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ON THE CHARACTERISTIC POLYNOMIAL OF THE PRODUCT OF TWO MATRICES

WILLIAM E. ROTH

The following theorem will be proved:

THEOREM. *If A and B are $n \times n$ matrices with elements in the field F , whose characteristic polynomials are*

$$a_0(x^2) - xa_1(x^2) \text{ and } b_0(x^2) - xb_1(x^2)$$

respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$ and $b_1(x)$ are elements in the polynomial domain, $F[x]$, of F ; and if the rank of $A - B$ does not exceed unity; then the characteristic polynomial of AB is $(-1)^n [a_0(x)b_0(x) - xa_1(x)b_1(x)]$.

The proof will be facilitated by two lemmas.

LEMMA I. *If the rank of an $n \times n$ matrix, D , with elements in F does not exceed unity, then there exist $1 \times n$ matrices R and S with elements r_i and s_i , $i = 1, 2, \dots, n$, in F such that $D = R^T S$, where R^T is the transpose of R .*

LEMMA II. *If $M = (m_{ij})$ is an $n \times n$ matrix with elements in $F[x]$ and if D is an $n \times n$ matrix as defined by Lemma I, then the determinant of $M + D$ is given by*

$$|M + D| = |M| + SM^A R^T,$$

where M^A is the adjoint of the matrix M and $D = R^T S$.

The validity of Lemma I is obvious. The rank of D does not exceed unity, hence every two of its rows (columns) are proportional and $D = (r_i s_j) = R^T S$, where $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ and r_i and s_i , $i = 1, 2, \dots, n$, are in F .

To prove Lemma II, let $D = R^T S$, where R and S are matrices established by Lemma I. The determinant of $M + D$ may be expressed as the sum of 2^n determinants. Of these $|M|$ is one; n others are $|M_i|$, $i = 1, 2, \dots, n$, where the matrix M_i is obtained from M by replacing its i th row by Sr_i of D ; and the remaining $2^n - 1 - n$ are zeros for their matrices are obtained by replacing two or more of the

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rows of M by corresponding rows of D . Upon expanding $|M_i|$ in terms of its i th row, we find

$$|M_i| = \left(\sum_{j=1}^n s_j M_{ji} \right) r_i,$$

where M_{ji} , $i, j=1, 2, \dots, n$, is the cofactor of m_{ij} of M . Consequently

$$|M - D| = |M| + \sum_{i=1}^n |M_i| = |M| + SM^A R^T,$$

where M^A is the adjoint of M . The lemma is proved.

We now take up the main theorem. By hypotheses

$$(1) \quad |xI - A| = a_0(x^2) - xa_1(x^2), \quad |xI - B| = b_0(x^2) - xb_1(x^2).$$

Let $(xI - A)^A = A_0(x^2) - xA_1(x^2)$, where $A_0(x^2)$ and $A_1(x^2)$ are polynomials in A and x^2I and are therefore commutative with A . Then

$$(xI - A)[A_0(x^2) - xA_1(x^2)] = [a_0(x^2) - xa_1(x^2)]I,$$

and consequently

$$(2) \quad a_0I = -AA_0 - x^2A_1, \quad a_1I = -AA_1 - A_0.^1$$

Since the rank of $A - B$ does not exceed unity, we may, according to Lemma I, let $A - B = D = R^T S$ and as a result of Lemma II

$$\begin{aligned} |xI - B| &= |xI - A + D| \\ &= |xI - A| + S(xI - A)^A R^T \\ &= a_0 - xa_1 + S(A_0 - xA_1)R^T, \end{aligned}$$

and by equation (1) it follows that

$$(3) \quad b_0 = a_0 + SA_0R^T, \quad b_1 = a_1 + SA_1R^T.$$

According to (2) and (3) and because A , A_0 , and A_1 are commutative, we find that

$$\begin{aligned} a_0b_0 - x^2a_1b_1 &= a_0(a_0 + SA_0R^T) - x^2a_1(a_1 + SA_1R^T), \\ &= a_0^2 - x^2a_1^2 + S(a_0A_0 - x^2a_1A_1)R^T, \\ (4) \quad &= a_0^2 - x^2a_1^2 - S[(AA_0 + x^2A_1)A_0 \\ &\quad - x^2(AA_1 + A_0)A_1]R^T, \\ &= a_0^2 - x^2a_1^2 - S(A_0^2 - x^2A_1^2)AR^T. \end{aligned}$$

¹ Here and in the following discussion we suppress the argument x^2 of polynomials until the final step in the proof.

Moreover since $B = A - D$

$$x^2I - AB = x^2I - A^2 + AD = x^2I - A^2 + AR^TS.$$

The rank of $AD = (AR^T)S$ does not exceed that of D , consequently by Lemma II we conclude that

$$|x^2I - AB| = |x^2I - A^2| + S(x^2I - A^2)^A AR^T.$$

It can be shown that

$$\begin{aligned} |x^2I - A^2| &= (-1)^n (a_0^2 - x^2 a_1^2), \\ (x^2I - A^2)^A &= (-1)^{n-1} (A_0^2 - x^2 A_1^2). \end{aligned}$$

Hence we have

$$\begin{aligned} |x^2I - AB| &= (-1)^n [a_0^2 - x^2 a_1^2 - S(A_0^2 - x^2 A_1^2) AR^T] \\ &= (-1)^n (a_0 b_0 - x^2 a_1 b_1), \end{aligned}$$

according to equation (4). If in this equation we replace x^2 by x , the proof of the theorem is complete.

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