leads to

$$
\begin{aligned}
A_{nn} &= 1 \\
A_{n,n-1} &= \binom{n}{2} \\
A_{n,n-2} &= 3\binom{n+1}{4} \\
A_{n,n-3} &= 15\binom{n+2}{6} + \binom{n}{4} \\
A_{n,n-6} &= 105\binom{n+3}{8} + 15\binom{n+1}{6} + 5\binom{n}{5}
\end{aligned}
$$

but there seems to be no simple general form.

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THE NUMBER OF PARTITIONS OF A SET

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Let S be a finite nonempty set with n elements. A *partition* of S is a family of disjoint subsets of S called "blocks" whose union is S. The number B_n of distinct partitions of S has been the object of several arithmetical and combinatorial investigations. The earliest occurrence in print of these numbers has never been traced; as expected, the numbers have been attributed to Euler, but an explicit reference to Euler has not been given, and Bell [7] doubts that it can be found in Euler's work. The properties of these numbers are periodically being rediscovered, as recently as 1962 (cf. [13]). Following Eric Temple Bell, we shall call them the *exponential numbers.* Bell [4, 5, 6, 7], used the notation ϵ_n; on the other hand, Jacques Touchard [29 and 30] used a_n to celebrate the birth of his daughter Ann; Becker and Riordan [3] used B_n in honor of Bell. We shall follow their choice.

A great many problems of enumeration can be interpreted as counting the number of partitions of a finite set; for example, the number of rhyme schemes for n verses, the number of ways of distributing n distinct things into n boxes (empty boxes permitted), the number of equivalence relations among n elements (cf. [8]), the number of decompositions of an integer into coprime factors when n distinct primes are concerned (cf. Bell [7]), the number of permutations of n elements with ordered cycles (cf. Riordan [27], page 77 ff.), the number of Borel fields over a set of n elements (cf. Binet and Szekeres [8]), etc., etc. Exponential numbers occur frequently in probability, and their theory is closely related to that of the Poisson-Charlier polynomials (see below).

Several explicit expressions for the exponential numbers are known, and can be found in [2, 3, 5, 6, 10, 13, 14, 15, 16, 22, 25, 29, 30, 32]. One of the simplest ways of describing the sequence B_n is by its exponential generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x-1}, \tag{1}$$

where we have set $B_0=1$ by convention. All known explicit formulas, however, except the one we shall derive, rely to a greater or lesser degree upon direct enumerations leading to nonimmediate recursions for the sequence B_n.

In this note we shall give a new formula for the exponential numbers (formula (4) below) which differs from the previous ones in that it relies least upon direct counting arguments, and which hinges instead upon some elementary considerations of a "functorial" nature. It is the author's conviction that formula (4), which we derive below, is the natural description of the exponential numbers. The basic idea is a general one, and can be applied to a variety of other combinatorial investigations. We shall see that it easily leads to quick derivations of the properties of the B_n.

Consider an auxiliary finite set U having u elements, $u>0$. We shall examine the structure of the set U^S of *functions* with domain S, a set with n elements, and range a subset of U. The basic fact is that there are u^n distinct such functions, as is evidenced by the most elementary of counting arguments. We shall now examine this set of functions in greater detail.

To every function $f\colon S\to U$ there is naturally associated a partition π of the set S, called the *kernel* of f, defined as follows. Two elements a and b of S are to belong to the same block of π, if and only if $f(a)=f(b)$.

How many distinct functions are there with a given kernel π? This question is easily answered. Indeed, let $N(\pi)$ denote the number of distinct blocks of the partition π. A function having kernel π must take distinct values on distinct blocks of π. Thus, such a function takes altogether $N(\pi)$ distinct values, and the number of distinct such functions equals the number of *one-to-one* functions from a set of $N(\pi)$ elements to the set U. Again, it is well known that such a number is $u(u-1)\cdots(u-N(\pi)+1)=(u)_{N(\pi)}$, and this expression is called the *factorial power* of the number u, with exponent $N(\pi)$.

Now, every function has a unique kernel. Therefore we have the following identity, valid for all integers $u>0$:

$$\sum_{\pi} (u)_{N(\pi)} = u^n, \tag{2}$$

where the sum on the left ranges over all partitions of π the set S.

We now come to the main idea. Let V be the vector space over the reals consisting of all polynomials in the single variable u. Any sequence of polynomials of degrees $0, 1, 2, \cdots$, is a basis for this vector space, in particular, the sequence $(u)_0=1$, $(u)_1$, $(u)_2$, $(u)_3$, $\cdots$. Since a linear functional L on V is uniquely determined by assigning the values it takes on an arbitrary basis, there exists a unique linear functional L on V such that

$$L(1) = 1, \qquad L((u)_k) = 1, \qquad\qquad k = 1, 2, 3, \cdots.$$

Applying L to both sides of (2) we obtain

$$\sum_{\pi} L((u)_{N(\pi)}) = L(u^n); \tag{3}$$

but, by the definition of L, the left side simplifies to a sum of as many ones as there are partitions of the set S. In other words, (3) simplifies to

$$B_n = L(u^n). \tag{4}$$

This formula is the explicit expression for the exponential numbers which we wanted to establish. Let us see now how it can be used to derive the other properties of the exponential numbers.

We begin by deriving the recursion formula for the numbers B_n,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{5}$$

Now, since $u(u-1)_n=(u)_{n+1}$, we have $L(u(u-1)_n)=1=L((u)_n)$. Since the polynomials 1, $(u)_n$ for $n=2, 3, \cdots$ form a basis for the vector space V, it follows from the linearity of L, that

$$L(up(u-1)) = L(p(u)) \tag{6}$$

for every polynomial p. In particular, for $p(u)=(u+1)^n$ we obtain

$$L(u^{n+1}) = L((u+1)^n),$$

but this is precisely formula (5), as we wanted to show.

Note that identity (6) for all polynomials p together with the initial condition $L(1)=1$ completely characterizes the linear functional L, as defined by (4), since the argument by which we have established (6) is reversible. We shall now use this fact in establishing the generating function (1) for the exponential numbers. To this end, let $g_n/n!$ be the nth coefficient in the Taylor series expansion of e^{e^x-1}:

$$\sum_{n=0}^{\infty} \frac{g_n}{n!} x^n = e^{e^x-1}.$$

There exists a unique linear functional M on V such that $M(u^n)=g_n$, and it will suffice to prove that $L=M$, to conclude that $g_n=B_n$. Now,

$$e^{e^x-1} = M(e^{xu}),$$

where $M(e^{xu})$ is *defined* as

$$\sum_{n=0}^{\infty} \frac{M(u^n)}{n!} x^n.$$

Differentiating, we get

$$e^x e^{e^x-1} = M\left(\frac{d}{dx} e^{xu}\right) = M(ue^{xu}), \tag{7}$$

whence $M(e^{x(u+1)})=M(ue^{xu})$. Expanding the functions $e^{x(u+1)}$ and e^{xu} into Taylor series in the variable x and comparing terms, we obtain $M((u+1)^n)=M(u^{n+1})$.

But, since the polynomials u^n form a basis for V, this implies at once property (6). Hence $M=L$.

Note that differentiating under M, as we have done in (7), does not require any continuity properties of the functional M: it is "purely formal."

There is another, more amusing derivation of the generating function directly from (4), which goes as follows:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{L(u^n)}{n!} x^n = L(e^{ux}).$$

Now, set $e^x=1+v$, and expand $(1+v)^u$ by the binomial theorem:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = L((1+v)^u) = L\left(\sum_{n=0}^{\infty} \frac{(u)_n}{n!} v^n\right) = \sum_{n=0}^{\infty} \frac{L((u)_n)}{n!} v^n$$
$$= e^v = e^{e^x-1}, \quad \text{q.e.d.}$$

In this derivation, it may at first seem puzzling (as suggested by R. D. Schafer) that infinite sums have been commuted with L, without discussing any continuity properties of L. The puzzle is solved as soon as it is noticed that all appearances of the symbol L can be completely eliminated, and the whole derivation amounts to the proof of an infinite sequence of identities relating the coefficients of two Taylor series. The use of L is just a speedy way of establishing these identities.

Next, we shall establish the remarkable formula of Dobinski [14]:

$$B_{n+1} = \frac{1}{e}\left(1^n + \frac{2^n}{1!} + \frac{3^n}{2!} + \frac{4^n}{3!} + \cdots\right). \tag{8}$$

We begin by noticing that the exponential series $e = \sum_{k=0}^{\infty} 1/k!$ can be trivially rewritten as $e = \sum_{k=0}^{\infty} (k)_n/k!$, where n is any nonnegative integer. In view of the definition of the linear functional L, this gives

$$L((u)_n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{(k)_n}{k!}.$$

Using again the fact that the polynomials $(u)_n$ form a basis for the vector space V, and that the functional L is linear, we infer at once that

$$L(p(u)) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{p(k)}{k!} \tag{9}$$

for any polynomial p. Dobinski's formula now follows by setting $p(u) = u^n$.

Dobinski's formula is particularly suited to the computation of B_n for large n, by an application of the Euler-Maclaurin summation formula (cf. [16] and [25]).

Identity (9) establishes an important property of the linear functional L, namely, that it is *positive definite* on the half-line $[0, \infty)$. We can therefore define a sequence of *orthogonal polynomials* relative to L, and the properties of classical systems of orthogonal polynomials (cf. Szegö [28]) will apply to this set. Such a set of polynomials, we shall now prove, is

$$h_n(u) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (u)_{n-k}, \tag{10}$$

where we use Touchard's notation h_n from [30].

We first note that (6) can be rewritten in more enlightening form by using operator notation. Let $E: p(u) \to p(u+1)$ be the shift operator, let $D: p(u) \to p'(u)$ be the derivative, and let $V: f(x) \to f(1)$ be the *linear* functional consisting in evaluating a function at $x = 1$. Then (6) can be rewritten for any integer $k \geq 0$, by iteration, as

$$L(E^k p(u)) = L(p(u) V D^k x^u),$$

where we have used the fact that $(u)_k = VD^k x^u$. It follows from linearity if g is any polynomial, that

$$L(g(E)p(u)) = L(p(u) V g(D) x^u).$$

Now set $g(x) = (1-x)^j$, giving $g(E) = (-1)^j \Delta^j$, the iterated difference operator. For this choice of g, we have evidently $Vg(D)x^u = (-1)^j h_j(x)$. Set $p(u) = h_n(u)$, and obtain

$$(-1)^{j+n}L(\Delta^j h_n(u)) = L(h_n(u)h_j(u)).$$

If $j>n$, $\Delta^j h_n$ vanishes identically, proving the orthogonality of the polynomials, and if $j=n$, we get $L(h_n(u)^2)=n!$, which gives the normalizing factors.

The polynomials h_n are the special case of the Poisson-Charlier polynomials (cf. Szegö [28], p. 34) obtained by setting $a=1$, in Szegö's notation. As remarked by Touchard [30], they are particularly useful for computation of the exponential numbers by recursion. Formulas for the first seven polynomials are given by Touchard [30].

These examples suffice to give an idea of the use of formula (4), and to support the contention that this formula gives the natural definition of the exponential numbers. Formula (4) has been suggested by the Blissard calculus techniques so useful in enumerative analysis, (cf. Riordan [27], Ch. 2 Section 4); by the systematic use of linear functionals we can give a rigorous foundation to this calculus, as well as extend its uses in some directions. We hope to implement these contentions in a future publication.

This work was begun under contract NSF-GP-149, continued under contract with the Office of Naval Research, and concluded while the author was a Fellow of the Sloan Foundation. The author wishes to thank the referee for several improvements in the text, and for valuable historical references.

The following bibliography contains all publications known to the author which study the exponential numbers. He will greatly appreciate any suggestions of omitted works.

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ON THE SPANS OF DERIVATIVES OF POLYNOMIALS

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1. Introduction. By the span of a polynomial all of whose roots are real, we shall mean the difference between its largest and smallest roots. We are interested in the following problem: *If the span of a polynomial $f(x)$ of degree n with real roots is given, how is the span of its k-th derivative maximized?* It will be sufficient to consider polynomials $f(x)=x^n+\cdots$ all of whose roots lie in the interval $-1\leqq x\leqq 1$. Then all of the roots of $f^{(k)}(x)$ lie in the same interval, and we try to maximize the difference between its largest and smallest roots.

We shall suppose throughout that $k\leqq n-2$, so that $f^{(k)}(x)$ will have more than one root. On the other hand, we see that the problem is trivial if $n\geqq 2k+2$. For in this case, we may put $k+1$ roots of $f(x)$ at each end point $x=\pm 1$. Then $f^{(k)}(x)$ will have a root at each end point, and will therefore have span 2. If $n>2k+2$, some of the roots of $f(x)$ are arbitrary, whereas for $n=2k+2$ the span of $f^{(k)}(x)$ is maximized only for $f(x)=(x-1)^{k+1}(x+1)^{k+1}$.

Thus the nontrivial cases of the problem are those with $k+2\leqq n\leqq 2k+1$. We shall show in Section 2 that in these cases *the span of $f^{(k)}(x)$ can be maximized only when all of the roots of $f(x)$ are at the end points $x=\pm 1$.*