PROOF OF GUDKOV'S HYPOTHESIS

V. A. Rokhlin

1. Introduction. We know that a real algebraic curve of degree m can possess not more than 1/2 (m − 1) (m − 2) + 1 components in RP^2. Curves with this maximum number of components exist for any m, do not possess singularities (neither real nor imaginary), and are termed M curves. In the case of an even m, all the components of a nonsingular real algebraic curve of degree m are distributed bilaterally in RP^2 and are divisible by even (contained within an even number of the other components) and by odd (other components); we denote by p the number of even components, and by n the number of the odd components.

Recently, Arnol'd [1] proved that $p - n \equiv k^2 \mod 4$ for any M curve of even degree 2k. In his paper, Arnol'd communicates that the stronger congruence $p - n \equiv k^2 \mod 8$ has been formulated by Gudkov as a hypothesis. It follows from Gudkov's [2] results that this congruence holds for k = 3 (for k = 1, 2, the congruence is obvious).

In the present note, we show that $p - n \equiv k^2 \mod 8$ for any M curve of any even degree 2k.

The proof rests upon the basic ideas set forth in Arnol'd's paper; however, use is made of more specialized facts in the topology of four-dimensional manifolds. Since all of these facts have not been published, they will be outlined in a separate section.

I would like to use this opportunity to thank V. I. Arnol'd for informing me about his work and for sharing his enthusiasm with me.

2. Brief Outline of Arnol'd's Paper. Let A be an arbitrary M curve of even degree 2k, and CA a complex curve defined by the same equation in CP^2 (so that $A = CA \cap RP^2$). We know that CA is an oriented closed surface of kind (k − 1) (2k − 1), that A divides CA in two, and that the two halves go into each other under complex conjugation. The projective plane RP^2 is also halved by curve A, A serving as the interface for the oriented portion RP^2_+ and the nonoriented portion RP^2_-. We will denote union of one of the halves of surface CA with RP^2_+ by $\mathfrak{A}$. The principal technical observation made by Arnol'd, which in the following will be referred to as Arnol'd's Lemma, is that this closed piecewise smooth surface realizes a zero of the group $H_2(CP^2; Z_2) = Z_2$ for an even k, and a value other than zero for an odd k.

Let Y be the branched double covering of manifold CP^2, branching along CA and possessing a natural complex structure, let τ: Y → Y be a nonidentical automorphism of the covering, and σ: Y → Y be a complex conjugation. The manifolds Fix τ and Fix σ of the fixed points of involutions τ and σ coincide with the inverse images of the manifolds CA and RP^2_+ for the covering Y → CP^2. From here, with the aid of Arnol'd's Lemma, it is not difficult to deduce that they realize the same element of group $H_2(Y; Z_2)$. Both of them are oriented, while the intersection indices of the elements t and s of the integral group $H_2(Y)$, which are realized by them, are defined by the formulas $t^2 = 2k^2$, $ts = 0$, $s^2 = 2(n - p)$. The class $w_2(y)$ is also easy to evaluate: it is zero for an odd k and is dual, in Poincaré's sense, to the class realized by the manifold Fix τ for an even k. Consequently, $(s + t)^2 \equiv 0 \mod 8$ for an odd k, and $s^2 \equiv t^2 \mod 8$ for an even k, which means that $p - n \equiv k^2 \mod 4$.

3. Topological Regression. Let X be an oriented closed smooth four-dimensional manifold with $H_1(X; Z_2) = 0$ and F its orientable (closed, two-dimensional) submanifold that realizes an element of group $H_2(X; Z_2)$, which is dual to $w_2(X)$. The compact two-dimensional submanifold P of manifold X is termed the membrane on F, if the intersection $P \cap F$ consists of the edge ∂P along which P does not touch on F,

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and of a finite number of points at which the intersection is transverse. If we construct a vector field on ∂P, which touches on F and does not touch on P, the attempt at continuing the vector field without touching on P leads to a two-dimensional obstruction with an intergral index that is independent of the field selected on ∂P, and which is termed the index of membrane P. It appears that the modulo-2 reduced sum of this index and of the number of points of which the intersection (Int P) $\cap$ F is composed does not change when any other membrane with the same edge ∂P is substituted for membrane P and, moreover, is defined by the element of group $H_1(F; Z_2)$, which is realized by the edge ∂P. The function ψ: $H_1(F; Z_2) \rightarrow Z_2$ thus generated satisfies the relation $\psi(\alpha+\beta)=\psi(\alpha)+\psi(\beta)+\alpha\beta$, and defines in the conventional manner the invariant Arf, which in the following will be denoted by Arf (F). (If $\alpha_1, \ldots, \alpha_q$; $\beta_1, \ldots, \beta_q$ is the canonical basis in $H_1(F; \mathbf{Z}_2)$, then $\operatorname{Arf}(F)=\varphi(\alpha_1)\varphi(\beta_1)+\ldots+\varphi(\alpha_q)\varphi(\beta_q)$; the canonicality of the basis $\alpha_1, \ldots, \alpha_q$; $\beta_1, \ldots, \beta_q$ means that $\alpha_1\beta_1=\ldots=\alpha_q\beta_q=1$, and that all the remaining intersection indices are zero.)

A fundamental property of the invariant Arf (F) is that it is related by the congruence $x(F)-\sigma(X) \equiv 8 \operatorname{Arf}(F) \bmod 16$ to the normal Euler number x(F) of surface F and to the signature $\sigma(X)$ of manifold X. In particular, Arf (F) is an invariant of an integral class of intrinsic homologies realized by surface F, and is an invariant of a class of intrinsic homologies of the pair X, F. It can be defined even without the assumption that $H_1(X; Z_2) = 0$, however, this general definition is less effective and is not required in the following. In the simplest case where F is a sphere, the preceding congruence reduces to the known theorem of Kervaire and Milnor [3].

4. Proof of The Congruence $p - n \equiv k^2 \bmod 8$. It is essential for the following that manifold Y (see Sec. 2) be simply connected. We assume first that k is odd. Then the manifold $F = \operatorname{Fix}(\tau \circ \sigma)$ of the fixed points of involution $\tau \circ \sigma$: $Y \rightarrow Y$ is orientable, and from what was asserted in Sec. 2, it is not difficult to deduce that it realizes a zero of group $H_2(Y; Z_2)$. Since, $w_2(Y) = 0$ at the same time, the invariant Arf (F) is defined and satisfies the congruence $x(F)-\sigma(Y)\equiv 8 \operatorname{Arf}(F) \bmod 16$, in which the Euler number x(F) (as can be easily computed) is $2(p - n - 1)$ and $\sigma(Y)$, as follows from the formula $\sigma(Y) = 2 - 2k^2$ (see, for example, [4]), or from equality $w_2(Y) = 0$, is divisible by 16. Thus, the congruence being proved is equivalent to the equality Arf (F) = 0. This equality is a corollary of the almost obvious fact that group $H_1(F; Z_2)$ possesses a canonical basis $\alpha_1, \ldots, \alpha_p$; $\beta_1, \ldots, \beta_p$ with $\psi(\beta_1)=\ldots=\psi(\beta_p)=0$; the classes $\alpha_1, \ldots, \alpha_p$ are realized by the inverse images of the even components of curve A in the case of the covering $Y \rightarrow CP^2$, while the classes $\beta_1, \ldots, \beta_p$ are realized by the inverse images of the p sections of manifold RP^2_-, which transform its components into topological circles and which join the even and odd components of curve A (that $\psi(\beta_i) = 0$ follows from symmetry considerations).

We now assume that k is even. We realize class $t_1 = t/k \in H_2(Y)$ (image of the generating group $H_2(CP^2)$ for the inverse Hopf homomorphism that corresponds to the covering $Y \rightarrow CP^2$) of submanifold F_1 that does not intersect Fix σ, and set $F = F_1 \cup \operatorname{Fix} \sigma$. Submanifold F realizes an element of group $H_2(Y; Z_2)$, that is dual to $w_2(Y)$, so that invariant Arf (F) is again defined and satisfies the congruence $x(F) - \sigma(Y) \equiv 8 \operatorname{Arf}(F) \bmod 16$. Simple calculations show that $x(F) = 2(n - p + 1)$, and that Arf (F) = 0. By substituting these values together with the value $\sigma(Y) = 2 - 2k^2$ into the preceding congruence, one can see that $p - n \equiv k^2 \bmod 8$. It is noteworthy that this reasoning holds also for an odd k; the proof of the equality Arf (F) = 0, however, is more complicated.

5. Concluding Remark: Arnol'd's Theorem and the Generalized Whitney Theorem. It is striking that the passage from manifold CP^2 to its branched double covering, proposed by Arnol'd (and used in Sec. 4), is quite unnecessary for proving Arnol'd's congruence $p - n \equiv k^2 \bmod 4$, and that this congruence may be derived directly from Arnol'd's lemma by making use of an elementary theorem in four-dimensional topology. This theorem generalizes the well-known Whitney theorem about the normal Euler numbers of closed surfaces in $\mathbf{R}^4$ (see [5]), and states that if F is a (closed two-dimensional) submanifold of an oriented smooth connected four-dimensional manifold X, which realizes an element of group $H_2(X; Z_2)$ that is dual to $w_2(X)$, then the Euler characteristic $\chi(F)$, the normal Euler number x(F), and the signature $\sigma(X)$ are related by the congruence $2\chi(F)+x(F)\equiv\sigma(X) \bmod 4$ (the submanifold F is not assumed to be orientable, and the Whitney theorem results for $X = S^4$). In order to prove Arnol'd's congruence for an odd k, it is sufficient to apply this theorem to a surface $\mathfrak{A}$ smoothed by natural means (CA and RP^2 do not touch on each other!), taking into consideration that for this surface $\chi=[2-(p+n)]+(p-n)=2-2n$, and $x = 2k^2 + (n - p)$, and that $\sigma(CP^2 = 1$. For an even k, the proof is carried out in the same fashion, except that $\mathfrak{A}$ is replaced by the union of the half-surface CA with RP^2_-.

LITERATURE CITED

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