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The Signature of Quasi-Nilpotent Fiber Bundles

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A quasi-nilpotent fibration, as defined in [HR], is a Serre fibration $F \to E \to B$ of connected spaces with the property that $\pi_1 B$ acts nilpotently on $H^*(F; A)$[1]. [This notion depends, of course, on the abelian group of coefficients used; in our theorem below, A will be the real numbers $\mathbf{R}$.]

An important class of quasi-nilpotent fibrations is that of the simple fibrations, namely those in which the action of the fundamental group is simple (=trivial). A number of results concerning fibrations, originally proved under the assumption that the fibration be simple, have been recently extended to the more general context. An example of such a result, useful in certain situations, is the Zeeman comparison theorem for the spectral sequence of a fibration [HR]. Another, much simpler, result is the multiplicativity of the Euler characteristic

$$\chi(E) = \chi(F) \cdot \chi(B),$$

where A is taken to be an arbitrary field, for example. The classical spectral sequence proof of this formula for simple fibrations (see, for example, [D]) generalizes readily to quasi-nilpotent fibrations. The multiplicative formula for χ does break down in general (loc. cit.) although it continues to hold if B has the homotopy type of a finite complex.

In this note, we study the analogous question of the multiplicativity of the signature. We henceforth assume that the spaces F, E and B are (connected) closed, oriented, differential manifolds and that the fibration $F \to E \to B$ is a coherently oriented differential fiber bundle. Moreover, we assume $A = \mathbf{R}$.

Theorem. *If $F \to E \to B$ is a quasi-nilpotent fiber bundle of the kind just described, then*

$$\sigma(E) = \sigma(F) \cdot \sigma(B),$$

σ denoting the signature.

[1] In [HR], the definition was actually formulated in terms of (integral) homology, rather than cohomology

[Examples of (non-simple) quasi-nilpotent fiber bundles are easy to construct. For instance, let $U(n, \Lambda)$, $\Lambda = \mathbf{Z}$ or $\mathbf{R}$, be the unipotent, upper triangular group with entries in Λ; thus $(u_{ij}) \in U(n, \Lambda)$ provided $u_{ii} = 1$, $u_{ij} = 0$ if $i > j$, $u_{ij} \in \Lambda$. Then

$$U(n, \mathbf{Z}) \to U(n, \mathbf{R}) \to U(n, \mathbf{R})/U(n, \mathbf{Z})$$

is clearly the universal principal $U(n, \mathbf{Z})$-bundle. Now $U(n, \mathbf{Z})$ acts nilpotently in a natural way on the torus $T^n([\mathrm{R}])$ and we may pass to the associated fiber bundle

$$T^n \to E \to U(n, \mathbf{R})/U(n, \mathbf{Z}).$$

The resulting π_1-action, being induced by the $U(n, \mathbf{Z})$-action on T^n, is plainly nilpotent.]

For simple fiber bundles, the above theorem was proved by Chern, Hirzebruch and Serre [CHS]. Their argument was based on the real cohomology Serre spectral sequence of the bundle, the hypothesis on the π_1-action being exploited to express the E_2-term in the usual way. About ten years later, Kodaira [K] and Atiyah [A] constructed examples to show that, in rough analogy with the Euler characteristic, the signature fails to be multiplicative in a general fiber bundle. Further, Atiyah derived an explicit formula, discussed below, expressing $\sigma(E)$ in terms of $\sigma(F)$, $\sigma(B)$ and a certain invariant of the action of $\pi_1 B$ on the middle-dimensional real cohomology of F. We use Atiyah's formula, together with some standard facts in the theory of real algebraic Lie groups,[2] to obtain a proof of our theorem.

Before proceeding, we point out a slight extension of the theorem.

Corollary. *Take the data given in the theorem except that we now assume only that $\pi_1 B$ admits a subgroup of finite index which acts nilpotently on $H^*(F; \mathbf{R})$. Then the same conclusion*

$$\sigma(E) = \sigma(F) \cdot \sigma(B)$$

continues to hold.

Proof. Let $\Gamma = \pi_1 B$ and suppose $\hat{\Gamma}$ is a subgroup of Γ of finite index r acting nilpotently on $H^*(F; \mathbf{R})$. Let $\hat{B} \xrightarrow{\varphi} B$ be the r-sheeted covering map corresponding to $\hat{\Gamma}$ and let $\hat{E} \to \hat{B}$ be the bundle induced via φ from $E \to B$ so that we have a commutative diagram (compare [S])

$$\begin{array}{ccccc} F & \longrightarrow & \hat{E} & \longrightarrow & \hat{B} \\ \| & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi} \\ F & \longrightarrow & E & \longrightarrow & B. \end{array}$$

Clearly, $\hat{E} \xrightarrow{\Phi} E$ is an r-sheeted covering map. By the well-known formula ([CHS]),

$$\sigma(\hat{B}) = r \cdot \sigma(B), \qquad \sigma(\hat{E}) = r \cdot \sigma(E);$$

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2 See, for example, [B]. By a real algebraic group, we shall understand its group of real-valued points

and by the theorem applied to the quasi-nilpotent fiber bundle $F \to \hat{E} \to \hat{B}$,

$$\sigma(\hat{E}) = \sigma(F) \cdot \sigma(\hat{B}).$$

The corollary follows.

Proof of Theorem. We begin by briefly reviewing Atiyah's formula. We may clearly assume that F is even-dimensional and set $k = \frac{1}{2}\dim F$. The cup product pairing induces on the real vector space $H^k(F;\mathbf{R})$ a nondegenerate bilinear form, symmetric if k is even, skew-symmetric if k is odd; further, the action of $\Gamma = \pi_1 B$ on $H^k(F;\mathbf{R})$ preserves this form. Hence we have a homomorphism

$$\pi\colon \Gamma \to G$$

where G is the subgroup of $Gl(n,\mathbf{R})$ ($n = \dim H^k(F;\mathbf{R})$) consisting of all matrices preserving this form. Atiyah then shows how to associate with π a certain virtual vector bundle $\sigma(\pi)$ over B, real if k is even, complex if k is odd. Viewed as a homotopy class of maps into $\mathbf{Z} \times BO$ (k odd) or $\mathbf{Z} \times BU$ (k even), the $\mathbf{Z}$-component of $\sigma(\pi)$ is always $\sigma(F)$. The BO-component, resp. BU-component, is determined by the composite

$$B \xrightarrow{c} B\Gamma \xrightarrow{B\pi} BG,$$

where $B \xrightarrow{c} B\Gamma$ is the canonical map; in particular, it will be (stably) trivial if $B\pi$ is nullhomotopic. The formula in question reads ([A])

$$\sigma(E) = \{ch(\sigma(\pi)) \cdot L(B)\} \cdot [B],$$

where $L(B)$ is the Hirzebruch polynomial of B.
In view of this formula and the preceding discussion, it suffices to show that $B\pi$ is nullhomotopic in the situation at hand.[3] The crux of the proof is contained in the following lemma.

Lemma. If Γ acts nilpotently on $H^k(F,\mathbf{R})$, there exists a contractible subgroup C of G such that $\pi(\Gamma) \subset C$.

Proof. From the nilpotency assumption, it follows that each element γ of $\pi(\Gamma)$ is unipotent, that is, satisfies the equation

$$(\gamma - 1)^n = 0.$$

Let $C = \overline{\pi(\Gamma)}$ be the algebraic hull of $\pi(\Gamma)$, that is, the real algebraic closure of $\pi(\Gamma)$ in the real algebraic Lie group G. It is clear that C is a real algebraic Lie group, each of whose elements satisfies the above equation and so is unipotent. We show that C is contractible. (Compare [Au].)

By [B; 4.9], C is conjugate to a subgroup (which we continue to denote by C) of $U(n,\mathbf{R})$. Now the Lie algebra $u(n,\mathbf{R})$ of $U(n,\mathbf{R})$ maps diffeomorphically onto $U(n,\mathbf{R})$ by the exponential map; furthermore, $\exp\colon u(n,\mathbf{R}) \to U(n,\mathbf{R})$ is algebraic (because the exponential series $1 + u + \frac{u^2}{2!} + \cdots$ terminates when u is a nilpotent matrix). Suppose $\gamma \in C$ and write

$$\gamma = \exp u, \qquad u \in u(n,\mathbf{R}).$$

[3] In fact, a slightly weaker assertion suffices. Namely suppose that $\tilde{\Gamma}$ is a subgroup of finite index in Γ and suppose $\tilde{\pi}\colon \tilde{\Gamma} \to G$ is the restriction of π. Then knowing only that $B\tilde{\pi}$ is nullhomotopic would imply, using the notation in the proof of the corollary, that $\sigma(\hat{E}) = \sigma(F) \cdot \sigma(\hat{B})$, hence that $\sigma(E) = \sigma(F) \cdot \sigma(B)$

Then, since C is algebraic and exp is algebraic, the group

$$\Lambda = \{\lambda \in \mathbf{R} / \exp \lambda u \in C\}$$

is also algebraic. As Λ contains infinitely many elements (namely the integers), it must be the whole of $\mathbf{R}$. Thus $u \in L(C)$, the Lie algebra of C, and so exp maps $L(C)$ diffeomorphically onto C. Hence C is contractible.

Returning now to the proof of the theorem, note that the factorization

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & G \\ & \searrow \quad \nearrow & \\ & C & \end{array}$$

of the lemma gives rise to a corresponding factorization

$$\begin{array}{ccc} B\Gamma & \xrightarrow{B\pi} & BG \\ & \searrow \quad \nearrow & \\ & BC. & \end{array}$$

Since BC is a contractible space, $B\pi$ is nullhomotopic, thereby completing the proof of the theorem.

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