so the higher indices, if they exist, will be all we've got.

REMARK: It will turn out that the grading operator ε does not always play such a crucial rôle in higher index theory. In fact, there will be two kinds of higher indices — 'even' indices, which depend on a grading ε just as the ordinary index does, and 'odd' indices which require no ε for their definition. To have a uniform notation we will therefore suppress mention of ε and regard it as incorporated where necessary into the definition of D; so we will from now on write $\mathrm{Ind}(D)$ where formerly we wrote $\mathrm{Ind}(D, \varepsilon)$.

Notes and references: The language of operators on Hilbert spaces, functional calculus, and spectral theory is treated in most introductory texts on functional analysis. A classic reference here is [**32**].

Some books on index theory which take a point of view similar to the one here are [**11**], [**70**] and [**90**]. For the original discussion of generalized Dirac operators on complete manifolds, see [**48**], which also contains much other material relevant to 'large scale index theory'. The original paper on the heat equation method is [**6**], with later improvements in [**41, 42**]. Finally, the foundational paper [**7**] must be mentioned in any list of references dealing with index theory.

CHAPTER 2

Coarse Geometry

In the previous lecture we used the index theorem to motivate the idea of viewing a manifold through successively blurrier lenses, and studying what geometry remained at the end of this 'coarsening' process. The first surprise of the theory is that any geometry remains at all.

Ideas of this sort have been around for quite a while. An early example occurs in Ahlfors' theory of covering surfaces [**2**], where he derives quantitative versions of the Picard theorem of complex analysis from consideration of 'average Euler characteristics' of open surfaces. In the 1960's, Mostow's proof of the rigidity theorem relied on the fact that 'coarse' maps preserve the ideal boundary of hyperbolic spaces (see [**78**]). Implicit here is the principle that coarse geometry is peculiarly appropriate to the study of spaces of negative curvature, a principle which led to Gromov's celebrated theory of hyperbolicity for metric spaces, which we will discuss later. Still more directly related to the material of these lectures are the ideas of (boundedly) *controlled topology*, in which one tries to carry out geometric constructions in a way which is 'small' when measured in some reference space $\mathcal{Z}$; the coarse structure of $\mathcal{Z}$ is exactly what is relevant here. Some references are [**26, 36, 112**]. Finally, the fact that a finitely generated discrete group carries a natural coarse structure has meant that 'coarse' questions have been intensively studied by geometric group theorists [**46**].

The coarse category

Here now are the formal definitions for the version of coarse geometry that we will be using. Let X and Y be metric spaces. A map $f\colon X \to Y$, not necessarily continuous, will be called a *coarse map* if

(a) (Uniform expansiveness) For each $R > 0$ there is $S > 0$ such that

$$d(x, x') \leqslant R \Rightarrow d(f(x), f(x')) \leqslant S.$$

(b) (Metric properness) For each bounded subset $B \subseteq Y$, the inverse image $f^{-1}(B)$ is bounded in X.

We will usually work with *proper metric spaces* — those in which closed bounded sets are compact (the terminology comes from the fact that the distance function to a fixed point is then a proper map to $\mathbb{R}$). If X and Y are such spaces and f is continuous, then metric properness is equivalent to ordinary properness.

Two coarse maps $f_0, f_1 \colon X \to Y$ are *coarsely equivalent* if there is a constant K such that

$$d(f_0(x), f_1(x)) \leqslant K$$

for all $x \in X$. We say that the *spaces* X and Y are coarsely equivalent if there are maps from X to Y and from Y to X whose composites (both ways round) are coarsely equivalent to the identity maps (on X and on Y). Finally, a *coarse structure* on X just means a coarse equivalence class of metrics.

Here are some examples.

EXAMPLE: Any complete Riemannian manifold is a proper metric space, so it can be equipped with a coarse structure. Our index theory for complete manifolds will depend functorially on this structure.

EXAMPLE: Let Γ be a finitely generated group, and let S be a finite generating set. Then there is a unique translation-invariant metric on Γ such that each element of S is at distance 1 from the identity and which is maximal among all metrics having that property; it is called the 'word length metric' and can be explicitly described by saying that the distance between γ_1 and γ_2 is equal to $|\gamma_1^{-1}\gamma_2|$, the length of the shortest word in the alphabet $S \cup S^{-1}$ that represents $\gamma_1^{-1}\gamma_2$. Although the word length metric itself is dependent on the choice of finite generating set S, an elementary argument shows that the coarse structure that it induces is not. So the group theory of Γ alone equips it with a uniquely determined coarse structure[1]. It is interesting to ask what restrictions on the coarse structure of Γ are imposed by the fact that it arises from a group structure[2].

These two examples are closer together than they may appear. Even though groups are discrete and manifolds are not, this sort of distinction is exactly the kind that is blurred by coarsening — for example, it is clear that the natural inclusion $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. More generally we have:

PROPOSITION 2.1: *Let M be a compact manifold, with fundamental group Γ and universal cover $\tilde{M}$. Then the map $\gamma \mapsto \gamma p$, $\Gamma \to \tilde{M}$, induced by any choice of basepoint p in $\tilde{M}$ is a coarse equivalence, with respect to any word metric on Γ and any Riemannian metric lifted from a Riemannian metric on $\tilde{M}$.*

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[1] It is sometimes necessary to distinguish between Γ as a group and Γ as a metric (or coarse) space; we will use the notation $|\Gamma|$ for the metric space underlying the group Γ.

[2] As a simple example, it is easy to see that the 'ends' of a group are invariants of the coarse structure. A discrete group can have only 0, 1, 2, or ∞ ends [**59**], whereas a general coarse space may have any number of ends. For more on these lines see [**15, 12**].

REMARK: We do not really require M to be a manifold here. It is enough that it should be a locally simply connected *path metric space* (that is, a space in which the distance between two points equals the infimum of the lengths of paths joining those points).

EXAMPLE: Let N be a compact (smooth) manifold. The *open cone* $\mathcal{O}N$ on N is the quotient space obtained from $N \times \mathbb{R}^+$ by identifying $N \times \{0\}$ to a point. It can be made into a proper metric space by choosing an arbitrary Riemannian metric g_N on N and then giving $\mathcal{O}N$ the (singular) Riemannian metric $g_{\mathbb{R}} + t^2 g_N$, where t is the $\mathbb{R}^+$ coordinate. One sees easily that the choice of Riemannian metric on N does not affect the coarse structure.

If N is embedded in M as a totally geodesic submanifold then $\mathcal{O}N$ is (isometric to) a subspace of $\mathcal{O}M$. Now we know that N can always be smoothly embedded in some high-dimensional S^{m-1}, and by suitably tweaking the metric on S^{m-1} near the normal bundle of N we can make this embedding totally geodesic. Switching back to the round metric on S^{m-1} only changes things by a coarse equivalence, and now $\mathcal{O}S^{m-1}$ is just ordinary Euclidean space $\mathbb{R}^m$. Thus $\mathcal{O}N$ is coarsely equivalent to the 'Euclidean' cone

$$\{tx \in \mathbb{R}^m : x \in N \subseteq S^{m-1},\ t \geqslant 0\}$$

and this can be used as a definition of the open cone for spaces N other than manifolds, so long as they are provided with an appropriate class of preferred embeddings into spheres.

The open cone contains copies of N on larger and larger scales. Thus, intuitively, we would imagine that the study of $\mathcal{O}N$ on large (but fixed) scales, as is done in coarse geometry, would be equivalent to the study of N itself on arbitrarily small scales — that is, the study of something like the topology of N. And this is, more or less, the case, although the $\mathbb{R}^+$-direction has introduced some extra stabilization. To be precise, let us recall that a map $f\colon X \to X'$ between metric spaces is said to be *Lipschitz* if there is a constant C such that $d(f(x_0), f(x_1)) < Cd(x_0, x_1)$ for all $x_0, x_1 \in X$; a *bi-Lipschitz homeomorphism* is a Lipschitz map with a Lipschitz inverse. The following proposition and its proof are due to Weinberger [**110**].

PROPOSITION 2.2: *Let N and N' be subspaces of some S^{m-1}. Suppose that $N \times S^1$ and $N' \times S^1$ are bi-Lipschitz homeomorphic, by a homeomorphism that makes the diagram*

$$\begin{array}{ccc} N \times S^1 & \longrightarrow & N' \times S^1 \\ & \searrow \quad \swarrow & \\ & S^1 & \end{array}$$

commute up to homotopy. Then $\mathcal{O}N$ and $\mathcal{O}N'$ are coarsely equivalent. In particular, this is so if N and N' are bi-Lipschitz homeomorphic.

PROOF: Take infinite cyclic covers to obtain a periodic bi-Lipschitz homeomorphism $N \times \mathbb{R} \to N' \times \mathbb{R}$. Now identify $N \times \mathbb{R}$ with $\mathcal{O}N$ (minus its vertex) by the map $(x, t) \mapsto 2^t x$. Do the same for N', transfer the given bi-Lipschitz homeomorphism to the cones, and extend by continuity to the vertex. This gives us a map $h\colon \mathcal{O}N \to \mathcal{O}N'$ which has the following properties

(a) h is Lipschitz when restricted to any compact subset of $\mathcal{O}N \setminus \{0\}$;

(b) $h(2x) = 2h(x)$ for all x.

and whose inverse has similar properties. But now h is Lipschitz. To see this, note that we would like to bound the ratio $d(h(x), h(x'))/d(x, x')$. We may take $|x| \geqslant |x'|$; if $|x| \leqslant 2|x'|$ the desired property follows by scaling by a power of 2 (using (b)) to get into the compact annulus between radii 1 and 4, then using (a); if $|x| > 2|x'|$ we use the crude estimate

$$\frac{d(h(x), h(x'))}{d(x, x')} \leqslant C\frac{|x| + |x'|}{|x| - |x'|} \leqslant 3C$$

where $C = \sup\{|h(x)| : |x| \leqslant 2\}$. Thus $\mathcal{O}N$ and $\mathcal{O}N'$ are bi-Lipschitz homeomorphic, hence certainly they are coarse equivalent. □

The conclusion of the theorem was stated in terms of coarse equivalence, but the proof produced a bi-Lipschitz homeomorphism. In fact one can see that these notions are equivalent for cones: rescaling a coarse equivalence between cones produces a sequence of maps which (by Ascoli's theorem) has a subsequence convergent to a bi-Lipschitz homeomorphism.

For high-dimensional manifolds one can go further: the condition of 2.2 is both necessary and sufficient for the existence of a coarse equivalence between the cones; moreover, by Sullivan's results [**105**], 'bi-Lipschitz homeomorphic' can be replaced by 'homeomorphic'. The proof of this result is due to Block and Weinberger [**16**]; in addition to Sullivan's results it requires the s-cobordism theorem for topological manifolds [**69**]. Here we simply want to make the point that the cone construction provides a way of encoding ordinary topology into coarse geometry; or, conversely, some problems in coarse geometry may be solvable by using the cone construction to convert them into ordinary problems at infinity. The appearance of *Lipschitz* homeomorphisms is at first a surprise, but it is unavoidable; one can give examples of homeomorphic metric spaces (not manifolds!) whose open cones are not coarsely equivalent. In lecture 9 we will develop a more relaxed notion of coarse equivalence which will to some extent allow us to circumvent this issue.

EXAMPLE: Further examples of coarse spaces may be obtained by warping the cone construction over a foliation or a group action. For example, suppose that N (above) carries an action of an infinite (finitely generated) discrete group Γ. We define $\mathcal{O}(N, \Gamma)$ to be the same space as $\mathcal{O}(N)$ but equipped with the maximal metric d that is dominated by the usual metric of $\mathcal{O}(N)$ and satisfies $d(x, \gamma x) \leqslant |\gamma|$ for all $\gamma \in \Gamma$. This is still a proper metric space and its coarse geometry encodes some information about the dynamics of the action of Γ on N. We won't use these 'shortcut cones' in this survey, but they seem to be interesting spaces for further study, especially insofar as they are related to 'foliated control' [**34**]. See [**96**] for a few remarks.

We defined coarse geometry as the study of metric spaces up to coarse equivalence. This is exactly parallel to the approach often taken in first courses on topology, whereby one defines topology as the study of metric spaces up to homeomorphism. Just as it subsequently proves useful to study topological spaces in the abstract, independent of the question as to whether the topology is generated by a metric, so it will be useful to us later to have an abstract and metric-independent notion of coarse structure. But we will postpone introducing this until lecture 10.

Coarse algebraic topology

What tools are available to investigate the coarse category? We would hope to obtain analogues to the tools of algebraic topology, such as homology and cohomology theories. In doing so, it is helpful to keep in mind a certain analogy between a 'coarse type' (that is, a coarse equivalence class of metric spaces) and a finitely generated group, and to imagine that we are trying to define some analogue of group (co)homology. Now the (co)homology of a group π can be defined in two ways: algebraically (for instance by writing down a specific chain complex) or topologically (by introducing the classifying space $B\pi$). Similarly, two procedures exist for defining coarse (co)homology. In the memoir [**95**] I constructed a cohomology theory on the coarse category by a combinatorial algebraic recipe. A more topological approach might proceed by first introducing the notion of a *coarsening*, which is the coarse analogue of the classifying space.

DEFINITION 2.3: *Let X be a metric space.*

(i) *X is called a* metric simplicial complex *if it is a simplicial complex equipped with a path metric which coincides on each simplex with the standard metric.*

(ii) *X has* bounded geometry *if it is coarsely equivalent to a discrete space which has the property that, for each $r > 0$, there is a uniform bound on the number of elements in a ball of radius r.*

(iii) *X is* uniformly contractible *if, for every $R > 0$ there is $S > 0$ such that, for every $x \in X$, the inclusion $B(x; R) \to B(x; S)$ is nullhomotopic.*

For example, Euclidean space $\mathbb{R}^n$ (triangulated sensibly) is a uniformly contractible bounded geometry metric simplicial complex. We notice that $\mathbb{R}^n$ is in some sense the 'topologically simplest' space in its coarse equivalence class. Thus $\mathbb{Z}^n$, or the infinite jail cell window, both have 'artificial' local topology which is not detected by the coarse structure; whereas the coarse structure and the fine structure of $\mathbb{R}^n$ are exactly the same. The following notion formalizes this idea of mapping coarse theory onto topology.

DEFINITION 2.4: *Let X be a metric space. A* coarsening EX *of X is a uniformly contractible bounded geometry metric simplicial complex equipped with a coarse equivalence $X \to EX$.*

EXAMPLE: $\mathbb{R}^n$ is a coarsening of $\mathbb{Z}^n$. More generally, if π is any group with a finite classifying space $B\pi$, then the universal cover of $B\pi$ is a coarsening of $|\pi|$. The E notation for coarsening comes from this example.

We did not use all the properties of the classifying space here. We needed to know only that $\widetilde{B\pi}$ was a contractible space admitting a cocompact π-action such that the map $\pi \to \widetilde{B\pi}$, $\gamma \mapsto \gamma p$, is a coarse equivalence. Now in **[10]** a certain space $\underline{E}\pi$ is studied from the perspective of K-homology. It is a universal example for *proper* π-actions (an action of π on X is *proper* if X can be covered by a family of π-invariant open sets each of which is of the form $\pi \times_F W$, where F is a finite subgroup of π and W is a space acted on by F). It is not hard to prove

PROPOSITION 2.5: *If there is a model for the universal space $\underline{E}\pi$ on which π acts cocompactly by isometries, then that model is a coarsening of $|\pi|$.*

Contrary to the impression which may have been given by the above examples, coarsenings do not always exist. However, when they do, they are unique up to proper homotopy equivalence. Moreover, any coarse map between coarse spaces induces a unique proper homotopy class of continuous maps between coarsenings. Thus coarsening becomes a functor from the coarse category to the proper homotopy category. For the proofs one constructs the desired maps by induction over simplices, using uniform contractibility to extend maps of the boundary of a simplex to maps of the whole simplex while maintaining overall metric control.

Let K_* be a locally finite generalized homology theory. Then, for a space X that admits a coarsening, we may define the associated *coarse homology* of X by

$$KX_*(X) = K_*(EX).$$

The functorial properties of coarsening show that KX_* is a functor on the coarse category. It can therefore be used to provide invariants analogous to those of classical algebraic topology.

EXAMPLE: Consider the natural inclusion $\mathbb{R}^n \to \mathbb{R}^n \times [0, \infty)$, which is a coarse map. We ask whether this inclusion admits a left inverse in the coarse category. To see that the answer is negative, consider the coarse homology theory HX_* associated to ordinary (locally finite) homology. Since both spaces are uniformly contractible, $HX_n(\mathbb{R}^n) = H_n^{lf}(\mathbb{R}^n) = \mathbb{Z}$, and $HX_n(\mathbb{R}^n \times [0, \infty)) = H_n^{lf}(\mathbb{R}^n \times [0, \infty)) = 0$. Thus, if the supposed left inverse existed, the identity map $\mathbb{Z} \to \mathbb{Z}$ would factor through the zero group, an obvious contradiction.

ASIDE: The notion of *locally finite homology* may not as familiar as its dual, *compactly supported cohomology*, so we take a moment to review it. Start by considering an infinite simplicial complex K. To define the usual homology, we consider chains that are finite formal linear combinations of simplices of K. To define *locally finite* homology, we allow *infinite, locally finite* formal linear combinations of such simplices. For example, if $K = \mathbb{R}$ triangulated in the obvious way, the 0-dimensional locally finite homology group is trivial (a single 0-simplex is the boundary of an infinite 1-chain consisting of all the simplices to the right of it), whereas the 1-dimensional locally finite homology group is non-trivial (a generator is the sum of all the 1-simplices).

Locally finite homology is functorial for *proper* maps and invariant under *proper* homotopies. Naturally, there are notions of locally finite generalized homology and the like. For our purposes the most important such theory will be the locally finite homology theory dual to K-theory, and this has a direct analytical description due to Kasparov (see lecture 5).

REMARK: Coarse homology and cohomology for *groups* π can be described in terms of group cohomology. In fact, suppose (for simplicity) that π is a group with a finite classifying space $B\pi$; then, as observed above, $E\pi$ (the universal cover of $B\pi$) is a coarsening of $|\pi|$. It is well-known, however, that the locally finite homology of $E\pi$ is simply the homology of $B\pi$ with coefficients in the π-module $\mathbb{Z}\pi$. Thus $HX_*(|\pi|) = H_*^{lf}(E\pi) = H_*(B\pi;\mathbb{Z}\pi) = H_*(\pi;\mathbb{Z}\pi)$. The dual cohomological result is also true. Thus we have

PROPOSITION 2.6: *For any finitely generated group π, $HX_*(|\pi|) = H_*(\pi;\mathbb{Z}\pi)$ and $HX^*(|\pi|) = H^*(\pi;\mathbb{Z}\pi)$.*

So far we have defined coarse homology only for spaces which admit coarsenings. However, the definition can be generalized to all spaces by means of the following construction, originated by E. Rips. Let $\mathfrak{U}$ be an open cover of a space X. Recall that the *nerve* $|\mathfrak{U}|$ of U is the simplicial complex with one vertex for each set $U \in \mathfrak{U}$, and such that a finite set of vertices span a simplex if and only if the corresponding open sets have non-empty intersection. Associated to such a cover are two numbers $R(\mathfrak{U})$, the least number such that any member of $\mathfrak{U}$ is included in a ball of radius R, and $r(\mathfrak{U})$, the greatest number such that any ball of radius r is included in a member of $\mathfrak{U}$. Choose a sequence $\mathfrak{U}_i$ of locally finite covers by relatively compact open sets such that $R(\mathfrak{U}_i) \leqslant r(\mathfrak{U}_{i+1}) < \infty$ for all i; this will be called a *coarsening sequence* of covers[3] for X. One can define maps $|\mathfrak{U}_i| \to |\mathfrak{U}_{i+1}|$ by choosing, for each open set of the cover $\mathfrak{U}_i$, an open set of the cover $\mathfrak{U}_{i+1}$ that contains it. Then one can show that if X has a coarsening,

$$KX_*(X) = \lim_{\rightarrow} K_*(|\mathfrak{U}_i|).$$

The right hand side of this equation may therefore be taken as the *definition* of coarse homology, and it now applies to *all* spaces X. The functorial properties still hold.

[3]Referred to in [**95**] as an *anti-Čech system* for X.

REMARK: One way to think of coarsening a space is as an 'identification of all pairs of points that are a finite distance apart'. This does not make literal sense — after all, *any* two points are a finite distance apart — but the Rips construction gives one way of understanding it: coarsening is the 'limit' of the complexes $|\mathfrak{U}_i|$, each of which can be thought of as obtained by 'smoothing out' X on some fixed large scale.

What do uniformly contractible spaces look like? Two classes of examples spring to mind. On the one hand, the open cone $\mathcal{O}N$, where N is a manifold, or any metric space in which small balls are contractible, is plainly uniformly contractible. On the other hand, suppose that V is a finite *aspherical* complex (that is, such that $\pi_i(V) = 0$ for $i \geqslant 2$). Then the universal cover $\tilde{V}$ is uniformly contractible. The study of examples suggests that uniformly contractible spaces in general, and universal covers in particular, should share the good coarse properties of open cones. For example, if V is a compact negatively curved manifold, the Cartan-Hadamard theorem provides us with the exponential map, a metrically controllable diffeomorphism of $\mathbb{R}^n = \mathcal{O}(S^{n-1})$ onto $\tilde{V}$, which can be used to transfer coarse properties from one to the other. As we will see, the 'coarse' statement that the universal cover of V looks like a cone has strong consequences for the 'ordinary' geometry and topology of V itself.

Notes and references: The expositions of coarse geometry that I know of all appear in papers devoted to applications of one kind or another: so, for example, the basic definitions given here may be found in [**16, 36, 44, 95, 112**] and (I have no doubt) in numerous other places. The approach to 'coarse algebraic topology' that is used here is based on [**57**]. That some form of bounded geometry constraint is necessary in the definition of a coarsening was shown by the example of [**31**].

There is an interesting distinction to be made between the coarse category and what has been called the *rough* category. The objects of this latter category are bounded geometry metric spaces, and the morphisms are those coarse maps that, instead of (b), satisfy the stronger condition

(c) (Uniform metric properness) For each $R > 0$ there is $S > 0$ such that the inverse image under f of a set of diameter R is a set of diameter at most S.

The fundamental example of a coarse map that is not a rough map is the projection $\mathcal{O}N \to \mathbb{R}^+$ from an open cone to a ray. The uniformity in the definition of the rough category allows a large number of new invariants to exist, based on constructions such as the averaging of a bounded function over a discrete amenable group. We will not develop rough geometry in detail in these lectures, but we will occasionally point out significant contrasts between the coarse and rough categories. The papers [**8, 9, 15, 16**] and the thesis [**72**] contain further discussion.

CHAPTER 3

C^*-Algebras

We will now relate coarse geometry to C^*-algebras. The motivation here comes from Connes' theory of noncommutative geometry [**28**], and we will begin by reviewing this.

Let H be a Hilbert space[1], $\mathfrak{B}(H)$ the collection of all bounded linear operators $H \to H$. Recall that $\mathfrak{B}(H)$ is equipped with an involution $*$, where the *adjoint* T^* of $T \in \mathfrak{B}(H)$ is determined by

$$\langle T^* x, y\rangle = \langle x, Ty\rangle.$$

Moreover, $\mathfrak{B}(H)$ is equipped with a norm

$$\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\},$$

and this norm induces a topology on $\mathfrak{B}(H)$.

DEFINITION 3.1: *A C^*-algebra of operators on H is a norm-closed subalgebra A of $\mathfrak{B}(H)$ such that $T \in A \Rightarrow T^* \in A$.*

We may define an (abstract) C^*-algebra to be an involutive normed algebra over $\mathbb{C}$ which is isomorphic to a C^*-algebra of operators on some Hilbert space; it is a theorem that this definition is equivalent to the more usual one (as a Banach algebra satisfying some additional conditions).

Notable examples of C^*-algebras are the algebras $C_0(X)$ of continuous functions, vanishing at infinity, on some locally compact Hausdorff space X. (To see that these are C^*-algebras by our definition, think of them as acting on $L^2(X, \mu)$ for an appropriate measure μ on X.) In fact, these are the *only* examples of commutative C^*-algebras; moreover, the 'nondegenerate' C^*-homomorphisms $C_0(X) \to C_0(Y)$ are just those induced from continuous and proper maps $Y \to X$. (These two statements constitute the *Gelfand-Naimark theorem*, whose proof will be found in any introductory text on Banach algebras.) Thus the study of commutative C^*-algebras is equivalent to the topology of locally compact Hausdorff spaces, and it is reasonable to think of the study of general C^*-algebras as some kind of 'noncommutative topology'.

[1] All Hilbert spaces considered will be separable.

This philosophy leads in particular to Connes' noncommutative notion of the quotient of a space by an equivalence relation. Let us illustrate it with a very simple example [**28**, page 85]. Consider a space $X = \{x_0, x_1\}$ with just two points. The associated C^*-algebra A may be regarded as the algebra of *diagonal* 2×2 matrices acting on the Hilbert space $\ell^2(X) = \mathbb{C} \oplus \mathbb{C}$. Now suppose we form the quotient by the equivalence relation that identifies the points x_0 and x_1. The classical topological construction corresponds to *restricting* to the subalgebra $\{\lambda I : \lambda \in \mathbb{C}\}$ of A. However, in noncommutative topology one instead *extends* A to a larger algebra containing off-diagonal matrix elements corresponding to the identifications that have been made. In this case we extend to the full matrix algebra $M_2(\mathbb{C}) \supset A$.

It can be shown in general that, when the classical quotient space is good (e.g. compact Hausdorff), then the classical and the noncommutative quotient constructions are (Morita) equivalent. But when the classical quotient is bad, it is often more appropriate to consider the noncommutative version — especially for the purposes of index theory.

EXAMPLE: Let Γ be a discrete group acting on a point. The 'standard quotient' is just a point, and forgets all the structure of Γ. The appropriate notion of noncommutative quotient in this case is the *reduced group C^*-algebra* $C^*_r(\Gamma)$, defined as follows. Consider the Hilbert space $\ell^2(\Gamma)$; Γ acts on this space by unitaries, and so there is an embedding of the group ring $\mathbb{C}\Gamma$ as a $*$-subalgebra of $\mathfrak{B}(\ell^2(\Gamma))$. The completion of $\mathbb{C}\Gamma$ in the norm of $\mathfrak{B}(\ell^2(\Gamma))$ is the algebra $C^*_r(\Gamma)$.

The link with coarse geometry comes about through the idea, which we have already mentioned, that coarsening a space can be thought of as 'identification of all pairs of points that are a finite distance apart'. As with the Rips construction, one should think of this in some limiting sense; one 'identifies' on scale R, then lets $R \to \infty$ and takes the limit. The noncommutative quotient construction is well adapted to this process. Specifically, if X is a *discrete* coarse space, we want to consider algebras of matrices parameterized by $X \times X$ which are zero outside some bounded neighbourhood of the diagonal. As in the simple example above, we have introduced off-diagonal matrix elements corresponding to identifications, but here we have required that all the identifications be on the same scale.

The general definition makes use of the language of X-modules and locally compact operators, which we have introduced in lecture 1. Recall that a Hilbert space H is called an *X-module* if there is given[2] a C^*-homomorphism $C_0(X) \to \mathfrak{B}(H)$. We defined an operator T on an X-module H to be *locally compact* if, for all $\varphi \in C_0(X)$, the operators $T\varphi$ and φT are compact on H.

In the next definition, $\mathrm{Supp}(\varphi) = \{x : \varphi(x) \neq 0\}$ denotes the support of φ; and, for subsets A and B of X, $d(A, B)$ denotes $\inf\{d(a,b) : a \in A, b \in B\}$.

[2]It follows from the Spectral Theorem that the given representation of the continuous functions $C_0(X)$ can be canonically extended to a representation of the bounded Borel functions; this will occasionally be useful.

DEFINITION 3.2: *Let X be a metric space. An operator T on an X-module H is said to have* finite propagation *if there is a constant $R > 0$ such that $\varphi T \psi = 0$ whenever $\varphi, \psi \in C_0(X)$ have $d(\mathrm{Supp}(\varphi), \mathrm{Supp}(\psi)) > R$. (The smallest such constant is the* propagation *of T.)*

Finite propagation is the continuous analogue of the condition, used above, that a matrix parameterized by $X \times X$ should vanish outside a bounded neighbourhood of the diagonal. Plainly, the locally compact, finite propagation operators on H form a $*$-subalgebra of $\mathfrak{B}(H)$.

DEFINITION 3.3: *For a coarse space X and an X-module H, we define $C^*(X; H)$ to be the C^*-algebra obtained as the closure in $\mathfrak{B}(H)$ of the locally compact, finite propagation operators.*

We will usually suppress mention of H. In fact, we will see that the choice of H is not very important, provided that we choose it to be non-degenerate and locally infinite-dimensional. The precise sense of this is given by the following definition:

DEFINITION 3.4: *An X-module H is said to be* adequate *if $\overline{C_0(X)H} = H$ and no nonzero element of $C_0(X)$ acts on H as a compact operator.*

It is useful to compose this construction with the K-theory functor for C^*-algebras. (See the appendix to this section for a short development of C^*-algebra K-theory.) One can show that the abelian groups $K_i(C^*X)$ depend only on X, and not on the choice of adequate X-module H. Moreover, the construction is functorial:

LEMMA 3.5: *Let X, Y be coarse spaces, $f: X \to Y$ a coarse map. Then f induces a functorial homomorphism $f_*: K_*(C^*X) \to K_*(C^*Y)$. Coarsely equivalent maps induce the same homomorphism.*

PROOF: Let H_X and H_Y be X and Y modules. An isometry $V: H_X \to H_Y$ is said to *cover* f if there is a constant $R > 0$ such that $\varphi V \psi = 0: H_X \to H_Y$ whenever $\psi \in C_0(X)$ and $\varphi \in C_0(Y)$ with $d(\mathrm{Supp}(\varphi), f(\mathrm{Supp}(\psi)) > R$. [If this relation is true for $R = 0$ we say V *exactly covers* f.] Provided that H_Y is adequate, there is always an isometry covering any coarse map. To see this notice that any Borel partition of X or Y leads, via the spectral theorem, to a direct sum decomposition of H_X or H_Y. We can partition Y into Borel pieces of uniform size and with non-empty interior, getting a direct sum decomposition of H_Y into infinite-dimensional summands. Now take the inverse image partition of X and map each summand of H_X isometrically into the corresponding summand of H_Y. This gives the desired V.

If V covers f, then $\mathrm{Ad}(V): \mathfrak{B}(H_X) \to \mathfrak{B}(H_Y)$, defined by $T \mapsto VTV^*$, is a homomorphism from $C^*(X)$ to $C^*(Y)$. We define f_* to be the map on K-theory induced by this homomorphism; it can be shown to be independent of the choice of V. The last statement of the lemma is true because if V covers f it also covers any map coarsely equivalent to f. □

REMARK: Since $C^*(X)$ is defined by completion, a general $T \in C^*(X)$ may not exactly have finite propagation. It is sometimes useful to have available a notion of 'approximate propagation': for $\varepsilon > 0$ we define the *ε-propagation* of T to be the smallest R such that $\|\varphi T \psi\| < \varepsilon$ whenever $\|\varphi\| \leqslant 1$, $\|\psi\| \leqslant 1$, and $d(\mathrm{Supp}(\varphi), \mathrm{Supp}(\psi)) > R$. Plainly, if $T \in C^*(X)$, then T has finite ε-propagation for each ε. If X is 'large-scale finite-dimensional' [**46, 114**], then the converse holds as well.

The coarse index

Let M be a complete Riemannian manifold, V any Hermitian vector bundle over M. Then $L^2(M; V)$ is an adequate M-module; we will usually think of $C^*(M)$ as defined using an M-module of this type.

PROPOSITION 3.6: *Let M be complete Riemannian, D a (generalized) Dirac operator on M. Let $h \in C_0(\mathbb{R})$. Then the operator $h(D)$ belongs to $C^*(M)$.*

PROOF: That $h(D)$ is locally compact is a version of elliptic regularity; we need to check the finite propagation condition. This is a consequence of the fact[3] that the operator e^{itD} has propagation $|t|$. This is called the finite propagation speed property of the Dirac wave equation — it tells us that a disturbance governed by the hyperbolic equation $\partial u/\partial t - iDu = 0$ travels with unit speed. Granted this property one can use Fourier analysis to write

$$h(D) = \frac{1}{2\pi} \int \hat{h}(t) e^{itD} dt,$$

where $\hat{h}$ is the Fourier transform of h. Thus, if $\hat{h}$ is compactly supported, $h(D)$ is a finite propagation operator. But functions with compactly supported Fourier transform are dense in $C_0(\mathbb{R})$, and the functional calculus map $h \mapsto h(D)$ is continuous, so the result follows. □

Now let $A = C^*(X) \subseteq \mathfrak{B}(H)$. Let $B = \mathfrak{M}(A) \subseteq \mathfrak{B}(H)$ be the *multiplier algebra* of A, that is the set of all operators S such that $ST \in A$ and $TS \in A$ for all $T \in A$. Then B is a C^*-algebra containing A as an ideal. Moreover, it follows from the same kind of analysis as was given above that for any chopping function χ, the operator $\chi(D)$ belongs to B, and $\chi(D)^2 - 1$ belongs to A. Thus the class

$$[\chi(D)] \in B/A$$

is a self-adjoint involution (element of square one) in the quotient algebra.

Involutions define elements of K-theory. In fact, an involution F in an algebra defines a projection $(1 + F)/2$, hence an element of K_0. However, in the 'even' case we recall that we are also given some extra data: D in fact acts on a *graded* vector bundle $S = S_0 \oplus S_1$, mapping sections of S_0 to sections of S_1. In this case $L^2(S_0)$ and $L^2(S_1)$ are both adequate M-modules, so we can find a *unitary*

[3] A proof may be found in [**25**], reproduced in [**90**], proposition 5.5.

operator $V\colon L^2(S_0) \to L^2(S_1)$ that covers the identity map $M \to M$, in the sense of lemma 3.5. (The choice of such a V does not matter.) Then

$$T = U^*\chi(D)\colon L^2(S_0) \to L^2(S_0) \qquad (*)$$

is an operator in B such that $TT^* - 1$ and $T^*T - 1$ are in A, so $[T]$ is unitary in B/A and defines an element of K_1. We therefore have defined a class $[\chi(D)] \in K_i(B/A)$, with $i = 1$ in the even (graded) case and $i = 0$ in the odd case.

Now recall that the short exact sequence of C^*-algebras

$$0 \to A \to B \to B/A \to 0$$

determines a 6-term cyclic exact sequence of K-theory groups, which includes a boundary map $\partial\colon K_i(B/A) \to K_{i-1}(A)$.

DEFINITION 3.7: *The class* $\partial[\chi(D)] \in K_*(A) = K_*(C^*X)$ *is called the* coarse index *of the operator* D*, and is denoted by* $\operatorname{Ind} D$.

EXAMPLE: Let us work all this out in the case of a *compact* manifold M. Here $A = \mathfrak{K}$, the algebra of compact operators, $B = \mathfrak{B}$, the algebra of all bounded operators, and so $B/A = \mathfrak{B}/\mathfrak{K}$ is the so-called *Calkin algebra.* It is known that $K_0(\mathfrak{K}) = \mathbb{Z}$, $K_1(\mathfrak{K}) = 0$, so only the even case is significant. Moreover, the construction of $\partial\colon K_1(\mathfrak{B}/\mathfrak{K}) \to K_0(\mathfrak{K})$ can be made very explicit in this case: if $u \in \mathfrak{B}/\mathfrak{K}$ is unitary, choose $U \in \mathfrak{B}$ projecting to u. U need not be unitary, but, since $UU^* - 1$ and $U^*U - 1$ are compact, U is a Fredholm operator. One finds that $\partial[u] \in K_0(\mathfrak{K}) = \mathbb{Z}$ is just the Fredholm index of U.

Here, since M is compact, any unitary at all from $L^2(S_0)$ to $L^2(S_1)$ covers the identity map. Thus we find that our coarse index is simply the Fredholm index of $\chi(D)$ considered as a map $L^2(S_0) \to L^2(S_1)$, in other words, it is the ordinary index of D.

Just like the ordinary index, our coarse index has some homotopy invariance and vanishing properties. For now, we will state only the vanishing theorem.

PROPOSITION 3.8: *Let M and D be as above. Suppose that* $\operatorname{Ind}(D) \neq 0$. *Then*

(a) *In the even case, 0 must belong to the spectrum of D;*

(b) *In the odd case, the spectrum of D must be the whole of* $\mathbb{R}$.

PROOF: Consider the even case, and use the notation above. If zero did not belong to Spectrum(D), then one could choose a chopping function χ such that $\chi(\lambda) = \pm 1$ for all $\lambda \in$ Spectrum(D). The operator T defined in $(*)$ above is then a unitary in B, hence the index lies in the image of the composite homomorphism

$$K_1(B) \to K_1(B/A) \to K_0(A).$$

But this composite is zero, because the six-term K-theory sequence is exact. A similar argument works in the odd case, but now we find that the spectrum cannot have any gaps at all; this is because one need no longer require that chopping functions be odd functions, because there is no grading that they have to preserve. □

COROLLARY 3.9: *Let M be a complete spin manifold. If M has a metric of uniformly positive scalar curvature (that is $\kappa \geqslant C > 0$), then the coarse index of the spinor Dirac operator vanishes.*

PROOF: The Lichnerowicz-Weitzenbock formula shows that the spectrum of D has a gap over the interval $(-C/4, C/4)$. □

The situation as regards the homotopy invariance of the coarse signature is rather less satisfactory. We will discuss in lecture 7 what the 'best possible' result in this regard might be, and what has so far been proved. Unfortunately, these are not the same.

Relative index theory

One frequently encounters the following situation: an operator is given on a complete manifold M, and it is known to be invertible on the complement of some subset Z of M. Or two operators are given, together with an isomorphism between their restrictions to the complement of Z. In such a situation one would like to be able to define an index 'supported on Z', which maps to the usual index under a forgetful map. To do this one needs to consider certain ideals in the C^*-algebra of a coarse space.

DEFINITION 3.10: *Let X be a metric space, $Z \subseteq X$, and let T be a finite propagation operator on some X-module. One says that T is* supported near Z *if there is a constant $R > 0$ such that $T\varphi = \varphi T = 0$ whenever $d(\mathrm{Supp}(\varphi), Z) > R$).*

One easily checks that the (locally compact) operators supported near Z form an ideal in the algebra of all (locally compact) finite propagation operators. Taking the closure we obtain an ideal in $C^*(X)$, which we will call $C^*_X(Z)$.

EXAMPLE: If Z is a point (or any compact subset) then $C^*_X(Z)$ is the ideal $\mathfrak{K}$ of compact operators.

The following proposition is stated for the spinor Dirac operator, since it is most useful in this case; but it has analogues for other generalized Dirac operators.

PROPOSITION 3.11: *Let M be a complete Riemannian spin manifold, and suppose that the scalar curvature of M is* uniformly *positive outside some subset $Z \subseteq M$. Then there is defined a 'relative index'*

$$\mathrm{Ind}_Z\, D \in K_*(C^*_X(Z))$$

of the Dirac operator, which maps to the absolute index (defined in the previous section) under the natural forgetful map.

The proof is a matter of choosing appropriate chopping functions and verifying that certain operators satisfy certain support conditions, and can once again be based on the finite propagation speed method. Notice that if Z is compact, we are saying that D is actually a Fredholm operator, with an index in $\mathbb{Z}$; this is a result of Gromov and Lawson [**48**].

REMARK: It is easy to verify [**95**, Chapter 5] that if M is non-compact then the natural homomorphism $K_*(\mathfrak{K}) \to K_*(C^*M)$ is zero. From the discussion above it follows that the hypothesis of the vanishing theorem 3.9 can be weakened to uniformly positive scalar curvature *outside a compact set.* Similar refined vanishing theorems can be obtained in other circumstances if one can calculate the K-theory of the relevant ideals $C^*_X(Z)$, as is done in some circumstances in lecture 9. However, we will not need these results.

Consider now the situation where two operators 'agree off Z'. More precisely, suppose given a metric space X and a subset $Z \subseteq X$. Suppose given also complete Riemannian manifolds M_i, subsets $Z_i \subseteq M_i$, coarse maps $f_i \colon M_i \to X$ such that $f(Z_i) \subseteq Z$, and generalized Dirac operators D_i on M_i (all these for $i = 1, 2$), such that the operators D_1 on $M_1 \backslash Z_1$ and D_2 on $M_2 \backslash Z_2$ are isomorphic. Then one can define a relative index

$$\operatorname{Ind}_r(D_1, D_2) \in K_*(C^*_X(Z))$$

which maps, under the forgetful map, to $f_{1*}(\operatorname{Ind} D_1) - f_{2*}(\operatorname{Ind} D_2) \in K_*(C^*X)$.

One way to do this is the following [**94**]. Form the algebra A obtained from a pull-back square

$$\begin{array}{ccc} A & \longrightarrow & C^*(X) \\ \downarrow & & \downarrow \\ C^*(X) & \longrightarrow & C^*(X)/C^*_X(Z) \end{array}$$

so that A consists of those pairs of operators in $C^*(X)$ which 'agree far from Z'. Then one can show by finite propagation speed arguments as above that the pair $(f_{1*}(\operatorname{Ind} D_1), f_{2*}(\operatorname{Ind} D_2))$ gives an element of $K_*(A)$. But there is a split short exact sequence

$$0 \to C^*_X(Z) \to A \to C^*(X) \to 0$$

and this gives a direct sum decomposition

$$K_*(A) = K_*(C^*_X(Z)) \oplus K_*(C^*(X))$$

on the level of K-theory.

THEOREM 3.12: (RELATIVE INDEX THEOREM) *The relative index defined above depends only on the geometry of Z_1 and Z_2 and the operators D_1 and D_2 on them.*

We have given an informal statement, since a formal one requires a somewhat indigestible notation. One needs to contemplate two sets of relative-index data as above, together with isomorphisms over the Z's between the relevant parts of one set and the relevant parts of the other. The conclusion is that the relative indices are the same.

Appendix: K-theory for C^*-algebras

Our object in this section is to review briefly the definitions and constructions of C^*-algebra K-theory. For fuller details consult [**14**] or [**109**].

Let A be a *unital* C^*-algebra. We let $P_r(A)$ denote the set of self-adjoint projections ($e = e^2 = e^*$) in the matrix algebra $M_r(A)$. We let $P(A)$ denote the inductive limit $\lim P_r(A)$ (under the obvious inclusion-by-zero maps). Consider $\pi_0 P(A)$; the operation of direct sum gives it the structure of an abelian semigroup. (The various choices that are implicit in the formation of direct sums wash out when we pass to homotopy classes.)

DEFINITION 3.13: *$K_0(A)$ is the Grothendieck group of the semigroup $\pi_0 P(A)$.*

We let $U_r(A)$ denote the set of unitaries ($uu^* = u^*u = 1$) in the matrix algebra $M_r(A)$. Let $U(A)$ be the direct limit (under the obvious inclusion-by-one maps). It is a topological group.

DEFINITION 3.14: *$K_1(A)$ is the group $\pi_0 U(A)$ of components of $U(A)$.*

We make a few remarks. Clearly, K_i, $i = 0, 1$, is a covariant functor of unital C^*-algebras and C^*-homomorphisms. If X is a compact space, then $K_i(C(X))$ is equal to the topological K-group $K^{-i}(X)$. This is clear for $i = 1$, and for $i = 0$ it follows from the identification of vector bundles over X with finite projective modules over $C(X)$. The C^*-algebra K_0 group is the same as the K_0-group as defined in algebraic K-theory [**75**] (the reason being that sufficiently close projections in a C^*-algebra are automatically conjugate), but the C^*-algebra K_1 group differs (already for $A = \mathbb{C}$) from the algebraic K_1 group. We will not make use of algebraic K-theory and will therefore not introduce any special notation here.

The requirement that our projections be self-adjoint and our invertibles be unitary can be dropped without changing the K-groups obtained. In fact, the space of all projections in a C^*-algebra is homotopy equivalent to the space of self-adjoint projections, and the group of all invertibles is homotopy equivalent to the group of unitaries.

One also wants to consider the K-theory of *non*-unital C^*-algebras (our algebra $C^*(X)$, for instance, is always non-unital). For this purpose one performs an algebraic analogue of the 1-point compactification which is used to define K-theory of locally compact spaces. Let J be a nonunital C^*-algebra, and let $J^+ = J \oplus \mathbb{C} = \{j + \lambda 1 : j \in J, \lambda \in \mathbb{C}\}$ with the obvious involution and multiplication law.

LEMMA 3.15: *J^+ can be made into a C^*-algebra in which J is a closed ideal.*

Now by analogy with the theory of locally compact spaces we define $K_i(J) = \operatorname{Ker}(K_i(J^+) \to K_i(\mathbb{C}))$. One can show without difficulty that if J already had a unit (which we had somehow failed to notice) then this definition does coincide with the preceding one.

K-theory for C^*-algebras has three characteristic properties. The first of these is *homotopy invariance:* Let A and B be C^*-algebras. A *homotopy* of C^*-homomorphisms from A to B is, by definition, a C^*-homomorphism from A to $C[0,1] \otimes B$ (the latter algebra can be described without recourse to tensor products simply as the continuous B-valued functions on $[0,1]$). Evaluation at 0 and at 1 of such a homotopy gives two C^*-homomorphisms from A to B, and the homotopy invariance property states that these two homomorphisms induce the same map on K-theory.

For the second property, the *stability* property, we need an important C^*-algebra, the algebra $\mathfrak{K} = \mathfrak{K}(H)$ of *compact* operators on a Hilbert space H. We recall that $\mathfrak{K}(H)$ is the norm closure of the algebra of finite rank operators. Thus $\mathfrak{K}$ is the inductive limit (in the sense of C^*-algebras) of the matrix algebras $M_n(\mathbb{C})$. More generally, for any C^*-algebra A, $A \otimes \mathfrak{K}$ is the inductive limit of the matrix algebras $M_n(A)$. Since matrices are already involved in the definition of K-theory, it is not surprising that $K_i(A) \cong K_i(A \otimes \mathfrak{K})$. This is the stability property.

The third characteristic property of C^*-algebra K-theory is *exactness.* We need to know about quotients of C^*-algebras. The following lemma is simple, but not completely trivial.

LEMMA 3.16: *Let A be a C^*-algebra, J a closed two-sided ideal. Then $J = J^*$, and the quotient A/J is a C^*-algebra also.*

This makes it possible to talk about *short exact sequences* of C^*-algebras. Let $0 \to J \to A \to A/J \to 0$ be such a short exact sequence. The exactness property says that there is a natural 6-term exact sequence of K-theory groups

$$\begin{array}{ccccc} K_1(J) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/J) \\ \uparrow & & & & \downarrow \\ K_0(A/J) & \longleftarrow & K_0(A) & \longleftarrow & K_0(J) \end{array}$$

The cyclic nature of the exact sequence is a version of the *Bott periodicity theorem* for C^*-algebra K-theory. It leads to the notational convention that, for any $n \in \mathbb{Z}$, $K_n(A)$ denotes $K_0(A)$ if n is even and $K_1(A)$ if n is odd.

REMARK: By applying this result to a suitable 'suspension sequence' one can prove the Bott periodicity theorem in a more familiar guise: the homotopy groups of the space $U(A)$ are periodic with period 2, and $\pi_i U(A) = K_{i+1}(A)$, $i \geqslant 0$.

A simple consequence of the exact sequence above is the 'Mayer-Vietoris' exact sequence for ideals in a C^*-algebra, which we will also use.

PROPOSITION 3.17: *Let I and J be (closed two-sided) ideals in a C^*-algebra A,* *and suppose that $I + J = A$. Then there is a cyclic exact sequence*

$$\begin{array}{ccccc} K_1(I \cap J) & \longrightarrow & K_1(I) \oplus K_1(J) & \longrightarrow & K_1(A) \\ \uparrow & & & & \downarrow \\ K_0(A) & \longleftarrow & K_0(I) \oplus K_0(J) & \longleftarrow & K_0(I \cap J) \end{array}$$

REMARK: Throughout this monograph we have made use of *complex* C^*-algebras and their K-theory. There is also a theory of *real* C^*-algebras and an associated 8-periodic *real* K-theory. There seems to be little difficulty in generalizing all our results to a real context, and this should be done in order to get the sharpest possible forms of the theorems relating analysis to surgery and to positive scalar curvature. But for simplicity we will not go into the details of this.

Notes and references: As already mentioned, [**109**] and [**14**] contain introductions to the subject of C^*-algebra K-theory. The book [**28**] is filled with profound applications of operator algebras to geometry. Versions[4] of the definition of $C^*(X)$ and of the coarse index first appear in [**91**], motivated by the problem of generalizing Connes' index theorem for measured foliations [**27**]. The present, hopefully final, form of the definitions comes from [**58**]. There is a version 'with coefficients' in [**53**].

Relative index theory has been developed from a number of points of view, starting with the paper [**48**]. See [**17, 21, 61**]. These articles consider the situation in which the relative index is an ordinary Fredholm index (in our notation, Z is compact). The generalization given here is based on [**94**].

There is a close analogy between the C^*-algebra $C^*(X)$ and the *bounded category over* X, $\mathfrak{C}(X; A)$, defined for any ring A to have objects free based A-modules equipped with a locally finite reference map from the basis to X, and morphisms A-module morphisms that satisfy a 'finite propagation condition' in X. See [**83, 88**] for this construction and some of its applications. An attempt to make a direct connection between the two notions is in [**82**].

[4] In our present understanding, this paper defines the 'rough index' rather than the 'coarse index'; but this distinction was not apparent at the time.

CHAPTER 4

An example of a higher index theorem

So far we have spoken rather abstractly about the 'higher index' $\mathrm{Ind}(D) \in K_*(C^*M)$, for an operator on a non-compact complete Riemannian manifold M, but we have not given any examples of such 'higher indices'. In fact, nothing that we have said rules out the rather depressing possibility that $K_*(C^*M)$ might be *zero* for all non-compact spaces M — a possibility which even seems plausible when we discover that the K-theory of the C^*-algebra of *all* locally compact operators on M actually *is* zero! Fortunately, however, the finite propagation condition saves the day. In this lecture we will give the first example of a non-trivial higher index theorem in coarse geometry — the so-called 'partitioned manifold' index theorem.

Odd operators

The index theorem that we will consider is one for elliptic operators on complete *odd-dimensional* manifolds. Here we review briefly the odd-dimensional counterparts of our standard examples, the Dirac and signature operators.

For the spinor Dirac operator, there is no great difference from the even-dimensional case. In fact, if M is an odd-dimensional complete Riemannian spin manifold, then the spinor bundle S over M and its Dirac operator D are still defined as before. It is now no longer the case that the spinor bundle decomposes as a direct sum $S = S_0 \oplus S_1$; consequently, the Dirac operator is an *ungraded* self-adjoint operator. As we have seen, this means that it will have a coarse index in $K_1(C^*M)$.

There is however a difference in the definition of the signature operator. Recall that on an even-dimensional manifold the signature operator was $D = d + d^*$ graded by a certain anticommuting involution ε constructed from the Hodge star operator. In the odd-dimensional case the analogous involution ε in fact *commutes* with D, so decomposes it into a direct sum of two self-adjoint operators. We want to take just *one* of these summands as the 'odd-dimensional signature operator'. To be definite, we say that the odd signature operator is the Dirac operator associated to the Clifford algebra of TM restricted to the $+1$-eigenspace of the action of the volume form. In terms of differential forms, this says that $D = *d \pm d*$ restricted to *even*-dimensional forms only.

We'll need to make use of a certain 'boundary' construction for generalized Dirac operators of this type. Specifically, let M be a manifold with boundary ∂M. Let S be a Clifford bundle over M; if M is even-dimensional we assume that S is graded by a grading operator ε. Clifford multiplication by $i = \sqrt{-1}$ times the inward-pointing unit normal to ∂M defines another involution ν on the restriction $S_{|\partial M}$. From these data we can construct a new Clifford bundle ∂S on ∂M,as follows:

(i) If M is even-dimensional, then $S_{\partial M}$ is equipped with two anticommuting involutions ε and ν. Thus $-i\varepsilon\nu$ is an involution and we take ∂S to be its $+1$-eigenspace. Since ν and ε both anticommute with the Clifford action of $T\partial M$, one sees that this is a Clifford bundle over ∂M;
(ii) If M is odd-dimensional, then $S_{\partial M}$ is a Clifford bundle graded by the involution ν. We take this as the definition of ∂S in this case.

We will say that the (generalized) Dirac operator for ∂S is the *boundary* of the (generalized) Dirac operator for S (this statement has a precise interpretation in terms of the boundary map for K-homology). Now we have the following table giving the boundaries of each of our standard operators in both the even and odd dimensional cases:

Operator on M	Boundary operator	
	M even	M odd
Spinor Dirac	Spinor Dirac	Spinor Dirac
Signature	Signature $\oplus$ Signature	Signature

The slogan 'the boundary of Dirac is Dirac' always holds good, but 'the boundary of signature is signature' is true only up to some power of 2. Eventually, this will require us to tensor by $\mathbb{Z}[\frac{1}{2}]$ when we compare surgery theory to analytic invariants.

Partitions of a noncompact manifold

To formulate a higher index theorem we need a procedure for constructing maps $K_*(C^*M) \to \mathbb{R}$. One natural hope might be that the algebra C^*M should admit a *trace*, that is a positive linear functional τ such that $\tau(ab) = \tau(ba)$ for all appropriate a and b. Such a trace is known to induce a map τ_* from K_0 to $\mathbb{R}$. In the context of rough geometry, this idea can be implemented for certain 'closed at infinity' manifolds [**91, 92**]. However, in coarse geometry, this fails because of the following lemma:

LEMMA 4.1: *There are no non-trivial traces on C^*M, for M non-compact.*

One proves this by showing that any such trace would give rise to a non-trivial shift-invariant positive linear map from the vector space of *all* integer valued

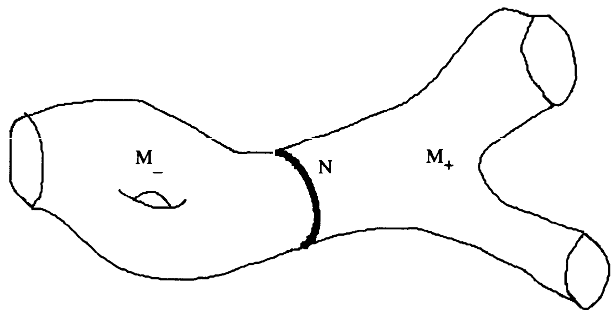

FIGURE 1. Partitioned manifold

functions on $\mathbb{N}$ to $\mathbb{R}$. But it is easy to see that no such map exists. We therefore seek a more sophisticated technique for constructing maps from K_*C^*M.

DEFINITION 4.2: *Let M be a manifold. A* partition *of M is a decomposition of M as the union $M_- \cup M_+$ of two submanifolds with boundary, such that $M_- \cap M_+ = \partial M_- = \partial M_+ = N$, where N is a compact codimension one submanifold of M.*

In other words, M is cut apart into two halves along the codimension one submanifold N (see figure 1).

Suppose that M is a complete Riemannian manifold with a partition N as above. We will construct a homomorphism $\varphi_N\colon K_1(C^*M) \to \mathbb{Z}$. Let H be the M-module on which $C^*(M)$ is defined, and let H_+ be the restriction of H to M_+; that is, H_+ is the range of the orthogonal projection operator P on H defined by multiplying by the characteristic function of M_+.

LEMMA 4.3: *For any operator $T \in C^*(M)$ the commutator $[T,P] = TP - PT$ is compact.*

PROOF: Without loss of generality we may assume that T has finite propagation R. Then, outside a $2R$-neighbourhood of N, the commutator $[T,P]$ is zero (because in that region T cannot see the difference between P and a multiple of the identity). Hence $[T,P]$ is a compactly supported operator, and it is locally compact because T is; so it is compact. □

Now let A be the subalgebra of $\mathfrak{B}(H)$ obtained by adjoining a unit to $C^*(M)$; then, by definition, $K_1(C^*(M)) = K_1(A)$. For $a \in M_n(A)$, define T_a to be the operator on $(H_+)^n$ obtained by first multiplying by a and then compressing to $(H_+)^n$ by the orthogonal projection $P \oplus \cdots \oplus P$. Now P commutes modulo compacts with A, and this implies that $T_aT_{a'} = T_{aa'}$, modulo compacts. In particular, if u is a unitary in $M_n(A)$, then T_u is a Fredholm operator, with parametrix T_{u^*}. Our map $\varphi_N\colon K_1(C^*(M)) \to \mathbb{Z}$ is then defined by

$$u \mapsto \operatorname{Index} T_u.$$

One verifies easily that this is well-defined on K-theory and is a homomorphism.

Now we can state the index theorem for partitioned manifolds.

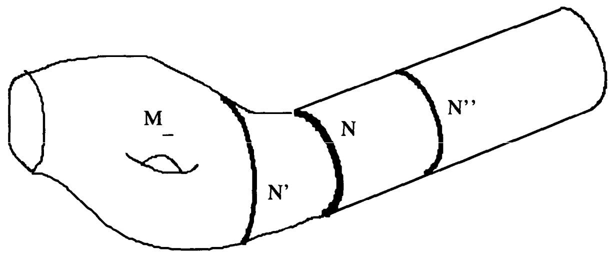

FIGURE 2. Reduction to the product case: the manifold $\hat{M}$.

THEOREM 4.4: *Let M be a partitioned manifold of odd dimension, and let $\varphi_N\colon K_1(C^*M) \to \mathbb{Z}$ be the homomorphism defined by the partition. Then for a generalized Dirac operator D on M,*

$$\varphi_N(\operatorname{Ind} D) = \operatorname{Ind} D_N,$$

where D_N is the boundary operator ∂D on N thought of as the boundary of M_+.

REMARK: Since ∂D is an elliptic operator on a compact manifold, its index can be calculated by the ordinary Atiyah-Singer index theorem and expressed in terms of the Pontrjagin classes of N. For example, in the case of the spinor Dirac operator the index is just the $\widehat{\mathcal{A}}$-genus of N.

Here is an outline of the proof, as simplified by Higson. One begins by checking the theorem in the case of a *product* manifold $M = \mathbb{R} \times N$. In this case one can analyse the Dirac operator on M by the method of separation of variables: it is a 'product' of a Dirac operator D_N on N with the Dirac operator id/dx of $\mathbb{R}$. By explicit calculation, one can then check that $\varphi_N(\operatorname{Ind} D) = \operatorname{Ind} D_N$. The remainder of the proof consists of a reduction to this case. We will need the notion of *bordism* between partitions of M: two partitions $M = M_- \cup M_+ = M'_- \cup M'_+$ are *bordant* if the symmetric difference $M_- \vartriangle M'_- = M'_+ \vartriangle M_+$ is compact. Now we have

LEMMA 4.5: *Bordant partitions define the same map $K_1(C^*M) \to \mathbb{Z}$.*

Indeed, let $u \in K_1(C^*M)$ and let T_u, T'_u be the Toeplitz operators corresponding to u and the two partitions. By definition, T_u and T'_u differ by an operator which is locally compact (because u is) and compactly supported (by definition of bordism), hence is compact. The index of a Fredholm operator is invariant under compact perturbations, so this proves the result.

The other key lemma for the reduction follows from the finite propagation speed arguments that we have used before.

LEMMA 4.6: $\varphi_N(\operatorname{Ind} D)$ *depends only on the geometry of M in an (arbitrarily small) neighbourhood of N.*

Granted these two lemmas, the reduction to the product case may be made as follows. Let $M = M_- \cup M_+$ be the given partition, with partitioning submanifold N. We can make N bordant to another partitioning submanifold N' contained in M_-, say, and by bordism invariance $\varphi_N(\operatorname{Ind} D) = \varphi_{N'}(\operatorname{Ind} D)$. Now, since the index is localized near N' which is disjoint from M_+, we may (without changing the index) replace M_+ by a copy of $N \times \mathbb{R}^+$ equipped with a product metric outside of some neighbourhood of $N \times \{0\}$. Let $\hat{M}$ denote the modified manifold so obtained (see figure 2). The manifold $\hat{M}$ is still partitioned by N', but now N' is bordant to a copy $N'' = N \times \{r\}$ of N sitting inside a part of $\hat{M}$ that is isometric to a product$N \times \mathbb{R}^+$. Thus, without changing the index, we may replace N' by N''. Finally, since N'' has a neighbourhood that is isometric to a product, the locality property of the index shows that we may (without changing the index) replace the whole of $\hat{M}$ by the product $N \times \mathbb{R}$. This completes the reduction.

EXAMPLE: In the original proof of the Atiyah-Singer index theorem [**80**], one key step was the proof that the analytical index is cobordism invariant: that is, if N and N' are compact manifolds which are equipped with generalized Dirac operators D_N and $D_{N'}$, and there is an (appropriately oriented) cobordism W between N and N', then $\operatorname{Ind} D_N = \operatorname{Ind} D_{N'}$. A simple approach to this result, which avoids the technicalities of elliptic boundary value problems, can be given using the higher index theorem. Let M be the non-compact manifold constructed by attaching cylinders $N \times \mathbb{R}^-$ and $N' \times \mathbb{R}^+$ to the ends of the cobordism W. Then both N and N' define partitions of M, and these partitions are bordant — they differ by the compact set W. Thus, if D denotes the Dirac operator on M, $\varphi_N(\operatorname{Ind} D) = \varphi_{N'}(\operatorname{Ind} D)$. But by the calculations above, $\varphi_N(\operatorname{Ind} D) = \operatorname{Ind} D_N$ and $\varphi_{N'}(\operatorname{Ind} D) = \operatorname{Ind} D_{N'}$.

This implication was observed by Higson [**51**].

EXAMPLE: Suppose that M is a spin manifold. The Lichnerowicz vanishing theorem (3.9) tells us that if M has uniformly positive scalar curvature, then $\operatorname{Ind} D = 0 \in K_1(C^*(M))$. Combining this with the higher index theorem, we find that if N is such that $\operatorname{Ind} D_N = \widehat{\mathcal{A}}(N) \neq 0$, then M cannot admit any complete metric of uniformly positive scalar curvature. In particular this holds for $N \times \mathbb{R}$. This result is due to Gromov and Lawson[1] [**48**].

Here is an an example which does not explicitly involve non-compact manifolds in its statement. Suppose that V is a compact spin manifold of dimension congruent to 1 modulo 4, and suppose given a homomorphism α from $\pi_1 V$ to $\mathbb{Z}$. Then we note

(i) Because the circle S^1 is the classifying space $B\mathbb{Z}$, the homomorphism α is induced by a map $f\colon V \to S^1$, unique up to homotopy;
(ii) Via the Hurewicz theorem, α gives a homomorphism $H_1(X;\mathbb{Z}) \to \mathbb{Z}$, and hence a cohomology class in $H^1(X;\mathbb{Z})$; this class is just the pull-back $f^*(s)$ of the standard generator $s \in H^1(S^1;\mathbb{Z})$.

[1] In fact they prove the stronger result that M has no complete metric of *positive* scalar curvature — uniformity is not required.

The number $\langle \widehat{\mathcal{A}}(V) \smile f^*(s), [V] \rangle$ is called the *higher $\widehat{\mathcal{A}}$-genus* corresponding to α. Now we have

PROPOSITION 4.7: *If the higher $\widehat{\mathcal{A}}$-genus corresponding to some α is non-zero, then V cannot have any metric of positive scalar curvature.*

PROOF: Notice that s is the cohomology class Poincaré dual to the homology class represented by a point p in S^1. Using transversality theory, choose f in its homotopy class to be transverse at p, so that $f^{-1}(p)$ is a codimension one submanifold N of V with trivial normal bundle. The higher $\widehat{\mathcal{A}}$-genus of V is then just the ordinary $\widehat{\mathcal{A}}$-genus of N. Let $\tilde{V}$ be the $\mathbb{Z}$-covering of V corresponding to α. Then $\tilde{V}$ is partitioned with N as partitioning submanifold. Since the $\widehat{\mathcal{A}}$-genus of N is non-zero, the higher index theorem shows that $\tilde{V}$ cannot have any metric of uniformly positive scalar curvature. But a metric of positive scalar curvature on V would lift to such a metric on $\tilde{V}$. □

EXAMPLE: In the preceding example we may replace the Dirac operator by the signature operator, and correspondingly replace the $\widehat{\mathcal{A}}$-genus by the $\mathcal{L}$-genus. Then we obtain the definition of the *higher signature* of the compact manifold V corresponding to a homomorphism $\alpha\colon V \to \mathbb{Z}$. Arguing as before and combining the higher index theorem with the homotopy invariance of the analytic signature[2] we obtain

PROPOSITION 4.8: *The higher signature of a compact manifold V, corresponding to some given homomorphism $\pi_1 V \to \mathbb{Z}$, is invariant under oriented homotopy equivalence.*

This result was known to Novikov in the early sixties, and it eventually led to the conjecture that all the higher signatures (arising from the classifying space $B\pi_1(V)$ rather than $B\mathbb{Z}$) must be homotopy invariants. Moreover, one can deduce from it that the class $L_k(V) \in H^{4k}(V;\mathbb{Q})$ is invariant under homeomorphisms of the $(4k+1)$-dimensional manifold V; this line of thought led to the proof of the topological invariance of the rational Pontrjagin classes [**79**]. We will look at an analytic approach to these results in lecture 7.

The K-theory of $C^*(|\mathbb{R}|)$.

A partitioned manifold is in reality nothing more than a manifold admitting a coarse 'control' map $c\colon M \to \mathbb{R}$ — the map can be easily constructed from the distance function to the partitioning submanifold. The partition of M is then pulled back, in the obvious sense, from the canonical partition of $\mathbb{R}$. We therefore see that the groups $K_*(C^*(|\mathbb{R}|))$ are[3] the universal receptacles for the higher indices of operators on partitioned manifolds, and it is of some interest to

[2] See [**63**], or our discussion in lecture 7.

[3] We use the notation $|\mathbb{R}|$ for the underlying metric space of $\mathbb{R}$, in order to avoid confusion with the group C^*-algebra.

compute exactly what these groups are. Notice that, since some higher indices are non-trivial, the group $K_1(C^*(|\mathbb{R}|))$ must be non-zero.

PROPOSITION 4.9: *The K-theory of $C^*(|\mathbb{R}|)$ is given as follows:*

$$K_i(C^*(|\mathbb{R}|)) = \begin{cases} 0 & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = 1 \end{cases}$$

PROOF: There are several ways to prove this proposition, and it will follow from our more general calculations in lecture 8. However a direct proof can also be given using the Pimsner-Voiculescu exact sequence for the K-theory of the crossed product of a C^*-algebra A with an action of $\mathbb{Z}$. Recall that if α is the generator of the $\mathbb{Z}$-action on A, then the crossed product $A \rtimes_\alpha \mathbb{Z}$ is generated by A together with a unitary U such that conjugation by U induces the automorphism α on A. The notion is relevant here because of the following observation: let A be the C^*-algebra $\ell^\infty(\mathbb{Z}; \mathfrak{K})$ of bounded, compact-operator-valued functions on $\mathbb{Z}$. Then A admits an obvious $\mathbb{Z}$-action by translation and we have

$$C^*(|\mathbb{R}|) = A \rtimes \mathbb{Z}.$$

To prove this one simply observes that both sides of the equation can be considered as given by doubly-infinite matrices of compact operators and the multiplication laws are the same.

Now Pimsner and Voiculescu [**84**] gave an exact sequence for the K-theory of a crossed product:

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes \mathbb{Z}) & \longleftarrow & K_1(A) & \longleftarrow & K_1(A) \end{array}$$

where the maps $K_i(A) \to K_i(A)$ are given by $1 - \alpha_*$. In our situation, $K_1(A) = 0$, and $K_0(A)$ is the abelian group $\mathbb{Z}^{\mathbb{Z}}$ of all two-way infinite sequences of integers, with α_* acting by translation, so that $1 - \alpha_*$ maps the sequence a_n to the sequence $a_n - a_{n+1}$. It is easy to see that $1 - \alpha_*$ is surjective on $\mathbb{Z}^{\mathbb{Z}}$, with kernel $\mathbb{Z}$ consisting of constant functions; and this gives the computation. □

REMARK: One can show that a generator of $K_1(C^*(|\mathbb{R}|))$ is given by the coarse index of the Dirac operator id/dx. Now this operator is invariant under translation by $\mathbb{R}$, so the whole of the definition of the index can be carried out inside the subalgebra of translation-invariant elements in $C^*(|\mathbb{R}|)$. But it is easy to see that this subalgebra is simply the group C^*-algebra $C^*_r(\mathbb{R})$ (compare 5.14), which by Fourier analysis is isomorphic to $C_0(\widehat{\mathbb{R}})$, the algebra of continuous functions vanishing at infinity on the dual group. The K-theory of the translation-invariant subalgebra can therefore be calculated, and we find

PROPOSITION 4.10: *The inclusion of the translation-invariant subalgebra $C^*_r(\mathbb{R})$ in $C^*(|\mathbb{R}|)$ induces an isomorphism on K-theory.*

It seems an interesting question whether a direct proof of this result (perhaps using some averaging procedure) can be found. It appears that such an argument could lead to a new proof of the Bott periodicity theorem.

REMARK: The analogue of proposition 4.10 is false for most groups ($\mathbb{Z}$ provides a simple example) but it is conceivably true for all contractible Lie groups; this would follow from the Baum-Connes conjecture [**10**].

REMARK: An intriguing contrast between coarse and rough geometry can be seen when one tries to duplicate the calculation of 4.9 for the C^*-algebra associated to the rough geometry of $\mathbb{R}$ (which is the C^*-closure of the algebra of 'uniform smoothing operators' on $\mathbb{R}$ defined in [**91**]). The algebra can still be written as a crossed product but the algebra A is smaller; the effect of the bounded geometry hypotheses is that $K_0(A)$ is now the subgroup of $\mathbb{Z}^{\mathbb{Z}}$ consisting of *bounded* sequences. Clearly, not every *bounded* sequence b_n can be written as $a_n - a_{n+1}$ for some *bounded* sequence a_n, so that $1-\alpha_*$ is now far from surjective, and so K_0 of the uniform algebra is quite large; in fact it is the elements of this K_0 group that are detected by the renormalized index theory of [**91**].

The methods used to calculate $K_*(C^*(X))$ for $X = |\mathbb{R}|$ generalize to the case $X = |\mathbb{R}^n|$, and one finds that $K_i(C^*(|\mathbb{R}^n|)$ is 0 if $i \neq n$ modulo 2, and is $\mathbb{Z}$ (generated by the coarse index of the Dirac operator) if $i = n$ modulo 2. Now suppose that $c\colon M \to \mathbb{R}^n$ is a coarse map from a complete Riemannian manifold M to $\mathbb{R}^n$. Then we have a homomorphism

$$c_*\colon K_n(C^*(M)) \to \mathbb{Z}$$

which is analogous to φ_N in the case $n = 1$. Corresponding to the partitioned manifolds index theorem we have the *$\mathbb{R}^n$-bounded index theorem*: If c is made transverse at $0 \in \mathbb{R}^n$, so that $c^{-1}(0)$ is a compact submanifold N, and if D is a Dirac operator on M, then $c_*(\operatorname{Ind} D) = \operatorname{Ind} D_N$, where the operator D_N on N is constructed from D by a procedure analogous to that used in the $n = 1$ case. In particular, if D is Dirac, then D_N is Dirac; if D is signature, then D_N is some power of 2 times signature.

Notes and references: The partitioned manifold index theorem appears in [**93**], and the simplified proof we have given comes from [**51**]. Analogous theorems have been stated by many other authors, see for example [**4**], [**21, 20**], [**71**], [**86**]. The computation of the K-theory of $C^*(|\mathbb{R}|)$ using the Pimsner-Voiculescu sequence has been part of the folklore of the subject for a long time; it was written down in [**113**]. The $\mathbb{R}^n$-bounded index theorem is in [**95**]; but the same geometry, differently connected to analysis, is the basis of the 'hyperspherical' ideas of [**48**].

CHAPTER 5

Assembly

In this lecture we will make a connection between the groups $K_*(C^*X))$, which are analytically defined coarse invariants of X, and some homology groups of X which belong to more classical algebraic topology. To be specific, these are the *locally finite K-homology* groups of X. Motivated by considerations of index theory, Atiyah [**5**] suggested that it might be possible to define these homology groups in terms of functional analysis, and such a definition was found by Kasparov [**64, 65**] (see also [**19**]). We begin by reviewing this definition.

Kasparov's K-homology

Let X be a (locally compact Hausdorff) space. Recall that a Hilbert space H is said to be an X-module if it is equipped with an action of the C^*-algebra $C_0(X)$ of continuous functions on X that vanish at infinity. Moreover, an operator T on H is called *locally compact* if Tf and fT are compact for all $f \in C_0(X)$.

DEFINITION 5.1: *Let X be a space. An even (or odd)* Fredholm module *for X consists of the following data: an X-module H and an operator U (or P) on H_X such that*

(i) *In the even case, $UU^* - 1$, $U^*U - 1$, and the commutator $[U, f]$ are locally compact operators, for all $f \in C_0(X)$;*
(ii) *In the odd case, $P - P^*$, $P - P^2$, and the commutator $[P, f]$ are locally compact operators, for all $f \in C_0(X)$.*

There are various natural equivalence relations (homotopy, unitary equivalence and so on) on such Fredholm modules, and there is also a natural notion of direct sum. Kasparov showed that under direct sum the equivalence classes form an abelian group: this group is denoted $K_0(X)$ in the even case and $K_1(X)$ in the odd case.

REMARK: We can easily see that Kasparov's groups $K_*(X)$ will be covariantly functorial for proper maps. For a proper map $Y \to X$ induces a C^*-homomorphism $C_0(X) \to C_0(Y)$, and hence makes every Y-module into an X-module.

In Kasparov's definition, the X-module H can be arbitrary[1]. However, it is possible to realize the whole of K-homology by operators on a single X-module, provided that it is *adequate* in the sense of 3.4. This follows from

THEOREM 5.2: (VOICULESCU [**107**]; BROWN, DOUGLAS AND FILLMORE [**19**]) *Let H be an adequate X-module, and let H' be any nondegenerate*[2] *X-module. Then the X-modules H and $H \oplus H'$ are 'essentially equivalent', in the sense that there is a unitary $U\colon H \to H \oplus H'$ which commutes modulo compact operators with the $C_0(X)$-actions on these two modules.*

Thus, by embedding H' as a direct summand in $H \oplus H'$ and then conjugating by U, one can obtain a Fredholm module on H from any Fredholm module on H'.

We now assume that for each space X a single adequate X-module H_X has been fixed once and for all.

DEFINITION 5.3: *An operator $T \in \mathfrak{B}(H_X)$ is called* pseudolocal *if $[T, f]$ is compact for all $f \in C_0(X)$.*

The pseudolocal operators on H_X form a C^*-algebra, which we denote $\Psi^0(X)$. The locally compact operators form an ideal in $\Psi^0(X)$; we denote this by $\Psi^{-1}(X)$.

REMARK: The Ψ notation arises from the following example, which was the one considered by Atiyah. Suppose that X is a manifold and that $H_X = L^2(X)$. Elliptic differential operators on X (such as the generalized Dirac operators we considered in the first lecture) are then invertible modulo 'small' (smoothing) error terms, but their formal inverses (*parametrices*) cannot be differential operators. The enlarged class of *pseudodifferential operators* is defined so as to contain all differential operators together with the parametrices of elliptic differential operators. It is a simple consequence of standard facts about pseudodifferential operators [**106**] that a pseudodifferential operator of order zero on X belongs to our algebra $\Psi^0(X)$, and a pseudodifferential operator of order less than zero belongs to $\Psi^{-1}(X)$. An elliptic pseudodifferential operator of order zero on X (meaning one that is invertible modulo operators of order less than zero) therefore gives an invertible in $\Psi^0(X)/\Psi^{-1}(X)$, that is, an even Fredholm module.

Kasparov gave the following characterization of pseudolocality, which is often easier to check:

LEMMA 5.4: *An operator T on H_X is pseudolocal if and only if fTg is compact for all $f, g \in C_0(X)$ having disjoint supports.*

---

[1] In fact, in the even-dimensional case Kasparov even allows U to be a map from one X-module to another! There is a simple 'infinite repetition' trick which allows one to turn Fredholm modules of this kind into ones where U acts on a single module.

[2] This means that $\overline{C_0(X)H} = H$.

The 'only if' is obvious here; the 'if' requires a partition of unity adapted to the decomposition of X into level sets of f.

Notice that, by definition, an even (or odd) Fredholm module gives rise to a unitary (or projection) in the quotient algebra $\Psi^0(X)/\Psi^{-1}(X)$. In fact Kasparov's definition of K-homology is equivalent to

DEFINITION 5.5: *We define the K-homology groups $K_i(X)$ of X by*

$$K_i(X) = K_{i+1}(\Psi^0(X)/\Psi^{-1}(X)).$$

To explain the terminology 'K-homology', recall that K-theory is a generalized cohomology theory (that is, it satisfies the Eilenberg-Steenrod axioms except for the dimension axiom). It is known abstractly that to every generalized cohomology theory there is associated a dual generalized homology theory (see [**1**] for discussion). Now one can prove that the functors $X \mapsto K_i(X)$, $i = 0, 1$, form a periodic generalized homology theory and that this is in fact 'the' dual theory to ordinary K-theory. We note explicitly that this is a *locally finite* generalized homology theory: it admits cycles with infinite support[3].

Since we will need it later, let us sketch the proof that K-homology satisfies the excision axiom. We need to show that if A and B are closed subsets of X, with $A \cup B = X$, then there is a cyclic Mayer-Vietoris exact sequence

$$\begin{array}{ccccc} K_1(A\cap B) & \longrightarrow & K_1(A)\cap K_1(B) & \longrightarrow & K_1(X) \\ \uparrow & & & & \downarrow \\ K_0(X) & \longleftarrow & K_0(A)\cap K_0(B) & \longleftarrow & K_0(A\cap B) \end{array}$$

For Y a closed subset of X, define the ideal $\Psi^0_X(Y) \subseteq \Psi^0(X)$ by

$$\Psi^0_X(Y) = \{T \in \Psi^0(X) : f_{|Y} = 0 \Rightarrow Tf \in \mathfrak{K}\}.$$

One can check that the map $\Psi^0(Y) \to \Psi^0(X)$ provided by functoriality has image in the ideal $\Psi^0_X(Y)$. In fact we have the following 'excision' lemma:

LEMMA 5.6: *Suppose that X is compact. Then the map $\Psi^0(Y) \to \Psi^0_X(Y)$ induces an isomorphism in K-theory. Whether or not X is compact, the corresponding map $\Psi^0(Y)/\Psi^{-1}(Y) \to \Psi^0_X(Y)/\Psi^{-1}(X)$ always induces an isomorphism in K-theory.*

PROOF: (SKETCH) We will sketch the construction of an inverse map on K-theory, assuming for notational simplicity that X and Y are compact. Let $s\colon C(Y) \to C(X)$ be a positive linear extension operator: that is, a linear map (not a homomorphism) which sends positive functions to positive functions and such that $s(f)$ is an extension of f for all $f \in C(Y)$. According to a theorem

[3]For example, let X be any discrete space. Let H_0 be some infinite-dimensional Hilbert space and let $U_0\colon H_0 \to H_0$ be a Fredholm operator of index one, unitary modulo compacts — for example the so-called 'unilateral shift'. Then $H = H_0 \otimes \ell^2 X$ (with the obvious $C_0(X)$-action), $U = U_0 \otimes 1$ give an even Fredholm module whose support is the whole of X.

of Stinespring [**103**], the induced positive linear map $s\colon C(Y) \to \mathfrak{B}(H_X)$ can be dilated to a representation $\Phi\colon C(Y) \to \mathfrak{B}(H_X \oplus H)$. This representation of Y is adequate, hence it is essentially equivalent to H_Y. Moreover, if we make $\Psi^0_X(Y)$ act on $H_X \oplus H$ by the direct sum of its given action on H_X and the zero action on H_Y, then $\Psi^0_X(Y)$ commutes modulo compacts with the action of $C(Y)$ via Φ. Thus we have obtained a map $\Psi^0_X(Y) \to \Psi^0(Y)$, and it can be shown that this is the inverse (at the level of K-theory) of the map previously considered. □

Now in our Mayer-Vietoris situation, one can check that $\Psi^0_X(A) \cap \Psi^0_X(B) = \Psi^0_X(A \cap B)$ and $\Psi^0_X(A) + \Psi^0_X(B) = \Psi^0(X)$. Taking the quotient of everything by $\Psi^{-1}(X)$, we may apply the Mayer-Vietoris sequence of 3.17 to get the required Mayer-Vietoris sequence for K-homology.

The assembly map

A fundamental justification for the analytic definition of K-homology is that an elliptic operator gives rise to a K-homology class. This is as it should be, since an elliptic operator can be paired with a vector bundle — an element of K-theory — to give an integer: one takes coefficients in the vector bundle and forms the index of the resulting operator. In our language the construction can be expressed as follows. Let D be a generalized Dirac operator on a complete n-manifold M, and let χ be a chopping function (1.5). Then one can show that $\chi(D) \in \Psi^0(M)$; use Kasparov's lemma (5.4) together with finite propagation speed arguments. Moreover, $\chi(D)^2 - 1 \in \Psi^{-1}(M)$. Thus $P = (1 + \chi(D))/2$ defines an (odd) Fredholm module (in the even-dimensional case we get an even Fredholm module by taking into account the grading of D). Thus we obtain a K-homology class which we denote by $[D] \in K_n(M)$.

It will be useful to know[4] that the K-homology of euclidean space $\mathbb{R}^n$ is simply a copy of $\mathbb{Z}$ in the dimension congruent to n mod 2, and is generated by the Dirac operator. (This is a special case of K-theory Poincaré duality.) In particular we have $K_0(\mathbb{R}) = 0$ and $K_1(\mathbb{R}) = \mathbb{Z}$ generated by the Dirac operator id/dx. Now recall that this is the same answer that we obtained in the last lecture for the K-theory groups of the C^*-algebra $C^*(|\mathbb{R}|)$ associated to the coarse structure. Is there some connection between K-homology and the K-theory of the coarse C^*-algebra? The answer involves the construction of the *assembly map* relating these groups.

It is convenient to introduce a 'finite propagation' version of $\Psi^0(X)$, corresponding to $C^*(X)$ which is a 'finite propagation' version of $\Psi^{-1}(X)$.

DEFINITION 5.7: *We define $D^*(X)$ to be the C^*-algebra of operators on H_X generated by all the finite propagation operators in $\Psi^0(X)$.*

---

[4]Beware that the inclusion of a point in Euclidean space is not a *proper* homotopy equivalence. It therefore need not, and in fact does not, induce an isomorphism of K-homology.

See 3.2 for the definition of a finite propagation operator. Just as $K_*(C^*(X))$ is functorial under coarse maps (3.5), one can show that $K_*(D^*(X))$ is functorial[5] under coarse maps which are also continuous.

LEMMA 5.8: *With the notation above*
(a) $C^*(X) = \Psi^{-1}(X) \cap D^*(X)$,
(b) $\Psi^0(X) = \Psi^{-1}(X) + D^*(X)$.

PROOF: The first one is obvious modulo analysis; $C^*(X)$ just is generated by things that are locally compact (in $\Psi^{-1}(X)$) and finite propagation (in $D^*(X)$). For the second we need to write any pseudolocal operator as the sum of a locally compact and a finite propagation operator. Let T be pseudolocal and choose a real $\varepsilon > 0$. Take a partition of unity φ_i^2 on X subordinate to a locally finite cover of X by open sets of diameter $\leqslant \varepsilon$. Consider the series

$$S = \sum_i \varphi_i T \varphi_i.$$

It can be shown that this series converges — not in the norm topology, but in the weaker *strong topology*. Explicitly, this is to say that for all $u \in H$, the series

$$Su = \sum_i \varphi_i T \varphi_i u$$

converges in H. The convergence is clear for compactly supported u — then the sum is just a finite one — so that all that needs to be checked is that then partial sums of the series defining S are bounded in the operator norm.

Clearly, S has propagation at most ε, since it is a sum of operators of propagation at most ε. On the other hand,

$$S - T = \sum_i [\varphi_i, T]\varphi_i$$

is a locally finite sum of compact operators. Hence it is locally compact. □

COROLLARY 5.9: *We have* $D^*(X)/C^*(X) \cong \Psi^0(X)/\Psi^{-1}(X)$. *Consequently,* $K_i(D^*(X)/C^*(X)) \cong K_{i-1}(X)$.

REMARK: It follows from the proof above that an element of $\Psi^0(X)/\Psi^{-1}(X)$ can be represented, not merely by an operator of *finite* propagation, but in fact by an operator of *arbitrarily small* finite propagation. This will be useful later.

---

[5]We are cheating slightly here. Recall (see the proof of 3.5) that we needed to use operators on adequate modules in order to define the functoriality of $K_*(C^*(X))$, in order to have enough 'room' to make certain embeddings of Hilbert spaces. It turns out that to define the functoriality of $K_*(D^*(X))$ one needs even more 'room' [**57**, Lemma 7.7]; for instance, the tensor product of an adequate X-module with an auxiliary infinite-dimensional Hilbert space will certainly be roomy enough. In what follows we will always assume that our modules are adequate in this more adequate sense.

Thus the long exact K-theory sequence associated to the short exact sequence of C^*-algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0$$

gives a cyclic exact sequence of groups

$$\begin{array}{ccccc} K_0(X) & \xrightarrow{A} & K_0(C^*X) & \longrightarrow & K_0(D^*X) \\ \uparrow & & & & \downarrow \\ K_1(D^*X) & \longleftarrow & K_1(C^*X) & \xleftarrow[A]{} & K_1(X) \end{array}$$

DEFINITION 5.10: *The indicated maps A are called* assembly maps.

REMARK: Notice that in this discussion the space X has played two rather different rôles: its local topological structure has been what is relevant for the discussion of pseudolocality, and its large scale metric structure has been relevant for the definition of finite propagation. It is frequently useful to separate these two structures out[6], so let us assume more generally that X is now a (locally compact second countable Hausdorff) space equipped with a reference map $c\colon X \to \mathcal{Z}$ to a proper metric space $\mathcal{Z}$; we require that c be proper, in the sense that the inverse image of a bounded (or compact) set should be relatively compact, but we do not require that c should be continuous. In this situation we can still define the algebras $C^*(X)$ and $D^*(X)$, where 'finite propagation' is defined using the metric in $\mathcal{Z}$ via the map c. (In the language of general coarse structures, see lecture 10, what we are doing here is pulling back the coarse structure via c from $\mathcal{Z}$ to X.) Moreover, corollary 5.9 continues to hold, and so we still have the exact sequence $(*)$. Note, however, that the algebra $C^*(X)$, when X is equipped with this pulled-back coarse structure, is just the same as $C^*(\mathcal{Z})$; so the assembly map in the context of a space X 'bounded over $\mathcal{Z}$' should be thought of as a map

$$A\colon K_*(X) \longrightarrow K_*(C^*(\mathcal{Z})).$$

Reverting to our consideration of a complete Riemannian manifold M, of dimension n, suppose that D is a generalized Dirac operator on M. Then we have seen that from D we can define

(a) the coarse index of D, $\operatorname{Ind} D$, belonging to $K_n(C^*M)$;

(b) the K-homology class of D, $[D]$, belonging to $K_n(M)$.

These are related by the assembly map.

PROPOSITION 5.11: *With the notation above, $A([D]) = \operatorname{Ind} D$.*

---

[6]Thanks to Shmuel Weinberger for insisting on this in numerous conversations.

PROOF: Consider the commutative diagram of C^*-algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(M) & \longrightarrow & \mathfrak{M}(C^*(M)) & \longrightarrow & \mathfrak{M}(C^*(M))/C^*(M) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C^*(M) & \longrightarrow & D^*(M) & \longrightarrow & D^*(M)/C^*(M) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \Psi^0(M)/\Psi^{-1}(M) & & \end{array}$$

which arises from embedding $D^*(X)$ in the multiplier algebra $\mathfrak{M}(C^*(X))$. We have defined $[D] \in K_n(M) = K_{n-1}(\Psi^0(M)/\Psi^{-1}(M))$ to be the K-theory class of $\chi(D)$, where χ is a chopping function; and $A[D]$ is the result of applying the boundary map in K-theory associated to the second row of the diagram above. But by our definition in lecture 3, $\operatorname{Ind} D$ is obtained by taking $[\chi(D)] \in K_*(\mathfrak{M}(C^*(M))/C^*(M)$ and applying the boundary map in K-theory associated to the first row. By naturality of the 6-term exact sequence in K-theory we get the same answer. $\square$

One should think of this result as showing that the definition of the coarse index, originally given only for the geometric operators of Dirac type, can be extended to all the 'generalized elliptic operators' defined by Fredholm modules over X.

Equivariant assembly

Suppose that X is equipped with a proper, cocompact action of some group Γ. (The case of greatest interest to us will be that of a Galois covering of some compact manifold V, associated to a homomorphism $\pi_1 V \to \Gamma$; here the action is free, but the more general case of proper actions can be accommodated without difficulty.) We would like to generalize the previous constructions to take into account the action of Γ.

DEFINITION 5.12: *Let H be an X-module. We say that it is a* covariant X-module *if it is equipped with a unitary action ρ of Γ, compatible with the action of Γ on X in the sense that for all $v \in H$, $f \in C_0(X)$, and $\gamma \in \Gamma$,*

$$(f \circ \gamma) \cdot v = \rho(\gamma) f \rho(\gamma)^* \cdot v.$$

For example, if X is equipped with a measure, the space of L^2 sections of a Hermitian Γ-vector bundle on X is a covariant X-module. Given any X-module H one can form a covariant X-module

$$\mathfrak{H} = \bigoplus_{\gamma \in \Gamma} H^\gamma$$

where H^γ is H with the $C_0(X)$-action shifted by γ, and Γ acts on $\mathfrak{H}$ by permuting the summands in the obvious way. If we form the algebra $C^*(X)$ using a covariant

X-module, then $\operatorname{Ad}\rho(\gamma)$ will map $C^*(X)$ to itself for all $\gamma \in \Gamma$, and so we will get an action of Γ on $C^*(X)$. Similarly we get an action of Γ on $D^*(X)$.

Since X is a proper, cocompact Γ-space it can be covered by finitely many open sets U, each of which is Γ-homeomorphic to $\Gamma \times_F W$, where W is acted on by the finite group F. Notice that a covariant X-module restricts to a module over each of the spaces U.

DEFINITION 5.13: *We say that a covariant X-module is* Γ-adequate *if X can be covered by finitely many open sets as above, such that on each $U = \Gamma \times_F W$, H is equivalent to the covariant U-module $\ell^2(\Gamma) \otimes H'_W$, where H'_W is an adequate W-module, Γ acts by translation on $\ell^2(\Gamma)$, and $C_0(U)$ acts via the proper map $\Gamma \times W \to U$.*

Such modules can always be found. We will assume in what follows that the algebras $C^*(X)$ and $D^*(X)$ have been formed using a Γ-adequate X-module.

If A is a C^*-algebra acted on by a group Γ, the elements fixed under Γ form a subalgebra A^Γ. We want to identify this subalgebra in the cases of $C^*(X)$ and $D^*(X)/C^*(X)$.

LEMMA 5.14: *In the above situation, $C^*(X)^\Gamma \cong C^*_r(\Gamma) \otimes \mathfrak{K}$. In particular, $K_i(C^*(X)^\Gamma) = K_i(C^*_r\Gamma)$.*

PROOF: Decompose X into finitely many (say m) Borel subsets with nonempty interior, each of which lies inside a 'coordinate patch' for the Γ-adequate X-module on which $C^*(X)$ is defined. Relative to this decomposition the elements T of $C^*(X)$ can be written as $m \times m$ matrices T_{ij}, whose entries are $\Gamma \times \Gamma$ matrices of compact operators. If T is translation invariant, so is each matrix entry T_{ij}; but a translation-invariant $\Gamma \times \Gamma$ matrix of compacts is just the same thing as an element of $C^*_r(\Gamma) \otimes \mathfrak{K}$. Thus

$$C^*(X)^\Gamma \cong M_m(C^*_r(\Gamma) \otimes \mathfrak{K}) \cong C^*_r(\Gamma) \otimes \mathfrak{K}$$

as required. □

LEMMA 5.15: *In the above situation, suppose additionally that the action is free, and let $V = X/\Gamma$ be a finite complex. Then*

$$K_i(D^*(X)^\Gamma / C^*(X)^\Gamma) = K_{i-1}(V).$$

PROOF: Let $\pi \colon X \to V$ be the covering map. Note that there is $\delta > 0$ such that for any subset $U \in V$ of diameter less than δ, $\pi^{-1}(U) = \Gamma \times U$. This gives a natural $1:1$ correspondence between operators of propagation less that $\delta/2$ on V and Γ-equivariant operators of propagation less than $\delta/2$ on X (lift using a partition of unity). However, as we remarked above, equivalence classes in the algebras D^*/C^* can be represented by operators with arbitrarily small propagation. Thus in fact, $D^*(X)^\Gamma/C^*(X)^\Gamma \cong \Psi^0(V)/\Psi^{-1}(V)$, and the result follows. □

REMARK: The corresponding statement in the case of proper actions is that $K_i(D^*(X)^\Gamma/C^*(X)^\Gamma) = K^\Gamma_{i-1}(X)$, the Γ-equivariant K-homology of X (see [**10**]).

Suppose now that Γ acts freely on X, with quotient V. Then associated to the short exact sequence of C^*-algebras

$$0 \to C^*(X)^\Gamma \to D^*(X)^\Gamma \to D^*(X)^\Gamma/C^*(X)^\Gamma \to 0$$

we obtain, as before, a cyclic exact sequence of groups

$$\begin{array}{ccccc} K_0(V) & \xrightarrow{A} & K_0(C^*_r\Gamma) & \longrightarrow & K_0(D^*(X)^\Gamma) \\ \uparrow & & & & \downarrow \\ K_1(D^*(X)^\Gamma) & \longleftarrow & K_1(C^*_r\Gamma) & \xleftarrow[A]{} & K_1(V) \end{array}$$

in which the maps A are, as before, called the assembly maps. Although we usually take X to be the universal covering of V here, the construction works just the same if X is any Galois covering space with group Γ.

DEFINITION 5.16: *Suppose that X is the universal cover of V. Then the algebra $D^*(X)^\Gamma$ will be denoted $S^*(V)$ and called the* structure algebra *of V. The cyclic exact sequence will be called the* analytic surgery exact sequence.

There are a number of standard conjectures about the behavior of the assembly map for good spaces V. In fact, for compact aspherical V, it is conjectured [**10**] that $K_*(S^*V) = 0$, that is, assembly is an isomorphism. The geometric consequences of this conjecture often use only the *injectivity* of the assembly map, which we state as

CONJECTURE 5.17: (ANALYTIC NOVIKOV CONJECTURE) *Suppose that Γ is a group such that $B\Gamma$ is represented by a finite complex. Then the assembly map*

$$K_*(B\Gamma) \to K_*(C^*_r\Gamma)$$

is injective.

Less optimistically, one could attempt to prove that the assembly map is injective after tensoring with $\mathbb{Q}$. Similar conjectures can be stated for the *Baum-Connes assembly map* $K^\Gamma_*(\underline{E}\Gamma) \to K_*(C^*_r\Gamma)$, in the case that Γ acts cocompactly on the universal space $\underline{E}\Gamma$ for proper actions.

In what follows we are going to deduce some cases of these conjectures about the equivariant assembly map from analogous results about the assembly map in coarse geometry. Here we give an example which may indicate the significance of the conjecture.

EXAMPLE: Suppose that M is a compact spin manifold, and that there is given a homomorphism $\alpha\colon \pi_1 M \to \Gamma$. Then we can construct a commutative diagram involving the assembly maps, looking like this

$$\begin{array}{ccc} K_*(M) & \xrightarrow{A} & K_*(C^*_r\Gamma) \\ \downarrow{\scriptstyle \alpha_*} & & \| \\ K_*(B\Gamma) & \xrightarrow{A} & K_*(C^*_r\Gamma) \end{array}$$

The A in the top row is the assembly map corresponding to the Γ-covering of M. By Lichnerowicz vanishing, if M admits positive scalar curvature, then $A[D] = 0$, where D is the homology class of the Dirac operator. Chasing the diagram, we find that if the Novikov conjecture holds for Γ, then $\alpha_*[D]$ vanishes (or is torsion, at least) in $K_*(B\Gamma)$. Via the Atiyah-Singer index theorem this translates into the statement that the higher $\widehat{\mathcal{A}}$-genera, that is the numbers $\langle \widehat{\mathcal{A}}(M) \smile \alpha^*(x), [M] \rangle$, $x \in H^*(B\Gamma; \mathbb{Q})$, must all vanish. Thus in the presence of the Novikov conjecture all the higher $\widehat{\mathcal{A}}$-genera are obstructions to positive scalar curvature. We proved this statement in the case $\Gamma = \mathbb{Z}$ in lecture 4, using the partitioned manifold index theorem.

Notes and references: Primary references for K-homology are [**19, 64**]. The approach via 'duality' which we have used here was initiated by Paschke [**81**] and perfected in [**50, 52**]. Another reference for this material should be [**54**], eventually.

The construction of the assembly map comes from [**57**] (see also [**115**]). The discussion of the equivariant case is folklore. (For the algebraic analogue, compare [**22**].)

For more detail on the assembly map in relation to positive scalar curvature, see [**97, 99, 104**].

CHAPTER 6

Surgery

In the last lecture we defined a certain exact sequence of C^*-algebra K-theory groups which we called the *analytic surgery exact sequence.* The justification for this terminology is an analogy between that exact sequence and the fundamental result about the topology of high-dimensional manifolds, the *surgery exact sequence* of Browder, Novikov, Sullivan, and Wall. In this lecture we will give a brief overview of various versions of surgery theory and the surgery exact sequence; in the next we will map the (topological) surgery exact sequence functorially to our 'analytic surgery' exact sequence. We start the discussion with the classical surgery theory of compact smooth manifolds.

Manifold structures

Let V be a connected space, say a finite CW-complex. A *manifold structure* on V is a homotopy equivalence $M \to V$, where M is a manifold[1]. The central project of surgery theory is to classify the manifold structures on V, up to some natural equivalence.

The most obvious (and interesting) definition of equivalence is to declare that two structures $M \to V$ and $M' \to V$ are *equivalent* if there is a homotopy commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

with h a diffeomorphism[2]. However for technical reasons it is more convenient to work with a slightly weaker definition. One says that the two structures above are *h-cobordant* if there is a cobordism W between them, equipped with a map

[1] One obtains different notions of manifold structure according to what sort of manifold is allowed here: smooth, piecewise-linear, topological, $\ldots$. Surgery theory works in all these categories. We will think of smooth manifold structures, unless otherwise stated. *All manifolds considered in this section will be compact.*

[2] If we are working with other kinds of manifold structure we should substitute 'piecewise-linear homeomorphism' or just 'homeomorphism' here as appropriate.

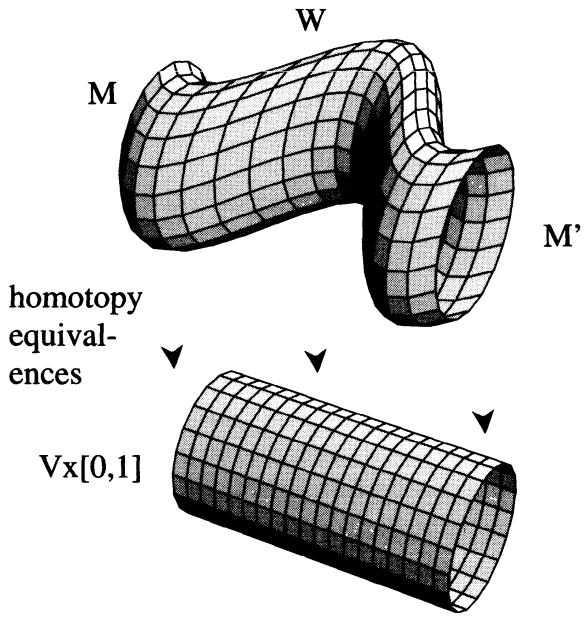

FIGURE 3. h-cobordism of manifold structures.

$W \to V \times [0,1]$ which is itself a homotopy equivalence and restricts to the given maps on the boundary pieces (see figure 3). Let $\mathcal{S}(V)$ be the set of h-cobordism classes of manifold structures on V; $\mathcal{S}(V)$ is called the (*smooth*) *structure set* of V. The project of surgery theory is to compute $\mathcal{S}(V)$.

REMARK: The *s-cobordism theorem* of Barden, Mazur, and Stallings [**68**] says that (in high dimensions) the difference between the two definitions of equivalence that we have contemplated above is measured by an algebraic K-theory invariant (Whitehead torsion) which is known to be zero in many cases (for example, if the fundamental group of V is free abelian). There are a number of other variations of the definition of structure and of equivalence; they are known as 'decorations' and the effect of a change of decoration can be quantified, as above, in terms of algebraic K-theory. Our definition of the structure set corresponds to the decoration 'h'.

EXAMPLE: Let $V = S^2 \vee S^2$ (two spheres identified at one point). Then $\mathcal{S}(V) = \emptyset$, since the homology of V does not obey Poincaré duality.

EXAMPLE: Let $V = S^n$, the n-sphere. If $n = 3$, the Poincaré conjecture is that $\mathcal{S}(V)$ has one element. The generalized Poincaré conjecture proved by Smale in the early sixties says that for $n \geqslant 5$, the *PL* manifold structure set of S^n has only one element; however, Milnor's work on exotic spheres showed that the *smooth* structure set of a sphere (the structure set as we have defined it) can have many elements.

As suggested by the first example above, there is an obvious necessary condition that V must satisfy in order for there to be any possibility that it should have manifold structures at all. Namely, its (co)homology must obey the Poincaré duality theorem. This means that there must be a fundamental class[3] $[V]$ in $H_n(V;\mathbb{Z})$ such that the cap product with this class induces isomorphisms $H^r(V;\mathbb{Z}) \to H_{n-r}(V;\mathbb{Z})$ for every r. In fact, this must even be true with Γ-twisted coefficients,where $\Gamma = \pi_1 V$ is the fundamental group. If V satisfies this condition it is said to be a *Poincaré space* of *formal dimension* n. Compact manifolds are Poincaré spaces by this definition, but so are many other spaces homotopically similar to manifolds: for example, if $(X, \partial X)$ and $(Y, \partial Y)$ are manifolds with boundary, and $f\colon \partial X \to \partial Y$ is an orientation-reversing homotopy equivalence, we may glue X to Y via the mapping cylinder of f, and thereby obtain a Poincaré space. Examples exist in which the resulting space cannot be given the structure of a manifold.

CONVENTION: In our discussion we will assume for simplicity that V is in fact an *oriented manifold*, so that it already has one canonical manifold structure and one wants to classify all the other possible structures on it. A structure on V may therefore be thought of as an *orientation-preserving* homotopy equivalence from an *oriented* manifold to V.

The fundamental result of surgery theory is an 'exact sequence'

$$\longrightarrow L_{n+1}(\Gamma) \cdots\!\!\!\!\longrightarrow \mathcal{S}(V) \longrightarrow \mathcal{N}(V) \xrightarrow{A} L_n(\Gamma)$$

for a Poincaré space V (of high dimension). Here the structure set $\mathcal{S}(V)$ has already been defined; we will define the other terms and the maps that appear.

Normal invariants. $\mathcal{N}(V)$ is the collection of homotopy classes of *normal invariants* for V. Recall that if M is a smooth submanifold of Euclidean space, a small tubular neighbourhood of M has the structure of a vector-bundle over M. For a Poincaré space V, a tubular neighbourhood is a bundle[4] over V in a weak sense (up to homotopy its boundary is a fibration over V with fiber a sphere); $\mathcal{N}(V)$ is the collection of honest vector bundle structures that can be put on a tubular neighbourhood of V compatibly with this homotopy bundle structure. In terms of classifying spaces, the homotopy-theoretic normal bundle of V is classified by a map from V to a certain space denoted BG; the process of passing from a vector bundle to its underlying spherical fibration defines a

[3]Our definition is in fact that of a Poincaré space with trivial *orientation character*: one can also define Poincaré spaces with other orientation characters (these characters correspond to the first Stiefel-Whitney class of a manifold, which is trivial if the manifold is orientable). For simplicity we will ignore this issue.

[4]This is called the *Spivak normal bundle*. Its structure depends only on the homotopy type of V.

forgetful map $BO \to BG$; and a normal invariant is a lifting

$$\begin{array}{ccc} & & BO \\ & \nearrow & \downarrow \\ V & \longrightarrow & BG \end{array}$$

as shown by the dotted arrow.

We are assuming that V is already a manifold, so one standard lifting is given already. Homotopy theory then shows that $\mathcal{N}(V)$ can be identified with the set of homotopy classes of maps from V to G/O, the homotopy fibre of $BO \to BG$.

REMARK: One can similarly define the notion of normal invariant for a *Poincaré pair* $(V, \partial V)$ (analogous to a manifold with boundary) but we will not go into the details here.

DEFINITION 6.1: *A normal map $M \to V$, where M is an oriented manifold, is a degree one map $f\colon M \to V$ fitting into a homotopy commutative diagram*

$$\begin{array}{ccc} M & \longrightarrow & BO \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow \\ V & \longrightarrow & BG \end{array}$$

where the top row classifies the stable normal bundle of M and the lower right triangle represents some normal invariant for V.

A transversality argument shows that every normal invariant comes from some normal map, and that two normal maps give the same normal invariant if and only if they are *normally cobordant* — meaning that their 'difference' is the boundary of a normal map to $V \times [0,1]$. Thus we can regard $\mathcal{N}(V)$ as made up of cobordism classes of maps from manifolds to V, equipped with some 'extra bundle data'. These extra bundle data are vital to the definition of surgery theory, but they will not enter into the map that we propose to construct from surgery to analysis[5], and so we will not treat them in detail. Notice however that a homotopy equivalence $M \to V$ is certainly a normal map, and an h-cobordism is a normal cobordism, so there is defined a forgetful map $\mathcal{S}(V) \to \mathcal{N}(V)$. This is the map appearing in the surgery exact sequence.

One should perhaps explain the word 'surgery' at this point. Given a normal map $f\colon M \to V$, one attempts to construct a normal cobordism of f to a homotopy equivalence by composing a series of elementary cobordisms, called surgeries, each of which should kill some designated element of the relative homotopy groups $\pi_*(f)$. (The simplest example of this procedure might arise if M had two connected components M_1 and M_2. Then one could form the connected sum $M_1 \# M_2$, which one can show is normally cobordant to the original M.) The main result of surgery theory, due to Wall, is that there is

[5]Essentially this is because, away from prime 2, the quadratic signature of a normal map is the difference of the symmetric signatures of the domain and target.

an obstruction to this procedure lying in a group $L_n(\Gamma)$, $\Gamma = \pi_1 V$, and that f can be surgered to a homotopy equivalence if and only if this obstruction vanishes.

The surgery obstruction groups. The groups $L_n(\Gamma)$ are defined algebraically, in terms of quadratic forms over the integral group ring $\mathbb{Z}\Gamma$. For our purposes, however, it is more convenient to work with an alternative geometric definition as cobordism groups [**108**, Chapter 9].

DEFINITION 6.2: *A* cycle *for $L_n(\Gamma)$ is a normal map of pairs $f\colon (M, \partial M) \to (X, \partial X)$, where M is an oriented manifold and X is an (oriented) Poincaré space of formal dimension n, together with a map $X \to B\Gamma$. It is also required that f should restrict to a homotopy equivalence $\partial M \to \partial X$.*

On these cycles we wish to impose an equivalence relation defined by cobordism. Unfortunately, since the cycles are manifolds with boundary, the cobordisms must inevitably be manifolds with corners. Specifically, we need to contemplate *manifold triads* in the sense of Wall. Such an object is a manifold W equipped with two 'boundaries' $\partial_1 W$ and $\partial_2 W$ of codimension 1, which themselves meet transversely on a 'corner' $\partial_{12} W$ of codimension two. (A helpful example to consider is the closed unit square $W = \{(x,y) : 0 \leqslant x, y \leqslant 1\}$ in the plane, with $\partial_1 W$ the union of the vertical boundary segments, $\partial_2 W$ the union of the horizontal boundary segments, and $\partial_{12} W$ the set of four corners.) There is an analogous notion of *Poincaré triad.*

DEFINITION 6.3: *(See figure 4.) A cycle $f\colon (M, \partial M) \to (X, \partial X)$ for $L_n(\Gamma)$ is* null-cobordant *if we can find a manifold triad W, a Poincaré triad Y, a map of triads $g\colon W \to Y$, and a map $Y \to B\Gamma$, such that*

(i) *$(\partial_1 W, \partial_{12} W) = (M, \partial M)$, $(\partial_1 Y, \partial_{12} Y) = (Y, \partial Y)$, and the restriction of g to $\partial_1 W$ is equal to f;*
(ii) *The restriction of g to $\partial_2 W$ is a homotopy equivalence;*
(iii) *The map $Y \to B\Gamma$ extends the given map $X \to B\Gamma$.*

We may now construct $L_n(\Gamma)$ as a group of equivalence classes of cycles in the usual kind of way: addition of cycles is defined by disjoint union, the inverse of a cycle is the same cycle with the reversed orientation, and two cycles are

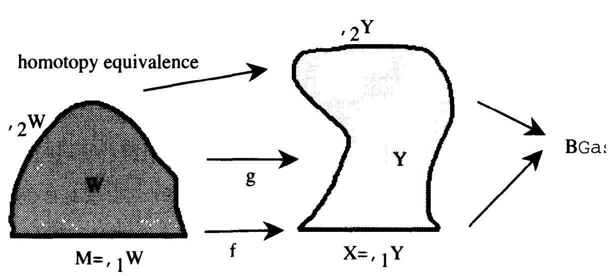

FIGURE 4. A null-cobordism for $L_n(\Gamma)$.

equivalent if their difference[6] is null-cobordant. It is not hard to verify that this procedure defines an abelian group, depending only on Γ.

Clearly there is a forgetful map $A\colon \mathcal{N}(V) \to L_n(\Gamma)$, which sends a normal map $M \to V$ to itself together with the classifying map $V \to B\Gamma$, considered as a cycle for $L_n(\Gamma)$. This map (the *assembly map*) is the one appearing in the surgery sequence. We say that the assembly map sends a normal map f to its *surgery obstruction.*

THEOREM 6.4: (FUNDAMENTAL SURGERY THEOREM) *A normal map f can be surgered to a homotopy equivalence if and only if its surgery obstruction vanishes.*

This is what we mean by saying that the surgery sequence is exact at $\mathcal{N}(V)$. In common with all the main results of surgery theory, the theorem is valid only in high dimensions ($\dim V \geqslant 5$) where there is enough room to carry out certain geometrical constructions.

We sketch the proof. If $f\colon M \to V$ is a structure on V then it defines an element of $L_n(\Gamma)$. I claim that this element is actually null-cobordant. For we can consider $W = M \times I$ as a manifold triad, with $\partial_1 W = M \times \{0\}$, $\partial_2 W = M \times \{1\}$, and $\partial_{12} W = \emptyset$. Because f is a structure (i.e. a homotopy equivalence), the map $f \times 1\colon M \times I \to V \times I$ satisfies the conditions for a null-cobordism of f. This proves that the maps $\mathcal{S}(V) \to \mathcal{N}(V) \to L_n(\Gamma)$ in the surgery sequence compose to zero.

The other direction of the proof depends on two key theorems. The first is that one obtains an equivalent definition of the L-group if one requires that the cycles have X connected and $X \to B\Gamma$ inducing an isomorphism on fundamental groups, and that the bordisms satisfy analogous conditions. Thus, suppose that we have a normal map $f\colon M \to V$ whose surgery obstruction vanishes. Then there is a null-cobordism $g\colon W \to Y$ of the normal map, with $M = \partial_1 W$ and $V = \partial_1 Y$, such that the induced map on fundamental groups $\pi_V \to \pi_1 Y$ is an isomorphism. The idea of the proof is now to do surgery on the whole cobordism W so as to produce a normal cobordism (of triads!) between it and a homotopy equivalence: restricting to ∂_1 we then get a normal cobordism between the given structure and a homotopy equivalence (see figure 5). The so-called (π-π)-*theorem* of Wall says that, if restriction to the boundary induces an isomorphism on fundamental groups (and if the dimension is sufficiently high) this surgery can *always* be carried out: there is no obstruction. This, then, completes the proof.

To discuss the remainder of the surgery exact sequence we must explain the dotted arrow from $L_{n+1}(\Gamma)$ to $\mathcal{S}(V)$. It is unknown whether the smooth structure set $\mathcal{S}(V)$ carries a group structure[7] and the dotted arrow denotes that the group $L_{n+1}(\Gamma)$ acts on the set $\mathcal{S}(V)$; exactness at this point means that the orbits of the group action are exactly the inverse images of elements of $\mathcal{N}(V)$ under the map $\mathcal{S}(V) \to \mathcal{N}(V)$. The dotted arrow is provided by the next result

[6]That is, one plus the inverse of the other.

[7]This *is* true for the topological manifold structure set.

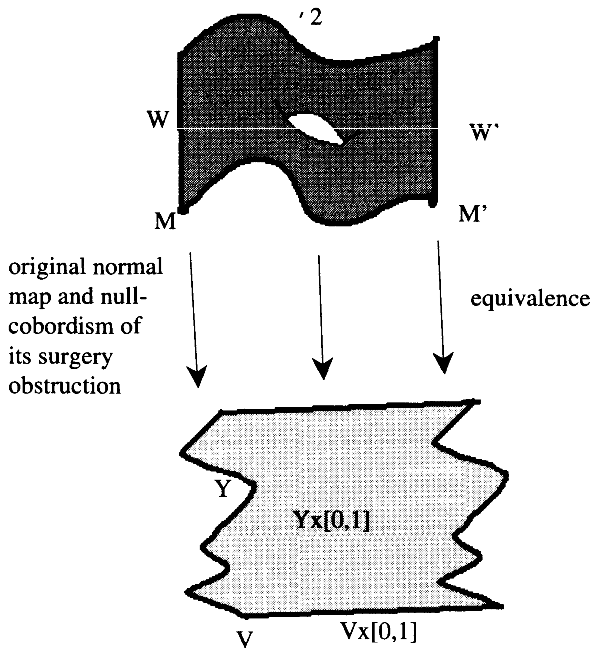

FIGURE 5. Illustrating the proof of the fundamental surgery theorem

PROPOSITION 6.5: (REALIZATION THEOREM) *Let V be a Poincaré space of formal dimension n and having fundamental group Γ, and let $f\colon M \to V$ be a manifold structure on V. Then any element $\alpha \in L_{n+1}(\Gamma)$ has a representative which is a map $h\colon W \to V \times [0,1]$, where W is a cobordism, $\partial_- W = M$ and $h_{|\partial_- W} = f$, and $h_{|\partial_+ W}$ gives another manifold structure on V.*

The action of α on $\mathcal{S}(V)$ is then defined to send the structure $M = \partial_- W \to V$ to the structure $\partial_+ W \to V$. These two structures are manifestly normal cobordant. Conversely, suppose that two structures on V are normal cobordant. The normal cobordism itself can then be considered as a normal map of pairs, with surgery obstruction in $L_{n+1}(\Gamma)$. Thus by the action of a suitable element of this group we can make the surgery obstruction zero. When we have done this, we can apply surgery to the cobordism itself, making it into an h-cobordism. This proves the exactness of the surgery sequence at $\mathcal{S}(V)$.

Novikov's conjecture

The surgery exact sequence takes a particularly concrete form if Γ is the trivial group. Then one can calculate the L-groups as follows:

n mod 4	0	1	2	3
$L_n(e)$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0

Moreover, if $f\colon M \to V$ is a normal map in dimension $4k$, then the surgery obstruction of f is just $\frac{1}{8}$ times the difference of the signatures of M and of V.

In particular, if $\mathrm{Sign}(M) = \mathrm{Sign}(V)$, then the given normal map can be surgered to a homotopy equivalence.

The homotopy groups of BG are finite. Therefore, "modulo finite groups" G/O is equivalent to the classifying space BO, and via the Chern character for real K-theory one obtains a map

$$[V, G/O] \otimes \mathbb{Q} \to H^*(V; \mathbb{Q}).$$

Under this isomorphism the class of a normal map $M \to V$ passes essentially to the total $\mathcal{L}$-class of M, and the assembly map is given by the Hirzebruch formula which expresses the signature of M in terms of the Pontrjagin classes (note that, rationally, the $\mathcal{L}$-class determines all the Pontrjagin classes and *vice versa*). The exactness of the surgery sequence can therefore be expressed as follows:

PROPOSITION 6.6: **[62]** *The rational Pontrjagin classes of a simply connected closed $4k$-dimensional manifold can be varied arbitrarily*[8] *by a homotopy equivalence, subject only to the single relation provided by the Hirzebruch signature theorem.*

A compact manifold M is said to be *rigid* if any homotopy equivalence $M' \to M$ is homotopic to a diffeomorphism, in other words, if $\mathcal{S}(M)$ has just one element. The notion of rigidity is of obvious importance in differential topology. For high-dimensional manifolds, rigidity is equivalent to the assembly map being an isomorphism.

The discussion above indicates that simply-connected manifolds with plenty of cohomology will tend not to be rigid. This, however, should not be a surprise. Since the assembly map relates the cohomology of V to something depending only on the fundamental group, the cohomology of a rigid manifold should have some close relation to the fundamental group. The natural hypothesis would be that V be an Eilenberg-Maclane space of type $K(\Gamma, 1)$, i.e., that V is *aspherical.*

CONJECTURE 6.7: (BOREL CONJECTURE) *A (high-dimensional) closed aspherical manifold must be rigid.*

The Borel conjecture is that the assembly map is an isomorphism. The *Novikov conjecture* is a somewhat weaker assertion:

CONJECTURE 6.8: (NOVIKOV CONJECTURE) *The assembly map for any $K(\Gamma, 1)$-space is rationally injective.*

Here we have made use of the result that for *any* space V, not necessarily a manifold or even a Poincaré space, there is a 'homological assembly map' $H_*(V; \mathbb{Q}) \to L_*(\pi_1 V) \otimes \mathbb{Q}$, such that if V *is* a manifold then this assembly map and the one we previously defined agree under the isomorphism $H^*(V; \mathbb{Q}) \to H_*(V; \mathbb{Q})$ provided by Poincaré duality.

Suppose now that M is an oriented manifold, with fundamental group Γ. Let $\alpha\colon M \to B\Gamma$ be the map classifying the universal cover of M, and let $x \in$

[8] Within the relevant cohomology group, of course!

$H^*(B\Gamma;\mathbb{Q})$. Recall that we have defined the *higher signature* of M corresponding to x to be the characteristic number

$$\langle \mathcal{L}(M) \smile \alpha^*(x), [M] \rangle$$

of M.

Novikov's conjecture implies (and can in fact be shown to be equivalent to) the statement that all the higher signatures of M are homotopy invariant. To see this consider the commutative diagram[9]

$$\begin{array}{ccccc} \mathcal{S}(M) & \longrightarrow & H_*(M;\mathbb{Q}) & \xrightarrow{A_M} & L_*(\Gamma)\otimes\mathbb{Q} \\ & & \downarrow{\scriptstyle \alpha_*} & & \| \\ & & H_*(B\Gamma;\mathbb{Q}) & \xrightarrow{A_\Gamma} & L_*(\Gamma)\otimes\mathbb{Q} \end{array}$$

A structure on M corresponds to its (Poincaré) dual $\mathcal{L}$-class $[M] \frown \mathcal{L}(M) \in H_*(M;\mathbb{Q})$. By exactness, the image of this class in $L_*(\Gamma)\otimes\mathbb{Q}$ is homotopy invariant. Assuming Novikov's conjecture, that A_Γ is injective, we find that the pushforward $\alpha_*([M] \frown \mathcal{L}(M)) \in H_*(B\Gamma;\mathbb{Q})$ is homotopy invariant. But it is easy to see that this statement is equivalent to the homotopy invariance of all the higher signatures.

We have seen that higher signatures can be defined analytically, as refined indices of the signature operator. This opens up the possibility of an analytic approach to the Novikov conjecture, which we will pursue in the next lecture.

Bounded surgery

We now discuss a more modern variant of surgery theory, which is directly connected with coarse geometry. This is *bounded surgery theory*[**36**].

Let $\mathcal{Z}$ be a metric space, the *reference space.* A *space bounded over* $\mathcal{Z}$ means a topological space equipped with a control map $c\colon M \to V$, which must be proper in the sense that the inverse image of a bounded set has compact closure. We do not require that c should be continuous. A *map* of such spaces[10] means a (continuous) map $f\colon V \to V'$ in the ordinary sense which has the property that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ & {\scriptstyle c}\searrow \quad \swarrow{\scriptstyle c'} & \\ & \mathcal{Z} & \end{array}$$

boundedly commutes; in other words, there is a constant $R > 0$ such that $d(c(x), c'(f(x))) < R$ for all $x \in V$. It is clear that the category of spaces over $\mathcal{Z}$ depends only on the coarse type of $\mathcal{Z}$, and that if $\mathcal{Z}$ is bounded then the

[9]Here we are using some results about the functoriality of the surgery exact sequence.

[10]For emphasis, we refer to this as a *bounded map.*

category of spaces over $\mathcal{Z}$ is just the category of compact spaces and continuous maps.

One can give 'bounded' versions of many of the notions of topology. In particular there are notions of 'bounded homotopy', 'bounded homotopy equivalence', and thus 'bounded structure set' $\mathcal{S}_b\begin{pmatrix} V \\ \downarrow \\ \mathcal{Z}\end{pmatrix}$: a bounded structure for V is a bounded homotopy equivalence $M \to V$, where M is a manifold bounded over $\mathcal{Z}$, and two such structures are equivalent if they are related by a bounded diffeomorphism.

REMARK: There are a number of reasons to study bounded structures, many of which stem from the applicability of infinite processes with some kind of control on their size to the study of *topological* manifolds (a striking example is the proof of Kirby's annulus theorem [**69**] by the methods of bounded topology, see [**36, 35**]). We will not go into this in these lectures, although the attempt to prove the topological invariance of the rational Pontrjagin classes by means of coarse index theory (see 7.14 and the remarks following) does belong to the same circle of ideas.

Now let V be a manifold bounded over $\mathcal{Z}$. (More generally, V could be any bounded Poincaré space, but we will not consider this situation.) Ferry and Pedersen produced a surgery theory to calculate the bounded structure set. A special case of their results is the following

THEOREM 6.9: *Suppose, in the above situation, that V is uniformly simply connected*[11]*. Then there are obstruction groups $L_{n,\mathcal{Z}}(e)$, depending only on the coarse geometry of $\mathcal{Z}$, which fit into a surgery exact sequence*

$$\longrightarrow L_{n+1,\mathcal{Z}}(e) \dashrightarrow \mathcal{S}_b\begin{pmatrix} V \\ \downarrow \\ \mathcal{Z}\end{pmatrix} \longrightarrow \mathcal{N}(V) = [V, G/O] \longrightarrow L_{n,\mathcal{Z}}(e)$$

Notice in this sequence that the normal invariant space is just the space of homotopy classes of maps $V \to G/O$, exactly as in the compact case; it does not involve the bounded structure of V. On the other hand the obstruction group depends only on the coarse geometry of $\mathcal{Z}$. Compare the remark on page 40.

Positive scalar curvature

We have tried to keep in mind throughout a certain analogy between problems related to homotopy equivalences of oriented manifolds (which have to do with the signature operator) and problems related to positive scalar curvature metrics on spin manifolds (which have to do with the spinor Dirac operator). Early in the study of metrics of positive scalar curvature it was observed by Gromov-Lawson [**47**] and by Schoen-Yau [**101**] that the method of surgery was also relevant to the study of positive scalar curvature; specifically, that surgeries of codimension $\geqslant 3$ could always be made in such a way as to preserve the property of having positive scalar curvature. Via some results of Wall [**108**, section 1A] identifying

[11] Definition by analogy with 'uniformly contractible' — see 2.3.

2-connected and spin bordism, this showed that the existence or otherwise of a positive scalar curvature metric on a high-dimensional spin manifold M depends only on the class in the spin bordism group $\Omega_n^{Spin}(B\Gamma)$, $\Gamma = \pi_1 M$, defined by M.

This observation has recently been extended to a full-blown surgery-like theory by Hajduk [**49**] and Stolz [**104**]. In translating between classical surgery theory and the 'surgery' theory of positive scalar curvature spin manifolds the following dictionary of analogies may be useful.

Classical surgery	Positive scalar surgery
Manifold	Manifold
Manifold structure	Positive scalar curvature metric
Normal invariant	Spin structure
Normal map of pairs	Spin manifold with positive scalar curvature on boundary
Wall group $L_n(\Gamma)$	Stolz group $R_n(\Gamma)$
Signature operator	Dirac operator
Homotopy invariance of signature	Lichnerowicz vanishing theorem
h-cobordism	Concordance of metrics
Whitehead torsion	???

REMARK: In giving the definitions one needs to consider Riemannian metrics on manifolds with boundary. We will say that such a metric is *conditioned* if it is a product metric on some collar neighbourhood of the boundary. Metrics on manifolds with boundary will usually be assumed to be conditioned.

First we define the 'structure set' computed by the theory.

DEFINITION 6.10: *Let V be a compact spin manifold. Two metrics g_0 and g_1 of positive scalar curvature are* concordant *if there is a (conditioned) metric of positive scalar curvature on $V \times [0,1]$ which restricts to the given metrics on the boundary.*

Let $PSC(V)$ denote the set of concordance classes of positive scalar curvature metrics on the closed[12] V. One wishes to give an effective description of this set.

To do this we define the groups $R_n(\Gamma)$. These are bordism groups analogous to $L_n(\Gamma)$. An object of $R_n(\Gamma)$ is an n-dimensional spin manifold V with boundary ∂V, equipped with a map $V \to B\Gamma$ and a metric of positive scalar curvature on ∂V; a bordism is such an object 'with corners'.

Notice that for any closed manifold V with fundamental group Γ, there is a natural map $\Omega_n^{spin}(V) \to R_n(\Gamma)$. Now a spin structure on V defines (tautologically) an element $[V]$ of $\Omega_n^{spin}(V)$ and we have

[12]There is also a version of the theory that works relative to a given positive scalar curvature metric on ∂V. For simplicity, we do not consider this.

THEOREM 6.11: (BORDISM THEOREM) *The spin manifold* V *carries a positive scalar curvature metric if and only if the spinor fundamental class* $[V]$ *maps to zero in* $R_n(\Gamma)$.

This can be regarded as an analogue of the fundamental surgery theorem: it states that the sequence

$$PSC(V) \to \Omega_n^{spin}(V) \to R_n(\Gamma)$$

is 'exact' at the middle term. In contrast to classical surgery theory, note that there is only one possible 'normal invariant' for a spin manifold V, namely its spinor fundamental class. The analogy with the surgery exact sequence would therefore lead us to expect that the group $R_{n+1}(\Gamma)$ should act freely and transitively on $PSC(V)$. This statement was proved by Stolz:

THEOREM 6.12: (CLASSIFICATION THEOREM) *Let* V *be a spin manifold with fundamental group* Γ. *Then there is a free and transitive action of* $R_{n+1}(\Gamma)$ *on* $PSC(V)$.

Both the Bordism and Classification theorems need the dimension restriction ($n \geqslant 5$) usual in surgery theory.

As we remarked, the groups $L_n(\Gamma)$ have an algebraic description in terms of the group ring $\mathbb{Z}\Gamma$, and this is the basis for their computation. Unfortunately the groups $R_n(\Gamma)$ have at present no such algebraic description. This makes computation difficult: according to [**104**], there is no example of a group Γ (even the trivial group) for which $R_n(\Gamma)$ is known up to isomorphism.

Notes and references: Classical references for surgery theory are [**18**] (in the simply-connected case) and [**108**]. A rapid introduction from a modern perspective is contained in the first part of [**112**]. An introductory treatise [**73**] is in preparation.

Information about the Novikov conjecture can be found in the books cited above. Relevant additional references are [**22, 29, 33, 66**] and the surveys [**37, 89, 100, 111**].

For bounded surgery see [**36**], or [**87**] for an algebraic perspective. Bounded surgery as we have described it is one formalization of the idea of 'controlled' geometric topology. I am not sufficiently knowledgeable to elucidate the history of this idea, but the influential paper [**85**] should perhaps be mentioned.

The surgery theory of positive scalar curvature manifolds is being actively developed at the time of writing. An exposition is in [**104**], and see also [**39, 49**].

CHAPTER 7

Mapping surgery to analysis

In this lecture we will try to define natural transformations from the various surgery theories mentioned last time to appropriate versions of the analytic surgery exact sequence of lecture 5. Recall that we showed that C^*X is an ideal in an algebra D^*X of pseudolocal, finite propagation operators. A principle which guides the construction of the maps from surgery to analysis is that an element of $K_*(D^*X)$ is a 'reason' for the truth of a vanishing or invariance theorem[1]. This is most transparent for positive scalar curvature, so we will start with a consideration of this case.

It is important that all our constructions should take into account the action of the fundamental group. We will therefore adopt the following convention: A discrete group Γ is fixed. All spaces X that are considered will be equipped with a homomorphism $\pi_1 X \to \Gamma$. The notation $C^*_\Gamma(X)$ will refer to the Γ-fixed subalgebra of $C^*(\tilde{X})$, where $\tilde{X}$ is the Γ-covering of X corresponding to the given homomorphism. We will adopt a similar notation for $D^*(X)$, and for the ideals $C^*_X(Z)$ and $\Psi^0_X(Y)$ defined in lectures 3 and 5 respectively. Notice that if V is compact and $\Gamma = \pi_1 V$, then $C^*_\Gamma(V)$ has the same K-theory as $C^*_r\Gamma$, and $D^*_\Gamma(V)$ is what we called in lecture 5 the 'structure algebra' S^*V.

Invariants of positive scalar curvature metrics

Let V be a compact spin manifold of dimension n, and let $\Gamma = \pi_1 V$. The *spinor fundamental class* $[V] \in K_n(V)$ is just the homology class of the Dirac operator on V (for some choice of Riemannian metric); using lemma 5.15, we may write this in terms of the Dirac operator D on $\tilde{V}$ as the class $[\chi(D)] \in K_{n+1}(D^*_\Gamma(V)/C^*_\Gamma(V))$, for some choice of chopping function χ.

Now suppose that V admits a metric g of positive scalar curvature. Then the induced metric on $\tilde{V}$ has uniformly positive scalar curvature, and so there is $\varepsilon > 0$ such that the interval $[-\varepsilon, \varepsilon] \subset \mathbb{R}$ does not meet the spectrum of D. As in 3.8, we may therefore choose a chopping function χ which is equal to 1 on the

[1]Ranicki defined the structure set algebraically, as a cobordism group of chain complexes which are locally Poincaré and globally contractible; see [**87**]. This is an algebraic version of the same idea.

positive part of the spectrum of D and equal to -1 on the negative part; let us call such a χ an *exact* chopping function for D. For such a chopping function, $\chi(D)$ is an exact involution, and it therefore defines a class in $K_{n+1}(D^*_\Gamma(V))$. It is clear that this class does not depend on the choice of exact chopping function χ, since any two such chopping functions can be joined by a linear homotopy.

DEFINITION 7.1: *The class thus defined will be called the* structure invariant *of the positive scalar curvature metric g, and will be denoted by $\rho(g) \in K_{n+1}(S^*(V))$.*

From the definitions, the image of $\rho(g)$ under the natural map $K_{n+1}(S^*V) \to K_n(V)$ is just the spinor fundamental class $[V]$. By the exactness of the analytic surgery sequence it follows that $A[V]$, the index of the Dirac operator in $K_n(C^*_r\Gamma)$, is zero. Thus we have shown that the invariant $\rho(g)$ for a positive scalar curvature metric g provides a quantitative 'reason' for the truth of the Lichnerowicz vanishing theorem.

Plainly the structure invariant depends only on the homotopy class of g within the space of positive scalar curvature metrics, since a continuous path of such metrics will produce a continuous path of elements of $S^*(V)$. In fact, more is true: the structure invariant depends only on the concordance class of g. To prove this we will first give a different interpretation of the structure invariant, which will also serve as a model to follow in the case of classical surgery theory.

DEFINITION 7.2: *Let X be a proper metric space. A (coarse)* end *E of X is a subset $E \subseteq X$ such that $N_R(E) \cap N_R(X \setminus E)$ is compact for all R. We say that E is a* cone-like end *with* boundary *N if there it is coarsely equivalent to open cone $\mathcal{O}N$.*

The notation $N_R(Y) = \{x \in X : d(x,Y) < R\}$ stands for the metric R-neighbourhood of Y. We're really only interested in the open cone near infinity, but the requirement of coarse equivalence takes care of that automatically.

Suppose now that X is a space with a cone-like end E with boundary N. Then we can compactify the end E by gluing on a copy of N at infinity, thus obtaining a new space X_N, which is $X \sqcup N$ with the topology defined by saying that a sequence $x_k \in E$ tends to a point p of N if and only if the corresponding sequence x'_k in $\mathcal{O}N$ tends to p in the natural compactification of $\mathcal{O}N$ by N. Even though x'_k is defined only up to coarse equivalence, one can readily verify that this is a well-defined topology. Notice now that an X-module H is also an X_N-module, since $C_0(X_N)$ may be identified with a certain algebra of bounded continuous functions on X.

The next result is a special case of something that will be proved more generally in 10.4.

LEMMA 7.3: *In the above situation, suppose that $T \in D^*(X)$. Then T is a pseudolocal operator on X_N.*

PROOF: We may as well assume that X is an open cone $\mathcal{O}N$. Let T be a pseudolocal, finite propagation operator on X; such operators are dense in $D^*(X)$. By Kasparov's lemma (5.4), we need to prove that if φ and ψ are continuous functions on X_N with disjoint supports, then $\varphi T\psi$ is compact.

Let R be the propagation of T. The functions φ and ψ are bounded functions on X, and the condition that their supports should be disjoint *in the compactification* implies that, in X, their supports are more than R apart outside some compact subset. Since T is pseudolocal on X, perturbation of φ and ψ by compactly supported functions changes $\varphi T\psi$ only by a compact operator; but by such a change we can make their supports more than R apart, and then $\varphi T\psi$ becomes zero. This proves the result. □

ADDENDUM: Suppose, in the above situation, that the inclusion map $N \to X_N$ induces an isomorphism on π_1. (Notice that this is the hypothesis of Wall's (π-π)-theorem.) Then the fundamental groups of X and X_N agree, so that the Γ-cover of X induced by a homomorphism $\pi_1 X \to \Gamma$ extends to a Γ-cover of X_N. The construction can be carried out on these covers, and therefore we obtain a map from $D^*_\Gamma(X)$ to the Γ-invariant pseudolocal operators on $\tilde{X}_N$.

COROLLARY 7.4: *Suppose that X has a cone-like end E with boundary N, and let $C^*_X(E)$ be the ideal in $C^*(X)$ consisting of operators supported near E (see* 3.10). *Then there is a natural map*

$$K_*(C^*_X(E)) \to K_*(D^*(N)).$$

In the (π-π) situation, we also get a map

$$K_*(C^*_{\Gamma,X}(E)) \to K_*(D^*_\Gamma(N)).$$

PROOF: The proposition shows that the identity map gives an inclusion $C^*_X(E) \to \Psi^0_{X_N}(N)$. Now apply lemma 5.6. □

Now we will give an alternative definition of the ρ-invariant. Suppose that V has a positive scalar curvature metric g. Form the 'trumpet space' $\mathrm{Tr}\, V$: topologically this is the product $V \times \mathbb{R}$, with the product metric near $-\infty$ and a cone-like metric near $+\infty$ (see figure 6). This space has a cone-like end E with boundary V. Moreover, since $\mathrm{Tr}\, V$ has positive scalar curvature away from E, the index of the Dirac operator on $\mathrm{Tr}\, V$ in fact belongs to the K-theory of the ideal $C^*_{\Gamma,\mathrm{Tr}\, V}(E)$, by 3.11.

PROPOSITION 7.5: *In the above situation, the invariant $\rho(g)$ is the image of* $\mathrm{Ind}\, D_{\mathrm{Tr}\, V} \in K_{n+1}(C^*_{\Gamma,\mathrm{Tr}\, V}(E))$ *under the map of corollary* 7.4.

The proof is a variant on the slogan that 'the boundary of Dirac is Dirac'.

Suppose now that V admits *two* positive scalar curvature metrics, g_0 and g_1. Then Gromov and Lawson defined a relative invariant $i(g_0, g_1) \in K_{n+1}C^*_r(\Gamma)$ as follows. Consider a metric g on $X = V \times \mathbb{R}$ defined to be the product metric of $(V, g_0) \times (-\infty, 0]$ and the product metric of $(V, g_1) \times [1, \infty)$ interpolated by any conditioned metric on $Y = V \times [0, 1]$. This open manifold has positive scalar

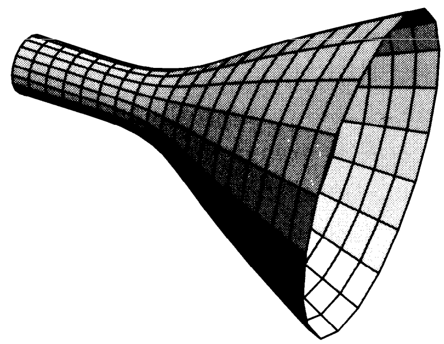

FIGURE 6. Trumpet space

curvature at infinity, so its Dirac operator has an index in $K_{n+1}(C^*_{\Gamma,X}(Y))$; and since $[0,1]$ is a compact set, $C^*_{\Gamma,X}(Y)$ can be identified with $C^*_r\Gamma$. Plainly the value of the invariant is unaffected by the choice of interpolating metric, since any two such metrics are homotopic by a homotopy supported near $[0,1]$.

PROPOSITION 7.6: *In the above situation, we have*

$$\rho(g_0) = v(i(g_0, g_1)) + \rho(g_1),$$

where $v\colon K_*(C^*_r\Gamma) \to K_*(S^*(V))$ *is the map appearing in the analytic surgery exact sequence.*

COROLLARY 7.7: *If* g_0 *and* g_1 *are concordant, then* $\rho(g_0) = \rho(g_1)$.

For it follows directly from the Lichnerowicz vanishing theorem that $i(g_0, g_1) = 0$ in this case.

PROOF: (OF THE PROPOSITION): We use the relative index theorem[2]. Consider the following four manifolds:

(i) M_1 is $\operatorname{Tr} V$ with the metric g_1 on the left-hand cylinder;
(ii) M_2 is $V \times \mathbb{R}$ with the product metric coming from the metric g_1 on V;
(iii) M_3 is $\operatorname{Tr} V$ with the metric g_0 on the left-hand cylinder;
(iv) M_4 is $V \times \mathbb{R}$ with the metric described above whose Dirac index is $i(g_0, g_1)$.

All four manifolds have natural maps to $\operatorname{Tr} V$. By the relative index theorem, we find that

$$\operatorname{Ind} D_1 - \operatorname{Ind} D_2 = \operatorname{Ind} D_3 - \operatorname{Ind} D_4$$

where D_i denotes the Dirac operator on manifold M_i. But $\operatorname{Ind} D_2 = 0$ by Lichnerowicz, so we get

$$\rho(g_1) - 0 = \rho(g_0) - v(i(g_0, g_1))$$

[2]Compare [**21**].

giving the result. □

Thus the ρ-invariant defines a map $PSC(V) \to K_{n+1}(S^*V)$. Moreover, the Gromov-Lawson invariant i gives a map from the Stolz group $R_{n+1}(\Gamma)$ to $K_{n+1}(C_r^*\Gamma)$ and the previous proposition should show that the following diagram commutes

$$\begin{array}{ccc} R_{n+1}(\Gamma) & \dashrightarrow & PSC(V) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \rho} \\ K_{n+1}(C_r^*\Gamma) & \dashrightarrow & K_{n+1}(S^*(V)) \end{array}$$

We will verify the analogous statement for classical surgery theory below.

Poincaré complexes

If M is a complete Riemannian manifold, then we have seen how to define a coarse signature for M: it is the index of the signature operator in the group $K_*(C^*M)$. Classically, however, the signature of a compact X can be defined whenever X is a Poincaré space, and in this section we will sketch an analogous construction of a 'coarse signature' for appropriate non-compact Poincaré spaces. The details will appear in [**55**].

Let X be a bounded geometry metric simplicial complex. Choose a reference point in each simplex of X — for example, its barycentre. Then the spaces $C_r^{\ell^2}(X)$ of ℓ^2 simplicial r-chains of X become X-modules (where we make a function on X act on a simplex via its value at the reference point), and the ℓ^2 simplicial chain complex

$$C_0^{\ell^2}(X) \xleftarrow{d} \cdots \xleftarrow{d} C_n^{\ell^2}(X)$$

becomes a complex of X-modules and finite propagation bounded operators. For simplicity of notation we will denote this complex simply by $\mathcal{C}_*(X)$. Similarly the dual complex of ℓ^2 cochains, $\mathcal{C}^*(X)$, is a complex of X-modules and finite propagation operators. In both cases the asserted boundedness of the operators is a consequence of bounded geometry.

REMARK: Notice that these X-modules are not 'adequate' in the sense of 3.4. In fact, they are locally finite-dimensional. Nevertheless, by Voiculescu's theorem they can be embedded in adequate X-modules, and this is enough for the purposes of index theory.

DEFINITION 7.8: *An (n-dimensional)* Hilbert-Poincaré *structure on X is a self-adjoint chain equivalence $T\colon \mathcal{C}^*(X) \to \mathcal{C}_{n-*}(X)$. If X is equipped with such a structure we will refer to it as a* Hilbert-Poincaré space.

To be precise, we require that T should be a chain equivalence in the category of complexes of X-modules and bounded finite propagation operators[3], and that

[3]In fact it is sufficient to require that T be a finite propagation operator and induce isomorphisms on 'unreduced' ℓ^2 (co)homology; 'unreduced' means that we take the kernel

it should be equal to its adjoint when we identify $\mathcal{C}^k(X)$ with $\mathcal{C}_k(X)$ via the ℓ^2 inner product.

EXAMPLE: A compact Poincaré space is a Hilbert-Poincaré space, and so is any covering space of such a space. A complete Riemannian manifold of bounded geometry can be triangulated so as to become a Hilbert-Poincaré space.

All the Hilbert-Poincaré spaces that we consider will be of the above two types, or will be obtained by gluing examples of these types together in various ways.

REMARK: (NONEXAMPLE) Suppose that X is a locally finite complex which is a Poincaré duality space in the usual sense that there is a fundamental class $[X] \in H_n^{lf}(X)$ which induces an isomorphism $H_c^r(X) \to H_{n-r}(X)$ by cap-product. Then it does not immediately follow that X has a Hilbert-Poincaré structure. In fact, while cap-product with the locally finite fundamental class defines a duality map on compactly supported cohomology in this case, there is no a priori reason to believe that this duality will satisfy the necessary norm estimates to pass to a bounded operator on the ℓ^2 cochain complex. The simplest sufficient condition that can be given is that X should be *both* a bounded geometry complex in the sense of 2.3 *and* a bounded Poincaré space in the sense of [**36**]; this combination of conditions is rather restrictive. This point will be important when we come to discuss bounded surgery theory.

The next lemma is a consequence of the fact that T is an equivalence.

LEMMA 7.9: *Let X be a Hilbert-Poincaré space. Then the operators $d + d^* \pm T$ are invertible (as maps from $\oplus\mathcal{C}^*(X)$ to $\oplus\mathcal{C}^*(X)$).*

ADDENDUM: The inverses of these operators belong to $C^*(X)$. (For the operators themselves are operators of finite propagation, and C^*-algebras are closed under the functional calculus.)

Now we can define the signature of a Hilbert-Poincaré space; for simplicity we will do this only when the formal dimension is *odd*. Then the operators $d+d^* \pm T$ preserve the even-odd grading of $\mathcal{C}^*(X)$. Consider the invertible operator

$$G = (d + d^* + T)(d + d^* - T)^{-1} \colon \mathcal{C}^{even}(X) \to \mathcal{C}^{even}(X).$$

This is an invertible operator on a locally finite-dimensional X-module, so it defines a class in $K_1(C^*(X))$. We call this class the *(Hilbert-Poincaré) signature* $\operatorname{Sign} X$ of X. There is an analogous definition in the even-dimensional case.

REMARK: The operator G need not be *unitary*. This will be significant later (see page 84).

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of d modulo the image of d, rather than modulo the closure of the image of d.

The signature of a Hilbert-Poincaré space has two key properties. First, in the case of a complete Riemannian manifold the Hilbert-Poincaré signature is equal to the coarse index of the signature operator. Second, the signature is cobordism invariant in a suitable sense. Here we use the language of bounded topology[4]. Suppose that X is equipped with a coarse map to a reference space $\mathcal{Z}$. One can define a notion of *Hilbert-Poincaré pair*, or Hilbert-Poincaré space with boundary: suppose that X is the boundary of such a space Y, which is also equipped with a coarse map to $\mathcal{Z}$ in such a way that the diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \mathcal{Z} & \end{array}$$

boundedly commutes. Then the conclusion is that the (image of) the Hilbert-Poincaré signature $\operatorname{Sign} X$ vanishes in $K_*(C^*(\mathcal{Z}))\otimes\mathbb{Z}[\frac{1}{2}]$. Notice the need to invert 2: this is because the (K-homology) boundary of the even-dimensional signature operator is *twice* the odd-dimensional signature operator.

REMARK: Notice that this implies that the $C^*_r(\pi_1 V)\otimes\mathbb{Z}[\frac{1}{2}]$ index of the signature operator on a compact manifold V is an invariant of oriented homotopy type[5]. Indeed, if two compact manifolds V_1 and V_2 are homotopy equivalent, then they are certainly π_1-equivariantly Poincaré cobordant. Passing to the universal cover and keeping track of the equivariance, we find that the equivariant signatures of the universal covers $\tilde{V}_1$ and $\tilde{V}_2$ agree. But these just are the equivariant indices of the signature operators. In the next section we will refine the argument, obtaining a quantitative reason for this invariance property, just as in the previous section the ρ-invariant gave a quantitative reason for the Lichnerowicz vanishing theorem.

A manifold structure invariant

Suppose that V is a closed manifold with fundamental group Γ. Let M_1 and M_2 be manifolds equipped with maps f_1, f_2 to V. In this section we propose to associate to any homotopy equivalence $h\colon M_1 \to M_2$ such that $f_2 h = f_1$ a *structure invariant* $\sigma(h) \in K_{n+1}(D^*_\Gamma V)\otimes\mathbb{Z}[\frac{1}{2}]$ which has the following properties:

(i) (Vanishing) If h is a diffeomorphism, $\sigma(h) = 0$;

(ii) (Additivity) Suppose that $h'\colon M_2 \to M_3$ is another homotopy equivalence, with $f_3\colon M_3 \to V$ such that $f_3 h' = f_2$. Then one has $\sigma(h'h) = \sigma(h') + \sigma(h)$.

(iii) (Signature) The image of $\sigma(h)$ under the natural map $K_{n+1}(D^*_\Gamma V) \to K_n(V)$ is, up to a possible factor of 2, the difference of the signature classes, $f_{2*}[D_{M_2}] - f_{1*}[D_{M_1}]$.

[4]See the section 'Bounded surgery' in the previous lecture.

[5]Kaminker and Miller [**63, 74**] proved this even without inverting 2.

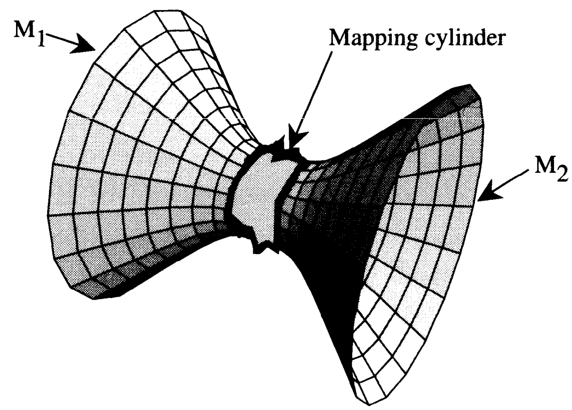

FIGURE 7. The space whose signature defines the structure invariant

Here is the definition, which is analogous to the second definition (7.5) of the positive scalar curvature structure invariant ρ. Let C_h denote the mapping cylinder of h. Form a Hilbert-Poincaré space W by gluing open conical ends on the two boundary components of C_h, diffeomorphic to $M_1 \times [1,\infty)$ and $M_2 \times [1,\infty)$ respectively; see the figure. By construction, W comes with a continuous coarse map $f = f_1 \cup_h f_2$ to the 'bicone' BV, which is just $V \times \mathbb{R}$ with a metric which is conical on both ends.

REMARK: Note that it is significant here that we have a homotopy equivalence between *manifolds*: we could not carry out this construction for a homotopy equivalence merely between *Poincaré spaces*. For the cone on a Poincaré space is not, in general, a Hilbert-Poincaré space.

On BV we have the following sequence of maps

$$K_{n+1}(C^*_\Gamma(BV)) \to K_{n+1}(D^*_\Gamma(BV)) \to K_{n+1}(D^*_\Gamma(V \times [0,1])) \to K_{n+1}(D^*_\Gamma(V)).$$

The first map is the one appearing in the analytic surgery exact sequence, induced by the inclusion $C^*_\Gamma(BV) \to D^*_\Gamma(BV)$. The second map comes from corollary 7.4. The third map is induced by the projection $V \times [0,1] \to V$.

DEFINITION 7.10: *The class $\sigma(h) \in K_{n+1}(D^*_\Gamma(V))$ is the image, under the composite map above, of $f_*(\operatorname{Sign} W)$, where $\operatorname{Sign} W \in K_{n+1}(C^*W)$ is the signature of the Hilbert-Poincaré space W.*

Notice that the vanishing property of $\sigma(f)$ is apparent from this definition. For, if h is a diffeomorphism, then C_h and hence W are manifolds, and thus the signature $\operatorname{Sign} W$ belongs to the image of the assembly map. Hence (since f is continuous), $f_*(\operatorname{Sign} W)$ belongs to the image of the assembly map for BW, hence it maps to zero in $K_{n+1}(D^*_\Gamma(BV))$ by exactness.

To check the additivity property, let W and W' be obtained from the mapping cylinders of h and h' by coning the ends as described above, and let W'' be similarly obtained from the mapping cylinder of $h'h$. Now consider the following spaces with their obvious control maps to BV:

$$A = W \sqcup W', \quad B = W'' \sqcup BM_2 \qquad \text{(disjoint unions)}.$$

Notice that $C_{h'h} = C_h \cup_{M_2} C_{h'}$. Thus, the space B is obtained from A by cutting off one end and gluing it back on 'the other way round'. We may therefore use the relative index theorem (a version of which applies to these signatures) to argue exactly as in the proof of 7.6 that the signature of A, in $K_{n+1}(C^*_\Gamma BV)\otimes\mathbb{Z}[\frac{1}{2}]$, equals the signature of B. However, the signature of A maps to $\sigma(h) + \sigma(h')$, whereas the signature of B maps to $\sigma(h'h) + \sigma(1_{M_2})$. Since $\sigma(1_{M_2}) = 0$ by the vanishing property, we have proved the result.

Finally, the signature property is once again a consequence of the principle that 'the boundary of Dirac is Dirac' — or rather, in this case, that 'the boundary of signature is signature, up to a factor of 2'. For details, the reader is referred to [**55**].

Now we will use the construction of the σ-invariant to map the surgery exact sequence to the analytic surgery exact sequence. Specifically, we will construct maps α, β, and γ which fit into a commutative diagram

$$\begin{array}{ccccccc}
L_{n+1}(\Gamma) & \cdots\to & \mathcal{S}(V) & \longrightarrow & \mathcal{N}(V) & \xrightarrow{A} & L_n(\Gamma) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
K_{n+1}(C^*_r\Gamma)\otimes\mathbb{Z}[\frac{1}{2}] & \to & K_{n+1}(D^*_\Gamma(V))\otimes\mathbb{Z}[\frac{1}{2}] & \to & K_n(V)\otimes\mathbb{Z}[\frac{1}{2}] & \xrightarrow{A} & K_n(C^*_r\Gamma)\otimes\mathbb{Z}[\frac{1}{2}]
\end{array}$$

where $n = \dim V$, the top row is the ordinary surgery exact sequence, and the bottom row is the analytic surgery exact sequence.

To define α, suppose that $f\colon M \to V$ is a structure on V. Then it has a structure invariant $\sigma(f)$, and we let $\alpha([f]) = 2^{-\lfloor (n+1)/2 \rfloor}\sigma(f) \in K_{n+1}(S^*V)\otimes\mathbb{Z}[\frac{1}{2}]$. The power[6] of 2 corrects for the fact that the boundary of the signature operator is not exactly equal to the signature operator. It is easy to see, using the vanishing and additivity properties of σ, that this map is h-cobordism invariant in the appropriate sense and therefore that α is well-defined on the structure set.

The map β sends a normal map $f\colon M \to V$ to the 'difference of signature operators' $2^{-\lfloor n/2 \rfloor}(f_*[D_M] - [D_V]) \in K_n(V)$. The map γ sends a normal map of pairs $(M, \partial M) \to (X, \partial X)$, defining an element of $L_n(\Gamma)$, to $2^{-\lfloor n/2 \rfloor}\operatorname{Sign} Z \in K_n(C^*_r\Gamma)\otimes\mathbb{Z}[\frac{1}{2}]$, where Z is the Poincaré space obtained by gluing M to X by means of the homotopy equivalence of their boundaries. Using the cobordism invariance of the signature one sees that β and γ are well-defined.

PROPOSITION 7.11: *The diagram of surgery exact sequences, above, commutes.*

PROOF: (SKETCH) The right-hand square commutes because the Hilbert-Poincaré signature of a manifold is the index of the signature operator. The middle square commutes because of the signature property of the σ-invariant. To

[6] The notation $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x.

show that the left-hand square commutes, recall that the top arrow is defined by Wall realization (Proposition 6.5): one constructs a cobordism from the standard structure on V to a new structure, which realizes a given surgery obstruction. Using the relative index theorem together with the additivity and invariance properties of the invariant σ, one can prove that this square commutes along the lines of the proof of Proposition 7.6. □

The following fact can be proved using the homotopy-theoretic description of $\mathcal{N}(V)$. (Compare our discussion of G/O modulo finite groups on page 52.)

LEMMA 7.12: *The map β becomes an isomorphism after tensoring with $\mathbb{Q}$.*

As a corollary of this fact and the surgery theory of simply-connected manifolds one can easily give explicit examples of structures f (fake complex projective spaces) for which $\sigma(f) \neq 0$.

Another corollary is the following

PROPOSITION 7.13: *The analytic Novikov conjecture* 5.17 *implies the topological Novikov conjecture* 6.8.

Assuming that $V = B\Gamma$ one can prove this simply by chasing the diagram relating the analytic and geometric surgery exact sequences. Even without this, however, we can prove that the analytic Novikov conjecture implies the homotopy invariance of the higher signatures: argue as in the example at the end of lecture 5, replacing the Lichnerowicz vanishing theorem by the homotopy invariance of the $K_*(C_r^*\Gamma)$-signature.

Bounded structures

Let us now see how far the above ideas can be applied to *bounded* surgery theory. We would like, in favorable situations, to be able to produce maps α, β, and γ, analogous to those above, and relating the bounded surgery exact sequence of theorem 6.9 to its analytic counterpart discussed on page 40. As above, the mapping ought to be obtainable if we can understand in a sufficiently canonical way a homotopy invariance property of the bounded analytic signature. What we would wish to be true is the following 'coarse' analogue of the Kaminker-Miller theorem:

CONJECTURE 7.14: *If smooth manifolds M and M' over $\mathcal{Z}$ are homotopy equivalent by a boundedly controlled orientation-preserving homotopy equivalence, then their coarse analytic signatures agree:*

$$\mathrm{Sign}(M) = \mathrm{Sign}(M') \in K_*(C^*(\mathcal{Z})).$$

REMARK: There is another, related, reason for interest in this conjecture. By a standard reduction (done by Novikov in the sixties) the conjecture for $\mathcal{Z} = \mathbb{R}^n$, together with the $\mathbb{R}^n$-bounded index theorem (page 34), implies Novikov's theorem that the rational Pontrjagin classes of a smooth manifold are invariant under homeomorphisms. Indeed, Novikov's reduction shows that the topological invariance of the Pontrjagin classes follows from the statement that if $W = M \times \mathbb{R}^n$, where M is a compact manifold, and $c\colon W \to \mathbb{R}^n$ is the obvious projection map, then, if W is endowed with an *arbitrary* smooth structure, and c is made transverse (for this smooth structure) at the point $0 \in \mathbb{R}^n$, then the signature of $c^{-1}(0)$ is equal to the signature of M. Now the $\mathbb{R}^n$-bounded index theorem tells us that the signature of $c^{-1}(0)$ made transverse is just the bounded analytic signature of W (up to a multiple of 2). To apply the conjecture, we need only to note that since W and $M \times \mathbb{R}^n$ are homeomorphic (with infinitely good control — the diagram of control maps commutes on the nose) they are certainly homotopy equivalent with bounded control; just smooth off the homeomorphism in a boundedly controlled way[7].

Unfortunately there is a basic difficulty in the way of proving the conjecture along the lines of the proof of the equivariant case, and it is this. We can certainly form the mapping cylinder of a bounded homotopy equivalence, and it is a bounded Poincaré cobordism. But it may not be a Hilbert-Poincaré space, nevertheless: the operators arising in the duality, though bounded in the geometrical sense of propagation, may not be bounded in the analytical sense of operators on Hilbert space. It is a delicate question whether the homotopy equivalence can be 'smoothed' sufficiently to obtain analytical boundedness without thereby destroying its geometrical boundedness.

Remark: This apparent trade-off between boundedness in two noncompact directions is somewhat reminiscent of the Heisenberg uncertainty principle and, more generally, of microlocal analysis. (Compare the remarks in [**112**, page 15] and also the introduction to [**93**].) There are some other indications which point in this direction, but, as yet, no systematic theory.

In order to circumvent this difficulty, the authors of [**82**] consider a different surgery theory, in which only those maps are allowed that induce bounded operators in both senses. For this surgery theory the analogue of the Kaminker-Miller theorem can be proved. Then, by computation, they show that in certain cases (precisely, whenever $X = \mathcal{O}N$ is an open cone on a polyhedron) the obstruction group for the 'bounded operator' version of bounded surgery is the same as the obstruction group for ordinary bounded surgery. Since $\mathbb{R}^n$ is a cone, the conjecture is in particular true for $\mathbb{R}^n$. However, [**82**] cannot be said to yield a new proof of the topological invariance of the Pontrjagin classes, since the computations that must be done to compare the 'bounded operator' and ordinary L-groups are strong enough by themselves to prove the required result, without bringing in questions of analysis.

[7] This idea that 'a homeomorphism is an infinitely controlled homotopy equivalence' is a basic one of controlled topology; it achieves a precise form in the α-approximation theorem of Chapman and Ferry [**24**].

Notes and references: The results of this lecture are mostly new, and details should appear in [**55**]. There are, however, other more algebraic approaches to the issue: maps from L-theory to C^*-algebra K-theory were first described in [**40**], using the algebraic definition of L-theory: the key idea is to apply the spectral theorem to separate the 'positive' and 'negative' parts of a quadratic form, as in Sylvester's theorem classifying quadratic forms over $\mathbb{R}$ by their signatures. For a survey of this material see [**100**].

CHAPTER 8

The coarse Baum-Connes conjecture

Let X be a metric (or coarse) space. In lecture 5 we have defined an 'assembly map'

$$A\colon K_*(X) \to K_*(C^*X)$$

from the K-homology of X to the K-theory of the C^*-algebra associated to the coarse structure of X. This corresponds to the assembly map in bounded surgery, although, as we saw at the end of the last lecture, there are difficulties in making a direct connection between these two topics.

When is the assembly map an isomorphism? The answer cannot be 'always' because the right-hand side is coarsely invariant whereas the left-hand side is not. However, recall our discussion of the Borel and Novikov conjectures: we should expect assembly to be an isomorphism only for spaces whose topology is closely related to their coarse geometry. Here the analogue of asphericity is *uniform contractibility*, defined in 2.3.

The coarse assembly map

Recall that in lecture 2 we discussed the notion of 'coarsening', which gives the best approximation to a homology theory by a 'coarse homology theory'. Let KX_* denote the coarse homology theory corresponding to K-homology.

PROPOSITION 8.1: *For any coarse space X there is defined a* coarse assembly map $A_\infty\colon KX_*(X) \to K_*(C^*(X))$, *which makes the diagram*

$$\begin{array}{ccc} K_*(X) & & \\ \big\downarrow c & \searrow^{A} & \\ KX_*(X) & \xrightarrow[A_\infty]{} & K_*(C^*(X)) \end{array}$$

commute.

PROOF: When X admits a coarsening (recall that this is a uniformly contractible bounded geometry metric simplicial complex EX equipped with a coarse equivalence $X \to EX$), this follows from the commutative diagram

$$\begin{array}{ccc} K_*(X) & \xrightarrow{A} & K_*(C^*(X)) \\ \downarrow & & \| \\ KX_*(X) =_{df} K_*(EX) & \longrightarrow & K_*(C^*(EX)) \end{array}$$

where the vertical maps are induced by coarsening $c\colon E \to EX$ and the right-hand one is an isomorphism because c is a coarse equivalence. If X does not admit a coarsening, one can define KX_* in terms of the nerves of a coarsening sequence of covers $\mathfrak{U}_i$, and the assembly map A_∞ becomes the direct limit of the assembly maps for the $|\mathfrak{U}_i|$. □

CONJECTURE 8.2: (COARSE BAUM-CONNES CONJECTURE) *For any bounded geometry space X, the coarse assembly map $A_\infty\colon KX_*(X) \to K_*(C^*(X))$ is an isomorphism.*

COROLLARY 8.3: (OF THE CONJECTURE) *For any uniformly contractible bounded geometry metric simplicial complex X, the usual assembly map $A\colon K_*(X) \to K_*(C^*X)$ is an isomorphism.*

In particular, suppose that V is a finite complex which is an Eilenberg-Mac Lane space of type $K(\Gamma, 1)$. Then the universal cover $\tilde{V} = E\Gamma$ is a uniformly contractible bounded geometry complex, and so the conjecture implies that $A\colon K_*(\tilde{V}) \to K_*(C^*(\tilde{V})) = K_*(C^*|\Gamma|)$ is an isomorphism.

REMARK: It may seem that the assembly map should be an isomorphism for all uniformly contractible spaces, irrespective of bounded geometry conditions. However, a remarkable example due to Dranishnikov, Ferry, and Weinberger [**31**] shows that this cannot be the case. The example is based on a construction of Dranishnikov [**30**] which yields a cell-like map $Z \to S^7$ which fails to induce an isomorphism on K-homology. A suitable warped coning of this construction produces a space X, uniformly contractible, and homeomorphic to $\mathbb{R}^8$, but which has the coarse type of a cone on a Dranishnikov space; which implies that coarsening does not induce an isomorphism $K_*(X) \to KX_*(X)$. The construction can be arranged so that $K_*(X) \to KX_*(X)$ is not injective; so the assembly map $K_*(X) \to K_*(C^*X)$ cannot be injective either.

REMARK: The construction of [**31**] leaves open the possibility that A_∞ may be an isomorphism for all spaces. However, Guoliang Yu[1] has recently produced an example of a space X of unbounded geometry for which A_∞ fails to be injective. Yu's space may be described as the disjoint union of an infinite sequence of spheres, with slowly increasing radii, more rapidly increasing spacing between each sphere and the next, and very rapidly increasing dimensions. This space is a complete spin manifold with uniformly positive scalar curvature, so the coarse index of the (spinor) Dirac operator is zero. However, one can compute the

[1] Personal communication.

group $KX_*(X)$ as a direct limit, and it turns out that the Dirac operator gives a nonzero element of this group. Thus, assembly fails to be injective in this case.

The interest of the coarse Baum-Connes conjecture is twofold. First, it is a purely coarse geometric statement. This means that, in seeking to prove it, there is no group structure which need be respected in seeking to simplify the problem. Second, the conjecture has consequences about open manifolds and also, via the universal covering, about compact ones. For instance we recall the standard conjecture [**48**] that no compact aspherical manifold can carry a metric of positive scalar curvature. It is well-known that this would follow from the analytic Novikov conjecture; however, it is also an immediate consequence of the coarse Baum-Connes conjecture. For consider the Dirac operator on the universal cover of such a manifold V, which is a uniformly contractible space[2]. Poincaré duality for K-homology tells us that the homology class of the Dirac operator represents the K-homology fundamental class for this open manifold; in particular, it certainly is not the zero class. Thus, if the conjecture holds, the index of the Dirac operator in $K_*(C^*(\tilde{V}))$ cannot be zero, and so, by Lichnerowicz, V cannot have positive scalar curvature.

The principle of descent

The fact that certain consequences of the analytic Novikov conjecture also flow from the coarse Baum-Connes conjecture is no coincidence. In this section we will discuss the process of 'descent', whereby the coarse Baum-Connes conjecture actually implies the analytic Novikov conjecture in certain cases. Specifically, we will prove

THEOREM 8.4: *Let Γ be a group which is classified by a finite complex. Suppose that the coarse Baum-Connes conjecture is true for the underlying metric space $|\Gamma|$ of Γ. Then the (analytic) Novikov conjecture is true for Γ, that is, the equivariant assembly map*

$$A_\Gamma \colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$$

is injective.

The proof will follow from a series of lemmas. Imagine that we have fixed a particular model for $B\Gamma$ which is a finite complex, and a corresponding model for $E\Gamma$ as a contractible finite free Γ-CW-complex.

DEFINITION 8.5: *Let G be any space on which Γ acts. The* homotopy fixed set *$G^{h\Gamma}$ of the action is defined to be*

$$G^{h\Gamma} = \mathrm{Maps}_\Gamma(E\Gamma, G),$$

the space of equivariant maps from $E\Gamma$ to G.

[2]In any metric lifted from V.

If $g \in G$ is fixed under the action of Γ, then the constant map $E\Gamma \to G$ with value g is equivariant. In this way we get a map from the ordinary fixed point set G^Γ to the homotopy fixed point set $G^{h\Gamma}$. Notice that if G is a C^*-algebra then so is $G^{h\Gamma}$ (with the supremum norm) and the natural map $G^\Gamma \to G^{h\Gamma}$ becomes a C^*-homomorphism.

The scheme of the proof is now the following. For any finite free Γ-CW-complex X we may consider the diagram

$$\begin{array}{ccc} K_*((D^*X)^\Gamma) & \xrightarrow{v} & K_*((D^*X/C^*X)^\Gamma) \\ \big\downarrow & & \big\downarrow h \\ K_*((D^*X)^{h\Gamma}) & \longrightarrow & K_*((D^*X/C^*X)^{h\Gamma}) \end{array} \qquad (\dagger)$$

By considering the homological properties of range and domain as functors on free Γ-CW-complexes, we will prove that the right-hand vertical map (denoted h) is always an isomorphism. On the other hand, the properties of homotopy fixed sets will show that if Γ satisfies the coarse Baum-Connes conjecture, then the bottom left-hand group is zero for $X = E\Gamma$. It will follow that for $X = E\Gamma$ the map v in the analytic surgery exact sequence

$$\begin{array}{ccccccc} \longrightarrow & K_{*+1}(S^*(B\Gamma)) & \xrightarrow{v} & K_*(B\Gamma) & \xrightarrow{A} & K_*(C^*_r\Gamma) & \longrightarrow \\ & \| & & \| & & \| & \\ & K_{*+1}((D^*X)^\Gamma) & \longrightarrow & K_{*+1}((D^*X/C^*X)^\Gamma) & \longrightarrow & K_*((C^*X)^\Gamma) & \end{array}$$

is zero, and therefore that the assembly map itself is injective.

LEMMA 8.6: *The functor $X \mapsto K_*((D^*X/C^*X)^{h\Gamma})$ is excisive*[3] *on the category of finite free Γ-CW-complexes, and the map h in the diagram $(\dagger)$ is a natural transformation of excisive functors.*

PROOF: One can follow exactly the outline of the proof of the Mayer-Vietoris sequence for K-homology (5.6), replacing all C^*-algebras and ideals by their 'homotopy fixed' versions. □

REMARK: There is a more highfalutin explanation for this result. Excisiveness can be expressed in terms of the functor $\mathcal{F}$ that sends a space X to the stable unitary group of the corresponding C^*-algebra by the statement that $\mathcal{F}$ transforms cofibrations into fibrations. Now one can show that if $P \to Q \to R$ is a fibration sequence in which the maps are equivariant, then the corresponding sequence of homotopy fixed sets $P^{h\Gamma} \to Q^{h\Gamma} \to R^{h\Gamma}$ is also a fibration sequence. The result then follows from the unequivariant excision properties of K-homology on X.

LEMMA 8.7: *If X consists of a single free Γ-simplex, then h is an isomorphism.*

[3] This means that it has a Mayer-Vietoris sequence.

PROOF: Suppose that $X = \Gamma \times \Delta^m$. Then (because of the localization property of D^*/C^*, that any element of this algebra has a representative with prescribed small propagation) the algebra $D^*(X)/C^*(X)$ is equal to the algebra $\mathrm{Maps}(\Gamma, D^*(\Delta^m)/C^*(\Delta^m))$, and similarly its stable unitary group is equal to $\mathrm{Maps}(\Gamma, U)$, where U is the stable unitary group of $D^*(\Delta^m)/C^*(\Delta^m)$. Recall that the K-theory groups of a C^*-algebra are simply the homotopy groups of its stable unitary group. But $U = \mathrm{Maps}(\Gamma, U)^\Gamma$ and $\mathrm{Maps}(\Gamma, U)^{h\Gamma}$ are homotopy equivalent, as the following argument shows:

$$\begin{aligned} \mathrm{Maps}(\Gamma, U)^{h\Gamma} &= \mathrm{Maps}_\Gamma(E\Gamma, \mathrm{Maps}(\Gamma, U)) \\ &= \mathrm{Maps}_\Gamma(\Gamma, \mathrm{Maps}(E\Gamma, U)) \\ &= \mathrm{Maps}(E\Gamma, U) \\ &\simeq U \end{aligned}$$

This proves the result for the case of a single simplex. □

PROPOSITION 8.8: *For any finite free Γ-CW-complex X, the map h in (†) above is an isomorphism*

PROOF: This follows from the previous two lemmas by a standard induction on the number of simplices, using the five lemma. □

To complete the argument we need to know that the bottom left-hand group in (†) is zero, when $X = E\Gamma$. Our assumption that the coarse Baum-Connes conjecture is true for $|\Gamma|$ says that D^*X has zero K-theory, and therefore that the stable unitary group of D^*X is (weakly) contractible. The result we want will therefore follow by applying to the stable unitary group the following general result on homotopy fixed sets.

LEMMA 8.9: *Suppose that Γ acts on U and that U is weakly contractible (in the sense that there is a point $1 \in U$ fixed under Γ such that the inclusion $1 \to U$ is a weak homotopy equivalence). Then $U^{h\Gamma}$ is weakly contractible.*

PROOF: Let us begin by proving that $U^{h\Gamma}$ is path connected. Let $f\colon E\Gamma \to U$ be a point of $U^{h\Gamma}$. We define a path connecting f to the constant map 1 by induction over the free Γ-simplices of $E\Gamma$, in increasing order of dimension. The induction step is therefore that $\Gamma \times \Delta$ is a free simplex and there is given an equivariant map $\Gamma \times \partial\Delta \times [0,1]$ connecting $f = f_0$ with $1 = f_1$ on the boundary of $\Gamma \times \Delta$. Considering a single component Δ of the Γ-simplex, we can (unequivariantly) extend the given data $\Delta \times \{0,1\} \cup \partial\Delta \times [0,1] \to U$ to a map $f_t\colon \Delta \times [0,1] \to U$, because U is weakly contractible. There is now one and only one way of extending this new f_t equivariantly over $\Gamma \times \Delta$, and by uniqueness this matches up with the given f_t on the boundary. This completes the induction step.

Now to prove that $U^{h\Gamma}$ is in fact weakly contractible, we apply the preceding argument to $V = \mathrm{Maps}(S^n, U)$. Since U is weakly contractible, so is V, hence $V^{h\Gamma}$ is path connected. But $\pi_0(V^{h\Gamma}) = \pi_n(U^{h\Gamma})$, completing the proof. □

I would now like to connect this presentation of the descent argument with that using Poincaré duality and the indices of elliptic families, as in [**60**]. We begin by remarking that the above argument can be made to work if the coarse assembly map is not an isomorphism; it need only be 'canonically' split injective. The index theory approach begins from the observation that higher index theorems can often be used to show this split injectivity.

Suppose for simplicity that V is a compact aspherical spin manifold with fundamental group Γ. By K-theory Poincaré duality, the K-homology of the contractible manifold $\tilde{V}$ is a copy of $\mathbb{Z}$ generated by the homology class of the Dirac operator. To split the coarse assembly map therefore requires only the computation of a single higher index. In particular if $\tilde{V}$ admits a contracting, degree-one map onto Euclidean space ("hyper-Euclidean"), then the $\mathbb{R}^n$-bounded index theorem provides such a splitting, by showing that the index in $K_*(C^*|\mathbb{R}^n|) = \mathbb{Z}$ of the Dirac operator is equal to one. Moreover, this copy of $\mathbb{Z}$ can be detected by pairing $K_*(C^*\tilde{V})$ with a Fredholm module coming from a suitable 'asymptotically flat vector bundle' over $\tilde{V}$ (see [**113**]).

Consider a 'families' version of this problem. Suppose therefore that there is given a bundle E over some compact manifold M, whose fibres are copies of $\tilde{V}$ and whose structural group is Γ. Associated to this bundle is a bundle of C^*-algebras with fibre $C^*\tilde{V}$, and a family of Dirac operators on the fibres has an index in the K-theory of the algebra $\mathfrak{A}$ of sections of this bundle of C^*-algebras. Notice that any operator on V gives such a family, since it lifts to a Γ-equivariant operator on $\tilde{V}$; so we get an index map $K_*(V) \to K_*(\mathfrak{A})$. Moreover, a family of asymptotically flat vector bundles gives a families index map

$$K_*(\mathfrak{A}) \to K^*(M)$$

generalizing the previous case where M is a point.

In particular we may consider the example $M = V$ where E is the balanced product $\tilde{V} \times_\Gamma \tilde{V}$. In this case the relevant higher index theorem states that the composite $K_*(V) \to K_*(\mathfrak{A}) \to K^*(V)$, obtained from a hyper-Euclidean structure on $\tilde{V}$ is simply K-theory Poincaré duality and is therefore an isomorphism. However, one can easily see that in this case $\mathfrak{A}$ is nothing other than the algebra $C^*(\tilde{V})^{h\Gamma}$ already alluded to. We therefore have a commutative diagram

$$\begin{array}{ccccc}
K_*(V) & \longrightarrow & K_*((D^*\tilde{V}/C^*\tilde{V})^{h\Gamma}) & & K^*(V) \\
\downarrow{\scriptstyle A_\Gamma} & & \downarrow & & \uparrow \\
K_*(C^*_r\Gamma) & \longrightarrow & K_*((C^*\tilde{V})^{h\Gamma}) & = & K_*(\mathfrak{A})
\end{array}$$

showing that A_Γ is split.

Notes and references: After foreshadowings in [**95, 58**], the coarse Baum-Connes conjecture was stated in detail in [**54, 115**]. The descent argument is widely known, see the discussions from various points of view in [**22, 38, 60**]; indeed, it is probably implicit in all approaches to the Novikov conjecture for torsionfree groups which depend on unequivariant geometric properties of the universal cover [**33, 64, 66, 77**]. The presentation here owes much to [**22**] and to conversations with Higson. The interpretation in terms of 'coarse index theory for families' I learned from Weinberger, some time around 1990.

The 'descent' argument presented here can be extended in a number of ways. For example, one can consider the universal space for proper actions $\underline{E}\Gamma$ (see page 14) in place of $E\Gamma$ itself. If there is a model for $\underline{E}\Gamma$ on which Γ acts cocompactly and which satisfies the coarse Baum-Connes conjecture, then there is a descent argument which gives the injectivity of the *Baum-Connes assembly map* $K_*^\Gamma(\underline{E}\Gamma) \to K_*(C_r^*\Gamma)$. See [**23**].

Another variation involves the introduction of 'coefficients' into the coarse Baum-Connes conjecture [**53**]. Let A be a separable C^*-algebra. Then one can define [**14, 65**] the notion of a *Hilbert A-module*: informally, this is 'a Hilbert space whose scalars are elements of A'. By replacing Hilbert spaces with Hilbert A-modules throughout the development, one obtains versions of $C^*(X)$, $D^*(X)$ 'with coefficients in A' and an assembly map 'with coefficients'. The techniques of the next chapter for proving the coarse Baum-Connes conjecture all are general enough, in fact, to prove the conjecture with arbitrary coefficients.

This is relevant to descent because of the following observation. Suppose that Γ_0 is a group with a finite classifying space for which the coarse Baum-Connes conjecture is true (with arbitrary coefficients). Descent then tells us that the Novikov conjecture is true for Γ_0. However, suppose that Γ is an (arbitrary) subgroup of Γ_0 and consider the coarse Baum-Connes conjecture with coefficients in $A = c_0(\Gamma_0/\Gamma)$, the algebra of functions vanishing at infinity on the coset space Γ_0/Γ. Then one can show that the restriction to Γ_0-fixed sets of the coarse assembly map with coefficients in A is simply the ordinary assembly map for the group Γ. Applying the descent argument we obtain the Novikov conjecture for Γ. In short, if these methods prove the Novikov conjecture for some group, they prove it also for every subgroup of that group.

CHAPTER 9

Methods of computation

In this lecture we will prove the coarse Baum-Connes conjecture for a number of interesting spaces. By the results of the previous lecture this will imply the Novikov conjecture for a wide range of groups. The key point is to develop computational techniques analogous to the usual ones of algebraic topology.

A coarse Mayer-Vietoris sequence

We begin with excision (equivalently, with Mayer-Vietoris sequences). Suppose X is a coarse space written as a union $X_1 \cup X_2$ of subspaces. We would hope that under favorable circumstances there should be a Mayer-Vietoris sequence

$$\begin{array}{ccccc} K_1(C^*(X_1 \cap X_2)) & \to & K_1(C^*(X_1)) \oplus K_1(C^*(X_2)) & \longrightarrow & K_1(C^*(X)) \\ \uparrow & & & & \downarrow \\ K_0(C^*(X)) & \longleftarrow & K_0(C^*(X_1)) \oplus K_0(C^*(X_2)) & \leftarrow & K_0(C^*(X_1 \cap X_2)) \end{array} \qquad (\ddagger)$$

It is clear that such a Mayer-Vietoris sequence cannot hold good in general, because it is often possible to replace X_1 and X_2 by coarsely equivalent subspaces X_1' and X_2' in such a way that $X_1 \cap X_2$ is *not* coarsely equivalent to $X_1' \cap X_2'$; for a concrete example of this, consider X to be the disjoint union of two parallel lines X_1 and X_2 in the plane, with the induced metric. This is simply the coarse analogue of the existence of non-excisive decompositions in ordinary algebraic topology[1]. What we need is a condition of 'coarse excisiveness', and the correct condition turns out to be the following one:

DEFINITION 9.1: [**58**] *A decomposition* $X = X_1 \cup X_2$ *is* coarsely excisive *if for every* R *there is an* S *such that* $N_R(X_1) \cap N_R(X_2) \subseteq N_S(X_1 \cap X_2)$.

Here $N_R(Y) = \{x \in X : d(x, Y) < R\}$ denotes the metric R-neighbourhood of Y.

THEOREM 9.2: [**58**] *The Mayer-Vietoris sequence* $(\ddagger)$ *is exact for coarsely excisive decompositions.*

[1]For example, there is no Mayer-Vietoris sequence for the decomposition of S^2 into the union of the closed northern hemisphere and the open southern hemisphere.

PROOF: Given $Y \subseteq X$, we let $C^*_X(Y)$ be the ideal of C^*X that is obtained as the norm closure of the set of all finite propagation operators T that are supported in some neighbourhood $N_R(Y)$ (see definition 3.10). This ideal can also be described as the C^*-ideal generated by the image of $C^*Y \to C^*X$, the inclusion-by-zero homomorphism which is obtained by regarding H_X as a Y-module, making a function on Y act on H_X via its extension by zero to a Borel function on X.

We notice that for each $n \in \mathbb{N}$ we may regard $C^*(N_n(Y))$ as a subalgebra of C^*X, in the same way as above. By construction, $C^*_X(Y)$ is an inductive limit

$$C^*_X(Y) = \varinjlim C^*(N_n(Y))$$

of these C^*-algebras. All these algebras have the same K-theory, since all the spaces $N_n(Y)$ are coarsely equivalent to Y; so, since K-theory commutes with inductive limits, we obtain

$$K_*(C^*_X(Y)) = \varinjlim K_*(C^*(N_n(Y))) = K_*(C^*Y).$$

Now we observe that $C^*X = C^*_X(X_1) + C^*_X(X_2)$ (a partition of unity argument), and that $C^*_X(X_1 \cap X_2) = C^*_X(X_1) \cap C^*_X(X_2)$ (modulo analytical details, this is an immediate consequence of the coarse excisiveness, which tells us that if an operator is supported near X_1 and near X_2 then it is supported near $X_1 \cap X_2$). The desired result therefore follows from 3.17, the Mayer-Vietoris sequence for ideals in a C^*-algebra. □

The decomposition $\mathbb{R} = \mathbb{R}^- \cup \mathbb{R}^+$ is coarsely excisive. We have computed the K-theory of $C^*(\mathbb{R})$ (lecture 4), and the $C^*(\{0\})$ is just the algebra of compact operators so its K-theory is also known. Since $\mathbb{R}^-$ and $\mathbb{R}^+$ are isometric, the groups $K_i(C^*(\mathbb{R}^-)) \oplus K_i(C^*(\mathbb{R}^+))$ cannot be non-trivial cyclic, and using the Mayer-Vietoris sequence we see that they must therefore be zero. The underlying geometric reason for this is brought out by the following notion.

DEFINITION 9.3: *We will say that a space X is* flasque *if it admits a self-map $s\colon X \to X$ such that*

(i) *s is coarsely equivalent to the identity map;*
(ii) *The powers of s eventually leave any compact set, that is, for each compact $K \subseteq X$ there is n_0 such that for all $n \geqslant n_0$, $s^n(X) \cap K = \emptyset$;*
(iii) *s is an isometry*[2] *of X into itself.*

Clearly $\mathbb{R}^+$ is flasque (consider the right shift) as is $\mathbb{R}^+ \times Y$ for any space Y. We have

PROPOSITION 9.4: *If X is flasque, then $K_*(C^*X) = 0$.*

[2]This condition is not coarsely invariant — the correct invariant condition is that the powers of s should be *uniformly* coarse ('equi-coarse'?) — but it is sufficient for our examples.

PROOF: Let $e \in (C^*X)^+$ be a projection representing a K-theory class. We will show how to prove that e represents the zero class. The general argument is similar.

Note that we may assume[3] that the X-module H on which C^*X is defined admits an isometry V which exactly covers s. Let $H' = \bigoplus^\infty H$, and on H' consider the projection

$$f = e \oplus V^*eV \oplus (V^*)^2eV^2 \oplus \cdots .$$

I claim that $f \in C^*(X, H')$; assume without loss of generality that e has finite propagation and is locally compact. Manifestly, then, f has finite propagation. To show that it is locally compact let φ be a compactly supported function on X. Note that because the powers of s eventually leave any compact set, all but finitely many terms in

$$f\varphi = e\varphi \oplus V^*eV\varphi \oplus \cdots$$

are zero. So $f\varphi$ is a finite sum of locally compact operators, hence it is compact.

Let $i\colon C^*(X,H) \to C^*(X,H')$ be the top left corner inclusion. Plainly we have $[f] = [i_*e] + [V^*fV]$ in $K_*(C^*(X,H'))$. But $[V^*fV] = [f]$ because V covers the map s which, by assumption, is coarsely equivalent to the identity. So $[i_*e] = 0$. However, i_* is an isomorphism (since both H and H' are adequate X-modules), so $[e] = 0$ in $K_*(C^*(X,H))$, as required. □

REMARK: This kind of argument is sometimes referred to as an 'Eilenberg swindle'.

These results can be used to compute $K_*(C^*X)$ when $X = \mathcal{O}Y$ is an open cone on a polyhedron Y, giving an analytic counterpart to [**83**]. We need to note

LEMMA 9.5: *The functor $Y \mapsto K_*(C^*(\mathcal{O}Y))$ is a reduced generalized homology theory (on the category of finite polyhedra).*

PROOF: If $f\colon Y \to Y'$ is a simplicial map, then it is Lipschitz, so $\mathcal{O}f\colon \mathcal{O}Y \to \mathcal{O}Y'$ is a coarse map, giving the functoriality. Functoriality for general continuous maps will follow via the simplicial approximation theorem once we have proved homotopy invariance. Excision follows from our Mayer-Vietoris results because the cone on any decomposition of Y into closed subsets is coarsely excisive. The theory is reduced since $\mathbb{R}^+ = \mathcal{O}(\text{point})$ is flasque. As for homotopy invariance, in the presence of excision this is equivalent to the statement that the homology groups of a closed cone $cY = Y \times [0,1]/Y \times \{0\}$ are zero; and this follows from the geometric observation that $\mathcal{O}(cY)$ is coarsely equivalent to $\mathcal{O}(Y) \times \mathbb{R}^+$, and hence is flasque. (To see this, embed Y in S^{n-1}, thought of as the equatorial sphere in S^n, and model cY as embedded in S^n by coning from the north pole. With this model one actually has

$$\mathcal{O}(cY) = \mathcal{O}(Y) \times \mathbb{R}^+$$

[3] If not, replace H by an infinite direct sum of copies of H, with $f \in C_0(X)$ acting through $(f, f \circ s, f \circ s^2, \dots)$, and take V to be the right shift.

as subsets of $\mathbb{R}^{n+1}$.) □

Note that the above argument in fact shows that the homology theory satisfies the *strong excision* axiom (excision for arbitrary closed pairs).

Now it is clear that $Y \mapsto K_*(\mathcal{O}Y)$, the K-homology of the open cone, is also a reduced generalized homology theory; by the usual long exact sequence of K-homology it is just the reduced K-homology of Y with dimension shifted by one. And the assembly map is a natural transformation of such generalized homology theories. To verify that the assembly map is an isomorphism for open cones in general, it therefore suffices to verify this in the case of the coefficient groups, that is in the case $X = \text{point} = \mathcal{O}(\emptyset)$; in this case both groups are $\mathbb{Z}$ in dimension zero, zero in dimension one, and the assembly map is an isomorphism. Thus by a Mayer-Vietoris argument we obtain

PROPOSITION 9.6: *The coarse Baum-Connes conjecture is true for open cones on polyhedra.*

As a special case, the coarse Baum-Connes conjecture is true for Euclidean space. This should be regarded as a coarse analytic counterpart to Shaneson's computation of the Wall L-groups of a free abelian group [**102**].

Coarse homotopy

In order to extend the range of spaces for which we can calculate, it is necessary to introduce an appropriate notion of 'coarse homotopy'. One might try to say that two maps $X \to Y$ are coarsely homotopic if they arise by evaluation at 0 and 1 from a coarse map $[0,1] \times X \to Y$. But this notion is too restrictive; because of the constant 'diameter' of $X \times [0,1]$ two maps will be coarsely homotopic in this sense if and only if they are coarsely equivalent. What one wants is a coarse map from $X \times [0,1]$ to Y where $X \times [0,1]$ is given a metric which 'opens out' as one goes to infinity in X. There are a number of slightly different ways of formalizing this notion. In [**56**] we used a metric on the product for which each copy $X \times \{t\}$ of X was isometrically embedded. However for the proofs it seems to be simpler to follow Yu [**115**] in making use of the notion of 'Lipschitz homotopy' due to Gromov [**46**]. Here is the definition that we will use.

DEFINITION 9.7: *Let X and Y be coarse spaces. A* Lipschitz homotopy *from X to Y is a coarse map $H\colon X \times \mathbb{R}^+ \to Y \times \mathbb{R}^+$, which is of the form $H(x,t) = (h_t(x), t)$, and which has the property that for each compact subset $K \subseteq X$ there is $t_K \in \mathbb{R}^+$ such that $h_t(x)$ is constant in t for $t \geqslant t_K$ and $x \in K$.*

A Lipschitz homotopy is thus a homotopy that runs at a constant speed, but possibly for longer and longer times as one goes further and further out in X. In particular the limit $h(\infty, x)$ is a well-defined function of x. If $f_0(x) = h(0,x)$ and $f_\infty(x) = h(\infty, x)$ are coarse maps[4] then we will say that they are *Lipschitz*

[4]Note that it does *not* follow from the definition of a Lipschitz homotopy that f_∞ is a coarse map, as easy examples show; this is an extra hypothesis.

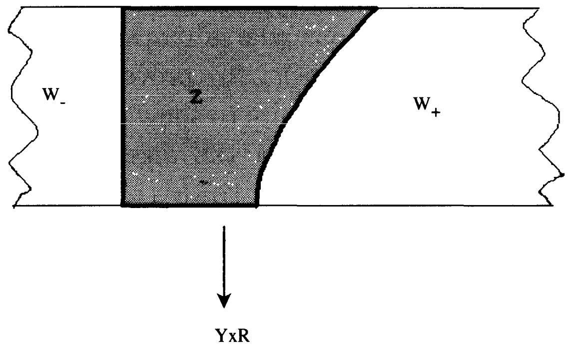

FIGURE 8. Lipschitz homotopy

homotopic. More generally, we will call them Lipschitz homotopic if they can be linked by a chain of Lipschitz homotopies as defined above — it being far from apparent that Lipschitz homotopies can be concatenated so as to define an equivalence relation.

THEOREM 9.8: *Lipschitz homotopic coarse maps $X \to Y$ induce the same homomorphism $K_*(C^*X) \to K_*(C^*Y)$.*

PROOF: Let

$$Z = \{(x,t) : 0 \leqslant t \leqslant t_{N_1(\{x\})}\}$$

be the space of the homotopy, so that H is constant in x outside Z (see figure 8. Let X_0 and X_∞ be the two boundary pieces of Z; X_0 is isometric with X, and X_∞ has a natural coarse map to X (the projection); H gives coarse maps $X_0 \to Y$ and $X_\infty \to Y$, which by hypothesis factor through the coarse maps f_0 and f_∞ from X to Y. As usual in proofs of homotopy invariance, it is sufficient to prove that the inclusions $X_0 \to Z$ and $X_\infty \to Z$ induce isomorphisms on the K-theory of the appropriate C^*-algebras.

Notice that Z can be considered as a subspace of $X \times \mathbb{R}$. Let W_- and W_+ be the parts of $X \times \mathbb{R}$ to the left and right of Z, that is,

$$W_- = \{(x,t) : t \leqslant 0\}, \qquad W_+ = \{(x,t) : t \geqslant t_{N_1(\{x\})}\}.$$

Consider the Mayer-Vietoris sequences associated to the following two coarsely excisive decompositions of $X \times \mathbb{R}$:

(a) $X \times \mathbb{R} = W_- \cup (Z \cup W_+)$;

(b) $X \times \mathbb{R} = (W_- \cup Z) \cup (Z \cup W_+)$.

Inclusion gives a natural transformation between the Mayer-Vietoris sequences. But the subspaces W_-, $W_- \cup Z$, and $Z \cup W_+$ are all flasque (by translation to the left or the right as appropriate), and so the associated K-theory groups are all zero. Hence, by the five lemma, inclusion induces an isomorphism on the K-theory groups associated to the intersections in the two decompositions, that is, an isomorphism $K_*(C^*(X_0)) \to K_*(C^*(Z))$. The argument for X_∞ is similar. □

REMARK: There is an analogue of this result for D^*X; namely, that continuously Lipschitz homotopic continuous coarse maps $X \to Y$ induce the same homomorphism $K_*(D^*X) \to K_*(D^*Y)$. To prove this, we may repeat the argument above, noting that we now require to show that $X_0 \to Z$ and $X_\infty \to Z$ induce isomorphisms on the K-theory of the algebras D^*. We know that these maps induce isomorphisms on the K-theory of C^*, and they also induce isomorphisms on K-homology because they are proper homotopy equivalences. Comparing the analytic surgery exact sequences for X_0 (or X_∞) and Z, and using the five lemma, we get the required isomorphism.

Consider an open cone $\mathcal{O}Y$ on some $Y \subseteq S^{n-1}$. Let $r\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a Lipschitz function that tends to infinity. Then one can define a coarse map $s_r\colon \mathcal{O}Y \to \mathcal{O}Y$, called the *radial shrinking* associated to r, by

$$s_r(x,t) = (x, r(t)).$$

We have the following simple result

LEMMA 9.9: *Any radial shrinking is Lipschitz homotopic to the identity map.*

PROOF: We use a linear homotopy at constant speed. □

A first application of this is to a direct proof of the topological invariance of $K_*(C^*\mathcal{O}Y)$ as a functor of Y, for any metrizable space Y. Let $f\colon Y \to Y'$ be a continuous map. As we have stressed, $\mathcal{O}f$ need not be a coarse map; however, it is always possible to choose a radial shrinking s_r such that $s_r \circ \mathcal{O}f$ is a coarse map. (The required shrinking function r is related to the *modulus of continuity* of f, that is the function which tells you how δ depends on ε in the definition of continuity.) By the Lipschitz homotopy invariance result above, we know that $(s_r \circ \mathcal{O}f)_*\colon K_*(C^*(\mathcal{O}Y)) \to K_*(C^*(\mathcal{O}Y'))$ does not depend on the choice of r, so this gives us a functorial $f_*\colon K_*(C^*(\mathcal{O}Y)) \to K_*(C^*(\mathcal{O}Y'))$, which will be an isomorphism if f is a homeomorphism. Thus we see that $Y \mapsto K_*(C^*(\mathcal{O}Y))$ is a homology theory on the category of compact metrizable spaces.

REMARK: This argument in fact shows that the 'Lipschitz homotopy type' of $\mathcal{O}Y$ depends only on the *homotopy type* of Y. One might ask whether there is any equivalence relation on coarse spaces that, when applied to open cones $\mathcal{O}Y$, corresponds *exactly* to homeomorphism on Y. I suspect that the answer is negative. Compare 2.2, and the paper [**16**] cited there.

As a second application, let us prove the coarse Baum-Connes conjecture for any complete simply connected Riemannian manifold X of non-positive sectional curvature. We need to observe that the conjecture is invariant under Lipschitz homotopy equivalence: if X is Lipschitz homotopy equivalent to a space for which the conjecture is true, then the conjecture is true for X. Indeed, $K_*(C^*X)$ is invariant under Lipschitz homotopy equivalence by our results above, and it is not hard to see that $KX_*(X)$ is invariant as well. In our case, by classical differential geometry the exponential map at a point $x \in X$ gives a distance-increasing diffeomorphism

$$\exp\colon \mathbb{R}^n \to X$$

whose inverse $\log\colon X \to \mathbb{R}^n$ is therefore a coarse map. The exponential map itself is not coarse, but one can find a radial shrinking s_r such that $\exp \circ s_r$ is coarse. The composites $\log \circ (\exp \circ s_r)$ and $(\exp \circ s_r) \circ \log$ are Lipschitz homotopic to the identity (by 'linear' homotopies in each case). Thus X is Lipschitz homotopy equivalent to $\mathbb{R}^n$ for which the coarse Baum-Connes conjecture is known, and therefore the coarse Baum-Connes conjecture is true for X. By descent theory (lecture 8) this means that the analytic Novikov conjecture is true for all groups which are fundamental groups of compact, non-positively curved manifolds — a result due to Mischenko.

Scaleable spaces

DEFINITION 9.10: *A coarse space X is* scaleable *if there is a continuous coarse map $s\colon X \to X$ that is continuously Lipschitz homotopic to the identity map and which satisfies $d(s(x), s(x')) \leqslant \frac{1}{2} d(x, x')$ for all $x, x' \in X$.*

The open cone $\mathcal{O}Y$ on any compact metrizable space Y is scaleable.

PROPOSITION 9.11: *If X is a scaleable space then the ordinary assembly map $A\colon K_*(X) \to K_*(C^*X)$ is an isomorphism.*

PROOF: We prove this by showing directly that D^*X has zero K-theory, by means of another Eilenberg swindle. Specifically, let H be the Hilbert space on which D^*X is defined and let $V\colon H \to H$ be an operator which exactly covers the rescaling map s. Because of the assumption that s is continuously Lipschitz homotopic to 1, $\mathrm{Ad}(V)$ induces the identity on $K_*(D^*X)$. To apply the same argument as in the proof of proposition 9.4, I need only show that if $e \in D^*(X; H)$, then the operator

$$f = e \oplus V^*eV \oplus (V^*)^2eV^2 \oplus \cdots$$

belongs to $D^*(X; H')$, where H' is an infinite direct sum of copies of H. This follows from Kasparov's lemma (5.4). We may assume that e has finite propagation, say r; then $(V^*)^neV^n$ has propagation $2^{-n}r$. Therefore, if $\varphi, \psi \in C_0(X)$ have disjoint supports, the operator $\varphi(V^*)^neV^n\psi$ is zero for all sufficiently large n. Thus $\varphi f \psi$ is a finite sum of compact operators, hence it is compact. So f is pseudolocal, and it is plainly of finite propagation. This proves the result. □

COROLLARY 9.12: *Let Y be any finite-dimensional compact metrizable space. Then the coarse Baum-Connes conjecture is true for $\mathcal{O}(Y)$.*

PROOF: We know that the assembly map A is an isomorphism for $\mathcal{O}(Y)$. One can also show that the coarsening map $c\colon K_*(\mathcal{O}Y) \to KX_*(\mathcal{O}Y)$ is an isomorphism in this case; notice that this is not a trivial assertion, since if Y is a 'bad' metric space, the Cantor set for example, then $\mathcal{O}Y$ need not be uniformly contractible. Since $A_\infty \circ c = A$, it follows that A_∞ is an isomorphism. □

The point, of course, is that Y may be a 'bad' metric space. This is important for the next section, in which we study hyperbolic metric spaces by relating them to the cones on their Gromov boundaries.

REMARK: An alternative approach to 9.11 for nice scaleable spaces can be given by making use of a recent theorem of Guoliang Yu [**114**]. This theorem, translated into our language, should imply that for each bounded geometry metric simplicial complex X there is a constant $\varepsilon_X > 0$ such that, if a self-adjoint projection or a unitary in D^*X has ε_X-propagation[5] less than ε_X, then it represents the zero class in $K_*(D^*X)$. Now if X is scaleable, then *any* class in $K_*(D^*X)$ has a representative which satisfies these propagation conditions, and we conclude that $K_*(D^*X) = 0$.

Yu's result is an analytic counterpart to the α-approximation theorem [**24**]. However, the analogy is not yet perfect. The following conjecture still lacks an analytic proof:

CONJECTURE 9.13: *Let V be a compact manifold, with fundamental group Γ. Then there is a constant ε_V such that, if $f\colon M \to V$ is an ε_V homotopy equivalence, then the structure invariant $\sigma(f) \in K_*(D^*_\Gamma(V))$ vanishes.*

This would give us the topological invariance of the rational Pontrjagin classes. The reason that the conjecture does not follow directly from Yu's theorem is that the structure invariant of a small homotopy equivalence is certainly represented by an invertible of small propagation, but this invertible may be rather far from being unitary.

Coarse Baum-Connes for hyperbolic spaces

Gromov's notion of *hyperbolicity* provides a coarse interpretation of negative sectional curvature. To give the definition let M be a *geodesic metric space.* By this I mean a metric space in which the distance between two points is equal to the infimum of the lengths of paths between them, and there is always a path attaining this infimum. A *geodesic* just means a path that minimizes distance between any two of its points.

Let T be a geodesic triangle (with vertices A, B, C and sides BC, CA, AB) in such a space. We define the *fatness* of T, $\varphi(T)$, as follows:

$$\varphi(T) = \inf\{d(a,b) + d(b,c) + d(c,a) : a \in BC, b \in CA, c \in AB\}.$$

Plainly the fatness of any triangle in $\mathbb{R}$ is zero, whereas triangles in $\mathbb{R}^n$, $n \geqslant 2$, can be arbitrarily fat. Less apparent is the following fact.

PROPOSITION 9.14: *No triangle in hyperbolic space H^n (of constant curvature -1) can be fatter than* 6.

[5] See page 20.

PROOF: We may restrict attention to H^2. The fatness of any triangle is at most 6 times the radius of the inscribed circle. Now the area of any hyperbolic triangle is at most π, so the area of the inscribed circle is also at most π; hence, since the area of a circle of radius r in H^2 is greater than the Euclidean area πr^2, the radius of the circle is at most one. □

REMARK: The constant 6 is obviously not best possible. This does not matter.

Gromov proved that the existence (although not of course the actual value) of an upper bound for the fatness of triangles is a coarsely invariant property of geodesic spaces. This makes the following definition sensible.

DEFINITION 9.15: *A coarse space is* hyperbolic *if it is coarsely equivalent to a geodesic metric space in which there is an upper bound for the fatness of geodesic triangles.*

Examples of hyperbolic metric spaces include complete simply connected Riemannian manifolds with sectional curvatures bounded above by a negative constant, free groups, and more generally suitable 'small cancellation' groups[6]. It can be shown that hyperbolic groups are common, in the sense that a group 'chosen at random' is likely to be hyperbolic.

Let M be a hyperbolic metric space, and consider the geodesic rays from a fixed point $O \in M$. An elementary argument shows that there is a constant C such that two such rays either remain within distance C for all time or else diverge exponentially.

DEFINITION 9.16: *The* Gromov boundary *of M is the collection of equivalence classes of geodesic rays, two such rays being equivalent if they remain within distance C.*

For example, the Gromov boundary of a complete, simply-connected Riemannian manifold with sectional curvature bounded from above by a negative constant is a sphere; the Gromov boundary of a free group is a Cantor set. In general the Gromov boundary is a finite-dimensional compact metric space. The metric is defined by saying that two geodesic rays are close in the metric if it takes a long time before they diverge significantly (say by more than $3C$). For details see [**43**].

The definition of the Gromov boundary of M makes it clear that there is a (coarsely defined) 'exponential map' $\mathcal{O}\partial M \to M$, which sends the pair $([\gamma], t)$ to $\gamma(t)$. By composing with a suitable radial contraction one can again make this into a coarse map. Moreover, an inverse 'logarithm' map can also be defined from M to $\mathcal{O}\partial M$; one sends $x \in M$ to $(d(x, O), [\gamma])$, where γ is a geodesic ray passing near[7] x. Of course the choice of such a geodesic ray is not unique, but,

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[6]A group Γ is a *small cancellation* group if it has a presentation in terms of generators and relators in which any common subwords between relators are comparatively short [**43**].

[7]This assumes that the exponential map is coarsely surjective; if this is not the case, an additional argument is required. See [**57**].

by definition of the metric in ∂M, the class $[\gamma]$ is defined in ∂M up to an error which becomes small as $d(x, O)$ becomes large; this can be arranged so that we get a coarsely well-defined map onto the cone. These rescaled exponential and logarithm maps are then Lipschitz homotopy inverses, so M is Lipschitz homotopy equivalent to the cone $\mathcal{O}\partial M$, for which we know that the coarse Baum-Connes conjecture holds. It follows that the conjecture holds for M. We state this formally.

PROPOSITION 9.17: *Let M be a hyperbolic metric space. Then the coarse Baum-Connes conjecture holds for M.*

Applying the descent theorem 8.4, we now know that the analytic Novikov conjecture holds for hyperbolic groups that are classified by a finite complex. This was first proved (modulo torsion) by Connes and Moscovici [**29**], using cyclic cohomology. Their methods appear to depend on rather different aspects of the theory of hyperbolic groups, and it is an interesting question what is the relation between the two approaches.

Notes and references: The Mayer-Vietoris theorem is in [**58**]. Most of the rest of this lecture is based on [**57**], where further details can be found. Lipschitz homotopy invariance of $K_*(C^*X)$ was proved by Yu in [**115**]; the analogous 'coarse homotopy invariance' property is in [**56**]. The proof given here comes from [**53**], and is based on unpublished earlier work of Ferry and Pedersen.

The theory of hyperbolic groups and metric spaces is expounded in [**45, 44, 43**].

CHAPTER 10

Coarse structures and boundaries

Especially in our discussion of the coarse Baum-Connes conjecture for hyperbolic spaces at the end of the last lecture, it became apparent that the conjecture is closely related to the existence of suitable 'ideal boundaries' of a space X. In this lecture we will approach matters directly from this perspective, which gives an alternative proof of very similar results.

Abstract coarse structures

Recall that in lecture 2 we mentioned that the coarse structure defined by a metric should be considered as a special case of a notion of 'abstract coarse structure' on a topological space. We will now formalize this.

DEFINITION 10.1: *Let X be a locally compact topological space. A* coarse structure *on X consists of a collection $\mathcal{E}$ of subsets of $X \times X$, called* entourages, *which have the following properties:*

(i) *The collection of entourages is closed under the operations of reflection in the diagonal, subset, union, and composition of relations*[1].
(ii) *Any compact subset of $X \times X$ is an entourage;*
(iii) *Entourages are proper relations: that is, if E is an entourage and $K \subseteq X$ is compact, so is $E \circ K = \{x : \exists x', (x, x') \in E, x' \in K\}$;*
(iv) *The union of all the entourages is $X \times X$.*

If the diagonal is an entourage the coarse structure is called unital.

We will only be concerned with unital coarse structures in this paper, and we will therefore usually drop the prefix. Two significant examples of coarse structures are the following:

(a) Suppose that X has a proper metric. Then we may define a coarse structure by declaring that the entourages are those sets E contained in some metric neighbourhood $N_R(\Delta)$ of the diagonal in $X \times X$. This is the *bounded coarse structure* associated to the metric.

[1]The composition of E_1 and E_2 is the set $\{(x, x'') : \exists x',\ (x, x') \in E_1, (x', x'') \in E_2\}$.

(b) Suppose that there is given a compactification of X, meaning an embedding of X as a dense open subset of some compact space C, with boundary $\partial X = C \setminus X$. Then we may define a coarse structure by declaring that the entourages are those subsets $E \subseteq X \times X$ such that, when they are considered as subsets of $C \times C$, their closures 'intersect infinity only in the diagonal', that is,

$$x \in C, y \in \partial X, \{(x,y),(y,x)\} \cap \overline{E} \neq \emptyset \Longrightarrow x = y.$$

This is called the *continuously controlled coarse structure* associated to the given compactification.

Every 'coarse' notion that we have used can be defined purely on the basis of this abstract notion of coarse structure. For example, a *coarse map* f between coarse spaces is just a proper map f such that $f \times f$ maps entourages to entourages. A coarse space is *uniformly contractible* if for every entourage E there is another entourage $E' \supseteq E$ having the property that if $V \subseteq X$ with $V \times V \subseteq E$, then the inclusion map $V \to E' \circ V$ is nullhomotopic. An operator T on an X-module has *finite propagation* if there is some entourage E such that $\varphi T \psi = 0$ whenever $\operatorname{Supp}\varphi \times \operatorname{Supp}\psi \cap E = \emptyset$. The algebras C^*X and D^*X can be defined just as before, and the critical lemma 5.9 remains true. Thus the assembly map can be defined, and we can state the coarse Baum-Connes conjecture in the form: for every uniformly contractible bounded geometry coarse space, the assembly map is an isomorphism.

EXAMPLE: In lecture 5 we defined algebras C^* and D^*, and an assembly map, for a space X equipped with a proper 'reference map' $X \to \mathcal{Z}$ to a metric space $\mathcal{Z}$. These can now be recognized as the algebras $C^*(X)$ and $D^*(X)$, where X is equipped with the coarse structure pulled back from the bounded coarse structure on $\mathcal{Z}$.

EXAMPLE: Suppose that a group Γ acts by homeomorphisms on the locally compact space X. There is a natural Γ-invariant coarse structure on X, whose entourages are just all subsets of sets

$$\bigcup_{\gamma \in \Gamma} \gamma K \times \gamma K,$$

where K is compact in X. This structure is unital if the action of Γ is cocompact (as is the case with all the actions we have considered so far). This structure will be called the *coarse structure induced by the* Γ*-action*; it coincides with the bounded coarse structure coming from a proper Γ-invariant metric, and is in an obvious sense the finest structure for which the action of Γ is equi-coarse.

DEFINITION 10.2: *Let X be a coarse space, and let C be a compactification of X. Then C will be called a* coarse compactification *of X, and $\partial X = C \setminus X$ will be called a* coarse corona *of X, if the identity map from X equipped with its given* *coarse structure to X equipped with the continuously controlled coarse structure arising from C is a coarse map.*

There is a dual formulation of this notion in terms of functions on X. Let X be a coarse space and let E be an entourage for X; then, for any continuous function $f\colon X \to \mathbb{C}$, we may define the *E-gradient* of f by

$$\nabla_E f(x) = \sup\{|f(x) - f(y)| : (x,y) \in E\}.$$

Then we have

PROPOSITION 10.3: *C is a coarse compactification of X if and only if, for every function f on X that extends continuously to C, $\nabla_E f \in C_0(X)$ for all entourages E.*

The proof is straightforward. This proposition makes it clear that there is a *universal* coarse compactification of any coarse space X, namely the maximal ideal space of the commutative C^*-algebra of all bounded functions f on X such that $\nabla_E f \in C_0(X)$ for all entourages E. This universal compactification is called the *Higson compactification*, and its boundary νX is the *Higson corona*. It is a compact Hausdorff space, but it is not metrizable except in trivial cases.

The assembly map for continuous control

Let us now discuss under what conditions we can prove that assembly is an isomorphism for continuously controlled coarse structures. We can in fact identify the whole analytic surgery exact sequence, in this case, with a more familiar object:

THEOREM 10.4: *Let X be a coarse space equipped with the continuously controlled coarse structure coming from a metrizable compactification C as above. Then the analytic surgery exact sequence for X is naturally identified with the reduced K-homology exact sequence of the pair $(C, \partial X)$, as in the diagram.*

$$\begin{array}{ccccccc}
\longrightarrow & K_{i+1}(D^*X) & \longrightarrow & K_i(X) & \longrightarrow & K_i(C^*X) & \longrightarrow \\
 & \| & & \|\text{excision} & & \| & \\
\longrightarrow & \tilde{K}_i(C) & \longrightarrow & K_i(C,\partial X) & \longrightarrow & \tilde{K}_{i-1}(\partial X) & \longrightarrow
\end{array}$$

PROOF: The second identification, as indicated, is just excision for K-homology. We will show how to make the first identification $K_{i+1}(D^*X) \cong \tilde{K}_i(C)$; the third one is similar to the first.

Notice that because X is a dense open subset of C, the adequate X-module H on which the algebras $C^*(X)$ and $D^*(X)$ act may also be considered as an adequate C-module, and so may be used to define the K-homology of C. By definition this is $K_i(C) = K_{i+1}(\Psi^0(C)/\Psi^{-1}(C))$; here, however, since C is compact, $\Psi^{-1}(C)$ is just the algebra of compact operators, so $K_{i+1}(\Psi^0(C))$ differs from $K_i(C)$ only by a possible $\mathbb{Z}$; in fact it is easy to check that

$K_{i+1}(\Psi^0(C))$ is exactly the *reduced* K-homology group $\tilde{K}_i(C)$. The proof will therefore be completed if we can identify the C^*-algebra $D^*(X)$ with $\Psi^0(C)$. Notice that these are, in fact, C^*-algebras of operators on the same Hilbert space.

For starters let $T \in D^*(X)$, so T is pseudolocal on X and of finite propagation, say supported within an entourage E. We want to prove[2] that T is pseudolocal on C. By Kasparov's lemma (5.4) it is enough for us to prove that if φ and ψ are continuous functions on C having disjoint supports, then $\varphi T \psi$ is compact. Now notice that since φ and ψ have disjoint supports, and the closure of E at infinity is the diagonal, the set

$$\operatorname{Supp}\varphi \times \operatorname{Supp}\psi \cap \overline{E}$$

is in fact a compact subset of $X \times X \subseteq C \times C$. Therefore we may write $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 \in C_0(X)$, $\operatorname{Supp}\varphi_0 \subseteq \operatorname{Supp}\varphi$, and

$$\operatorname{Supp}\varphi_1 \times \operatorname{Supp}\psi \cap \overline{E} = \emptyset.$$

By the finite propagation condition on T, $\varphi T \psi = \varphi_0 T \psi$, and since T is pseudolocal on X this is equal modulo compacts to $T\varphi_0\psi = 0$.

Conversely suppose that $T \in \Psi^0(C)$. We want to approximate T by finite propagation operators. It is not hard to see that the condition of finite propagation on an operator S is equivalent to the following requirement: given any compact $K \subseteq \partial X$ and any open $U \subseteq C$ with $K \subseteq U$, there exists an open $V \subseteq C$, $K \subseteq V \subseteq U$, such that S and S^* don't propagate from outside U to inside V. (This is the condition of continuous control as used in [**3, 22**].) Given $\varepsilon > 0$ I will show how to perturb T by at most ε so that it satisfies this condition for some fixed K and U; the proof is then completed by an induction over a suitable countable basis (using metrizability), replacing ε by $2^{-n}\varepsilon$, and passing to the limit.

So, now, consider U and K as given, and fix an open set W such that $K \subseteq W \subseteq \overline{W} \subseteq U$. Then $\chi_{C\setminus U} T \chi_W$ is a compact operator, because T is pseudolocal on C. Take a sequence V_n of open sets contained in W and with $\bigcap V_n = K$. Then the characteristic functions of V_n, considered as operators on H, tend to zero in the strong topology; therefore, the operators $\chi_{C\setminus U} T \chi_{V_n} = \chi_{C\setminus U} T \chi_W \chi_{V_n}$ tend to zero in norm. Let $V = V_n$ where n is chosen so that $\|\chi_{C\setminus U} T \chi_V\| < \varepsilon$. The operator $\chi_U T \chi_V + T\chi_{X\setminus V}$ then does not propagate from outside U to inside V, and differs from T by at most ε in norm. □

COROLLARY 10.5: *If X has a continuously controlled coarse structure arising from a* contractible *compactification C, then the assembly map for X is an isomorphism.*

[2]This is a generalization of lemma 7.3.

How is this related to the coarse Baum-Connes conjecture? In this situation the conjecture would imply that the assembly map is an isomorphism for a space X which is uniformly contractible (in the continuously controlled sense) and satisfies, perhaps, an appropriate 'bounded geometry' hypothesis.

LEMMA 10.6: *Let X be an absolute neighbourhood retract*[3] *(ANR) equipped with a compactification C. Suppose that*

(i) *X is uniformly contractible for the continuously controlled coarse structure;*

(ii) *C is a finite-dimensional metric space.*

Then C is a contractible ANR.

COROLLARY 10.7: *The coarse Baum-Connes conjecture holds for a space X of the kind described in the Lemma.*

Probably, the coarse Baum-Connes conjecture holds for any reasonable space with a continuously controlled structure coming from a finite-dimensional compactification, but I did not check this in detail.

PROOF: (OF THE LEMMA)[4] Clearly X is contractible. But by [**13**, Proposition 2.1], which applies to our situation because of uniform contractibility, the compactification C is an ANR and $\partial X = C \setminus X$ is a Z-set in C. Thus $X \to C$ is a homotopy equivalence, and the result follows. □

Let us now apply the descent theory of chapter 8. Suppose therefore that X is a contractible space equipped with a compactification C, and suppose that a discrete group Γ acts cocompactly on X (so that $X/\Gamma = B\Gamma$ is compact) and that the action extends continuously to C. The algebras C^*X, D^*X and so on are then equipped with a Γ-action. The key lemmas 5.14 and 5.15 remain true in this situation, provided that C is a coarse compactification of the coarse structure on X induced by the Γ-action; this condition is often expressed by saying that 'compact sets become small at infinity under translation'. The descent theory goes through without change. We therefore obtain the following result, which parallels the main theorem of [**22**]:

THEOREM 10.8: *Let Γ be a group classified by a finite complex $B\Gamma$. Suppose that the corresponding universal space $E\Gamma$ admits a contractible metrizable compactification to which the Γ-action extends and for which compact sets become small at infinity. Then the analytic Novikov conjecture is true for Γ.*

REMARK: Recall that any coarse space has a universal compactification, the Higson compactification. If the total space of the Higson compactification of a uniformly contractible space were itself contractible, this would imply the Novikov conjecture; and in [**95**] I was rash enough to conjecture that this might be the case. However, Keesling [**67**] showed that this conjecture was

[3] For example, a locally finite simplicial complex.

[4] Thankd to Steve Ferry for discussion of this result.

overoptimistic, by proving that the one-dimensional Čech cohomology of the Higson corona of $\mathbb{R}$ is an uncountably generated group.

It is still possible that the Higson compactification can be put to use in the study of the Novikov conjecture. In all known examples of spaces $X = E\Gamma$ the inclusion of X into its Higson compactification $\bar{X}$ induces the zero map $H^*_c(X;\mathbb{Q}) \to H^*(\bar{X};\mathbb{Q})$ (using Čech cohomology). This at least implies the rational injectivity of the coarse assembly map. However, at present it seems that one needs more refined compactifications taking into account extra geometric structure before this approach to Novikov can be made to work.

Notes and references: Most of the material in this lecture comes from [**53**]. The notion of an 'abstract coarse structure' is surely known to workers in controlled topology (compare [**3**]), though I do not know of anywhere that something like definition 10.1 is written down.

The relationship between the K-theory of $C^*(X)$ and the K-homology of a coarse compactification first surfaced in [**50**] (see also [**98, 100**]). The theory of the Higson corona is developed in detail in [**95**].

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