Geometriae Dedicata **73:** 295–315, 1998.

Gauss–Bonnet's Theorem and Closed Frenet Frames

PETER RØGEN
Department of Mathematics, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark. e-mail: Peter.Roegen@mat.dtu.dk

(Received: 15 October 1997)

Abstract. Our main result is that integrated geodesic curvature of a (nonsimple) closed curve on the unit two-sphere equals a half integer weighted sum of the areas of the connected components of the complement of the curve. These weights that gives a spherical analogy to the winding number of closed plane curves are found using Gauss–Bonnet's theorem after cutting the curve into simple closed sub-curves. If the spherical curve is the tangent indicatrix of a space curve we obtain a new short proof of a formula for integrated torsion presented in an unpublished manuscript by C. Chicone and N. J. Kalton. Applying our result to the principal normal indicatrix we generalize a theorem by Jacobi stating that *a simple closed principal normal indicatrix of a closed space curve with nonvanishing curvature bisects the unit two-sphere* to nonsimple principal normal indicatrices. Some errors in the literature are corrected.

We show that a factorization of a knot diagram into simple closed sub-curves defines an immersed disc with the knot as boundary and use this to give an upper bound on the unknotting number of knots.

Mathematics Subject Classifications (1991): 53A04, 53A05.

Key words: integral geometry, total geodesic curvature, total curvature, total torsion, index of spherical curves, generalization of Jacobi's theorem, unknotting number.

1. The Frenet Apparatus

We generally consider curves C in Euclidean three-space with nonvanishing curvature. This ensures that each curve has Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ satisfying the cross product relations $\mathbf{t} = \mathbf{n} \times \mathbf{b}$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ and the Frenet formulas

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' &= -\tau\mathbf{n} \end{aligned}$$

where $\kappa > 0$ is the curvature and τ is the torsion of C and primes indicate differentiation with respect to arc length, s, of C. A curve with Frenet frame defines three curves on the unit two-sphere, namely, the spherical indicatrices of the tangent vectors, of the principal normal vectors, and of the binormal vectors. Denote these curves by $\Gamma_{\mathbf{t}}$, $\Gamma_{\mathbf{n}}$, and $\Gamma_{\mathbf{b}}$ respectively. The lengths of these indicatrices are

$$|\Gamma_{\mathbf{t}}| = \kappa(C) = \int_C \kappa(s)\,\mathrm{d}s$$

$$|\Gamma_{\mathbf{n}}| = \omega(C) = \int_C \sqrt{\kappa^2(s) + \tau^2(s)}\,\mathrm{d}s$$

$$|\Gamma_{\mathbf{b}}| = |\tau|(C) = \int_C |\tau(s)|\,\mathrm{d}s.$$

It is not standard to use the notation, $|\tau|(C)$, for the total torsion of a space curve – but it is convenient as $\tau(C)$ hereby can denote the integral of the torsion of C with respect to arc-length, s, i.e.,

$$\tau(C) = \int_C \tau(s)\,\mathrm{d}s.$$

Our main topic is closed curves on the unit two-sphere that arise as indicatrices of space curves. This explains

DEFINITION 1. We say that the tangent indicatrix, $\Gamma_{\mathbf{t}}$, of a regular curve $\mathbf{r}$: $[0, L] \to \mathbb{R}^3$, is closed if $\mathbf{t}(0) = \mathbf{t}(L)$ and all derivatives of $\Gamma_{\mathbf{t}}$ agree in this point. With similar definitions of closed principal normal and binormal indicatrices we say that a space curve has closed Frenet frame if its indicatrices are closed.

An example of a nonclosed curve with closed Frenet frame is a suitable piece of a circular Helix. This explains half of the title of this manuscript. I order to use Gauss–Bonnet's theorem on closed Frenet frames we have to factorize nonsimple curves into simple sub-curves. Due to an error in the literature the next section is devoted to this factorization.

2. Factorization of Closed Curves

In this section we consider closed continuous curves $\Gamma\colon \mathbb{S}^1 \to \mathbb{T}$ in a topological space $\mathbb{T}$. We assume that the curves are not constant in any interval of $\mathbb{S}^1$.

Assume that there exists a closed interval $I \in \mathbb{S}^1$ such that when we identify the endpoints of I then $\Gamma \mid I$ is a simple closed curve in $\mathbb{T}$. By an elementary factorization of the curve Γ we mean a splitting of Γ into the above simple closed curve S and into the rest of Γ, denoted by $\tilde{\Gamma}$. For an elementary factorization of Γ we write $\Gamma \to S + \tilde{\Gamma}$, where $S = \Gamma \mid I$ (identifying the endpoints of I) is a simple closed sub-curve of Γ and $\tilde{\Gamma} = \Gamma|(\mathbb{S}^1 \backslash \mathrm{int}(I))$ (identifying the endpoints of $\mathbb{S}^1$ minus the interior of I) is the rest of Γ.

DEFINITION 2. Let $\Gamma\colon \mathbb{S}^1 \to \mathbb{T}$ be a closed continuous curve in a topological space $\mathbb{T}$. If there exists a finite number of elementary factorizations

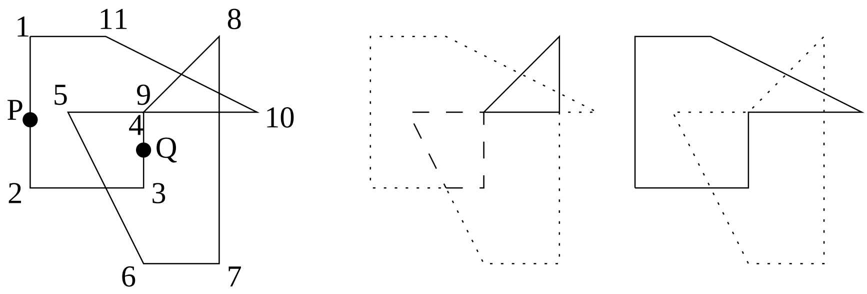

Figure 1. A curve with two different 'Umlaufszahlen'.

$$\Gamma = \Gamma_0 \to S_1 + \Gamma_1 \to S_1 + S_2 + \Gamma_2 \to \dots$$
$$\to S_1 + S_2 + \dots + S_{n-1} + \Gamma_{n-1}$$

such that Γ_{n-1} is a simple closed curve S_n, then we say the curve Γ possesses a simple closed sub-curve factorization (scs-factorization). In this case we simply write $\Gamma \to S_1 + \dots + S_n$ and we say that the scs-factorization is of order n.

The scs-factorizations are obviously preserved under homeomorphism of the curve. This is basically why we give the definition in a general topological space.

By the simple closed sub-curve number (scs-number), $\mathrm{scsn}(\Gamma)$, of Γ we mean the minimum of the orders of the scs-factorizations that the curve Γ possesses. If the curve Γ does not possess an scs-factorization, then we set $\mathrm{scsn}(\Gamma) = +\infty$.

Remark 3. In [10] an analogue to our scs-number, which is crucial for this paper, is defined using an algorithm. This algorithm starts in an arbitrary point P_0 on the curve Γ_0 and traverses the curve until the first pre-traversed point P_1 is reached. Then the simple closed sub-curve S_1 from the first time P_1 lies on the curve Γ_0 to the second time P_1 lies on the curve Γ_0 is excluded from Γ_0. Now, apply the elementary factorization $\Gamma_0 \to S_1 + \Gamma_1$ and mark Γ_1 with the starting point P_1. If this iterative process stops after l steps, then Γ_0 (or more correctly the pair (Γ_0, P_0)) is said to have 'Umlaufszahl' l.

In [10] it is claimed that *the 'Umlaufszahl' is independent of the starting point.* This is false! On Figure 1 is a curve with 'Umlaufszahl' 3 if traversion starts at the point P and 'Umlaufszahl' 2 if traversion starts at the point Q. On Figure 2 we show a curve with finite 'Umlaufszahl' if traversion starts at the point P and infinite 'Umlaufszahl' if traversion starts at the point Q.

It is possible for a closed curve to have infinite scs-number. On Figure 3 are shown two such curves⋆. To give a sufficient condition for a closed curve to possess

⋆ In [10] it is claimed that a *closed spherical curve with continuous geodesic curvature has finite 'Umlaufszahl'*. The curve on the right-hand side on Figure 3 contradicts this statement.

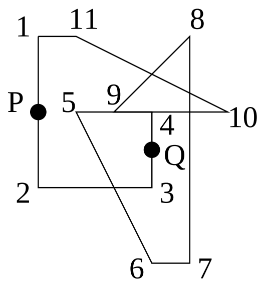

Figure 2. A curve with finite and infinite 'Umlaufszahlen'.

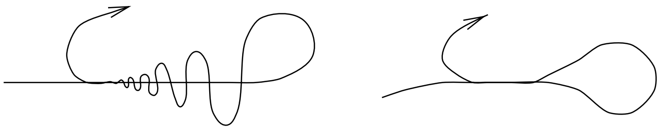

Figure 3. Pieces of curves with no scs-factorization.

an scs-factorization we need some notation. Let Γ be a non simple closed curve and let $P \in \Gamma(\mathbb{S}^1)$ be a point of self-intersection of Γ. If the inverse image of the point P under the map Γ, $\Gamma^{-1}(P) \subset \mathbb{S}^1$, consists of m parameter values then we say that the point P has multiplicity m and that the curve Γ has $(m-1)$ self-intersections in the point P. For a closed curve Γ with self-intersection points $P_1, \ldots, P_n$, each of multiplicity m_j, we let $s(\Gamma) = \sum_{j=1}^{n}(m_j - 1)$ denote the number of self-intersections of Γ.

THEOREM 4. *Let $\Gamma: \mathbb{S}^1 \to \mathbb{T}$ be a continuous closed curve with only finitely many self-intersections. Then Γ possesses at least one scs-factorization and any scs-factorization of Γ has order less than or equal to the number of Γ's self-intersections plus one. In particular the simple closed sub-curve number of Γ fulfills $csn(\Gamma) \leqslant 1 + s(\Gamma)$.*

Proof. If Γ is simple there is nothing to prove. Assume that Γ is not simple but that $s(\Gamma) = n$ is finite. Apply an elementary factorization $\Gamma \to S_1 + \Gamma_1$ to Γ. As the multiplicity of the point in which S_1 and Γ_1 are glued together is one less for Γ_1 than for Γ we have that $s(\Gamma_1) \leqslant n - 1$. Hence, after at most n elementary factorizations we obtain an scs-factorization of Γ. $\square$

To give a sufficient condition for a closed curve on the unit 2-sphere to possess an scs-factorization we use the terminology that a closed regular C^1-curve only has transversal self-intersections if no pair of tangents to the curve with the same base point are parallel.

LEMMA 5. *Let Γ be a closed regular C^1-curve on the unit 2-sphere. If Γ only has transversal self-intersections then $scsn(\Gamma)$ is finite.*

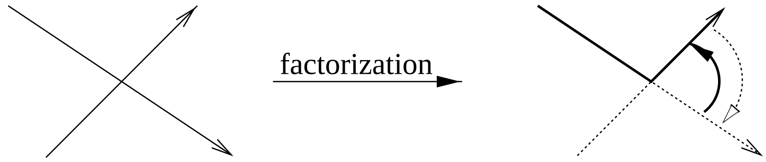

Figure 4. Exterior angles.

Proof. By transversality and compactness the curve Γ has only finitely many self-intersection points. Since Γ has finite length and Γ is a regular C^1-curve each self-intersection point has finite multiplicity. Now Theorem 4 applies. $\square$

Note, that Lemma 5 implies that it is a generic property for regular C^1-curves to possess scs-factorizations.

3. An Index Formula on $\mathbb{S}^2$

In this section we give an index of points in the complement of a closed regular C^1-curve on the unit 2-sphere and prove that the integral over the 2-sphere of this index equals the integrated geodesic curvature of the curve in case the curve is of type C^2.

Let $\Gamma \to S_1 + S_2 + \cdots + S_n$ be a simple closed sub-curve factorization of a closed regular C^1-curve on the unit 2-sphere. This scs-factorization is obtained by $(n-1)$ elementary factorizations, each cutting away a simple closed curve defined on an interval, $[a, b]$. We call the point $\Gamma(a) = \Gamma(b)$ a cutting point. If the tangents $\Gamma'(a)$ and $\Gamma'(b)$ are linearly independent we call the cutting point a transversal cutting point. If all cutting points of the scs-factorization are transversal cutting points, then we say that the scs-factorization is a transversal scs-factorization.

THEOREM 6. *Let Γ be a closed regular C^2-curve on the unit 2-sphere with a transversal scs-factorization $\Gamma \to S_1 + S_2 + \cdots + S_n$. Denote the area of the positive* resp. *negative turned component of the complement of each simple closed sub-curve S_i by $\mu(\Omega_i^+)$* resp. *$\mu(\Omega_i^-)$. Then the integral of the geodesic curvature of Γ, $\kappa_g(\Gamma)$, with respect to arc-length satisfies $\kappa_g(\Gamma) = \frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+))$.*

Remark 7. If a curve only has transversal self-intersections, then the curve does possess scs-factorizations, by Lemma 5, and all its scs-factorizations are transversal. Note, that the formula given in Theorem 6 is true for all transversal scs-factorizations of a curve.

Proof. Let $\Gamma \to S_1 + S_2 + \cdots + S_n$ be a transversal scs-factorization of Γ and let $\mu(\Omega_1^+), \ldots, \mu(\Omega_n^+)$ resp. $\mu(\Omega_1^-), \ldots, \mu(\Omega_n^-)$ be the areas of the positive resp. negative turned components of the complements of these simple closed sub-curves. Let $\alpha_{ij} \in (-\pi, \pi)$ be the exterior angle (see Figure 4) between the tangents to S_i at the cutting point between S_i and S_j. If the i'th and the j'th sub-curve do not have a mutual cutting point, or $i = j$, then we set $\alpha_{ij} = \alpha_{ji} = 0$. Note, that for all $1 \leqslant i, j \leqslant n$ the exterior angles fulfil $\alpha_{ij} = -\alpha_{ji}$. Gauss–Bonnet's theorem for the i'th sub-curve, S_i, gives

$$\int_{S_i} \kappa_{g,\Gamma}\, \mathrm{d}\sigma + \sum_{j=1}^{n} \alpha_{ij} = 2\pi - \mu(\Omega_i).$$

By our scs-factorization of Γ we get

$$\begin{aligned}\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma &= \sum_{i=1}^{n} \int_{S_i} \kappa_{g,\Gamma}\, \mathrm{d}\sigma = \sum_{i=1}^{n} \left(2\pi - \mu(\Omega_i^+) - \sum_{j=1}^{n} \alpha_{ij} \right) \\ &= 2\pi n - \sum_{i=1}^{n} \mu(\Omega_i^+) - \sum_{i,j=1}^{n} \alpha_{ij} = 2\pi n - \sum_{i=1}^{n} \mu(\Omega_i^+).\end{aligned}$$

Let $\tilde{\Gamma}$ denote Γ with reversed orientation. By reversing the orientation of all the simple closed sub-curves in the scs-factorization of Γ we obtain an scs-factorization of $\tilde{\Gamma}$. This gives us

$$-\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma = \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}}\, \mathrm{d}\tilde{\sigma} = 2\pi n - \sum_{i=1}^{n} \mu(\Omega_i^-).$$

Hence,

$$\begin{aligned}\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma &= \frac{1}{2} \left(\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma - \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}}\, \mathrm{d}\tilde{\sigma} \right) \\ &= \frac{1}{2} \left(\sum_{i=1}^{n} \mu(\Omega_i^-) - \sum_{i=1}^{n} \mu(\Omega_i^+) \right). \qquad \square\end{aligned}$$

One of the ingredients that make the proof of Theorem 6 work is that the sum of the exterior angles vanishes. Assume that this is not the case and let $\sum_{i,j=1}^{n} \alpha_{ij} = a \neq 0$. Using Gauss and Bonnet's Theorem with the first orientation of the curve Γ we get

$$\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma = 2\pi n - \sum_{i=1}^{n} \mu(\Omega_i^+) - a.$$

With reversed orientation all exterior angles change sign and hence,

$$-\int_{\Gamma} \kappa_{g,\Gamma}\, \mathrm{d}\sigma = \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}}\, \mathrm{d}\tilde{\sigma} = 2\pi n - \sum_{i=1}^{n} \mu(\Omega_i^-) + a.$$

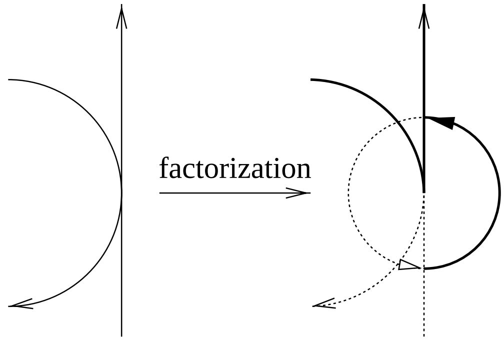

Figure 5. Exterior angles with the same sign.

Subtracting the last equation from the first equation and dividing by two we get

$$\int_0^l \kappa_{g,\Gamma}\,\mathrm{d}\sigma = \frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+)) - a.$$

We conclude, that the equation in Theorem 6 is only valid if the sum of the exterior angles vanishes. As shown on Figure 5 there are nontransversal self-intersections with both exterior angles equal to π. So for a curve with this kind of nontransversal self-intersections the rhs. and the lhs. of the equation in Theorem 6 differ by an integral multiple of 2π. If we smoothen the curve on Figure 1 and consider the two scs-factorizations of this curve, then the alternating sum of areas, $\frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+))$, for the two scs-factorizations exactly differ by 2π. This is found using a little Linear Algebra remembering that the sum of the seven unknown areas equals 4π.

If instead of the unit 2-sphere we consider any topological 2-sphere we have a theorem similar to Theorem 6 where the areas $\mu(\Omega_i^\pm)$ are exchanged by the integral of Gaussian curvature over the corresponding sets. At first sight this looks like a generalization of Gauss and Bonnet's theorem to nonsimple closed curves. But given a closed curve Γ on the unit 2-sphere there is a topological 2-sphere, M, in $\mathbb{R}^3$ and a simple closed curve γ on M such that the image of the surface normal to M along γ equals Γ. Note, that it is necessary that M is a topological 2-sphere since we have to use Gauss and Bonnet's theorem in both orientations of the curve γ, i.e., both components of the complement of γ on M have to be disks⋆. Now the normal image of M gives the formula in Theorem 6.

⋆ Such a surface M can be constructed as follows: We can assume that Γ only has transversal double points and that in a neighbourhood of each double point the curve Γ lies on two great circles. In this neighbourhood we choose a piece of the cylinder orthogonal to each great circle to lie on our surface M. Lifting one of the great circles fixing the rulings of the cylinder preserves the normal image. Hence, we have a surface M and a simple curve, γ, on M such that the surface normal along γ equal the prescribed curve Γ on the unit 2-sphere. By choosing over- and under- crossings such that γ is unknotted γ bounds a disk on M. By reversing the orientation of Γ, and thus also on γ, we also have that the other complement of γ on M is a disk. Hence, M is a topological 2-sphere and γ is a simple closed curve on M such that the image of the surface normal to M along γ equals Γ.

We now prove that in the alternating sum of areas in Theorem 6 each connected component of the complement is counted a half integer number of times independent of the scs-factorization. To do this we need some notation. Let $\alpha\colon [0,l] \to \mathbb{R}^2$ be a plane, continuous closed curve. Recall that the index or winding number of the plane curve α relative to a point p_0 is a map $\mathrm{Index}(\alpha, \mathrm{p}_0)\colon \mathbb{R}^2 \backslash \alpha([0,\mathrm{l}]) \to \mathbb{Z}$ defined on the complement of the curve α, into the integers. By continuity, Index is constant on each connected component of the complement of α and it counts the number of times the plane curve α wraps around each connected component. If a plane curve $\alpha\colon [0,l] \to \mathbb{R}^2$ is a closed regular C^1-curve then its rotation index, $\mathrm{Index}_{\mathrm{R}}(\alpha)$, is the number of complete turns given by the tangent vector field along the curve. The index and the rotation index of plane curves can be found in e.g. [4] pp. 392–393.

The following theorem (Theorem 9) is implicitly used in [3] but first formulated in [8]. This theorem gives a spherical analogy to the index of plane curves. To state this theorem let $-P$ and P be a pair of antipodal points on the unit 2-sphere and denote the stereographic projection from the unit 2-sphere onto the tangent plane of $\mathbb{S}^2$ at P, $T_P\mathbb{S}^2$, by $\Pi_P\colon \mathbb{S}^2 \backslash \{-P\} \to T_P\mathbb{S}^2$.

DEFINITION 8. Let $\Gamma\colon [0,l] \to \mathbb{S}^2$ be a closed regular curve of type C^1 on the unit 2-sphere. Denote the complement of $\Gamma([0,l])$ by Ω. Let $-P \in \Omega$ and let P be its antipode. Define the map $\mathrm{Ind}_{\Gamma,-P}\colon \Omega\backslash\{-P\} \to \mathbb{Z}/2$ by

$$\mathrm{Ind}_{\Gamma,-P}(Q) = \tfrac{1}{2}\mathrm{Index}_{\mathrm{R}}(\Pi_P(\Gamma)) - \mathrm{Index}(\Pi_P(\Gamma), \Pi_P(Q)), \quad Q \in \Omega\backslash\{-P\}.$$

The condition $-P \in \Omega$ in Definition 8 ensures that the stereographic projection of the curve Γ is a closed curve. One could define maps from $\Omega\backslash\{-P\}$ using any expression in the rotation index and the winding number, but the linear combination used in Definition 8 is, up to a multiplicative constant, the only linear combination giving

THEOREM 9. *The map* $\mathrm{Ind}_{\Gamma,-P}\colon \Omega\backslash\{-P\} \to \mathbb{Z}/2$ *defined in Definition* 8 *is independent of the point* $-P \in \Omega$ *used to define it. Hereby, we have a well-defined map* $\mathrm{Ind}_\Gamma\colon \Omega \to \mathbb{Z}/2$ *from the complement of any closed regular curve of type* C^1 *on the unit* 2*-sphere into* $\mathbb{Z}/2$.

A direct proof of Theorem 9 can be found in [8] pp. 26–29. Here Theorem 9 will follow from Lemma 10 which gives a reformulation of the map Ind_Γ.

LEMMA 10. *Let* Γ *be a closed regular* C^1*-curve on the unit* 2*-sphere. With notation as in Theorem* 6, *we for* Q *in the complement of* Γ *have that* $\mathrm{Ind}_\Gamma(Q) = \frac{1}{2}(\sharp\{i \mid Q \in \Omega_i^-\} - \sharp\{i \mid Q \in \Omega_i^+\})$ *for all transversal scs-factorizations of* Γ.

Proof of Lemma 10 *and Theorem* 9. Let Γ be a closed regular C^1-curve on the unit 2-sphere and let $\Gamma = S_1 + S_2 + \cdots + S_n$ be a transversal scs-factorization of Γ. Denote the positive resp. negative turned component of the complement of S_i

by Ω_i^+ resp. Ω_i^-. Consider a stereographic projection of the unit 2-sphere such that the image of Γ is closed under this projection. See Definition 8.

To simplify notation the stereographic projection of a set, denoted by a capital letter, will be denoted by the corresponding small letter. Hence, q is the projection of the point Q, $\gamma \to s_1 + s_2 + \cdots + s_n$ is the projection of the curve $\Gamma \to S_1 + S_2 + \cdots + S_n$, and $\omega_i^\pm$ is the projection of the set $\Omega_i^\pm$. As Γ and γ are homeomorphic their scs-factorizations are in one-to-one correspondence. Furthermore, let bc_i be the bounded component of s_i's complement and let ubc_i be the unbounded component of s_i's complement. As each s_i is simple

$$\mathrm{Index}(q, s_i) = \begin{cases} \nu_i, & \text{if } q \in bc_i, \\ 0, & \text{if } q \in ubc_i, \end{cases}$$

where $\nu_i = 1$ if s_i runs in the positive direction and $\nu_i = -1$ if s_i runs in the negative direction. Let, in analogue to the proof of Theorem 6, α_{ij}, $1 \leqslant i, j \leqslant n$ be the jump of the tangent vectors to the sub-curve s_i at the cutting point between the sub-curve s_i and the sub-curve s_j. If we let $\mathrm{Index}_{\mathrm{R}}(\gamma)|s_i$ denote the contribution to $\mathrm{Index}_{\mathrm{R}}(\gamma)$ coming from s_i, then by Hopf's Umlaufsatz we have

$$\mathrm{Index}_{\mathrm{R}}(\gamma)|s_i + \frac{1}{2\pi}\sum_{j=1}^{n}\alpha_{ij} = \nu_i.$$

For the contribution to the map Ind from Definition 8 we have

$$\begin{aligned} \mathrm{Ind}_\Gamma(Q)|s_i &= \tfrac{1}{2}\mathrm{Index}_{\mathrm{R}}(\gamma)|s_i - \mathrm{Index}(q, s_i) \\ &= \frac{1}{2}\nu_i - \frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} - \begin{cases} \nu_i, & \text{if } q \in bc_i \\ 0, & \text{if } q \in ubc_i \end{cases} \\ &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \tfrac{1}{2}\nu_i \begin{cases} -1, & \text{if } q \in bc_i \\ +1, & \text{if } q \in ubc_i \end{cases} . \end{aligned}$$

In case ω_i^+ is the bounded component of s_i's complement $\nu_i = +1$, as Ω_i^+ is defined to be the positive turned component of S_i's complement. Hereby,

$$\begin{aligned} \mathrm{Ind}_\Gamma(Q)|s_i &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\begin{cases} -1, & \text{if } q \in \omega_i^+ \\ +1, & \text{if } q \in \omega_i^- \end{cases} \\ &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\begin{cases} -1, & \text{if } Q \in \Omega_i^+ \\ +1, & \text{if } Q \in \Omega_i^- \end{cases} . \end{aligned}$$

In case ω_i^+ is the unbounded component of s_i's complement $\nu_i = -1$. Hereby,

$$\mathrm{Ind}_\Gamma(Q)|s_i = -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} - \frac{1}{2}\begin{cases} -1, & \text{if } q \in \omega_i^- \\ +1, & \text{if } q \in \omega_i^+ \end{cases}$$

$$= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\begin{cases} -1, & \text{if } Q \in \Omega_i^+ \\ +1, & \text{if } Q \in \Omega_i^- \end{cases}.$$

The following calculation completes the proof of Lemma 10.

$$\begin{aligned}
\mathrm{Ind}_\Gamma(Q) &= \sum_{i=1}^{n} \mathrm{Ind}_\Gamma(Q) \mid s_i \\
&= -\frac{1}{4\pi}\sum_{i,j=1}^{n}\alpha_{ij} + \tfrac{1}{2}(\sharp\{i \mid Q \in \Omega_i^+\} - \sharp\{i \mid Q \in \Omega_i^-\}) \\
&= \tfrac{1}{2}(\sharp\{i \mid Q \in \Omega_i^+\} - \sharp\{i \mid Q \in \Omega_i^-\}).
\end{aligned}$$

The right-hand side of the above equation is clearly independent of the stereographic projection used to define the left hand side, $\mathrm{Ind}_\Gamma(Q)$. Hence, the map Ind_Γ is well-defined as a map from the complement of a regular closed C^1-curve into the half integers. This proves Theorem 9. □

COROLLARY 11 (of Lemma 10). *All transversal scs-factorizations of a regular C^1-curve on the unit 2-sphere have either odd or even degrees.*

Proof. Given a regular C^1-curve on the 2-sphere the map Ind_Γ either takes integer values or values equal to one half plus integers*. If Ind_Γ (takes integer values/takes values equal to one half plus integers) then Lemma 10 implies that each transversal scs-factorizations of the curve has (even/odd) degree. □

Combining Theorem 6 and Lemma 10 we get our main result

THEOREM 12. *Let Γ be a closed regular C^2-curve on the unit 2-sphere and let Ind_Γ be as in Theorem 9. Then the integrated geodesic curvature of Γ, $\kappa_g(\Gamma)$, fulfills*

$$\kappa_g(\Gamma) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)\, \mathrm{d}A.$$

Proof. By Theorem 6, Remark 7, and Lemma 10 the desired formula is true for all regular C^2-curves with only transversal self-intersections. As the right-hand side of the equation is well-defined for all regular closed C^2-curves the formula, by continuity, is true for all regular closed C^2-curves. □

Remark 13. Kroneckers Drehziffer. Let Γ be a closed curve on the unit 2-sphere. Let $Q \in \mathbb{S}^2$ be a point such that neither Q nor its antipodal point $-Q$ lie on Γ. Now

* It is easily checked that Ind_Γ takes values equal to one half plus integers for all regular C^1-curves Γ in the equators regular homotopy class and that Ind_Γ takes integer values for all regular C^1-curves Γ in the other (the double-covered equators) regular homotopy class. Confer [11].

the stereographic projection, $\Pi_Q(\Gamma)$, of Γ into $T_Q\mathbb{S}^2$ is a closed curve avoiding the origin, $\mathbf{0}$, of $T_Q\mathbb{S}^2$. Hereby, $\Pi_Q(\Gamma)$ has a well-defined winding number with respect to the origin, $\mathrm{Index}(\mathbf{0}, \Pi_Q(\Gamma))$. As the author reads [2] this winding number is called Kroneckers Drehziffer, $k(Q, \Gamma)$. In [2] p. 83 it is stated that the integral of continuous geodesic curvature of Γ, $\kappa_g(\Gamma)$, fulfil

$$2\kappa_g(\Gamma) = -\int_{Q\in\mathbb{S}^2} k(Q, \Gamma)\,\mathrm{d}A.$$

As $\mathrm{Index}(\mathbf{0}, \Pi_Q(\Gamma)) = -\mathrm{Index}(\mathbf{0}, \Pi_{-Q}(\Gamma))$ for all pairs of antipodal points, Q and $-Q$, not touching Γ the sphere integral $\int_{\mathbb{S}^2} k(Q, \Gamma)\,\mathrm{d}A$ equals zero for all closed spherical curves. It appears there is a mistake in [2].

In [5] p. 53 the Kroneckers Drehziffer integral formula is mentioned as the dual (in the sense of spherical dual curves) of the Crofton formula for length of spherical curves. If we add (or subtract) the index, Ind_Γ, of a spherical curve Γ and the index, $\mathrm{Ind}_{-\Gamma}$, of its antipodal curve $-\Gamma$ (according to the orientation chosen on $-\Gamma$), then we obtain an index of Γ and its antipodal curve with the property that Kroneckers Drehziffer is claimed to have in [5]. It could be interesting to check if this antipodal curve pair weight is the correct dual of the Crofton formula for length of spherical curves.

4. Definition of Integrated Geodesic Curvature

The map Ind, given by Definition 8, is defined on closed regular C^1-curves on the unit 2-sphere. We thus give

DEFINITION 14. Let Γ be a closed regular C^1-curve on the unit 2-sphere then the integrated geodesic curvature of Γ, $\kappa_g(\Gamma)$, is defined by

$$\kappa_g(\Gamma) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)\,\mathrm{d}A.$$

Consider a continuous closed spherical curve possessing more than one scs-factorization. As the two components of the complement of a simple closed sub-curve, Ω^- and Ω^+, are open sets they are Lebesgue measurable. Hence, for a fixed scs-factorization the real number $\frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+))$ is well-defined. By lack of transversality we have to take this number modulo 2π. But it is unknown to the author *if integrated geodesic curvature can be defined, modulo* 2π*, on closed continuous scs-factorizeable curves on the unit* 2*-sphere using the expression* $\frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+))$.

5. Closed Spherical Indicatrices

In this section we apply Theorem 12 to closed spherical curves given as the tangent indicatrix, principal normal indicatrix, binormal indicatrix, or Darboux indicatrix of a space curve. Hereby we obtain a new and short proof of a formula for integrated torsion of space curves due to C. Chicone and N. J. Kalton and we generalize a classical theorem by Jacobi.

THEOREM 15. *Let* $\Gamma_{\mathbf{t}}\colon [0,l] \to \mathbb{S}^2$ *be a regular* C^2*-curve on the unit* 2*-sphere. Then the integral of torsion for any space curve* C*, with* $\Gamma_{\mathbf{t}}$ *as spherical tangent indicatrix, equals the integral of the geodesic curvature of* $\Gamma_{\mathbf{t}}$*. Furthermore, if* $\Gamma_{\mathbf{t}}$ *is closed then*

$$\tau(C) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q)\,\mathrm{d}A.$$

Remark 16. In analogy with Definition 14 we have a natural definition of integrated torsion of a regular C^2-space curve with nonvanishing curvature and closed tangent indicatrix given by the equation in Theorem 15.

Proof. Let $\Gamma_{\mathbf{t}}\colon [0,l] \to \mathbb{S}^2$ be a closed regular C^2-curve on the unit 2-sphere parametrized by arc-length σ. Any space curve with $\Gamma_{\mathbf{t}}$ as spherical tangent indicatrix can be written as

$$\mathbf{r}(s) - \mathbf{r}(0) = \int_0^s \Gamma_{\mathbf{t}}(\sigma(t))\,\mathrm{d}t, \qquad s \in [0, L],$$

where L is the length of this space curve, s is its arc-length, and $\sigma\colon [0,L] \to [0,l]$ is a nondecreasing C^1-map given by $\sigma = \sigma(s)$.

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \Gamma_{\mathbf{t}}(\sigma(s))$$

$$\mathbf{t}' = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma}(\sigma(s))\frac{\mathrm{d}\sigma}{\mathrm{d}s}(s) = \kappa_{\mathbf{r}}(s)\mathbf{n}(s).$$

Hence, $\frac{\mathrm{d}\sigma}{\mathrm{d}s}(s) = \kappa_{\mathbf{r}}(s)$ is the curvature of the space curve at the point $\mathbf{r}(s)$ and $\mathbf{n}(s) = \frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma}(\sigma(s))$. Hereby,

$$\mathbf{n}' = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \frac{\mathrm{d}^2\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma^2}(\sigma(s))\frac{\mathrm{d}\sigma}{\mathrm{d}s}(s),$$

giving torsion of the space curve as

$$\begin{aligned}\tau_{\mathbf{r}}(s) &= \mathbf{n}'\cdot\mathbf{b} = \mathbf{n}'\cdot\left(\Gamma_{\mathbf{t}} \times \frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma}\right)(\sigma(s)) \\ &= \frac{\mathrm{d}\sigma}{\mathrm{d}s}(s)\left[\Gamma_{\mathbf{t}}\frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma}\frac{\mathrm{d}^2\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma^2}\right](\sigma(s)).\end{aligned}$$

Here $[\cdot]$ is the triple scalar product in $\mathbb{R}^3$. As

$$\kappa_{g,\Gamma_{\mathbf{t}}}(\sigma) = \left[\Gamma_{\mathbf{t}}\frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma}\frac{\mathrm{d}^2\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma^2}\right](\sigma)$$

is the geodesic curvature of $\Gamma_{\mathbf{t}}$ we obtain

$$\begin{aligned}\int_0^L \tau_{\mathbf{r}}(s)\,\mathrm{d}s &= \int_0^L \left[\Gamma_{\mathbf{t}} \frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma} \frac{\mathrm{d}^2\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma^2}\right](\sigma(s))\frac{\mathrm{d}\sigma}{\mathrm{d}s}(s)\,\mathrm{d}s \\ &= \int_0^l \left[\Gamma_{\mathbf{t}} \frac{\mathrm{d}\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma} \frac{\mathrm{d}^2\Gamma_{\mathbf{t}}}{\mathrm{d}\sigma^2}\right](\sigma)\,\mathrm{d}\sigma \\ &= \int_0^l \kappa_{g,\Gamma_{\mathbf{t}}}\,\mathrm{d}\sigma = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q)\,\mathrm{d}A,\end{aligned}$$

where the last equality follows by Theorem 12 in case $\Gamma_{\mathbf{t}}$ is closed. □

The formula for integrated torsion given in Theorem15 is due to C. Chicone and N. J. Kalton. Their proof can be found in [3] and in [8] pp. 18–36. This proof is by induction on the number of connected components of the complement of the tangent indicatrix. By homotopying the stereographic projection of the tangent indicatrix and inserting needles to change its Index they use Green's Formula to transform the torsion integral such that the index-formula from Definition 8 is recognizable. Observing that the integral over the unit 2-sphere appearing in the Chicone–Kalton formula depends only on the closed tangent indicatrix and not on the space curve itself, lead to the wish of finding a proof of this formula reflecting this fact. Having found such a proof we now give similar formulas for closed principal normal and binormal indicatrices of space curves.

THEOREM 17. *Let C be a regular curve in $\mathbb{R}^3$ of type C^4 with nonvanishing curvature and closed Frenet frame. Denote the spherical indicatrix of the principal normal vector and of the binormal vector by $\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{b}}\colon [0, l] \to \mathbb{S}^2$. Then*

$$\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}}}(Q)\,\mathrm{d}A = 0.$$

If furthermore C has nonvanishing torsion then the total curvature of C fulfil

$$\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{b}}}(Q)\,\mathrm{d}A = \kappa(C).$$

Remark 18. As the lengths of the curves $\Gamma_{\mathbf{t}}, \Gamma_{\mathbf{b}}\colon [0, l] \to S^2$ are $\kappa(C)$ and $|\tau|(C)$, respectively this theorem gives an 'almost duality' between total curvature and total torsion for space curves with closed Frenet-frames. In fact, we have the following identities

$$|\Gamma_{\mathbf{t}}| = \kappa(C) \qquad \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q)\,\mathrm{d}A = \tau(C)$$

$$|\Gamma_{\mathbf{n}}| = \omega(C) \qquad \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}}}(Q)\,\mathrm{d}A = 0$$

$$|\Gamma_{\mathbf{b}}| = |\tau|(C) \qquad \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{b}}}(Q)\,\mathrm{d}A = \kappa(C).$$

Proof. Let the curve C be as in the theorem and let $\mathbf{r}\colon [0,l] \to \mathbb{R}^3$ be an arc-length parametrization of C. Let $\mathbf{t}_{\mathbf{r}}(s)$, $\mathbf{n}_{\mathbf{r}}(s)$, $\mathbf{b}_{\mathbf{r}}(s)$, $\kappa_{\mathbf{r}}(s)$, and $\tau_{\mathbf{r}}(s)$ denote the tangent vector, the principal normal vector, the binormal vector, the curvature, and the torsion of the curve C at the point $\mathbf{r}(s)$. Define a curve $\mathbf{x}\colon [0,l] \to \mathbb{R}^3$ by

$$\mathbf{x}(s) = \int_0^s \mathbf{n}_{\mathbf{r}}(s)\,\mathrm{d}s, \quad \text{for } s \in [0,l].$$

Note, that $\mathbf{x}$ is an arc-length parametrization of a regular curve of type C^3. In order to calculate the torsion, $\tau_{\mathbf{x}}(s)$, of the curve parametrized by $\mathbf{x}$ we find

$$\begin{aligned}
&\mathbf{x}' = \mathbf{n}_{\mathbf{r}}\\
&\mathbf{x}'' = -\kappa_{\mathbf{r}}\mathbf{t}_{\mathbf{r}} + \tau_{\mathbf{r}}\mathbf{b}_{\mathbf{r}} \quad (\neq \mathbf{0} \Rightarrow \kappa_{\mathbf{x}}(s) > 0)\\
&\mathbf{x}''' = -\kappa_{\mathbf{r}}'\mathbf{t}_{\mathbf{r}} - \kappa_{\mathbf{r}}^2\mathbf{n}_{\mathbf{r}} + \tau_{\mathbf{r}}'\mathbf{b}_{\mathbf{r}} - \tau_{\mathbf{r}}^2\mathbf{n}_{\mathbf{r}}\\
&[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] = \begin{vmatrix} 0 & -\kappa_{\mathbf{r}} & -\kappa_{\mathbf{r}}' \\ 1 & 0 & -\kappa_{\mathbf{r}}^2 - \tau_{\mathbf{r}}^2 \\ 0 & \tau_{\mathbf{r}} & \tau_{\mathbf{r}}' \end{vmatrix} = \tau_{\mathbf{r}}'\kappa_{\mathbf{r}} - \kappa_{\mathbf{r}}'\tau_{\mathbf{r}}.
\end{aligned}$$

Hence,

$$\tau_{\mathbf{x}} = \frac{\tau_{\mathbf{r}}'\kappa_{\mathbf{r}} - \kappa_{\mathbf{r}}'\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}^2 + \tau_{\mathbf{r}}^2} = \frac{\kappa_{\mathbf{r}}^2 \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)}{\kappa_{\mathbf{r}}^2 + \tau_{\mathbf{r}}^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Arctan}\left(\frac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)\right).$$

Using Theorem 15 we as $\kappa_{\mathbf{x}} > 0$ get

$$\begin{aligned}
&\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}_{\mathbf{r}}}}(Q)\,\mathrm{d}A\\
&\quad = \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}_{\mathbf{x}}}}(Q)\,\mathrm{d}A = \int_0^l \tau_{\mathbf{x}}(s)\,\mathrm{d}s = \left[\mathrm{Arctan}\left(\frac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)\right]_0^l = 0.
\end{aligned}$$

This proves the first part of the theorem. To prove the last part of the theorem let the curve $\mathbf{y}\colon [0,l] \to \mathbb{R}^3$ be given by

$$\mathbf{y}(s) = \int_0^s \mathbf{b}_{\mathbf{r}}(s)\,\mathrm{d}s, \quad \text{for } s \in [0,l].$$

Note, that $\mathbf{y}$ is an arc-length parametrization of a regular curve of type C^3. In order to calculate the torsion, $\tau_{\mathbf{y}}(s)$, of the curve parametrized by $\mathbf{y}$ we find

$$\mathbf{y}' = \mathbf{b}_{\mathbf{r}}$$

$$\mathbf{y}'' = -\tau_{\mathbf{r}}\mathbf{n}_{\mathbf{r}} \quad (\neq \mathbf{0} \Rightarrow \kappa_{\mathbf{y}} = |\tau_{\mathbf{r}}| > 0)$$
$$\mathbf{y}''' = -\tau_{\mathbf{r}}'\mathbf{n}_{\mathbf{r}} + \kappa_{\mathbf{r}}\tau_{\mathbf{r}}\mathbf{t}_{\mathbf{r}} - \tau_{\mathbf{r}}^2\mathbf{b}_{\mathbf{r}}$$
$$[\mathbf{y}'\mathbf{y}''\mathbf{y}'''] = \begin{vmatrix} 0 & 0 & \kappa_{\mathbf{r}}\tau_{\mathbf{r}} \\ 0 & -\tau_{\mathbf{r}} & -\tau_{\mathbf{r}}' \\ 1 & 0 & \tau_{\mathbf{r}}^2 \end{vmatrix} = \tau_{\mathbf{r}}^2\kappa_{\mathbf{r}}.$$

Hence,

$$\tau_{\mathbf{y}}(s) = \frac{\tau_{\mathbf{r}}^2\kappa_{\mathbf{r}}}{\tau_{\mathbf{r}}^2} = \kappa_{\mathbf{r}}(s),$$

for all $s \in [0, l]$, which together with Theorem 15 imply the last statement in the theorem as

$$\int_{Q\in S^2} \operatorname{Ind}_{\Gamma_{\mathbf{b_r}}}(Q)\,\mathrm{d}A$$
$$= \int_{Q\in S^2} \operatorname{Ind}_{\Gamma_{\mathbf{t_y}}}(Q)\,\mathrm{d}A = \int_0^l \tau_{\mathbf{y}}(s)\,\mathrm{d}s = \int_0^l \kappa_{\mathbf{r}}(s)\,\mathrm{d}s = \kappa(C). \qquad \square$$

Remark 19. Let $\mathbf{r}$ be an arc-length parametrization of a regular C^4-curve with isolated points of either zero curvature or zero torsion. If the curve allows a C^2 binormal vector field, i.e., a vector field orthogonal to both $\mathbf{r}'$ and $\mathbf{r}''$, then with this new definition of the Frenet frame, the Frenet's formulas are still valid but curvature can be negative. See e.g. [5] Section 2. Denote (as in [10]) the index of the plane curve $(\kappa(s), \tau(s))$ with respect to origin, $(0, 0)$, by the nutation, $\nu(C)$, of the curve C. Allowing negative curvature of space curves the first equation in Theorem 17 is replaced by

$$\int_{\Gamma_{\mathbf{n}}} \kappa_{g,\Gamma_{\mathbf{n}}}\,\mathrm{d}\sigma$$
$$= \int_{Q\in S^2} \operatorname{Ind}_{\Gamma_{\mathbf{n}}}(Q)\,\mathrm{d}A = \int_C \frac{\mathrm{d}}{\mathrm{d}s}\left(\operatorname{Arctan}\left(\frac{\tau}{\kappa}\right)\right)\,\mathrm{d}s = 2\pi\nu(C).$$

Combining the first equation in Theorem 17 with Lemma 10 we obtain a generalization of a theorem by Jacobi (1842) that states: *A simple closed principal normal indicatrix of a regular C^4-curve with nonvanishing curvature bisects the unit* 2*-sphere**.

THEOREM 20. *Let $\Gamma_{\mathbf{n}}$ be the closed principal normal indicatrix of a regular C^4-curve with nonvanishing curvature. If $\Gamma_{\mathbf{n}}$ possesses a transversal* scs*-factorization*

* In [10] there is another generalization of this theorem but that generalization uses the previous mentioned 'Umlaufszahl' which not is well-defined. See also the footnote on page 53 in [5] and the review by S. B. Jackson [Math. Rev. Vol. 8 (1947) p. 226] on [10].

then the sum of the areas of the positive turned complements of the sub-curves equals the sum of the areas of the negative turned complements of the sub-curves.

Proof. Under the assumptions taken in the theorem we have that

$$0 = \int_{\Gamma_{\mathbf{n}}} \kappa_{g,\Gamma_{\mathbf{n}}} \,\mathrm{d}\sigma = \frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+)),$$

where the first equality is given in the proof of Theorem 17 and the last equality uses Theorem 6. □

Another spinoff is the classical

COROLLARY 21. *Let $C\colon \mathbb{S}^1 \to \mathbb{R}^3$ be a closed regular C^3-curve lying on a sphere of radius r. Then the integrated torsion of C is zero. Or equivalently: A tangent indicatrix of a closed spherical regular C^3-curve has integrated geodesic curvature zero.*

Proof. It is sufficient to prove the theorem in case of the unit 2-sphere. Thus let C be a closed regular C^3-curve on the unit 2-sphere and let $\tilde{C}$ be a space curve with nonvanishing curvature which has C as tangent indicatrix. See *e.g.* the proof of Theorem 17. Now the tangent indicatrix of C, $\Gamma_{\mathbf{t},C}$ prescribe the same curve as the principal normal indicatrix of $\tilde{C}$, $\Gamma_{\mathbf{n},\tilde{C}}$. Using Lemma 15 and Theorem 17 we get $\tau(C) = \kappa_g(\Gamma_{\mathbf{t},C}) = \kappa_g(\Gamma_{\mathbf{n},\tilde{C}}) = 0$. □

It is noteworthy that the generalization of Jacobi's theorem and the well-known fact that closed spherical curves has integrated torsion equal to zero in fact are equivalent. Again we note that the generalization of Jacobi's theorem, and hereby Jacobi's theorem in particular, are not to be considered as 'space curve theorems' as they are implied by the fact that the principal normal indicatrix is the tangent indicatrix of a closed spherical curve.

The method used in the proof of Theorem 17 provides a wealth of integral formulas as follows. Let C be a regular space curve of type at least C^3 with nonvanishing curvature and closed Frenet frame and let X be a unit vector field along the curve C such that X is closed relative to C's Frenet frame. In coordinates that is – if C is given by $\mathbf{r}\colon [0, l] \to \mathbb{R}^3$ then X given by $s \mapsto \alpha(s)\mathbf{t}(s) + \beta(s)\mathbf{n}(s) + \gamma(s)\mathbf{b}(s)$ has to be a closed regular curve of type C^1 regarded as a curve on the unit 2-sphere. Hence, (α, β, γ) must describe a closed curve* on the unit 2-sphere. Now, integration of the geodesic curvature of X gives a new integral formula using Theorem 15.

* In the proof of Theorem 17 these 'curves' were the points $(0, 1, 0)$ and $(0, 0, 1)$.

In order to obtain interesting integral formulas the vector field along the curve must have geometric meaning. An example of such a vector field is the Darboux vector field

$$D(s) = \frac{\tau(s)\mathbf{t}(s) + \kappa(s)\mathbf{b}(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}.$$

This vector field is the direction of $\mathbf{n} \times \mathbf{n}'$ and gives e.g. the direction of the rulings on the surface known as the rectifying developable of the curve. This ruled surface is the ruled surface with zero Gaussian curvature on which the curve is a geodesic curve. See e.g. [9]. Using this vector field and a straightforward calculation we get

COROLLARY 22 (of Theorem 15). *Let C be a regular curve in $\mathbb{R}^3$ of type C^4 with nonvanishing curvature and closed Frenet frame and let D be the Darboux vector field along the curve C. Considering D as a closed curve on $\mathbb{S}^2$ we have*

$$\int_{Q \in S^2} \mathrm{Ind}_D(Q)\, \mathrm{d}A = \omega(C) = |\Gamma_{\mathbf{n}}|.$$

6. The Tennis Ball Theorem and the Four Vertex Theorem

The tennis ball theorem [1] p. 53 states: *A closed simple smooth spherical curve dividing the sphere into two parts of equal areas has at least four inflection points* (*points with zero geodesic curvature*). It is natural to note that a nonsimple curve on S^2 that bisects S^2 in the sense of Theorem 20, need only have two inflection points. An example is a curve of the shape of the figure eight on S^2. Here we draw a connection between the tennis ball theorem and the four vertex theorem for C^3 closed convex simple space curves which restricted to spherical closed curves states: *Any simple closed spherical C^3-curve has at least four vertices* (*points with zero torsion*), see [6].

LEMMA 23. *Let Γ be a regular closed spherical C^3-curve and let $\Gamma_{\mathbf{t}}$ be its tangent indicatrix. If the number of vertices $V(\Gamma)$ and the number of inflection points $I(\Gamma)$ of Γ both are finite then $V(\Gamma) \geqslant I(\Gamma)$ and $V(\Gamma) = I(\Gamma_{\mathbf{t}})$ where $I(\Gamma_{\mathbf{t}})$ is the number of inflection points of $\Gamma_{\mathbf{t}}$.*

Proof. Let Γ be as in the lemma. We can assume that Γ lies on the unit 2-sphere. As the curvature, κ, and the geodesic curvature, κ_g, of Γ fulfil $1 + \kappa_g^2 = \kappa^2$ the curvature has global minima precisely in the inflection points of Γ. By the equation $\kappa\kappa_g\tau = \kappa'$, which is easily derived from [12] equation (6) p. 365, all other local extremas of the curvature of Γ lie in vertices of Γ. Assuming that $V(\Gamma)$ and $I(\Gamma)$ both are finite Rolles theorem gives $V(\Gamma) \geqslant I(\Gamma)$.

In the proof of Theorem 15 we found $\tau = \kappa\kappa_{g,\Gamma_{\mathbf{t}}}$, where $\kappa_{g,\Gamma_{\mathbf{t}}}$ is the geodesic curvature of the tangent indicatrix of Γ, $\Gamma_{\mathbf{t}}$. As the spherical curve Γ has nonvanishing curvature we conclude that $V(\Gamma) = I(\Gamma_{\mathbf{t}})$. □

The tennis ball theorem, the 'spherical' four vertex theorem, and their connecting Lemma 23 give

THEOREM 24. *Let* Γ *be a regular closed spherical* C^3*-curve and let* $\Gamma_{\mathbf{t}}$ *be its tangent indicatrix. If* Γ *or* $\Gamma_{\mathbf{t}}$ *is simple or* Γ *is the iterated tangent indicatrix of a simple spherical curve, then* Γ *has at least four vertices and* $\Gamma_{\mathbf{t}}$ *has at least four inflection points.*

Proof. Let Γ_0 be a sufficiently smooth closed spherical curve and let $\Gamma_1, \Gamma_2, \ldots$ be its iterated tangent indicatrices. By Lemma 23 we have

$$I(\Gamma_0) \leqslant V(\Gamma_0) = I(\Gamma_1) \leqslant V(\Gamma_1) = I(\Gamma_2) \leqslant \cdots .$$

If Γ_n, $n \geqslant 0$, is simple then $V(\Gamma_n) \geqslant 4$, by the four vertex theorem. Hence, $V(\Gamma_i) \geqslant 4$ for $i \geqslant n$ and $I(\Gamma_j) \geqslant 4$ for $j \geqslant n+1$. If Γ_n, $n \geqslant 1$, is simple then by Theorem 20 and the proof of Corollary 21 Γ_n bisects the unit 2-sphere. Hence, $I(\Gamma_n) \geqslant 4$ by the tennis ball theorem and $V(\Gamma_i) \geqslant 4$ for $i \geqslant n-1$ and $I(\Gamma_j) \geqslant 4$ for $j \geqslant n$. □

7. Topological Bounds for Geodesic Curvature

In this section we consider closed curves on the 2-sphere with *only transversal self-intersections*. The (integral) formulas for integrated geodesic curvature of a spherical curve Γ presented here,

$$\int_\Gamma \kappa_g(\sigma)\,\mathrm{d}\sigma = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)\,\mathrm{d}A = \frac{1}{2}\sum_{i=1}^{n}(\mu(\Omega_i^-) - \mu(\Omega_i^+)),$$

give some topological bonds on integrated geodesic curvature. Let $\max(\mathrm{Ind}_\Gamma)$ denote the maximal value of Ind_Γ on the complement of Γ and let $\min(\mathrm{Ind}_\Gamma)$ denote the minimal value of Ind_Γ. Recall that $\mathrm{scsn}(\Gamma)$ is the simple closed sub-curve number of Γ and finally that $\mathrm{s}(\Gamma)$ is the number of self-intersections of Γ as defined in Section 2. As each component of Γ's complement have area less than 4π we get the inequalities

$$-2\pi(s(\Gamma)+1) \leqslant -2\pi\ \mathrm{scsn}(\Gamma) \leqslant 4\pi\ \min(\mathrm{Ind}_\Gamma) < \int_\Gamma \kappa_g(\sigma)\,\mathrm{d}\sigma,$$

$$\int_\Gamma \kappa_g(\sigma)|, \mathrm{d}\sigma < 4\pi\ \max(\mathrm{Ind}_\Gamma) \leqslant 2\pi\ \mathrm{scsn}(\Gamma) \leqslant 2\pi(s(\Gamma)+1).$$

By Theorem 17 the total curvature of a regular C^4-curve, C, with nonvanishing curvature and nonvanishing torsion equals the integrated geodesic curvature of its binormal indicatrix, $\Gamma_{\mathbf{b}}$. This gives the inequalities

$$\begin{aligned}\kappa(C) &= \int_{\Gamma_{\mathbf{b}}} \kappa_g(\sigma)\,\mathrm{d}\sigma < 4\pi\ \max(\mathrm{Ind}_{\Gamma_{\mathbf{b}}}) \\ &\leqslant 2\pi\,\mathrm{scsn}(\Gamma_{\mathbf{b}}) \leqslant 2\pi(s(\Gamma_{\mathbf{b}})+1).\end{aligned}$$

In the following we use some results on total curvature of closed space curves that all can be found in [7]. The first result is due to W. Fenchel who in 1929 proved that a closed space curve has total curvature $\geqslant 2\pi$, where equality holds if and only if the curve is planar and convex. As we have assumed nonvanishing torsion we have a strict inequality. The other result is due to J.W. Milnor and states that if we let K be a knotted knot type and define the curvature of a knot type $\kappa(K)$ as the greatest lower bound of the total curvature of its representatives, then $\kappa(K) = 2\pi\mu$, where μ is an integer $\geqslant 2$ (the crookedness of the knot type defined as the minimal number of local maxima in any direction of any representative of the knot type K – thus the crookedness equals the bridge number of the knot type). This greatest lower bound is never attained for knotted space curves. By the inequalities

$$2\pi\mu < \kappa(C) < 2\pi \operatorname{scsn}(\Gamma_{\mathbf{b}}) \leqslant 2\pi(s(\Gamma_{\mathbf{b}}) + 1),$$

where $\mu = 1$ corresponds to unknotted space curves, we get

THEOREM 25. *Let C be a regular C^3 representative of a knot type K with crookedness $\mu(K)$. If C has both nonvanishing curvature and torsion then the binormal indicatrix of C, $\Gamma_{\mathbf{b}}$, has simple closed sub-curve number scsn$(\Gamma_{\mathbf{b}}) \geqslant \mu(K) + 1$ and Γ has at least μ self-intersections* (*in the sense of Section* 2).

EXAMPLE 26. The standard shadow of the trefoil knot has three self-intersections but it can not be obtained as a stereographic projection of the binormal indicatrix of a knotted space curve – since this shadow has simple closed sub-curve number equal to two.

8. Knots and SCS-factorization

Recall the construction of Seifert surfaces from a knot diagram. Firstly the knot diagram is 'factorized' into a number of disjoint simple closed curves, the so-called Seifert circles. Each Seifert circle bounds a disc and these disjoined discs are glued together by half twisted bands given by the crossings in which the 'factorizations' have taken place. The constructed Seifert surface is an embedded orientable surface with the knot as its only boundary curve.

Let KD be a knot diagram and let $KD \to S_1 + \cdots + S_n$ be an scs-factorization of the shadow of this knot diagram. Using the Seifert construction on an scs-factorization of a knot diagram we also get an orientable surface, with the knot as its only boundary curve. In the following we call such a surface an scs-surface. The simple closed sub-curves in an scs-factorization may intersect each other. Hence, an scs-surface may, and generally will, have self-intersections – but each disc is embedded.

THEOREM 27. *All scs-surfaces are immersed discs*.

Proof. Let KD be a knot diagram of a knot K and let $KD \to S_1 + \cdots + S_n$ be an scs-factorization of the shadow of this knot diagram. By the Seifert construction the scs-surface has only one boundary curve, the knot, and the scs-surface is orientable. The genus, g, of a surface constructed by the Seifert construction can be found in any textbook on knots and $2g$ equals the number of half twisted bands minus the number of discs plus one. The Seifert construction from an scs-factorization $KD \to S_1 + \cdots + S_n$ replaces the n simple closed sub-curves by n discs and the $(n-1)$ cutting points by $(n-1)$ half twisted bands. Hence, the genus of the scs-surface S is $g(S) = \frac{1}{2}((n-1) - n + 1) = 0$ and the scs-surface is topologically a disc. □

Start with an scs-surface of a knot and deform the surface until it is an embedding. Now the boundary of the embedded surface is unknotted by Theorem 27. Hence, by removing the self-intersections of the scs-surface we unknot the boundary curve. On the other hand, change a knot diagram, KD, to a knot diagram of the unknot, UD, by a number of crossing changes. The two knot diagrams KD and UD possess the same scs-factorizations. If KD possesses an scs-factorization such that these changes of crossings do not take place in cutting points, then the self-intersections of the corresponding scs-surface are removed. Therefore, scs-surfaces are intimately connected with the unknotting of knots. Here we improve a standard upper bound on the unknotting number.

THEOREM 28. *Let KD be a knot diagram of a knot K and let $KD \to S_1+\cdots+S_n$ be an scs-factorization of order n of the shadow of this knot diagram. If $c(KD)$ is the number of crossings in KD and $u(K)$ is the unknotting number of the knot K then $u(K) \leqslant \frac{1}{2}(c(KD) - n + 1)$.*

Proof. Let K, $KD \to S_1 + \cdots + S_n$, and $u(K)$ be as in the theorem and let S be the scs-surface defined by the scs-factorization. Recall, that the scs-factorization is obtained by $(n-1)$ elementary factorizations $KD \to S_1 + \Gamma_1 \to S_1 + S_2 + \Gamma_2 \to \cdots \to S_1 + S_2 + \cdots + S_n$. Consider the elementary factorization $KD \to S_1 + \Gamma_1$. Let m_1 denote the number of crossings between S_1 and Γ_1 not counting the cutting point between S_1 and Γ_1. As S_1 and Γ_1 are closed curves and they only have transversal intersections m_1 is even. By changing at most half of the m_1 crossings we can bring S_1 to lie entirely above Γ_1 or entirely below Γ_1. Hence, a disc spanned by S_1 need not intersect Γ_1 after at most $m_1/2$ crossing changes in the knot diagram KD.

Doing these changes of crossings for each elementary factorization we can unknot the diagram KD by changing at most half of the crossings not used as cutting points. Hence, the knot K can be unknotted by use of at most $(c(KD) - (n-1))$ changes. □

The orders of scs-factorizations of a knot diagram are generally changed when the knot diagram is changed by Reidemeister moves. It is therefore natural to define a knot invariant by attaching to each knot the minimal order of all scs-factorizations of any knot diagram of the knot. This gives a measure of complexity of the knot type – but due to the quite surprising Theorem 29 this measure only detects knottedness.

THEOREM 29. *The unknot is the only knot with a knot diagram* (*a simple curve*) *that can be factorized into one simple closed curve. Any knotted knot has a knot diagram that can be factorized into two simple closed curves*.

Proof (Sketch). The first part of the theorem is obvious. To prove the last part let KD be a knot diagram with an scs-factorization $KD \to S_1 + \cdots + S_n$ of order $n \geqslant 3$ and let S be a corresponding scs-surface. The surface S consists of n discs, $D_1, \ldots, D_n$, each of which is embedded and these discs are connected by $(n-1)$ half-twisted bands. Lets assume that there is only one half-twisted band attached to D_1 connecting D_1 and D_2. If D_1 and D_2 intersects then this intersection can be pushed into $D_3, \ldots, D_n$ without changing the knot type of the boundary of S. The part of the surface S given by D_1 and D_2 and their connecting band is now embedded allowing us to consider it as one embedded disc. We can now flatten out the $(n-1)$-disc surface and obtain a knot diagram with an scs-factorization of order $(n-1)$. Hence, by changing the knot diagram we can reduce the order of scs-factorizations until order two is reached. □

Acknowledgements

I thank my supervisors Professor V. L. Hansen and Docent S. Markvorsen for the encouragement and the discussions that have led to the results presented here.

References

1. Arnold, V. I.: *Topological Invariants of Plane Curves and Caustics*, Univ. Lecture Ser. 5, Amer. Math. Soc., Providence, 1994.
2. Blaschke, W.: *Integralgeometrie, Systeme von Diff. Gl.*, Chelsea Publ., New York, 1949.
3. Chicone, C. and Kalton, N. J.: *Flat Embeddings of the Möbius Strip in* $\mathbb{R}^3$, Department of Mathematics, University of Missouri, Columbia, MO 65211, 1984.
4. Do Carmo, M. P.: *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, 1976.
5. Fenchel, W.: On the differential geometry of closed space curves, *Bull. Amer. Math. Soc.* **57** (1950), 44–54.
6. Romero Fuster, M. C. and Sedykh, V. D.: On the number of singularities, zero curvature points and vertices of a simple convex space curve, *J. Geom.* **52**(1) (1995), 168–172.
7. Milnor, J. W.: On the total curvature of knots, *Ann. Math.* **52**(2) (1950), 248–257.
8. Randrup, T. and Røgen, P.: *How to Twist a Knot*. Master's thesis, Mathematical Institute, Technical University of Denmark, 1995. (Available at http://www.mat.dtu.dk/publications/masterthesis-1995-02.html).
9. Randrup, T. and Røgen, P.: Sides of the Möbius strip, *Arch. Math.* **66**(6) (1996), 511–521.
10. Scherrer, W.: Über das Hauptnormalenbild einer Raumkurve, *Comment. Math. Helv.* **19** (1947), 115–133.
11. Solomon, B.: Tantrices of spherical curves, *Amer. Math. Month.* **103**(1) (1996), 30–39.
12. Wong, Y.: A global formulation of the condition for a curve to lie in a sphere, *Monatsh. Math.* **67** (1963), 363–365.