ON THE CHARACTERISTIC POLYNOMIAL OF THE PRODUCT OF TWO MATRICES

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The following theorem will be proved:

THEOREM. *If A and B are $n\times n$ matrices with elements in the field F, whose characteristic polynomials are*

$$a_0(x^2)-xa_1(x^2) \text{ and } b_0(x^2)-xb_1(x^2)$$

respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$ and $b_1(x)$ are elements in the polynomial domain, $F[x]$, of F; and if the rank of $A-B$ does not exceed unity; then the characteristic polynomial of AB is $(-1)^n[a_0(x)b_0(x)-xa_1(x)b_1(x)]$.

The proof will be facilitated by two lemmas.

LEMMA I. *If the rank of an $n\times n$ matrix, D, with elements in F does not exceed unity, then there exist $1\times n$ matrices R and S with elements r_i and s_i, $i=1, 2, \cdots, n$, in F such that $D=R^TS$, where R^T is the transpose of R.*

LEMMA II. *If $M=(m_{ij})$ is an $n\times n$ matrix with elements in $F[x]$ and if D is an $n\times n$ matrix as defined by Lemma I, then the determinant of $M+D$ is given by*

$$|M+D| = |M| + SM^AR^T,$$

where M^A is the adjoint of the matrix M and $D=R^TS$.

The validity of Lemma I is obvious. The rank of D does not exceed unity, hence every two of its rows (columns) are proportional and $D=(r_is_j)=R^TS$, where $R=(r_1, r_2, \cdots, r_n)$ and $S=(s_1, s_2, \cdots, s_n)$ and r_i and s_i, $i=1, 2, \cdots, n$, are in F.

To prove Lemma II, let $D=R^TS$, where R and S are matrices established by Lemma I. The determinant of $M+D$ may be expressed as the sum of 2^n determinants. Of these $|M|$ is one; n others are $|M_i|$, $i=1, 2, \cdots, n$, where the matrix M_i is obtained from M by replacing its ith row by Sr_i of D; and the remaining 2^n-1-n are zeros for their matrices are obtained by replacing two or more of the

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rows of M by corresponding rows of D. Upon expanding $|M_i|$ in terms of its ith row, we find

$$|M_i| = \left(\sum_{j=1}^{n} s_j M_{ji}\right) r_i,$$

where M_{ji}, $i, j=1, 2, \cdots, n$, is the cofactor of m_{ij} of M. Consequently

$$|M - D| = |M| + \sum_{i=1}^{n} |M_i| = |M| + SM^A R^T,$$

where M^A is the adjoint of M. The lemma is proved.

We now take up the main theorem. By hypotheses

$$(1) \qquad |xI - A| = a_0(x^2) - xa_1(x^2), \qquad |xI - B| = b_0(x^2) - xb_1(x^2).$$

Let $(xI-A)^A = A_0(x^2) - xA_1(x^2)$, where $A_0(x^2)$ and $A_1(x^2)$ are polynomials in A and x^2I and are therefore commutative with A. Then

$$(xI - A)[A_0(x^2) - xA_1(x^2)] = [a_0(x^2) - xa_1(x^2)]I,$$

and consequently

$$(2) \qquad a_0 I = -AA_0 - x^2 A_1, \qquad a_1 I = -AA_1 - A_0.$$[1]

Since the rank of $A-B$ does not exceed unity, we may, according to Lemma I, let $A-B=D=R^TS$ and as a result of Lemma II

$$\begin{aligned} |xI - B| &= |xI - A + D| \\ &= |xI - A| + S(xI - A)^A R^T \\ &= a_0 - xa_1 + S(A_0 - xA_1)R^T, \end{aligned}$$

and by equation (1) it follows that

$$(3) \qquad b_0 = a_0 + SA_0R^T, \qquad b_1 = a_1 + SA_1R^T.$$

According to (2) and (3) and because A, A_0, and A_1 are commutative, we find that

$$(4) \qquad \begin{aligned} a_0b_0 - x^2a_1b_1 &= a_0(a_0 + SA_0R^T) - x^2a_1(a_1 + SA_1R^T), \\ &= a_0^2 - x^2a_1^2 + S(a_0A_0 - x^2a_1A_1)R^T, \\ &= a_0^2 - x^2a_1^2 - S[(AA_0 + x^2A_1)A_0 \\ &\qquad - x^2(AA_1 + A_0)A_1]R^T, \\ &= a_0^2 - x^2a_1^2 - S(A_0^2 - x^2A_1^2)AR^T. \end{aligned}$$

[1] Here and in the following discussion we suppress the argument x^2 of polynomials until the final step in the proof.

Moreover since $B=A-D$

$$x^2I - AB = x^2I - A^2 + AD = x^2I - A^2 + AR^TS.$$

The rank of $AD=(AR^T)S$ does not exceed that of D, consequently by Lemma II we conclude that

$$| x^2I - AB | = | x^2I - A^2 | + S(x^2I - A^2)^A AR^T.$$

It can be shown that

$$| x^2I - A^2 | = (-1)^n(a_0^2 - x^2 a_1^2),$$

$$(x^2I - A^2)^A = (-1)^{n-1}(A_0^2 - x^2A_1^2).$$

Hence we have

$$| x^2I - AB | = (-1)^n[a_0^2 - x^2 a_1^2 - S(A_0^2 - x^2A_1^2)AR^T]$$

$$= (-1)^n(a_0b_0 - x^2 a_1b_1),$$

according to equation (4). If in this equation we replace x^2 by x, the proof of the theorem is complete.

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