TWO-DIMENSIONAL SUBMANIFOLDS OF FOUR-DIMENSIONAL MANIFOLDS

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§1. Introduction

1.1. Formulation of the Problem. The terminology of this paper is that of differential topology. In particular, smooth manifolds are called manifolds, submanifolds of smooth manifolds in the sense of differential topology are called submanifolds, and smooth O(SO)-fibers are called O(SO)-fibers.

The problem to which this paper is devoted may be formulated as follows. Let X be a connected closed four-dimensional manifold, and let ξ be an element of the (integer) homology group $H_2(X)$. We assume that the action of trivial factors is excluded, so that $H_1(X) = 0$. What is the minimal genus of the (oriented connected closed two-dimensional) submanifold realizing class ξ?

To answer this question we must learn how to find effective upper and lower bounds for this minimal genus. We shall deal here with lower bounds.

1.2. Results to be Found in the Literature. The author is aware of only three publications containing such bounds, and these for special cases only. The case is cited in [5] when the realizing submanifold cannot be a sphere. In Kervaire and Milnor [3], this example is generalized to the theorem: if the class of ξ, reduced modulo 2, is the Poincaré dual to the Stiefel-Whitney class $w_2(X)$, and if $\xi\xi-\sigma(X) \not\equiv 0 \bmod 16$ ($\xi\xi$ is the self-linkage index of class while ξ, $\sigma(X)$ is the signature of manifold X), then the realizing submanifold cannot be a sphere (this theorem is also true without the assumption that $H_1(X) - 0$, if manifold X is orientable). It is proven in Tristram [8] that if $X = S^2 \times S^2$ and $\xi = n_1\xi_1 + n_2\xi_2$, where ξ_1, ξ_2 are the natural generators of group $H_2(S^2 \times S^2)$* while n_1 and n_2 are non-zero integers with $(n_1, n_2) \neq 1$, then the realizing submanifold cannot be a sphere. It is also asserted there that the realizing submanifold cannot be a sphere in the case when $X = \mathbf{CP}^2$ and $\xi = n\xi_0$, where ξ_0 is a generator of the group $H_2(\mathbf{CP}^2)$ and n is an integer with $|n| > 2$.

The Kervaire-Milnor formulation generalizes a theorem of the author according to which the signature of an oriented closed four-dimensional manifold with $w_2 = 0$ is divisible by 16, and is quite simply derived from this theorem. Tristram's proof is based on the well-known connection between the problem at hand and the theory of links, and makes use of the algebraic invariants of links. It is probable that this method could be productive of further bounds, but it seems more promising to exploit this connection in the reverse direction, for example, for the use in link theory of the bounds to be found in the present paper.

1.3. Principal Result of the Paper. The principal result of the present paper is contained in the following theorem.

BASIC THEOREM. Let A be an oriented connected closed two-dimensional submanifold of connected closed four-dimensional manifold X with $H_1(X) = 0$, realizing class $\xi \in H_2(X)$, and let g be the genus of surface A. If ξ is divisible by 2 then

$$g \geqslant \left| \frac{\xi\xi}{4} - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2},$$

where $b_2(x)$ is the two-dimensional Betti number of manifold X. If ξ is divisible by odd number h, a power of a prime, then

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$$g \geqslant \left| \frac{(h^2-1)\xi\xi}{4h^2} - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2}.$$

The last inequality also remains valid in the case when h is an arbitrary odd number dividing ξ if group $\pi_1(X\setminus A)$ is finite.

The proof is based on the study of finite-sheeted cyclic ramified coverings of manifold X with branches along A. It turns out that with such a covering the signature of the covering manifold can be computed in terms of $\sigma(X)$, $\xi\xi$, and the number of sheets of the covering and, in particular, does not depend on g, while its two-dimensional Betti number, in the most interesting cases, increases linearly with g. Since the two-dimensional Betti number of a covering manifold cannot be less than its signature, we obtain a lower bound for g. In many cases this bound is strengthened by the available information on the action of the group of automorphisms of the coverings in the cohomologies of the covering manifold (the deepest information is provided by the signature formula of At'ya-Zinger [1], §6). I might add that a similar method was recently by Massey [4] in the proof of the Whitney hypothesis on normal Euler numbers of nonoriented surfaces in R^4; see, 1.5.

1.4. Examples. If $X = CP^2$ and $\xi = n\xi_0$ (where, as in section 1.2, ξ_0 is a generator of group $H_2(CP^2)$, and n is an integer) then, according to the basic theorem, $g \geqslant \frac{1}{4}n^2 - 1$ when n is even, and $g \geqslant \frac{h^2-1}{4h^2}n^2 - 1$ when $n \equiv 0 \mod h$, where h is a power of an odd prime. On the other hand, ξ is realized when $n \neq 0$ by algebraic curves of genus $1/2(|n|-1)(|n|-2)$, and a realization of a submanifold of lower genus is unknown for any $n \neq 0$. When $|n| \leq 4|$ this information provides an exact value of the minimal genus of the realizing surface (what is new, of course, is only the exact value $g = 3$ for $n = 4$). For larger values of $|n|$, the upper bound is almost twice as large as the lower one.

If $X = S^2 \times S^2$ and $\xi = n_1\xi_1 + n_2\xi_2$ (as in section 1.2, ξ_1, ξ_2 are natural generators of the group $H_2(S^2 \times S^2)$, and n_1 and n_2 are integers) then, on the basis of the theorem, $g \geqslant \frac{1}{2}|n_1n_2| - 1$ for even n_1 and n_2, and $g \geqslant \frac{h^2-1}{2h^2}|n_1n_2| - 1$ when $n_1, n_2 \equiv 0 \mod h$, where h is a power of an odd prime. On the other hand, ξ is realized, when $n_1 \neq 0$ and $n_2 \neq 0$, by algebraic curves of genus $(|n_1|-1)(|n_2|-1)$, and no realizations by submanifolds of lower genus are known for any $n_1 \neq 0$, $n_2 \neq 0$. Exact values of the minimal genus of the realizing surface are obtained from this information only in the case when n_1 and n_2 are even and one of them equals 2 or -2, and in the trivial case when $|n_1| \leq 1$ or $|n_2| \leq 1$. For large values of $|n_1|$, $|n_2|$, the upper bound is again almost twice as large as the lower bound.

In these two examples the results of the present paper disclose information which is contained in the Kervaire-Milnor theorem. This is not the case if $X = CP^2 \# CP^2$ and $\xi = 3\xi_1 + \xi_2$, where ξ_1, ξ_2 are natural generators of group $H_2(CP^2 \# CP^2)$. By virtue of the Kervaire-Milnor theorem, the class of ξ is not realized by a sphere, while our theorems say nothing on this point.

1.5. The Non-orientable Case. Our basic theorem takes the following form for the case when submanifold A is non-orientable.

THEOREM. Let A be a non-orientable connected closed two-dimensional submanifold of connected closed four-dimensional manifold X with $H_1(X) = 0$, and let g be the genus of surface A (i.e., $2-\chi(A)$, where χ is the Euler characteristic). If A realizes the zero element of group $H_2(X; Z_2)$, then

$$g \geqslant \left| \frac{a}{2} - \sigma(X) \right| - b_2(X),$$

where a is the ("torsion") normal Euler number of surface A.

The simplest corollary of this theorem is that the normal Euler number of a non-oriented connected closed two-dimensional submanifold of sphere S^4 does not exceed twice the genus of this submanifold. As a conjecture, this statement had been published in 1941 by Whitney [9]. Recently, Massey [4] published his proof of it, differing only slightly from that contained in the present paper. Partial results bearing on the Whitney conjecture and deriving from the Kervaire-Milnor theorem are to be found in [6].

§ 2. Basic Construction

2.1. Initial Data. We assume as given: oriented connected closed four-dimensional manifold X with $H_1(X) = 0$; its oriented connected closed two-dimensional submanifold A realizing the nonzero class of $\xi \in H_2(X)$; integer $m > 1$ dividing the class of ξ. These conventions will be altered only in § 7.

We shall denote by n the largest natural number dividing ξ; the fraction n/m will be denoted by μ; the self-linkage index of class ξ (the normal Euler number of surface A) will be denoted by α; g will denote the genus of surface A; the fundamental group $\pi_1(X\setminus A)$ will be denoted by Π.

We remark that it follows from the condition $H_1(X) = 0$ that (by virtue of the Poincaré duality) Tor $H_2(X) = 0$, $H_3(X) = 0$.

2.2. Manifold U. Let T be a (closed) cylindrical neighborhood of surface A in X, and let U be its closed complement.

A neighborhood of T has the structure of an SO(2)-fiber over A with fiber D^2 and Euler number α, while its boundary ∂T has the structure of the associated fibration with fiber S^1. The fibers of these fibrations over marked points of surface A will be denoted by D and C. The complement U is an oriented connected compact four-dimensional manifold with boundary $\partial U = \partial T$. It is a deformation retract of manifold $X\setminus A$, so that the fundamental group $\pi_1(U)$ can be identified with Π.

Group $H_1(U)$ is isomorphic to $\mathbf{Z}_n$ and the embedding homomorphism $H_1(C) \to H_1(U)$ is an epimorphism.

Proof. Study of the homology sequences of the pair X, A shows that $H_3(X, A) = 0$ and Tor $H_2(X, A) \cong \mathbf{Z}_n$. Since the embedding $(X, A) \to (X, T)$ is a homotopy equivalence then, in these relationships, the pair X, A can be replaced by the pair X, T, after which the excision permits the replacement in them of pair X, T by the pair U, ∂U. Thus, $H_3(U, \partial U) = 0$, Tor $H_2(U, \partial U) \cong \mathbf{Z}_n$, and, by virtue of the Poincaré-Lefschetz duality and the formula of universal coefficients,

$$H_1(U) \cong H^3(U, \partial U; \mathbf{Z}) \cong \mathrm{Hom}(H_3(U, \partial U), \mathbf{Z}) \oplus \mathrm{Ext}(H_2(U, \partial U), \mathbf{Z}) \cong \mathbf{Z}_n.$$

Consider the embedding homomorphisms

$$\alpha : H_1(\partial U) \to H_1(T), \quad \beta : H_1(\partial U) \to H_1(U), \quad \gamma : H_1(C) \to H_1(\partial U).$$

It follows from the equation $H_1(X) = 0$ (by virtue of the exactness of the additive sequences of triads of X, T, and U) that the homomorphism $\alpha \oplus \beta$: $H_1(\partial U) \to H_1(T) \oplus H_1(U)$ is an epimorphism. In particular, for any $u \in H_1(U)$ there exists $u_1 \in H_1(\partial U)$ with $(\alpha \oplus \beta)(u_1) = (0, u)$, i.e., with $\alpha(u_1) = 0$, $\beta(u_1) = u$, and this means that $\beta(\mathrm{Ker}\,\alpha) = H_1(U)$. But $\mathrm{Ker}\,\alpha = \mathrm{Im}\,\gamma$ (this is obvious from the commutative diagram

$$\begin{array}{ccccc} H_2(D, C) & \xrightarrow{\partial} & H_1(C) & & \\ \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ H_2(T, \partial T) & \xrightarrow{\partial} & H_1(\partial T) & \xrightarrow{\alpha} & H_1(T), \end{array}$$

in which the lower row is exact, while the upper boundary homomorphism and the left vertical homomorphism, corresponding to an embedding are isomorphisms). Thus, $\mathrm{Im}(\beta \circ \gamma) = \beta(\mathrm{Im}\,\gamma) = H_1(U)$, while $\beta \circ \gamma$ is also the embedding homomorphism of interest to us $H_1(C) \to H_1(U)$.

2.3. Covering p: $V \to U$. We denote by ${}_iG$, where G is a group and i a natural number, the set of those $x \in G$, for which x^i is contained in the commutant [G, G]. It is clear that ${}_iG$ is a normal divisor containing [G, G]. If group $H_1(G) = G/[G, G]$ is isomorphic to $\mathbf{Z}_{i \cdot j}$ (with some natural j), then the group $G/{}_iG$ is isomorphic to $\mathbf{Z}_j$.

For us, the role of G will be played by the group $\pi_1(U) = \Pi$, and the roles of i and j by the numbers $\mu = n/m$ and m. Since $H_1(\Pi) = H_1(U) \cong \mathbf{Z}_n$, then $\Pi/{}_\mu\Pi \cong \mathbf{Z}_m$.

Let us construct the covering p: $V \to U$ with $p_*\pi_1(V) = {}_\mu\Pi$. This is a regular m-sheeted covering, and V is an oriented connected compact four-dimensional manifold with boundary $\partial V = p^{-1}(\partial U)$. It follows from the fact that the embedding homomorphism $H_1(C) \to H_1(U)$ is an epimorphism that the composition of the embedding homomorphism $\pi_1(C) \to \pi_1(U) = \Pi$ and the projection $\Pi \to \Pi/{}_\mu\Pi$ is also an epimorphism, and from this it follows that the preimage $p^{-1}(C)$ of neighborhood C is connected. (We have used here the evident general theorem: let spaces Γ, $\Gamma_0 \subset \Gamma$ and Δ be linearly connected, and let ψ: $\Delta \to \Gamma$ be a regular covering; if the composition of the embedding homomorphism $\pi_1(\Gamma_0) \to \pi_1(\Gamma)$ and the projection $\pi_1(\Gamma) \to \pi_1(\Gamma)/\psi_*\pi_1(\Delta)$ is an epimorphism, then the set $\psi^{-1}(\Gamma_0)$ is linearly connected.) Thanks to this connectedness of manifold ∂V we can be confident of the structure of the SO(2)-fibration over A with fiber S^{-1}. The projection of this fibration is the composition of contraction $\partial V \to \partial U$ of covering p (this contraction, as p itself, is an m-sheeted covering), and the projection $\partial U \to A$. The Euler number of this fibration equals a/m.

2.4. Manifold Y. Let W be the total manifold of the associated SO(2)-fibration with fiber D^2. We shall denote the null section of this fibration by B. The boundary ∂W can be identified with ∂V and, using this identification, we splice V and W in oriented connected closed four-dimensional manifold Y (its orientation is defined by the orientation of V).

The covering p: $V \to U$ is naturally continued to the mapping P: $Y \to X$ which may be called a ramified covering with ramifications along A. Over $X \setminus A$ this is a true covering, with manifold $B = P^{-1}(A)$ being diffeomorphically mapped by P onto A.

The automorphisms of this ramified covering (i.e., the diffeomorphisms θ: $Y \to Y$ such that $P \circ \theta = P$) constitute a group, which we shall denote by π. This group is canonically isomorphic to group $\Pi/_{\mu}\Pi$ of the automorphisms of covering p: $V \to U$ and is therefore isomorphic to Z_m. The set of its fixed points is precisely B. On the complement $Y \setminus B$ it acts freely and, on the fibers of fibration $\partial V \to A$, as a rotation group. It is clear that in π there is exactly one automorphism which rotates these fibers by angle 360°/m (we consider the fibers to be oriented in accordance with the orientations of A and ∂V), and that this automorphism is a generator of group π. It will be denoted by t.

§3. The Betti Number of Manifold Y

3.1. The Group $H_1(V)$. Since the covering p: $V \to U$ is defined by the condition $p_*\pi_1(V) = {}_{\mu}\Pi$ and the homomorphism p_*: $\pi_1(V) \to \Pi$ is a monomorphism, group $\pi_1(V)$ is isomorphic to ${}_{\mu}\Pi$. Consequently,

$$H_1(V) \cong H_1({}_{\mu}\Pi). \tag{1}$$

We are principally interested in the case when this group is finite, i.e., when $b_1(V) = 0$. For example, this is the case when group Π itself is finite. A less obvious condition, necessary for the sequel, will be specified in section 3.5. Here we shall only note that, whatever the group G, subgroup ${}_iG$ and, with it, subgroup $[{}_iG, {}_iG]$, increases when the number i is replaced by its multiple. Therefore, if i_1 is divisible by i then, from the finiteness of groups $G/{}_iG$ and $H_1({}_iG)$, follows the finiteness of group $H_1({}_{i_1}G)$. In particular, if the group of (1) is finite then it remains finite upon replacement of the number m by its divisors.

3.2. Formulas for $b_1(Y)$ and $b_2(Y)$. The following equations are valid

$$b_1(Y) = b_1(V), \tag{2}$$

$$b_2(Y) = mb_2(X) + 2(m-1)g + 2b_1(V). \tag{3}$$

In particular, if group $H_1({}_{\mu}\Pi)$ is finite then

$$b_2(Y) = mb_2(X) + 2(m-1)g.$$

Proof. The homology sequence of the pair W, ∂W (with rational coefficients) shows that the embedding homomorphism $H_1(\partial W; Q) \to H_1(W; Q)$ is an isomorphism. This fact allows us to derive Eq. (2) from the additive sequence of triads Y, V, and W (with rational coefficients).

We can compute the Euler characteristic of manifold Y by the obvious formula

$$\chi(Y) = m\chi(X) - (m-1)\chi(A). \tag{4}$$

Since $b_3(X) = b_1(X) = 0$ and $b_3(Y) = b_1(Y) = b_1(V)$, then $\chi(X) = 2 + b_2(X)$ and $\chi(Y) = 2 - 2b_1(V) + b_2(Y)$. Substituting these values of $\chi(X)$ and $\chi(Y)$, and the value $\chi(A) = 2 - 2g$ into formula (4), we obtain (3).

3.3. LEMMA. Let τ be an automorphism of infinite, finitely-generated, Abelian group F, with $\tau^j = 1$. If j is a power of prime q, then the order of factor group F/Im $(1-\tau)$ is either infinite or is divisible by q.

Proof. Factorization into periodic subgroups shows that it suffices to consider the case when Tor F = 0. In this case, the order of factor group F/Im $(1-\tau)$, if it is finite, equals the value of the characteristic polynomial of automorphism τ at point 1. This characteristic polynomial has the form

$$(\lambda - 1)^{p_0} \prod_{r=1}^{k} \left(\sum_{s=0}^{q-1} \lambda^{sq^{r-1}} \right)^{p_r}, \tag{5}$$

where $\rho_0, \rho_1, \ldots, \rho_r$ are non-negative integers with positive sum. Since the finiteness of factor group $F/\mathrm{Im}\,(1-\tau)$ is equivalent to the equation $\rho_0 = 0$, it remains to remark that when $\rho_0 = 0$ the value of polynomial (5) at point $\lambda = 1$ equals $q^{\rho_1 + \cdots + \rho_k}$.

3.4. Sufficient Condition for Finiteness of Group $H_1({}_iG)$. Let i be a natural number, j a power of a prime, and G a group with $H_1(G) \cong Z_{i \cdot j}$. If group $H_1({}_iG)$ is finitely-generated, then it is finite.*

Proof. It suffices to consider the case when the numbers i and j are relatively prime: in the general case they can be replaced by using the remark in section 3.1, by the numbers $i/(i, j)$, $j \cdot (i, j)$, where (i, j) is the greatest common divisor of the numbers i and j. Group G acts in G as a group of inner automorphisms, and this defines the action of group $G/{}_iG$ in $H_1({}_iG)$. Let τ be a generator of group $G/{}_iG$. Consider the sequence

$$0 \to H_1({}_iG)/\mathrm{Im}\,(1-\tau) \to H_1(G) \to G/{}_iG \to 0,$$

in which the second homomorphism on the left is defined by the embedding ${}_iG \to G$, while the third is a projection. A simple check shows that this sequence is exact (it is a special case of a well-known sequence of covering theory; see, for example, [2], chapter 16, §9). Consequently, group $H_1({}_iG)/\mathrm{Im}\,(1-\tau)$ is isomorphic to Z_1 which, by virtue of Lemma 3.3 is impossible if group $H_1({}_iG)$ is infinite.

3.5. COROLLARY. If m is a power of a prime then group $H_1({}_\mu\Pi)$ is finite.

§4. Algebraic Interlude: τ-Signature

4.1. Definition of τ-Signature. Let L be a finite-dimensional real vector space and τ a linear transformation of space L such that $\tau^m = 1$ (m is an arbitrary natural number). Furthermore, let f be a (real) quadratic form on L, invariant under τ. We set $l = [m/2]$, $\zeta = e^{2\pi i/m}$ and we define the polynomials $E_0, \ldots, E_l$ by the formula

$$E_r(\lambda) = \begin{cases} \lambda - 1, & \text{if} \quad r = 0, \\ (\lambda - \zeta^r)(\lambda - \zeta^{-r}), & \text{if} \quad 1 \leqslant r < m/2, \\ \lambda + 1, & \text{if} \quad r = m/2. \end{cases}$$

Obviously,

$$\lambda^m - 1 = \prod_{r=0}^{l} E_r(\lambda),$$

and this expansion corresponds to an expansion of space L in the direct sum of subspaces $L(r) = \mathrm{Ker}\, E_r(\tau)$. We denote, for $r = 0, \ldots, l$, by $\alpha(r)$, the signature of form f on $L(r)$, and we set

$$\sigma(f, \tau) = \sum_{r=0}^{l} \alpha(r) \cos \frac{2r\pi}{m}.$$

This real algebraic number is called the τ-signature of form f.

It is clear that $\sigma(f, \tau)$ does not depend on the arbitrary choice of m (i.e., it is not changed when m is multiplied by a natural number). Since $\mathrm{Ker}\, E_r(\tau^{-1}) = \mathrm{Ker}\, E_r(\tau)$, then $\sigma(f, \tau^{-1}) = \sigma(f, \tau)$. If $\tau = 1$, then $\sigma(f, \tau)$ coincides with the ordinary signature $\sigma(f)$ of form f. In every case, $|\sigma(f, \tau)| \leq \dim L$.

We now define the numbers $\alpha'(0), \ldots, \alpha'(m-1)$ by the formula

$$\alpha'(r) = \begin{cases} \alpha(r), & \text{if} \quad r = 0, \frac{m}{2}, \\ \frac{\alpha(r)}{2}, & \text{if} \quad 1 \leqslant r < \frac{m}{2}, \\ \frac{\alpha(m-r)}{2}, & \text{if} \quad \frac{m}{2} < r \leqslant m-1. \end{cases}$$

Obviously, $\sigma(f, \tau) = \sum_{r=0}^{m-1} \alpha'(r)\,\zeta^r$.

*There is a special case of this theorem in Massey [4]. I am indebted to S. A. Yuzbinskii for a useful discussion of this group of questions.

4.2. Orthogonality of Subspaces L(r). Subspaces L(0), . . ., L(l) are pairwise orthogonal with respect to form f.

It is necessary to show that if $u \in L(r)$, $v \in L(r_1)$ and $r < r_1$, then $(u, v) = 0$. There are four cases to distinguish: $1 \le r < r_1 < m/2$; $0 = r < r_1 < m/2$; $1 \le r < r_1 = m/2$ (m even); $r = 0$, $r_1 = m/2$ (m even).

In the first case we perform scalar multiplication of the equation $\tau^2 u - (\zeta^r + \zeta^{-r})\tau u + u = 0$ by τv, and replace $(\tau^2 u, \tau v)$ by $(\tau u, v)$ and $(\tau u, \tau v)$ by (u, v). This leads to the relationship $(\tau u, v) + (u, \tau v) = (\zeta^r + \zeta^{-r})(u, v)$, and, exactly the same, $(\tau u, v) + (u, \tau v) = (\zeta^{r_1} + \zeta^{-r_1})(u, v)$. Consequently, $(\zeta^r + \zeta^{-r})(u, v) = (\zeta^{r_1} + \zeta^{-r_1})(u, v)$, and since $\zeta^r + \zeta^{-r} \neq \zeta^{-r_1}$, then $(u, v) = 0$.

In the three other cases, the proof is analogous, but simpler.

4.3. Certain Relationships. It follows from the pairwise orthogonality of spaces L(r) that

$$\sigma(f, \tau^s) = \sum_{r=0}^{m-1} \alpha'(r)\zeta^{rs} = \sum_{r=0}^{l} \alpha(r)\cos\frac{2rs\pi}{m} \quad (s = 0, \ldots, m-1), \tag{6}$$

and, in particular,

$$\sigma(f) = \sum_{r=0}^{m-1} \alpha'(r) = \sum_{r=0}^{l} \alpha(r). \tag{7}$$

Since

$$\sum_{r=0}^{m-1} \zeta^{rs}\zeta^{-rs_1} = \begin{cases} m, & \text{if } s = s_1, \\ 0, & \text{if } s \neq s_1, \end{cases}$$

the formulas in (6) are readily inverted. We obtain

$$\alpha'(r) = \frac{1}{m}\sum_{s=0}^{m-1} \sigma(f, \tau^s)\zeta^{-rs} \quad (r = 0, \ldots, m-1) \tag{8}$$

and, in particular,

$$\alpha(0) = \frac{1}{m}\sum_{s=0}^{m-1} \sigma(f, \tau^s). \tag{9}$$

Formulas (6) and (8) show that the set of integers $\{\alpha'(r)\}_{r=0}^{m-1}$ and the set of algebraic numbers $\{\sigma(f, \tau^s)\}_{s=0}^{m-1}$ define one another uniquely. Since $\sigma(f, \tau^s) = \sigma(f, \tau^{m-s})$, the sets $\{\alpha(r)\}_{r=0}^{l}$, $\{\sigma(f, \tau^s)\}_{s=0}^{l}$ also uniquely define one another. We note that these sets constitute an essential part of the invariants of the triple L, τ, f: in order to obtain the complete set of its invariants we need to adjoin to them the number $\beta(r) = \dim L(r)$ $(r = 0, \ldots, l)$ or (sometimes more convenient) the numbers $\beta'(0), \ldots, \beta'(m-1)$, defined by the formula

$$\beta'(r) = \begin{cases} \beta(r), & \text{if } r = 0, \frac{m}{2}, \\ \frac{\beta(r)}{2}, & \text{if } 1 \leqslant r < \frac{m}{2}, \\ \frac{\beta(m-r)}{2}, & \text{if } \frac{m}{2} < r \leqslant m-1. \end{cases}$$

We can present formula (9) in the form

$$\sigma(f) = m\alpha(0) - \sum_{s=1}^{m-1} \sigma(f, \tau^s)$$

and substitute this value of $\sigma(f)$ in formula (8). We get

$$\alpha'(r) = \alpha(0) - \frac{1}{m}\sum_{s=1}^{m-1} \sigma(f, \tau^s)(1 - \zeta^{-rs}) = \alpha(0) - \frac{2}{m}\sum_{s=1}^{m-1} \sigma(f, \tau^s)\sin^2\frac{rs\pi}{m}. \tag{10}$$

Formula (10) provides the numbers, needed in the sequel, $\alpha'(1), \ldots, \alpha'(m-1)$ in terms of $\alpha(0)$ and $\sigma(f, \tau), \ldots, \sigma(f, \tau^{m-1})$.

§5. Action of Group π in $H^2(Y;\ \mathbf{R})$

5.1. Notation. In this section we study the automorphism $t^*: H^2(Y;\ \mathbf{R}) \to H^2(Y, \mathbf{R})$, defined by the diffeomorphism $t: Y \to Y$ (see, 2.4). This study is comprehended in the algebraic scheme of the previous section if one takes $H^2(Y;\ \mathbf{R})$ for L, automorphism t^* for τ, and the quadratic form defined on $H^2(Y;\ \mathbf{R})$ by cohomological multiplication as f. This triple L, τ, f corresponds to the subspace $L(r) = \operatorname{Ker} E_r(\tau)$ and the numbers $\beta(r)$ ($\beta'(r)$), and $\alpha(r)$ ($\alpha'(r)$); we will now compute them.

In addition to the ramified covering $P: Y \to X$, constructed on the basis of triple X, A, m, we need to consider the ramified covering $P_h: Y_h \to X$, constructed in the same way, but on the basis of the triple X, A, h, in which X and A are the same, but h is a divisor of m. The objects pertaining to covering $P_h: Y_h \to X$, will be denoted the same as the corresponding objects appertaining to covering $P: X \to X$, but with subscript h.

5.2. Computation of $\beta(0)$ and $\alpha(0)$. Consider the homomorphism $P^*: H^2(X;\ \mathbf{R}) \to H^2(Y;\ \mathbf{R})$, defined by the mapping $P: Y \to X$. Since P is a projection of manifold Y on the space of orbits of group $\pi \cong Z_m$, then P^* isomorphically maps $H^2(X;\ \mathbf{R})$ on the set of elements of group $H^2(Y;\ \mathbf{R})$ which are fixed with respect to π (see, for example, [7], p. 38), i.e., on $L(0) = \operatorname{Ker} E_0(\tau)$. Moreover, $(P^*x)^2[Y] = mx^2[X]$ for $x \in H^2(X;\ \mathbf{R})$, so that the signature $\sigma(f/\operatorname{Im} P^*)$ equals the signature $\sigma(X)$ of manifold X. Thus,

$$\beta(0) = b_2(X), \tag{11}$$

$$\alpha(0) = \sigma(X). \tag{12}$$

5.3. LEMMA. If $m = \omega h$ then

$$\beta_h(\rho) = \beta(\omega\rho) \quad \left(\rho = 0, \ldots, \left[\frac{h}{2}\right]\right),\ \beta'_h(\rho) = \beta'(\omega\rho) \quad (\rho = 0, \ldots, h-1). \tag{13}$$

Proof. We set $\nu = n/h$. Since $\nu = \omega\mu$ then

$$p_* \pi_1(V) = {}_\mu\Pi \subset {}_\nu\Pi = (p_h)_* \pi_1(V_h)$$

(cf., 3.1). This inclusion defines the ω-sheeted covering $\tilde{p}: V \to V_h$ which is naturally continued to the mapping $\tilde{P}: Y \to Y_h$. This latter can be considered as the projection of manifold Y on the space of orbits of a cyclic group of order ω, generated by the diffeomorphism t^h, so that homomorphism $\tilde{P}^*: H^2(Y_h;\ \mathbf{R}) \to H^2(Y;\ \mathbf{R})$ isomorphically maps $H^2(Y_h;\ \mathbf{R})$ on the set of vectors of space $H^2(Y;\ \mathbf{R})$ which are fixed with respect to τ^h (see, 5.2), i.e., on the orthogonal sum of subspaces $L(r)$ with $r \equiv 0 \bmod \omega$. To these facts we add the commutative diagram

$$\begin{array}{ccc} H^2(Y;\mathbf{R}) & \xleftarrow{\tau^\omega} & H^2(Y;\mathbf{R}) \\ \uparrow{\scriptstyle \tilde{P}^*} & & \uparrow{\scriptstyle \tilde{P}^*} \\ H^2(Y_h;\mathbf{R}) & \xleftarrow{\tau_h} & H^2(Y_h;\mathbf{R}), \end{array}$$

showing that $\tilde{P}^* L_h(\rho) \subset L(\omega\rho)$. Thus, $\tilde{P}^*$ isomorphically maps $L_h(\rho)$ on $L(\omega\rho)$, whence also follows (13).

5.4. Computation of $\beta'(r)$ with $r > 0$. If group $H_1({}_\mu\Pi)$ is finite then, when $r > 0$,

$$\beta'(r) = b_2(X) + 2g. \tag{14}$$

Proof. If the numbers $e^{2\pi i r/m}$ and $e^{2\pi i r_1/m}$ are conjugate over Q, then, obviously, $\beta'(r) = \beta'(r_1)$. Therefore it follows from the equation $\beta'(0) + \beta'(1) + \ldots + \beta'(m-1) = b_2(Y)$ that

$$\beta(0) + \sum_{d|m,\ d<m} \varphi\left(\frac{m}{d}\right) \beta'(d) = b_2(Y),$$

where φ is the Euler function (summation is over all divisors of m less than m). Exactly the same for any divisor $h > 1$ of m,

$$\beta_h(0) + \sum_{d|h,\ d<h} \varphi\left(\frac{h}{d}\right) \beta'_h(d) = b_2(Y_h). \tag{15}$$

According to section 3.1, from the finiteness of group $H_1({}_\mu\Pi)$ follows the finiteness of group $H_1({}_\nu\Pi)$ with $\nu = r/h$, and, therefore,

$$b_2(Y_h) = hb_2(X) + 2(h-1)g$$

(see, 3.2). Moreover, in accordance with sections 5.2 and 5.3, $\beta_h(0) = b_2(X)$, $\beta'_h(\rho) = \beta'\left(\frac{m}{h}\rho\right)$. Substituting these values of $b_2(Y_h)$, $\beta_h(0)$ and $\beta'_h(\rho)$ into (15), we see that the numbers $\beta'(d)$ satisfy the system of linear algebraic equations

$$\sum_{d|h,\, d<h} \varphi\left(\frac{h}{d}\right)\beta'\left(\frac{m}{h}d\right) = (h-1)(b_2(X)+2g) \qquad (h\,|\,m,\ h>1)$$

(in which both the number of unknowns and the number of equations are one less than the number of divisors of number m). As shown by the obvious induction on m, this system is uniquely solvable, and it is clear that it is satisfied by the values in (14).

5.5. The At'ya-Zinger Formula. A decisive contribution to the following calculations is made by the formula

$$\sigma(f, \tau^s) = \frac{a}{m\sin^2\frac{s\pi}{m}} \qquad (s = 1, \ldots, m-1), \tag{16}$$

which is an adaptation to our situation of a special case of the general signature formulas of At'ya and Zinger. These general formulas and their proofs are to be found in [1], §6. We require only formula (16).

5.6. Computation of $\alpha'(r)$ with $r > 0$. Knowing $\alpha(0)$ and $\sigma(f, \tau), \ldots, \sigma(f, \tau^{m-1})$, we can compute $\alpha'(r)$ with $r = 1, \ldots, m-1$ by formula (10). We find:

$$\alpha'(r) = \sigma(X) - \frac{2a}{m^2}\sum_{s=1}^{m-1}\frac{\sin^2\frac{rs\pi}{m}}{\sin^2\frac{s\pi}{m}}.$$

As will be shown in the next section,

$$\sum_{s=1}^{m-1}\frac{\sin^2\frac{rs\pi}{m}}{\sin^2\frac{s\pi}{m}} = r(m-r) \text{ when } m \geqslant 2 \text{ and } 0 \leqslant r \leqslant m. \tag{17}$$

Thus,

$$\alpha'(r) = \sigma(X) - \frac{2ar(m-r)}{m^2}. \tag{18}$$

5.7. Proof of Formula (17). We set $\psi_s(r) = \sin^2\frac{rs\pi}{m}\Big/\sin^2\frac{s\pi}{m}$ and denote the left side of Eq. (17), i.e., the sum $\psi_1(r) + \ldots + \psi_{m-1}(r)$, by $\psi(r)$, and the right side of (17), i.e., $r(m-r)$, by $\psi'(r)$. Simple calculations show that the second difference $\Delta^2\psi_s(r) = \psi_s(r+2) - 2\psi_s(r+1) + \psi_s(r)$ equals $2\cos(2(r+1)s\pi/m)$, so that, when $r = 0, \ldots, m-2$,

$$\Delta^2\psi(r) = \Delta^2\psi_1(r) + \ldots + \Delta^2\psi_{m-1}(r) = -2.$$

But we also have $\Delta^2\psi'(r) = -2$ (and for any r) and, since $\psi(0) = \psi'(0)$ and $\psi(1) = \psi'(1)$, then $\psi(r) = \psi'(r)$ for $r = 0, \ldots, m$.

5.8. Signature of Manifold Y. Signature $\sigma(Y)$ of manifold Y is no other than the signature $\sigma(f)$ of form f. We can compute it by formula (7), substituting the value of $\alpha'(r)$ from formula (18). After simple calculations we obtain

$$\sigma(Y) = m\sigma(X) - \frac{(m^2-1)a}{3m}. \tag{19}$$

Formula (19) was first obtained by the author independently of the At'ya-Zinger formula on the basis of the corresponding variant of cobordism theory. This theory permits the general computation of the signature $\sigma(Y)$ to be reduced to its computation for concrete complex-algebraic ramified coverings, which can be performed by traditional means. Fortunately, when m = 2, 3, formula (16) is easily derivable from formula (19). A simple general proof of formula (16) is not known to the author.

§6. Bounds on Genus g

6.1. Basic Inequality. The invariants of ramified covering P: Y → X considered in the previous sections can be divided into two groups: in the first belong the numbers $b_2(Y)$ and $\beta(r)$ ($\beta'(r)$) and, in the second, the numbers $\sigma(Y)$ and $\alpha(r)$ ($\alpha'(r)$). The invariants of the second group depend on class $\xi \in H_2(X)$ realized by surface A, but not on this surface itself. The invariants of the first group depend on the genus g of surface A. Obviously, $\beta(r) \geq |\alpha(r)|$, which provides a series of lower bounds for g. The best of these, at least in the case when group $H_1({}_{\mu}\Pi)$ is finite, is obtained from the inequality $\beta(l) \geq |\alpha(l)|$, i.e., from the inequality

$$b_2(X) + 2g \geqslant \left| \frac{2l(m-l)}{m^2} a - \sigma(X) \right|. \tag{20}$$

Considering the various possibilities for m, we arrive at the following formulations of sections 6.2-6.4.

6.2. The Case When ξ is Divisible by 2. If ξ is divisible by 2 then

$$g \geqslant \left| \frac{a}{4} - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2}. \tag{21}$$

This bound is obtained from inequality (20) for m = 2 (that group $H_1({}_{\mu}\Pi)$ is finite follows from 3.5).

6.3. The Case When ξ is Divisible by q^k. If ξ is divisible by the power q^k of odd prime q then

$$g \geqslant \left| \frac{q^{2k}-1}{4q^{2k}} a - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2}. \tag{22}$$

This bound is obtained from inequality (20) when $m = q^k$ (the finiteness of group $H_1({}_{\mu}\Pi)$ follows from section 3.5).

6.4. A Broader Formulation. If ξ is divisible by odd number h and group $H_1({}_{\nu}\Pi)$ with $\nu = n/h$, is finite, then

$$g \geqslant \left| \frac{h^2-1}{4h^2} a - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2}. \tag{23}$$

This bound is obtained from inequality (20) when m = h.

6.5. Signature Bound. The bounds just presented are apparently the best that can be extracted from the computations of the preceding major section. The following bound (24), while known to be worse than these if m > 3, still has the advantage that it is based, not on the At'ya-Zinger formula (16), but only on formula (19) for the signature $\sigma(Y)$. It is obtained from the evident inequality $b_2(Y) - \beta(0) \geq |\sigma(Y) - \alpha(0)|$. By substituting into this the values of $\beta(0)$, $\alpha(0)$ and $\sigma(Y)$ from formulas (11), (12), and (19), we find:

$$g \geqslant \left| \frac{m+1}{6m} a - \frac{\sigma(X)}{2} \right| - \frac{b_2(X)}{2}. \tag{24}$$

The lower m is, the better this bound. Since finiteness of group $H_1({}_{\mu}\Pi)$ is provided, if m is the least prime divisor of number n, Ineq. (24) is valid on the sole condition that ξ is divisible by m.

When m = 2, inequality (24) provides the same result as Theorem 6.2 and, when m = 3, the same as Theorem 6.3.

§7. The Non-Orientable Case

7.1. Basic Construction. In this section we assume given an oriented connected closed four-dimensional manifold X with $H_1(X) = 0$ and its non-orientable connected closed two-dimensional submanifold A realizing the zero element of the group $H_2(X; Z_2)$. The genus of surface A will be denoted by g, its normal Euler number by a.

Let U be the closed complement of a cylindrical neighborhood of surface A in X. This is an oriented connected compact four-dimensional manifold whose boundary has the natural structure of an O(2)-fibration over A with fiber S^1 and Euler number a (the one-dimensional Stiefel-Whitney class of this fibration coincides with the one-dimensional Stiefel-Whitney class $w_1(A)$ of surface A). Repeating, with the appropriate changes, the computations of section 2.2., we see that group $H_1(U)$ is isomorphic to Z_2. Consequently, the two-sheeted covering p: V → U with $p_*\pi_1(V) = [\pi_1(U), \pi_1(U)]$ is defined and, as in section 2.3, V is an oriented connected compact four-dimensional manifold with boundary $\partial V = p^{-1}(\partial U)$. Repeating the subsequent computations (with the obvious changes) of section 2.2, we see that the embedding homomorphism $H_1(C) \to H_1(U)$, where C is a fiber of the aforementioned fibration over the indicated points of surface A, is an epimorphism. Consequently, the preimage $p^{-1}(C)$ of neighborhood C is connected (see, 2.3) and manifold ∂V has the natural structure of an O(2)-fibration over A with fiber S^1. The Euler number of this fibration equals $a/2$ (and the one-dimensional Stiefel-Whitney class coincides with $w_1(Z)$).

Let W be the total manifold of the associated O(2)-fibration with fiber D^2. As in section 2.4, we splice V and W in oriented connected closed four-dimensional manifold Y, and continue the covering $p: V \to U$ to the ramified covering $P: Y \to X$ with ramifications along A. The automorphisms of this ramified covering constitute a group isomorphic to Z_2. We denote its generator by t.

7.2. Bound on Genus g. It follows from Theorem 3.4 that group $H_1(V)$ is finite, i.e., that $b_1(V) = 0$. Repeating the computations of section 3.2 we conclude from this that $b_1(Y) = 0$ and

$$b_2(Y) = 2b_2(X) + g. \tag{25}$$

As in section 5.2, we define the triple L, τ, and f, taking $H^2(Y; R)$ for L, the automorphism $t^*: H^2(Y; R) \to H^2(Y; R)$ for τ, and the quadratic form defined on $H^2(Y; R)$ by cohomological multiplication for f. This triple corresponds to the numbers $\beta(0)$, $\alpha(0)$, $\beta(1)$, $\alpha(1)$, which can be computed by the formulas

$$\beta(0) = b_2(X), \tag{26}$$

$$\alpha(0) = \sigma(X), \tag{27}$$

$$\beta(1) = b_2(X) + g, \tag{28}$$

$$\alpha(1) = \sigma(X) - \frac{a}{2}. \tag{29}$$

Formulas (26) and (27) are proven just as in section 5.2, after which formula (28) is derived from the relationship $\beta(0) + \beta(1) = b_2(Y)$ and formulas (25) and (26), while formula (29) is derived from formula (27) and the At'ya-Zinger formula (16) which, as applied to this situation, means that $\alpha(0) - \alpha(1) = a/2$. Finally,

$$\sigma(Y) = \alpha(0) + \alpha(1) = 2\sigma(X) - \frac{a}{2}.$$

The best bound on genus g which can be derived from these computations is contained in the inequality $\beta(1) \geq |\alpha(1)|$. We obtain:

$$g \geq \left|\frac{a}{2} - \sigma(X)\right| - b_2(X).$$

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