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Author(s): William E. Roth
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ON THE CHARACTERISTIC POLYNOMIAL OF THE PRODUCT OF SEVERAL MATRICES

WILLIAM E. ROTH

We shall prove two theorems.

THEOREM I. *If A is an $n\times n$ matrix with elements in the field F, if R and S_i, $i=1, 2, \cdots, r$, are $1\times n$ matrices with elements in F, and $D_i=R^TS_i$, where R^T is the transpose of R, and if the characteristic polynomial of $A_i=A+D_i$ is*

$$|xI - A_i| = m_{i0} + m_{i1}x + m_{i2}x^2 + \cdots + m_{i,r-1}x^{r-1},$$

where $m_{i,j-1}$, $i, j=1, 2, \cdots, r$, are polynomials in x^r with coefficients in F, then the characteristic polynomial of the product $P=A_1A_2\cdots A_r$ is given by $(-1)^{(r-1)n}|\Delta(x)|$, where

$$(1) \qquad \Delta(x^r) = \begin{pmatrix} m_{10}, & m_{1,r-1}x^{r-1}, & m_{1,r-2}x^{r-2}, & \cdots, & m_{11}x \\ m_{21}x, & m_{20}, & m_{2,r-1}x^{r-1}, & \cdots, & m_{22}x^2 \\ m_{32}x^2, & m_{31}x, & m_{30}, & \cdots, & m_{33}x^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{r,r-1}x^{r-1}, & m_{r,r-2}x^{r-2}, & m_{r,r-3}x^{r-3}, & \cdots, & m_{r,0} \end{pmatrix}.$$

This proposition has been proved by the writer [1] for the case $r=2$. Recently Parker [2] generalized that result and Goddard [3] gave an alternate proof of it and extended his method to the product of three matrices. This latter result does not come under the theorem above. Schneider [5] proved the theorem for the case $A_iA_j=0$, $i<j$, $i, j=1, 2, \cdots, r$.

Capital letters and expressions in bold faced parentheses will indicate matrices with elements in the field F or in $F(\omega)$, the extension of F by the adjunction of ω a primitive rth root of unity to it, and in $F(x)$ the polynomial domain of $F(\omega)$. The direct product of B and C is $(b_{ij}C)=B\langle C\rangle$. The product indicated by Π will run from 1 to r.

If R is not zero a nonsingular matrix Q with elements in F exists such that $QR^T=(1, 0, \cdots, 0)^T$; as a result

$$QD_iQ^I = (1, 0, \cdots, 0)^TS_iQ^I = E_i,$$

where Q^I is the inverse of Q and E_i has nonzero elements in only the first row. Now let

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$$QA_kQ^I = M_k = Q(A + D_k)Q^I = M + E_k, \tag{2}$$

where Q is the matrix defined above and $QAQ^I = M = (m_{ij})$. Consequently $M_k = (m_{ij}^{(k)})$, where $m_{ij}^{(k)} = m_{ij} + e_{ij}^{(k)}$ and $m_{ij}^{(k)} = m_{ij}$ for $i > 1$. That is, the matrices M_k differ only in the elements of their first rows. As a result the elements of the first columns of the adjoints $[xI - M_k]^A$ and $[xI - M]^A$ of $xI - M_k$ and $xI - M$ respectively are identical for $k = 1, 2, \cdots, r$, since all these matrices agree in the elements of their last $n-1$ rows and for the same reason

$$N_k(x) = [xI - M_k][xI - M]^A,$$

$$= \begin{pmatrix} m_k(x) & * & * & \cdots & * \\ 0 & m(x) & 0 & \cdots & 0 \\ 0 & 0 & m(x) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & m(x) \end{pmatrix}, \tag{3}$$

$k = 1, 2, \cdots, r$, where asterisks indicate nonzero elements in $F(x)$ and $|xI - M| = m(x)$.

Let

$$W = (\omega_{ij}) = (\omega^{(i-1)(1-j)}), \qquad i, j = 1, 2, \cdots, r;$$

then

$$| W\langle I_k \rangle | = | W |^k, \tag{4}$$

where I_k is the identity matrix of order k. The determinantal equation holds because $W\langle I_k \rangle$ can be transformed by the interchange of rows and corresponding columns to the direct sum $W \dotplus W \dotplus \cdots \dotplus W$ of k summands.

We shall operate in $F(x)$ on the matrix

$$M(x) = (\delta_{ij}xI - \delta_{i+1,j}M_i) \qquad (\delta_{r+1,1} = 1)$$

$$= \begin{pmatrix} xI & -M_1 & 0 & \cdots & 0 \\ 0 & xI & -M_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -M_{r-1} \\ -M_r & 0 & 0 & \cdots & xI \end{pmatrix}.$$

If we multiply this matrix on the right by one whose first row is I, M_1x^{-1}, $M_1M_2x^{-2}, \cdots, M_1M_2 \cdots M_{r-1}x^{-r+1}$, whose second row is 0, I, $M_2x^{-1}, \cdots, M_2M_3 \cdots M_{r-1}x^{-r+2}$, and whose last row is

$0, 0, 0, \cdots, I$, we find that the determinant of the product is $|x^rI-M_rM_1M_2\cdots M_{r-1}|$ and is therefore equal to $|x^r-P|$. The proof of Theorem I will consist in showing that

$$(5)\qquad |M(x)| = (-1)^{(r-1)n}|\Delta(x^r)|.$$

We now proceed to establish this equation.

$$\begin{aligned} M(x)W\langle I\rangle &= (\delta_{ij}xI - \delta_{i+1,k}M_i)(\omega^{(k-1)(1-j)}I),\\ &= (\omega^{(i-1)(1-j)}xI - \omega^{i(1-j)}M_i),\\ &= (\omega^{1-j}\{\omega_{ij}[\omega^{j-1}xI - M_i]\}).\end{aligned}$$

The number ω^{1-j} is a common multiplier of the $n\times n$ matrices in the jth column of the $nr\times nr$ matrix in right member above. Consequently the determinant of this matrix has the factor $\pi\omega^{(1-k)n}=\omega^{-r(r-1)n/2}$ $=\omega^{r(r-1)n/2}=(-1)^{(r-1)n}$. The determinantal equation obtained from the matric equation above is as a result:

$$(6)\qquad |M(x)|\,|W|^n = (-1)^{(r-1)n}|\omega_{ij}[\omega^{j-1}xI - M_i]|.$$

According to (3) the product

$$(7)\qquad (\omega_{ik}[\omega^{k-1}xI - M_i])(\delta_{kj}[\omega^{j-1}xI - M]^A) = (\omega_{ij}N_i(\omega^{j-1}x)).$$

The $nr\times nr$ matrix of the right member of this equation can be transformed by the interchange of corresponding rows and columns to a similar one having the form

$$\begin{pmatrix} (\omega_{ij}m_i(\omega^{j-1}x)), & * & ,\cdots, & * \\ 0, & (\omega_{ij}m(\omega^{j-1}x)), & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0 & ,\cdots, & (\omega_{ij}m(\omega^{j-1}x)) \end{pmatrix},$$

where asterisks represent $r\times r$ matrices with elements in $F(x)$ and the zeros are $r\times r$ zero matrices. The determinant of this matrix is

$$(8)\qquad |W|^{n-1}[\textstyle\prod m(\omega^{j-1}x)]^{n-1}|(\omega_{ij}m_i(\omega^{j-1}x))|$$

for each of the matrices $(\omega_{ij}m(\omega^{j-1}x))$ has $m(\omega^{j-1}x)$ as a divisor of all elements in the jth column. The determinant of the direct sum $(\delta_{ij}[\omega^{j-1}xI-M]^A)$ in equation (7) is $[\prod m(\omega^{j-1}x)]^{n-1}$; consequently the determinantal equation which follows from (7) and (8) is

$$\begin{aligned} &|\omega_{ij}[\omega^{j-1}xI - M_i]|\,[\textstyle\prod m(\omega^{j-1}x)]^{n-1}\\ &\qquad = |W|^{n-1}[\textstyle\prod m(\omega^{j-1}x)]^{n-1}|(\omega_{ij}m_i(\omega^{j-1}x))|,\end{aligned}$$

or

$$(9)\qquad |\omega_{ij}[\omega^{j-1}xI - M_i]| = |W|^{n-1}|(\omega_{ij}m_i(\omega^{j-1}x))|,$$

where the determinant of the left member is that of an $nr\times nr$ matrix and those in the right members are of order r. From (6) and (9) we have $|M(x)|=(-1)^{(r-1)n}|(\omega_{ij}M_i(\omega^{j-1}x))|/|W|$. It remains to be shown that the right member here is $(-1)^{(r-1)n}|\Delta(x^r)|$; this is easily accomplished by multiplying $\Delta(x^r)$ in (1) on the right by W. Herewith equation (5) is established and the proof of Theorem I is completed.

COROLLARY. *Under the hypotheses of Theorem* I *and if B is an $n\times n$ matrix with elements in F and if $B_i=B+S_i^TR$ and*

$$|xI - B_i| = n_{i0} + n_{i1}x + n_{i2}x^2 + \cdots + n_{i,r-1}x^{r-1};$$

then the characteristic polynomial of $B_1B_2\cdots B_r$ is given by $(-1)^{(r-1)n}|\Delta'(x)|$ where

$$\Delta'(x^r) = \begin{pmatrix} n_{r,0}, & n_{r,r-1}x^{r-1}, & \cdots, & n_{r,1}x \\ n_{r-1,1}x, & n_{r-1,0}, & \cdots, & n_{r-1,2}x^2 \\ \cdot & \cdot & \cdot & \cdot \\ n_{1,r-1}x^{r-1}, & n_{1,r-2}x^{r-2}, & \cdots, & n_{1,0} \end{pmatrix}.$$

This case can be made to come under Theorem I for $B_i^T=B^T+R^TS_i$, where B_i^T now satisfies the conditions imposed upon A_i. Moreover $|xI-B_i|=|xI-B_i^T|$. Since $(B_1B_2\cdots B_r)^T=B_r^TB_{r-1}^T\cdots B_1^T$, it follows that in $\Delta(x^r)$ of (1) we must replace the elements $m_{i,j-1}x^{j-1}$ by $n_{r-i+1,j-1}x^{j-1}$ in forming the matrix $\Delta'(x^r)$ above. This proves the corollary.

THEOREM II. *If D_i, $i=1, 2, \cdots, r$, are $n\times n$ matrices with elements in F, each of which is nilpotent and commutative with the others and with A, which also has elements in F, then the characteristic polynomials of $A_i=A+D_i$, $i=1, 2, \cdots, r$, are given by*

$$|xI - A| = m_0 + m_1x + m_2x^2 + \cdots + m_{r-1}x^{r-1}, \tag{10}$$

where m_{i-1}, $i=1, 2, \cdots, r$, are polynomials in x^r with coefficients in F, and the characteristic polynomial of the product $P=A_1A_2\cdots A_r$ is $(-1)^{(r-1)n}|\bar{\Delta}(x)|$, where

$$\bar{\Delta}(x^r) = \begin{pmatrix} m_0, & m_{r-1}x^{r-1}, & m_{r-2}x^{r-2}, & \cdots, & m_1x \\ m_1x, & m_0, & m_{r-1}x^{r-1}, & \cdots, & m_2x^2 \\ m_2x^2, & m_1x, & m_0, & \cdots, & m_3x^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{r-1}x^{r-1}, & m_{r-2}x^{r-2}, & m_{r-3}x^{r-3}, & \cdots, & m_0 \end{pmatrix}.$$

According to a theorem by Frobenius [4], the determinant of the matrix $B+C$ is equal to that of B if B and C are commutative matrices and C is nilpotent. This establishes equation (10) as giving the characteristic polynomial of A_i, $i=1, 2, \cdots, r$. By Theorem I the determinant

$$|xI - A^r| = (-1)^{(r-1)n} |\bar{\Delta}(x)|.$$

We shall proceed by induction. Let $P_i = A_1 A_2 \cdots A_i$, then

$$|xI - P_1 A^{r-1}| = |xI - (A + D_1)A^{r-1}| = |xI - A^r - D_1 A^{r-1}|.$$

Now the matrix $D_1 A^{r-1}$ is nilpotent and commutative with $xI - A^r$ consequently the determinants above are equal to $|xI - A^r|$. We assume that

$$|xI - P_i A^{r-i}| = |xI - A^r|;$$

then

$$|xI - P_{i+1} A^{r-i-1}| = |xI - P_i A^{r-i} - P_i D_{i+1} A^{r-i-1}|.$$

Here $P_i D_{i+1} A^{r-i-1}$ is commutative with $xI - P_i A^{r-i}$ and is nilpotent because D_{i+1} is nilpotent and commutative with both P_i and A; hence by Frobenius' theorem

$$|xI - P_{i+1} A^{r-i-1}| = |xI - P_i A^{r-i}| = |xI - A^r|.$$

Consequently

$$|xI - P| = |xI - A^r| = (-1)^{(r-1)n} |\bar{\Delta}(x)|,$$

and the theorem is proved.

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Mississippi City, Miss.