Local Poincaré Duality

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Abstract of the Dissertation

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This work is part of a project which aims to describe algebraic structures on the chains and cochains of closed manifolds that characterize those manifolds up to homeomorphism. Once knows from the rational homotopy theory of Quillen and Sullivan, and from the more recent work of Mandell, that the homotopy type of a simply connected space is determined by algebraic structure on the cochains of the space. There is an informational gap, however, between the homotopy type of a manifold and its homeomorphism type, as there are nonhomeomorphic manifolds which have the same homotopy type. Moreover, the surgery theory of Browder, Novikov, Sullivan, and Wall tells us that not every space satisfying Poincaré duality has the homotopy type of a manifold.

We represent a Poincaré duality space as a chain complex with a fixed basis satisfying certain axioms. We use the combinatorial data of the basis to to define an algebraic notion of locality, which we use to describe manifold structures. Our main result is that in dimensions greater than 4, simply-connected closed topological manifold structures in the homotopy type of a suitable based chain complex are in one-to-one correspondence with choices of local inverse to the Poincaré duality map up to algebraic bordism. The proof relies on Ranicki's algebraic formulation of surgery theory.

We expect that the theory of based chain complexes and algebraic locality developed here can be extended to encode the E_∞ algebra structure on the cochains of a space.

To my first mathematics teacher

My Mother

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CHAPTER 1

Based Chain Complexes and Polyhedral Spaces

In this chapter we consider categories of chain complexes with a fixed basis. We will restrict our attention to chain complexes which are *regular*, which means that basis elements are homology cells. We will show that a regular chain complex determines a simplicial complex, and this determination is both functorial and unique up to homotopy. All of the definitions and proofs of the various properties of regular chain complexes are inspired by the analogous arguments which are familiar from combinatorial topology; see for example [**RS72**].

1.1. Based Chain Complexes

A based chain complex is a nonnegatively graded chain complex of free $\mathbb{Z}$-modules together with a fixed basis.

D 1.1 (Based Chain Complex). A *based chain complex* (C, B, ∂)

(1) an $\mathbb{N}$-graded finite set $B = \amalg_{k \in \mathbb{N}} B_k$
(2) a degree -1 differential $\partial_k : C_k \longrightarrow C_{k-1}$, where C_k is the free $\mathbb{Z}$-module generated by the set B_k.

A *based subcomplex* of a based chain complex (C, B, ∂) is a subcomplex generated by a subset of B.

Let us fix some notation. We will often suppress the basis and differential in our notation and simply use the symbol C to mean a free chain complex with fixed basis B and differential ∂. We call the generators $x \in B_k$ the *k-cells* of C, and we call the 0-cells the *vertices*. This integer k is called the *degree* or *dimension* of x and is denoted $|x|$. The *k-skeleton* of C is the based subcomplex of C generated by the subset $\amalg_{j \leq k} B_j$. We denote the k-skeleton $C^{(k)}$ and its basis $B^{(k)}$. Since B is finite, there exists some minimal n such that $B_k = \emptyset$ for $k > n$. We call this n the *dimension* of C.

To any based chain complex C we associated a poset $(B, \preceq)$.

D 1.2 (Associated Poset). Let C be a based chain complex. Let

$$\langle\, , \rangle : C \otimes C \longrightarrow R$$

be the pairing on C where the generators $x \in B$ are defined to be orthogonal. Now define a partial order $\preceq$ on B by setting $x \preceq y$ if and only if one of the following holds:

(1) $x = y$
(2) there is a sequence

$$\{y = z_0, z_1, \dots, z_k = x\}$$

such that for $0 \le i \le k$,

$$\langle \partial z_i, z_{i+1} \rangle \neq 0$$

If $x \preceq y$, we say that x is a *face* of y and y is a *coface* of x.

Equivalently, we could define $\preceq$ by saying that $x \preceq y$ if $x = y$ or if x appears with nonzero coefficient in the formula for ∂y, and then extend this relation to be transitive. We will use the notation $x \prec y$ and sometimes $x \precneqq y$ to indicate that $x \preceq y$ and $x \neq y$.

E 1.3. The simplicial chain complex of a simplicial complex is a based chain complex. More generally, the cellular chain complex of a cell complex is a based chain complex. In each case the associated poset is the poset of cells, and $\preceq$ is the usual face relation.

D 1.4. Given a subset $E \subset B$, the *closure* of E is the set

$$\overline{E} := \{x \in B \quad | \quad x \prec e \text{ for some } e \in E\}$$

A subset $E \subset B$ is *closed* if $\overline{E} = E$. Observe that $\overline{E}$ generates the minimal subcomplex of C which contains every element of E.

Similarly, given a chain c of C, let $C(\overline{c})$ denote the minimal based subcomplex of C containing each $x \in B$ such that $x \preceq y$ for some y with $\langle c, y \rangle \neq 0$.

If x is a cell of C, then $C(\overline{x})$ is the subcomplex of C with basis

$$\{y \in x \quad | \quad y \preceq x\}$$

We call this chain complex the closure of x. If x is a cell of C, ∂x is a chain, and there is a based chain complex $C\left(\overline{\partial x}\right)$ with basis

$$\{y \in B \quad | \quad y \preceq x, y \neq x\}$$

We call this chain complex the *boundary* of x, and denote it $C(\dot{x})$. If E and E' are subsets of B and $E \subset E'$, we use the notation $C\left(\overline{E'}, \overline{E}\right)$ to denote the chain complex $\frac{C(\overline{E'})}{C(\overline{E})}$. This is a based chain complex with basis $\overline{E'} \setminus \overline{E}$.

D 1.5. Let $\mathbb{Z}$ denote the based chain complex with a single generator degree 0, and let C be a based chain complex. The *augmentation map* of C is the linear map $\varepsilon : C \to \mathbb{Z}$ given on on generators by the formula:

$$\varepsilon(y) = \begin{cases} 1 & |y| = 0 \\ 0 & |y| > 0 \end{cases}$$

Given a cell $x \in B$, let

$$\varepsilon_x : C(\overline{x}) \longrightarrow \mathbb{Z}$$

denote the restriction of e to $C(\overline{x})$.

Because we wish based chain complexes to model geometric objects like cell complexes, we must introduce an additional axiom. We now restrict our attention to based chain complexes where the closure of each cell is a homology cell, and the closure of the boundary of each cell is a homology sphere.

D 1.6 (Regular chain complexes). Let C be a based chain complex. We say that C is *regular* if

(1) The augmentation map

$$\varepsilon : C \longrightarrow \mathbb{Z}$$

is a chain map

(2) for each $x \in B$, $\varepsilon_x : C(\overline{x}) \longrightarrow \mathbb{Z}$ induces an isomorphism on homology.

R 1.7. Regularity is a hereditary property of based chain complexes. That is to say, if C is regular, so any based subcomplex of C.

P 1.8. *Let C be a regular chain complex and let x be an n-cell of C. Then $C(\dot{x})$ is a regular chain complex which has the homology of an $(n-1)$-dimensional sphere, and the chain ∂x is a representative of the $(n-1)$-dimensional homology class.*

P . This claim follows from looking at the short exact sequence

$$0 \longrightarrow C(\dot{x}) \longrightarrow C(\overline{x}) \longrightarrow C(\overline{x}, \dot{x}) \longrightarrow 0$$

By hypothesis $C(\overline{x})$ has homology only in degree 0, and $C(\overline{x}, \dot{x})$ is a based chain complex with a single generator in degree n. Since $C(\dot{x})$ has generators only in degrees 0 through $n-1$, the claim follows. □

E 1.9. The simplicial chain complex of a simplicial complex is a regular chain complex. The cellular chain complex of a regular cell complex is a regular chain complex.The second example is the reason for the name.

E 1.10. Consider the chain complex of cellular chains on a CW-decomposition of the circle with one 0-cell and one 1-cell. This is a based chain complex with one generator e in degree 1, one generator v in degree 0, and zero differential. This chain complex is not regular, as $H_1(\overline{e}) \cong \mathbb{Z}$.

We now establish a basic fact about regular chain complexes.

P 1.11. *If (C, B, ∂) is a regular based chain complex and $e \in B_1$, then e has exactly two faces v_1 and v_0, such that $\partial e = v_1 - v_0$. Thus $\overline{e}$ is isomorphic to the simplicial chain complex of the standard interval, and the isomorphism is canonical up to sign.*

P . By assumption, $H_0(\overline{e}) \cong \mathbb{Z}$ and $H_1(\overline{e}) \cong 0$. Thus e must have at least one 0-face.

Suppose e has only one face v_0. Then $\partial e = kv_0$ for some $k \in \mathbb{Z}$. If $k = 0$, then $H_1(\overline{e}) \cong \mathbb{Z}$. If $k \neq 0$, then $H_0(\overline{e}) \cong \mathbb{Z}/k\mathbb{Z}$.

Suppose e has more than two faces $\{v_0, ..., v_n\}$. Then ∂e is some word $\alpha = \sum k^i v_i$ in these faces, and $H_0(\overline{e}) \cong \mathbb{Z}^n/\alpha$. If $n > 2$ this quotient cannot be isomorphic to $\mathbb{Z}$.

Thus e has exactly two faces v_1 and v_0 , and $\partial e = k_1 v_1 + k_0 v_0$. Since the augmentation map $\varepsilon_x : \overline{x} \longrightarrow \mathbb{Z}$ is a chain map:

$$\begin{aligned} 0 &= \partial\varepsilon(e) \\ &= \varepsilon(\partial e) \\ &= k_1\partial(v_1) + k_0\varepsilon(v_0) \\ &= k_1 + k_0 \end{aligned}$$

Thus $\partial e = kv_1 - kv_0$ for some $k \in \mathbb{Z}$. We compute that $H_0(\overline{e}) \cong \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ so $k = \pm 1$. □

1.2. Morphisms of Regular Chain Complexes

We now describe two categories of regular chain complexes.

D 1.12 (Subdivision). Let B be a regular chain complex and let $x \in B$ and a cell of C. An *elementary subdivision of* x is an augmented chain map $s_x : C \to C'$, where C' is a regular chain complex with a decomposition of its basis B' as follows:

$$B' = (B \setminus \{x\}) \amalg B''$$

Let $B''_{|x|}$ denote the cells of B'' which have the same dimension as x. This data must satisfy:

(1) For all $y \in B$ such that $y \neq x$, $s_x(y) = y$

(2) For some choice of signs $\epsilon : B'' \to \{-1, +1\}$,

$$s_x(x) = \sum_{z \in B''_{|x|}} \epsilon(z) z$$

(3) For each $w \in B''$, $w \in C\left(\overline{s_x(x)}\right)$

(4) s_x is an augmented chain map which is quasi-isomorphism.

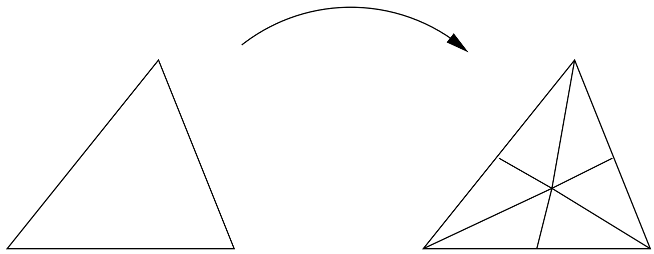
A augmented chain map of regular chain complexes is a *subdivision* if it is a composition of elementary subdivisions.

Note that a subdivision of regular chain complexes is by definition always a quasi-isomorphism.

E 1.13. Let Δ_n denote the standard-simplex, and let Δ'_n denote the barycentric subdivision of the standard n-simplex. These two simpicial complexes are triangulations of the same topological space, and the identity map between them is a linear map which sends each vertex of Δ_n to the corresponding vertex of Δ'_n. This map induces a chain map

$$b : \Delta_n \longrightarrow \Delta'_n$$

which is a subdivision of regular chain complexes. This example, shown in Figure 1.1, motivates several of the constructions which follow. Note that b is *not* a simplicial map, as it does not map simplices to simplices.



F 1.1. The map of geometric simplicial complexes which sends a simplex to its barycentric subdivision induces a subdivision map of regular based chain complexes

D 1.14 (Cellular Map). An augmented chain map $f : C \to C'$ is *cellular* if it is induced by a map of posets in the following sense: There is a map $\hat{f} : B \to B'$ such that for $x \in B$

$$|\hat{f}(x)| \leq |x| \tag{1}$$

Here $|\hat{f}(x)|$ denotes the degree of $\hat{f}(x)$ in B' and $|x|$ denotes the degree of x in B. such that

$$f(x) = \begin{cases} \epsilon_x \hat{f}(x) & |\hat{f}(x)| = |x| \\ 0 & |\hat{f}(x)| \neq |x| \end{cases} \tag{2}$$

for some choice of signs $\epsilon : B \to \{-1, +1\}$

E 1.15. A cellular map of cell complexes induces a cellular chain map on cellular chain complexes. This example is the reason for the name.

D 1.16. Let REG denote the category with objects the regular chain complexes and morphisms all compositions of subdivisions and cellular maps. Let CEL denote the subcategory with objects the regular chain complexes and morphisms the cellular maps.

R 1.17. Note that the identity map is a cellular map, so both REG and CEL are categories. More generally, any isomorphism of regular chain complexes which is induced by an isomorphism of bases is a cellular map.

1.3. The Barycentric Subdivision

In this section we describe one of two subdivisions that will be of interest to us, the barycentric subdivision. All of the definitions a proofs are inspired by the familiar geometric barycentric subdivision of simplicial complexes and cell complexes.

1.3.1. The Conical Subdivision of a Cell. We start by introducing an elementary subdivision, called the conical subdivision. This subdivision replaces a cell by the cone on the boundary of the cell. Given a graded set B, the *suspension* of B is the graded set ΣB, where $(\Sigma B)_k := B_{k+1}$. There is an evident bijection $\sigma : B \to \Sigma B$ which a element $x \in B$ of degree k to the corresponding element $x \in \Sigma B$ of degree $k + 1$.

D 1.18 (Conical Subdivision). Let C be regular chain complex and let x be a cell of C. If $|x| = 0$, then the *conical subdivision of* x is the isomorphism $s_x : C \to C$ which relabels x by c_x. (This triviality is necessary make notation consistent in the sequel.) If $|x| > 0$ then the *conical subdivision of* x is the map $s_x : C \to C'$, where s_x and C' are defined as follows. C' is the free chain complex with basis

$$B' := (B \setminus x) \amalg \Sigma\dot{x} \amalg \{c_x\}$$

Here $\Sigma\dot{x}$ is the suspension of the graded set

$$\dot{x} = \{\, y \in B \quad | \quad y \leq x, y \neq x \,\}$$

The set $\{c_x\}$ is a singleton in degree 0. [1] The map s_x is defined as follows:

$$s_x(y) := y \qquad\qquad y \neq x$$
$$s_x(y) := (-1)^{|x|-1}\sigma\partial x \quad y = x$$

The differential $d : C' \to C'$ is defined for $y \in B \setminus \{x\}$ and $\sigma y \in \Sigma\dot{x}$ as follows:

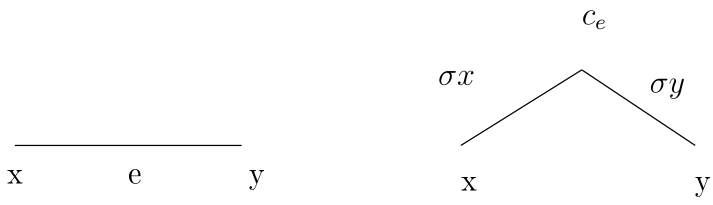
$$d(c_x) := 0$$
$$d(y) := s_x(\partial y)$$
$$d(\sigma y) := \begin{cases} (-1)^{|y|}y + \sigma\partial y & |x| > 0 \\ y - c_x & |y| = 0 \end{cases}$$

The notation $d(y) = s_x(\partial y)$ means that we replace each instance of x in the expression ∂x with $s_x(x)$.

P 1.19. *If C is a regular chain complex and $x \in C$, the conical subdivision $s_x : C \to C'$ is an elementary subdivision of x.*

[1] The letter 'c' stands for 'cone'.

The proposition follows from a series of lemmas. All of these lemmas are intuitively clear, because we are simply carving out a homology cell and replacing it with the cone on the boundary homology sphere. A low dimensional example is shown in Figure 1.2.



F 1.2. The conical subdivision of the edge e of an interval.

L 1.20. *If C is regular and C' is defined as in Definition 1.18, C' is a based chain complex.*

P . We check that $d^2 = 0$. It suffices to check that $d^2(B') = 0$.

For $y \in B$ such that x is not a codimension 1 or codimension 2 face of y,

$$d^2(y) = \partial^2 y = 0$$

For $y \in B$ such that x is a codimension 1 face of y,

$$\partial y = \alpha + \epsilon x$$

for some chain α with $\langle \alpha, x \rangle = 0$ and some coefficient ϵ. Thus

$$\begin{aligned} d^2(y) &= d\,(\alpha + \epsilon s_x(x)) \\ &= d\left(\alpha + \epsilon(-1)^{|x|+1}\sigma\partial x\right) \\ &= \partial\alpha + \epsilon(-1)^{|x|+1}\left((-1)^{|x|-1}\partial x + \sigma\partial^2 x\right) \\ &= \partial(\alpha + \epsilon x) \\ &= \partial^2 y \\ &= 0 \end{aligned}$$

For $y \in B$ such that x is a codimension 2 face of y, $\partial^2 y = \alpha + \epsilon x$ for some chain α with $\langle \alpha, x \rangle = 0$ and some coefficient ϵ. Moreover, $\alpha = 0$ and $\epsilon = 0$ because $\partial^2 = 0$. Thus

$$d^2(y) = \alpha + \epsilon s_{(}x) = 0$$

For $\sigma y \in \Sigma\dot{x}$ with $|y| > 1$

$$\begin{aligned} d^2\sigma y &= d((-1)^{|y|}y + \sigma(\partial y)) \\ &= (-1)^{|y|}\partial y + (-1)^{|\partial y|}\partial y + \sigma(\partial^2 y) \\ &= 0 \end{aligned}$$

For $\sigma y \in \Sigma\dot{x}$ with $|y| = 1$, we know from Proposition 1.11 that $\partial y = v_1 - v_0$ for some $v_1, v_0 \in B_0$. Thus we compute:

$$\begin{aligned} d^2\sigma y &= d((-1)^{|y|}y + \sigma(\partial y)) \\ &= -\partial y + d\sigma(v_1 - v_0) \\ &= -(v_1 - v_0) + (v_1 - c_x) - (v_0 - c_x) \\ &= 0 \end{aligned}$$

Finally $d(c_x) := 0$. □

L 1.21. *If C is regular and C' is as defined as in Definition 1.18, $s_x : C \to C'$ is an augmented chain map.*

P . The map s_x takes 0-cells to 0-cells and thus commutes with augmentation. To check that s_x is a chain map, we compute:

$$\begin{aligned} ds_x(x) &= d\left((-1)^{|x|-1}\sigma\partial x\right) \\ &= (-1)^{|x|-1}\left((-1)^{|\partial x|}\partial x + \sigma\partial\partial x\right) \\ &= \partial x \\ &= s_x(\partial x) \end{aligned}$$

If $y \in B$ and $y \neq x$,

$$d(s_x(y)) = d(y) := s_x(\partial y)$$

□

L 1.22. *If $y \prec x$, then the based chain complex $C(\overline{\sigma y}) \subset C'$ has basis*

$$\{w \in B | w \prec y\} \amalg \Sigma\overline{y} \amalg \{c_x\} \tag{3}$$

P . By definition, $C(\overline{\sigma y})$ is the based subcomplex of C' with basis

$$\{w \in B' \quad | \quad w \prec \sigma y\} \tag{4}$$

We must show that the sets (3) and (4) are equal.

If $w \in B$ and $w \prec y$, then $w \prec \sigma y$ because $y \prec \sigma y$. If $\sigma w \in \Sigma\overline{y}$, then $w \prec y$ by definition of $\Sigma\overline{y}$. We claim that

$$w \prec y \Rightarrow \sigma w \prec \sigma y \tag{5}$$

For if $w \prec y$, then by definition there is a sequence

$$w = w_0 \prec \ldots \prec w_k = y$$

such that w_i appears with nonzero coefficient in the formula for ∂w_{i+1}. Looking at the formula for $d\sigma y$:

$$d\sigma y = (-1)^{|y|}y + \sigma\partial y$$

we see that σw_{k-1} appears with nonzero coefficient in the formula for $d\sigma y$. Similarly, each σw_i appears with nonzero coefficient in the formula for $d\sigma w_{i+1}$, so $\sigma w \prec \sigma y$.

Finally, y has at least one zero face v. Then $c_x \prec \sigma v$, and by the previous argument $\sigma v \prec \sigma y$. We have shown that (3) is a subset of (4).

Suppose $\alpha \in B'$ and $\alpha \prec \sigma y$. Then by definition of B', either $\alpha = w$ for some $w \in B$, $\alpha = \sigma w$ for some $w \prec x$, or $\alpha = c_x$. If $\alpha = c_x$ there is nothing to show.

We claim that if $w \in B$,

$$w \prec \sigma y \Rightarrow w \prec y \tag{6}$$

We argue by induction on the dimension of y. If $|y| = 0$, then the only cell of B which is a face of σy is y itself. Suppose $|y| = k$ and $w \prec \sigma y$. If $w = y$, then certainly $w \prec y$. If $w \precneqq y$, then by looking at the formula for $d\sigma y$ we see that either $w \prec y$ or $w \prec \sigma z$ for some z which appears in the formula for ∂y. If the first case $w \prec y$, and in the second case $w \prec \sigma z$ and $|z| = k - 1$. Thus by the inductive hypothesis $w \prec z \prec y$.

We claim that

$$\sigma w \prec \sigma y \Rightarrow w \prec y \tag{7}$$

Once again we argue by induction on the dimension of y. The case $|y| = 0$ is vacuous, for if $|y| = 0$, then the only cell of B which is a face of σy is y itself. Suppose $|y| = k$ and $\sigma w \prec \sigma y$. Since

$$d\sigma y = (-1)^{|y|} y + \sigma\partial y$$

we see that $\sigma w \prec \sigma z$ for some z which appears in the formula for ∂y. Thus by the inductive hypothesis $w \prec z \prec y$. We have shown that (4) is a subset of (3). □

L 1.23. *The based chain complex $C\left(\overline{s_x(x)}\right)$ has basis*

$$\{y \in B | y \precneqq x\} \amalg \Sigma\dot{x} \amalg \{c_x\} \tag{8}$$

P . The proof is completely analogous to that of Lemma 1.22. Recall that $s_x(x) = \sigma\partial x$, so $C\left(\overline{s_x(x)}\right)$ is the based chain complex with basis

$$\{y \in C' \quad | \quad y \prec \sigma z \text{ for some } z \text{ with } \langle \partial x, z\rangle \neq 0\} \tag{9}$$

We will show that the sets (8) and (9) are equal.

If $y \in B$ and $y \precneqq x$, then $y \prec z$ for some z with $\langle \partial x, z\rangle \neq 0$. By definition

$$d\sigma z := (-1)^{|z|} z + \sigma\partial z$$

so $z \prec \sigma z$, and therefore $y \prec \sigma z$. If $\sigma y \in \Sigma\dot{x}$, then by definition of $\dot{x}$, $y \prec z$ for some z with $\langle \partial x, z\rangle \neq 0$. Thus by (5), $\sigma y \prec \sigma z$. Finally, there exists a 0-cell v such that $v \prec x$. Then $c_x \prec \sigma v \in \Sigma\dot{x}$. We have shown that (8) is a subset of (9).

If $y \in B'$, then either $y \in B \setminus \{x\}$, $y \in \Sigma\dot{x}$, or $y = c_x$. Suppose that $y \prec \sigma z$ for some z with $\langle \partial x, z\rangle \neq 0$. If $y \in \Sigma\dot{x}$ or $y = c_x$ there is nothing to show. If $y \in B \setminus \{x\}$, and

$y \prec \sigma z$ for some $z \prec x$, then by (6) $y \prec z \prec x$. We have shown that (9) is a subset of (8). $\square$

L 1.24. *If C is regular and C' is as defined as in Definition 1.18, the augmentation map*

$$\varepsilon : C\left(\overline{s_x(x)}\right) \longrightarrow \mathbb{Z}$$

has a chain homotopy inverse.

P . First we must check that ε is a chain map, that is to say that $\varepsilon d : (C\left(\overline{s_x(x)}\right))_1 \to \mathbb{Z}$ is the zero map. We check on the list of generators given in the statement of Lemma 1.23. For a generator $y \in (B \setminus \{x\})_1$, $\varepsilon dy = e\partial y = 0$ because ε is a chain map on B. For a generator $\sigma y \in (\Sigma\dot{x})_1$,

$$\begin{aligned} \varepsilon dy &= \varepsilon(y - c_x) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Next we define a map $\tau : \mathbb{Z} \to C\left(\overline{s_x(x)}\right)$ by $\tau(1) = c_x$. We claim that τ is a chain homotopy inverse of ε. Observe that $\varepsilon\tau = 1$ on the nose. We define a chain homotopy

$$H : C\left(\overline{s_x(x)}\right) \longrightarrow C\left(\overline{s_x(x)}\right)$$

by defining H on the generating set of Lemma 1.23.

$$\begin{aligned} H(y) &= (-1)^{|y|}\sigma y \\ H(\sigma y) &= 0 \\ H(c_x) &= 0 \end{aligned}$$

We claim that

$$dH + Hd = 1_{C\left(\overline{s_x(x)}\right)} - \tau\varepsilon$$

We must check several cases.

(1) For $y \in B \setminus \{x\}$ with $|y| > 0$,

$$\begin{aligned} (dH + Hd)y &= d((-1)^{|y|}\sigma y) + H(\partial y) \\ &= (-1)^{|y|}(-1)^{|y|}y + (-1)^{|y|}\sigma(\partial y) + (-1)^{|\partial y|}\sigma(\partial y) \\ &= y \\ &= (1 - \tau\varepsilon)y \end{aligned}$$

(2) For $\sigma y \in \Sigma\dot{x}$ with $|y| > 0$,

$$\begin{aligned}(dH + Hd)\sigma y &= 0 + H\Big((-1)^{|y|}y + \sigma(\partial y)\Big)\\ &= (-1)^{|y|}(-1)^{|y|}y\\ &= y\\ &= (1 - \tau\varepsilon)y\end{aligned}$$

(3) For $y \in B \setminus \{x\}$ with $|y| = 0$,

$$\begin{aligned}(dH + Hd)y &= d\sigma y + 0\\ &= y - c_x\\ &= (1 - \tau\varepsilon)y\end{aligned}$$

(4) For $\sigma y \in \Sigma\dot{x}$ with $|y| = 0$,

$$\begin{aligned}(dH + Hd)\sigma y &= 0 + H(y - c_x)\\ &= \sigma y\\ &= (1 - \tau\varepsilon)\sigma y\end{aligned}$$

(5) Finally for c_x,

$$\begin{aligned}(dH + Hd)c_x &= 0\\ &= c_x - c_x\\ &= (1 - \tau\varepsilon)c_x\end{aligned}$$

□

C 1.25. *The restriction of s_x to the closure of x*

$$s_x : C(\overline{x}) \longrightarrow C\Big(\overline{s_x(x)}\Big)$$

is a quasi-isomorphism.

P . Consider the commutative diagram:

$$\begin{array}{ccc} C(\overline{x}) & \xrightarrow{s_x} & C\Big(\overline{s_x(x)}\Big) \\ & \searrow^{\varepsilon} & \downarrow{\scriptstyle \varepsilon} \\ & & \mathbb{Z} \end{array}$$

Since both augmentation maps are quasi-isomorphisms and the diagram commutes,

$$s_x : C(\overline{x}) \longrightarrow C\Big(\overline{s_x(x)}\Big)$$

is a quasi-isomorphism as well. □

L 1.26. *If C is regular and C' is as defined as in Definition 1.18, C' is a regular chain complex.*

P . We must check that for each cell y of C',

$$\varepsilon : C(\overline{y}) \longrightarrow \mathbb{Z}$$

is a quasi-isomorphism.

If $y \in B \setminus \{x\}$ and $x \not\preceq y$, then consider the following commutative diagram:

$$\begin{array}{ccc} C(\overline{y}) & \xrightarrow{s_x} & C(\overline{y}) \\ & \searrow^{\varepsilon_y} & \downarrow{\scriptstyle \varepsilon'_y} \\ & & \mathbb{Z} \end{array}$$

The map ε_y is a quasi-isomorphism, and since $x \not\preceq y$, s_x restricted to the closure of y is the identity. The composition of a quasi-isomorphism and the identity is a quasi-isomorphism, so

$$\varepsilon'_y : C(\overline{y}) \longrightarrow \mathbb{Z}$$

is a quasi-isomorphism.

If $y \in B$ and $x \prec y$ then $C(\overline{x})$ is a based subcomplex of $C(\overline{y})$. Thus s_x induces a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\overline{x}) & \longrightarrow & C(\overline{y}) & \longrightarrow & C(\overline{y}, \overline{x}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s_x} & & \downarrow{\scriptstyle s_x} & & \downarrow{\scriptstyle s_x} & & \\ 0 & \longrightarrow & C\big(\overline{s_x(x)}\big) & \longrightarrow & C\big(\overline{s_x(y)}\big) & \longrightarrow & C\big(\overline{s_x(y)}, \overline{s_x(x)}\big) & \longrightarrow & 0 \end{array}$$

The vertical map on the left is a quasi-isomorphism by Corollary 1.25. The based chain complexes $C(\overline{y}, \overline{x})$ and $C\big(\overline{s_x(y)}, \overline{s_x(x)}\big)$ both have bases isomorphic to

$$\{z \in B \quad | \quad z \not\preceq x\}$$

and

$$s_x : C(\overline{y}, \overline{x}) \longrightarrow C\big(\overline{s_x(y)}, \overline{s_x(x)}\big)$$

is an isomorphism. Thus

$$s_x : C(\overline{y}) \longrightarrow C\big(\overline{s_x(y)}\big)$$

is a quasi-isomorphism by the Five Lemma.

If $\sigma y \in \Sigma\dot{x}$, the map

$$\varepsilon : C(\overline{\sigma y}) \longrightarrow \mathbb{Z}$$

has a chain homotopy inverse. The proof of this claim is completely analogous to the proof of Lemma 1.24. The chain homotopy inverse

$$\tau : \mathbb{Z} \longrightarrow C(\overline{\sigma y})$$

is defined by $\tau(1) = c_x$. Once again, $\varepsilon\tau = 1$ on the nose, and $\tau\varepsilon$ is chain homotopic to 1. The chain homotopy H is defined by the same formula as in the proof of Lemma 1.24. To check that

$$dH + Hd = 1 - \tau\varepsilon$$

we must check this formula on the generators of $C(\overline{\sigma y})$. We use the description of the generators given by Lemma 1.22, and from here the proof is a copy of the proof of Lemma 1.24.

Finally

$$\varepsilon : C(\overline{c_x}) \longrightarrow \mathbb{Z}$$

is an isomorphism. □

P P 1.19. Let C be a regular chain complex and let x be a cell of C. Let

$$s_x : C \longrightarrow C'$$

be the conical subdivision of x. We are now ready to prove that s_x is an elementary subdivision in the sense of Definition 1.12. The basis for C' is

$$(B \setminus x) \amalg \Sigma\dot{x} \amalg \{c_x\}$$

which is a decomposition of the form

$$(B \setminus x) \amalg B''$$

By Lemma 1.21, s_x is an augmented chain map. By Lemma 1.26, C' is a regular chain complex. The conical subdivision map s_x is defined so that satisfies conditions (1) an (2) of Definition 1.12. Condition (3) states that for each

$$w \in B'' = \Sigma\dot{x} \amalg \{c_x\}$$

the cell w is a generator of $C\left(\overline{s_x(x)}\right)$. By Lemma 1.23, the set

$$\{y \in B | y \lneqq x\} \amalg \Sigma\dot{x} \amalg \{c_x\}$$

is a basis for $C\left(\overline{s_x(x)}\right)$. Thus condition (3) is satisfied.

It remains to check that $s_x : C \to C'$ is a quasi-isomorphism. Since $C\left(\overline{s_x(x)}\right)$ is a based subcomplex of C', s_x induces a map of short exact sequences as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(\overline{x}) & \longrightarrow & C & \longrightarrow & C(B, \overline{x}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s_x} & & \downarrow{\scriptstyle s_x} & & \downarrow{\scriptstyle s_x} & & \\
0 & \longrightarrow & C\left(\overline{s_x(x)}\right) & \longrightarrow & C' & \longrightarrow & C\left(B', \overline{s_x(x)}\right) & \longrightarrow & 0
\end{array}$$

The left hand vertical arrow is a quasi-isomorphism by Corollary 1.25.

The based chain complex $C(B, \overline{x})$ has basis

$$B \setminus \overline{x} = \{y \in B \quad | \quad y \nleq x\}$$

The based chain complex $C\left(B', \overline{s_x(x)}\right)$ has basis

$$\begin{aligned} B' \setminus \overline{s_x(s)} &= ((B \setminus \{x\}) \amalg \Sigma\dot{x} \amalg \{c_x\}) \setminus (\{y \in B | y \lneqq x\} \amalg \Sigma\dot{x} \amalg \{c_x\}) \\ &= B \setminus \overline{x} \end{aligned}$$

Thus the right hand vertical arrow is an isomorphism of based chain complexes.

We conclude that the middle vertical arrow $s_x : C \to C'$ is a quasi-isomorphism by the Five Lemma. □

1.3.2. The Barycentric Subdivision. We define the barycentric subdivision of a regular chain complex by iterating the conical subdivision.

D 1.27 (Barycentric Subdivision). Let C be a regular chain complex of dimension n. We define

$$s^{(k)} : C \to C_{(k)}$$

to be the subdivision map which is the composition of the elementary subdivision maps s_x for each k-cell x of C. (Note that since each map s_x modifies C only in the interior of x, these elementary subdivision maps all commute.) Then we define the *barycentric subdivision* of C to be the composition

$$C \xrightarrow{s^{(0)}} C_{(0)} \xrightarrow{s^{(1)}} C_{(1)} \xrightarrow{s^{(2)}} C_{(2)} \ldots \xrightarrow{s^{(n)}} C_{(n)} = C'$$

Since this map is a composition of elementary subdivisions, it is a subdivision map of regular chain complexes; we denoted it by $s : C \to C'$.

R 1.28. We can give an inductive formula for s. If x is a k-cell of C and $k > 0$, then

$$s^{(k)}(x) := (-1)^{|x|-1} \sigma_x s^{(k-1)} \partial_{C_{(k-1)}} x$$

where $\sigma_x y$ denotes the element of $\Sigma\dot{x}$ corresponding to y. Thus

$$s(x) = \begin{cases} (-1)^{|x|-1} \sigma_x s \partial x & |x| > 0 \\ c_x & |x| = 0 \end{cases}$$

The barycentric subdivision is a functor from the category CEL to itself.

D 1.29 (Barycentric Subdivision of a Cellular Chain Map). Let $f : C \to D$ be a map of regular chain complexes induced by a cellular map $\hat{f} : B \to E$ of posets. That is to say, let f be given by the formula

$$f(x) = \begin{cases} \epsilon_x \hat{f}(x) & |\hat{f}(x)| = |x| \\ 0 & |\hat{f}(x)| \neq |x| \end{cases} \tag{10}$$

for some choice of signs $\epsilon : B \to \{-1, +1\}$

We define a cellular map

$$f_s : s(C) \longrightarrow s(D)$$

inductively on the k-skeletons $s(C^{(k)})$. First we define the underlying poset map

$$\hat{f}_s : s(B) \longrightarrow s(E)$$

On $s(C^{(0)}$, the barycentric subdivision of the 0-skeleton of C, we define

$$\begin{aligned} \hat{f}_s : s(B_0) &\longrightarrow (E_0) \\ c_x &\mapsto c_{\hat{f}(x)} \end{aligned}$$

The barycentric subdivision of the k-skeleton of C has basis

$$s\left(B^{(k-1)}\right) \amalg (\amalg_{|x|=k} \Sigma\dot{x}) \amalg (\amalg_{|x|=k} c_x) \tag{11}$$

where $s\left(B^{(k-1)}\right)$ is a basis for $s(C^{(k-1)})$, the barycentric subdivision of the $(k-1)$-skeleton of C. Suppose inductively that $\hat{f}_s$ has been defined on $s\left(B^{(k-1)}\right)$. Then extend $\hat{f}_s$ over the k-skeleton by:

$$c_x \mapsto c_{\hat{f}(x)} \tag{12}$$

$$\sigma_x y \mapsto \begin{cases} \sigma_{\hat{f}(x)} \hat{f}_s(y) & c_{\hat{f}(x)} \notin \overline{\hat{f}_s(y)} \\ \hat{f}_s(y) & c_{\hat{f}(x)} \in \overline{\hat{f}_s(y)} \end{cases} \tag{13}$$

Here $\sigma_x y$ denotes the element of $\Sigma\dot{x}$ corresponding to y. The poset map $\hat{f}_s$ defines a chain map in the usual way:

$$f_s(x) = \begin{cases} \hat{f}_s(x) & |\hat{f}(x)| = |x| \\ 0 & |\hat{f}(x)| \neq |x| \end{cases}$$

P 1.30. *The barycentric subdivision s is a functor from CEL to CEL. That is to say, given a cellular map $f : C \to D$ of regular chain complexes, the map $f_s : s(C) \to s(D)$ of Definition 1.29 is a cellular chain map.*

P. There are a number of details to check. In each case we will argue by induction on the barycentric subdivision of k-skeleton of C.

First let us check that $\hat{f}_s$ is a poset map. Since $\hat{f}_s(c_x) = c_{\hat{f}(x)}$, $\hat{f}_s$ is a poset map on the 0-skeleton. Suppose inductively that $\hat{f}_s$ is a poset map on $s(B^{(k-1)})$, and consider the basis (11) for $s(C^{(k)})$. Suppose $\sigma_x y$ is a cell of $s(C^{(k)})$ which is not a cell of $s(C^{(k-1)})$, and let $\alpha \leq \sigma_x y$. If $\alpha \in s(C^{(k-1)})$, then by (6), $\alpha \leq y$. Thus $\hat{f}_s(\alpha) \leq \hat{f}_s(y)$ by the inductive hypothesis. Looking at (13), we see that $\hat{f}_s(y) \leq \hat{f}_s(\sigma_x y)$, so

$$\hat{f}_s(\alpha) \leq \hat{f}_s(\sigma_x y)$$

If α is a cell of $s(C^{(k)})$ which is not a cell of $s(C^{(k-1)})$, then either $\alpha = \sigma_x z$ for some $z \leq y$ or $\alpha = c_x$. If $\alpha = c_x$, then looking at (13) we see that

$$\hat{f}_s(c_x) = c_{\hat{f}(x)} \leq \hat{f}_s(\sigma_x y)$$

If $\alpha = \sigma_x z$ then there are 3 cases. If $c_{\hat{f}(x)} \in \overline{\hat{f}_s(\sigma_x z)} \preceq \overline{\hat{f}_s(\sigma_x y)}$, then

$$\hat{f}_s(\sigma_x z) = \hat{f}_s(\sigma_x y) = \hat{f}_s(y)$$

If $c_{\hat{f}(x)} \notin \overline{\hat{f}_s(\sigma_x z)}$ but $c_{\hat{f}(x)} \in \overline{\hat{f}_s(\sigma_x y)}$, then

$$\hat{f}_s(\sigma_x z) = \sigma_{\hat{f}(x)} \hat{f}_s(z)$$
$$\hat{f}_s(\sigma_x y) = \hat{f}_s(y)$$

Since $z \preceq y$, $\hat{f}_s(z) \preceq \hat{f}_s(y)$ by the inductive hypothesis. Since $c_{\hat{f}(x)} \preceq \hat{f}_s(\sigma_x y)$ as well,

$$\hat{f}_s(\sigma_x z) = \sigma_{\hat{f}(x)} \hat{f}_s(z) \preceq \hat{f}_s(y) = \hat{f}_s(\sigma_x y)$$

Finally, if $c_{\hat{f}(x)} \notin \overline{\hat{f}_s(\sigma_x y)}$, then

$$\hat{f}_s(\sigma_x z) = \sigma_{\hat{f}(x)} \hat{f}_s(z)$$
$$\hat{f}_s(\sigma_x y) = \sigma_{\hat{f}(x)} \hat{f}_s(y)$$

Since $z \preceq y$, $\hat{f}_s(z) \preceq \hat{f}_s(y)$ by the inductive hypothesis. Thus by (5),

$$\hat{f}_s(\sigma_x z) \preceq \hat{f}_s(\sigma_x y)$$

We have shown that $\hat{f}_s$ is a poset map. Next we check that $\hat{f}_s$ does not increase degree. By definition $\hat{f}_s$ maps 0-cells to 0-cells. Suppose that for $(k-1)$-cells α

$$|\hat{f}_s(\alpha)| \leq |\alpha|$$

Let $\sigma_x y$ be a k-cell of $s(C)$. Then looking at (13) we see that

$$\begin{aligned} |\hat{f}_s(\sigma_x y)| &\leq |\sigma_{\hat{f}(x)} \hat{f}_s(y)| \\ &= |\hat{f}_s(y)| + 1 \\ &\leq |y| + 1 \\ &= |\sigma_x y| \end{aligned}$$

Finally we check that f_s is a chain map. To do so we need the following Lemma.

L 1.31. *Let $\alpha \in s(B)$. If $|\hat{f}_s(\alpha)| = |\alpha|$, then $|\hat{f}_s(w)| = |w|$ for all w with* $\langle \partial\alpha, w\rangle \neq 0$.

The contrapositive of Lemma 1.31 is the following statement: if there exists a cell $w \in s(B)$ such that $\langle \partial\alpha, w\rangle \neq 0$ and $|\hat{f}_s(w)| < |w|$, then $|\hat{f}_s(\alpha)| < |\alpha|$. We prove the contrapositive by induction on the degree of α. If $|\alpha| = 0$ or $|\alpha| = 1$, then the statement is vacuous. Suppose the statement is true for cells of degree less then k, and let $\alpha \in s(B)$ such that $|\alpha| = k > 1$. Then $\alpha = \sigma_x y$ for some $y \in s(B)$ with $|y| = k - 1$. By definition

$$d(\sigma_x y) = (-1)^{|y|} + \sigma\partial y$$

Thus the codimension 1 faces of $\sigma_x y$ are y and the $k-2$ cells z such that $\langle \partial y, z\rangle \neq 0$. If $|\hat{f}_s(y)| < |y|$, then

$$\begin{aligned} |\hat{f}_s(\sigma_x y)| &\leq |\hat{f}_s(y)| + 1 \\ &< |y| + 1 \\ &= |\sigma_x y| \end{aligned}$$

If $|\hat{f}_s(z)| < |z|$ for some codimension 1 face of y, then $|\hat{f}_s(y)| < |y|$ by the inductive hypothesis. Then by the previous argument $|\hat{f}_s(\sigma_x y)| < |\sigma_x y|$. We conclude that, as claimed, if $|\hat{f}_s(w)| < |w|$ for some codimension 1 face of α, then $|\hat{f}_s(\alpha)| < |\alpha|$.

We can now show that f_s is a chain map. Let α be a 1-cell of $s(B)$. The $\alpha = \sigma_x c_y$ for some 0-cell y of B. If $|\hat{f}_s(\alpha)| < |\alpha|$, then $f_s(\alpha) = 0$, so certainly $d f_s(x) = f_s d(x)$. If $|\hat{f}_s(\alpha)| = |\alpha|$ then we compute:

$$\begin{aligned} d f_s(\alpha) &= d f_s(\sigma_x c_y) \\ &= d\left(\sigma_{\hat{f}(x)} \hat{f}_s(c_y)\right) \\ &= d\left(\sigma_{\hat{f}(x)} c_{\hat{f}(y)}\right) \\ &= c_{\hat{f}(y)} - c_{\hat{f}(x)} \\ &= \hat{f}_s(c_y - c_x) \\ &= \hat{f}_s d(\sigma_x c_y) \end{aligned}$$

Suppose that f_s commutes with d for all cells of degree less than k, and let α be a k-cell of $s(B)$ with $k > 1$. Then $\alpha = \sigma_x y$ for some $k-1$-cell y of $s(B)$. If $|\hat{f}_s(\alpha)| < |\alpha|$, then $f_s(\alpha) = 0$, so certainly $d f_s(x) = f_s d(x)$. If $|\hat{f}_s(\alpha)| = |\alpha|$, then by Lemma 1.31, $|\hat{f}_s(y)| = |y|$ and moreover $|\hat{f}_s(z)| = |z|$ for every codimension 1 face z of y. Then we compute:

$$\begin{aligned} d f_s(\alpha) &= d f_s(\sigma_x y) \\ &= d\left(\sigma_{\hat{f}(x)} \hat{f}_s(y)\right) \\ &= (-1)^{|\hat{f}_s(y)|} \hat{f}_s y + \sigma_{\hat{f}(x)} \partial(\hat{f}_s y) \\ &= (-1)^{|f_s(y)|} \hat{f}_s y + \sigma_{\hat{f}(x)} \partial(f_s y) \\ &= (-1)^{|f_s(y)|} \hat{f}_s y + \sigma_{\hat{f}(x)} f_s(\partial y) \\ &= (-1)^{|f_s(y)|} \hat{f}_s y + \sigma_{\hat{f}(x)} \hat{f}_s(\partial y) \\ &= f_s\left((-1)^{|y|} + \sigma_x \partial y\right) \\ &= f_s d(\sigma_x y) \end{aligned}$$

□

P 1.32. *The barycentric subdivision functor is natural in the following sense. If $f : C \to D$ is a cellular map of regular chain complexes, then the following diagram commutes.*

$$\begin{array}{ccc} C & \xrightarrow[s]{} & s(C) \\ \downarrow f & & \downarrow f_s \\ D & \xrightarrow{s} & s(D) \end{array}$$

P . Let x be a cell of C. If $|\hat{f}(x)| < |x|$, then $f(x) = 0$ and the diagram certainly commutes. Suppose $|\hat{f}(x)| = |x|$. If $|x| = 0$, then

$$\begin{aligned} sf(x) &= c_{f(x)} \\ &= f_s(c_x) \\ &= f_s(s(x)) \end{aligned}$$

Suppose that the diagram commutes for all cells in the $(k-1)$-skeleton of C and let $|x| = k$. Then we compute:

$$\begin{aligned} sf(x) &= (-1)^{|f(x)|-1}\sigma_{\hat{f}(x)}s\,(\partial f(x)) \\ &= (-1)^{|f(x)|-1}\sigma_{\hat{f}(x)}sf(\partial x) \\ &= (-1)^{|x|-1}\sigma_{\hat{f}(x)}f_s s(\partial x) \\ &= f_s\left((-1)^{|x|-1}\sigma_x s(\partial x)\right) \\ &= f_s(s(x)) \end{aligned}$$

□

1.3.3. The Nerve. We have defined both the set of cells of the barycentric subdivision and the boundary map by iterating conical subdivisions. It will be more convenient to have a “closed-form” description of this subdivision.

D 1.33 (Nerve). Let C be a based chain complex with basis B. The *nerve* of C is the based chain complex ηC with basis

$$\eta B_k \coloneqq \{\{x_0, \ldots, x_k\} \subset B \quad | \quad x_0 \lneqq \ldots \lneqq x_k\}$$

The differential is given by the formula

$$d_\eta(x_0 \prec x_1 \prec \ldots \prec x_k) \coloneqq \sum_{i=0}^{k}(-1)^{i+1}\{x_0 \prec \ldots \prec \hat{x}_i \prec \ldots \prec x_k\}$$

If C is regular, the barycentric subdivision that we have constructed by iterated cones is isomorphic to the nerve.

P 1.34. *Let C be a regular chain complex, let $s(C)$ denote the barycentric subdivision of C, and let ηC denote the nerve of C. Then $s(C)$ and ηC are isomorphic after relabeling generators. To be more explicit, the chain map $b : s(C) \to \eta C$ constructed in the proof below is an isomorphism.*

P . We will inductively define a chain map which is a bijection of generating sets from barycentric subdivision of the k-skeleton of C to the nerve of the k-skeleton of C.

The 1-skeleton of $s(C)$ has basis

$$s(B^{(0)}) \amalg (\amalg_{|x|=1} \Sigma \dot{x}) \amalg (\amalg_{|x|=1} c_x)$$

If x is a 1-cell of C, then by Proposition 1.11, x has two 0-faces v and w. Thus for each 1-cell x, the graded set $\Sigma \dot{x}$ consists of two generators in dimension 1, $\sigma_x v$ and $\sigma_x v$. We define a map

$$b : s\left(C^{(1)}\right) \longrightarrow \eta(C^{(1)})$$

by mapping generators as follows:

$$\begin{aligned} v \in B_0 &\mapsto v \\ \sigma_x v \in \Sigma \dot{x} &\mapsto v \prec x \\ c_x &\mapsto x \end{aligned}$$

This map is a bijection of the generating sets. We check that it is a chain map.

$$\begin{aligned} d_\eta b(\sigma_x v) &= d_\eta (v \prec x) \\ &= v - x \\ &= b(v - c_x) \\ &= bd(\sigma_x v) \end{aligned}$$

Now suppose that we have defined

$$b : s(C^{(k-1)}) \longrightarrow \eta(C^{(k-1)})$$

such that b is a chain map which is a bijection of generating sets, and such that for cells x of $s(C^{(k-1)})$

$$b(\dot{x}) \subset \eta(\dot{x}) \tag{14}$$

The k-skeleton $s(C^{(k)})$ has basis

$$s\left(B^{(k-1)}\right) \amalg (\amalg_{|x|=k} \Sigma \dot{x}) \amalg (\amalg_{|x|=k} c_x)$$

where $s\left(B^{(k-1)}\right)$ is a basis for the barycentric subdivision of the $(k-1)$-skeleton of C. Let

$$i_x : \eta C(\dot{x}) \longrightarrow \eta C(\overline{x})$$

be the linear map defined on generators by:

$$y_0 \prec \ldots \prec y_l \mapsto y_0 \prec \ldots \prec y_l \prec x$$

We define a map

$$b : (C_{(k)})^{(k)} \longrightarrow \eta(C^{(k)})$$

by mapping the generators as follows:

$$y \in s\left(B^{(k-1)}\right) \mapsto b(y) \tag{15}$$

$$\sigma_x y \in \Sigma\dot{x} \mapsto i_x(b(y)) \tag{16}$$

$$c_x \mapsto x \tag{17}$$

We claim that b is a well-defined bijection. For $\sigma_x y \in \Sigma\dot{x}$, $b(y) \in \eta\dot{x}$ by assumption (14). Thus $b(y) = y_0 \prec \ldots \prec y_l$ for some $y_l \prec x$, and it makes sense to define

$$b(\sigma_x y) = i_x(b(y)) = y_0 \prec \ldots \prec y_l \prec x$$

By the inductive hypothesis, b is already defined and a bijection on $s\left(B^{(k-1)}\right)$. The cells of $\eta(C^k)$ that are not cells of $\eta(C^{(k-1)})$ are precisely those of the form

$$y_0 \prec \ldots \prec y_l \prec x$$

such that $|x| = k$. Consider such a cell. If $l > 0$, then by the inductive hypothesis there exists some $y \in s\left(B^{(k-1)}\right)$ such that

$$b(y) = y_0 \prec \ldots \prec y_l$$

Thus

$$b(\sigma_x y) = y_0 \prec \ldots \prec y_l \prec x$$

If $l = 0$, then

$$b(c_x) = x$$

Thus all such cells are in the image of b and b is surjective.

To see that b is injective, observe that (15) is injective by the inductive hypothesis, and similarly (16) is injective because b is injective on $s\left(B^{(k-1)}\right)$. (17) is injective because $y = x$ if and only if $c_x = b_y$.

We check that b is a chain map. The map already commutes with d on $s\left(B^{(k-1)}\right)$ by the inductive hypothesis. Suppose $y \in s\left(B^{(k-1)}\right)$ and $b(y) = y_0 \prec \ldots \prec y_l$. Then:

$$\begin{aligned}
d_\eta b(\sigma_x y) &= d_\eta(y_0 \prec \ldots \prec y_l \prec x) \\
&= (-1)^l(y_0 \prec \ldots \prec y_l) + \sum_{i=0}^{l}(-1)^{i+1} y_0 \prec \ldots \hat{y}_i \ldots \prec y_l \prec x \\
&= (-1)^{|y|} b(y) + i_x(d_\eta b(y)) \\
&= (-1)^{|y|} b(y) + i_x(bdy) \\
&= (-1)^{|y|} b(y) + b(\sigma_x dy) \\
&= bd(\sigma_x y)
\end{aligned}$$

Finally we must check that b satisfies (14). Suppose z is a cell of $C_{(k)}$; we must check that

$$b(C(\dot{z})) \subset \eta C(\dot{z})$$

Let α be a face of z. If $\alpha \in s\left(B^{(k-1)}\right)$, the $b(\alpha) \in \eta C(\overline{z})$ by the inductive hypothesis. If $\alpha = \sigma_x y$ and $\alpha \prec z$, then $x \prec z$. Thus

$$b(\sigma_x y) = i_x(b(y)) \prec z$$

Finally, if $\alpha = c_x$ and $c_x \prec z$, then $b(c_x) = x \prec z$.

We conclude that b is an isomorphism of regular chain complexes. □

The nerve is a functor on the category of regular chain complexes and cellular maps.

D 1.35. Let $f : C \to D$ is a cellular map of regular chain complexes induced by a map $\hat{f} : B \to E$ of posets. The *nerve* of $\hat{f}$ is the poset map

$$\hat{f}_\eta : \eta B \longrightarrow \eta E$$

which maps

$$x_0 \prec \ldots \prec x_k \mapsto \hat{f}(x_0) \prec \ldots \prec \hat{f}(x_k)$$

An expression of the form

$$\hat{f}(x_0) \prec \ldots \prec \hat{f}(x_k)$$

will not designate an element of ηE if $\hat{f}x_j = \hat{f}x_{j+1}$ for some j. In this case we define the expression to mean the element of ηE which results from omitting repeats. The poset map $\hat{f}_\eta$ induces a chain map

$$f_\eta : \eta C \longrightarrow \eta D$$

in the usual way:

$$f_\eta(x) = \begin{cases} \hat{f}_\eta(x) & |\hat{f}_\eta(x)| = |x| \\ 0 & |\hat{f}_\eta(x)| \neq |x| \end{cases} \tag{18}$$

P 1.36. *The map f_η is a cellular map of regular chain complexes.*

P . Note that the faces of the cell $x_0 \prec \ldots \prec x_k$ are precisely the subsets of $\{x_0, \ldots x_k\}$. If

$$x_{i_0} \prec \ldots \prec x_{i_j} \quad \leqq \quad x_0 \prec \ldots \prec x_k$$

then

$$\begin{aligned} \hat{f}_\eta(x_{i_0} \prec \ldots \prec x_{i_j}) &= \hat{f}(x_{i_0}) \prec \ldots \prec \hat{f}(x_{i_j}) \\ &\leqq \hat{f}(x_0) \prec \ldots \prec \hat{f}(x_k) \\ &= \hat{f}_\eta(x_0 \prec \ldots \prec x_k) \end{aligned}$$

Thus $\hat{f}_\eta$ is a poset map.

The map f_η is defined so that

$$f_\eta(x_0 \prec \ldots \prec x_k) = f_\eta(x_0) \prec \ldots \prec f_\eta(x_k)$$

where the expression on the right is defined to be 0 if $f(x_j) = f(x_{j+1})$ for any j. We check that f_η is a chain map:

$$\begin{aligned}
d f_\eta(x_0 \prec \ldots \prec x_k) &= d\left(f_\eta(x_0) \prec \ldots \prec f_\eta(x_k)\right) \\
&= \sum_{i=0}^{k} (-1)^{i+1} f_\eta(x_0) \prec \ldots \widehat{f_\eta(x_i)} \prec f_\eta(x_k) \\
&= f_\eta\left(\sum_{i=0}^{k} (-1)^{i+1} x_0 \prec \ldots \hat{x}_i \prec x_k\right) \\
&= f_\eta d(x_0 \prec \ldots \prec x_k)
\end{aligned}$$

□

P 1.37. *The isomorphism $b : s(C) \to \eta C$ of Proposition 1.34 is natural with respect to cellular maps. If $f : C \to D$ is a cellular map of regular chain complexes, then the following diagram commutes:*

$$\begin{array}{ccc}
s(C) & \xrightarrow{\ b\ } & \eta C \\
\downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_\eta} \\
s(D) & \xrightarrow{\ b\ } & \eta(D)
\end{array}$$

P . We argue by induction on the skeletons $s(C)$. The 0-cells of $s(C)$ are of the form c_x, where x is a cell of C.

$$\begin{aligned}
b f_s(c_x) &= b(c_{f(x)}) \\
&= f(x) \\
&= f_\eta(x) \\
&= f_\eta(b(c_x))
\end{aligned}$$

Suppose that the diagram commutes on the $(k-1)$-skeleton of $s(C)$. A k-cell of $s(C)$ is of the form $\sigma_x y$, where x is a k-cell of C and y is a $(k-1)$-cell of $s(C)$. If $f(x) = 0$, then the diagram clearly commutes. If not:

$$\begin{aligned}
b f_s(\sigma_x y) &= b\left(\sigma_{\hat{f}(x)} f_s(y)\right) \\
&= i_{\hat{f}(x)} b f_s(y) \\
&= i_{\hat{f}(x)} f_\eta b(y) \\
&= f_\eta\,(i_x b(y)) \\
&= f_\eta b(\sigma_x y)
\end{aligned}$$

□

From now on, we will not distinguish between the barycentric subdivision $s(C)$ and the nerve ηC of a regular chain complex, as we regard them as two different combinatorial descriptions of the same subdivision of C. We will use the notation C' to refer to the barycentric subdivision of C.

1.4. Geometric Realization

In this section, we show that every regular chain complex is naturally chain equivalent to the simplicial chain complex of a simplicial complex.

Let SIM be the following category. The objects of SIM are finite simplicial complexes equipped with a partial order on the set of vertices such that the set of vertices of each simplex is totally ordered. The morphisms of SIM are simplicial maps. To be more explicit, a map $f : K \to L$ is a set map from the set of vertices of K to the set of vertices of L such that image of the set of vertices of each simplex of K is a simplex of L. Let $\mathcal{S} : SIM \to CEL$ be the simplicial chain functor. Note that added structure of a partial order on the vertices of a simplicial complex such that the vertices of each simplex is totally ordered allows us to define a functor from simplicial complexes to chain complexes. Moreover, the chain complex of a simplicial complex is a regular based chain complex, and the basis is the set of simplices. A simplicial map induces a cellular map between the associated chain complexes.

Let PL be the category of PL spaces and PL maps, as defined in Rourke and Sanderson [**RS72**]. Roughly, a PL space is one which can be given the local structure of a complex of convex cells in Euclidean space. In particular, SIM embeds as a subcategory of PL.

D 1.38 (Geometric Realization). We define a functor $\mathcal{G} : CEL \to SIM$. Let C be a regular chain complex with basis B. Then $\mathcal{G}(C)$ is the simplicial complex ηB, the nerve of the poset of C. Let $f : C \to D$ be a cellular map induced by a poset map $\hat{f} : B \to E$. Then $\mathcal{G}(f)$ is the simplicial map

$$\hat{f}_\eta : \eta B \longrightarrow \eta E$$

P 1.39. *The geometric realization functor $\mathcal{G}$ is indeed a functor from CEL to SIM.*

P . Let C be a regular chain complex with basis B. A vertex of ηB is simply a singleton set $\{x\}$ for some $x \in B$. Then there is a partial order on the set of vertices of ηB where we set

$$\{x\} \leq \{y\} \quad \Leftrightarrow \quad x \leqq y$$

For each simplex

$$x_0 \prec \ldots \prec x_k$$

of ηB, the vertices are totally ordered with respect to this partial order:

$$\{x_0\} \leq \cdots \leq \{x_k\}$$

Thus ηB is indeed on object of SIM.

If $f : C \to D$ is a cellular map induced by a poset map $\hat{f} : B \to E$, then $\eta\hat{f} : \eta B \to \eta E$ is a indeed simplicial map. □

L 1.40. *Let (C, B, ∂) be a regular chain complex. There is a natural isomorphism between the regular chain complexes ηC and $\mathcal{SG}(C)$.*

The proof is nearly a tautology, since ηC and $\mathcal{SG}(C)$ are based chain complexes each with basis the nerve of the poset B.

P . The based chain complex ηC has basis

$$\eta B = \{\{x_0, \ldots, x_k\} \subset B \quad | \quad x_0 \lneqq \ldots \lneqq x_k\}$$

The based chain complex $\mathcal{SG}(C)$ has basis the set of simplices of $\mathcal{G}(C)$, which is the set

$$\{\{x_0, \ldots, x_k\} \subset B \quad | \quad x_0 \lneqq \ldots \lneqq x_k\}$$

Let $\hat{\imath}$ denote the identity map between these posets. Then $\hat{\imath}$ induces a chain map $i : \eta C \to \mathcal{SG}(C)$ which is isomorphism.

Let $f : C \to D$ be a cellular map, and consider the following diagram.

$$\begin{array}{ccc} \eta(C) & \xrightarrow{i} & \mathcal{SG}(C) \\ \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle \mathcal{SG}(f)} \\ \eta(D) & \xrightarrow{i} & \mathcal{SG}(D) \end{array}$$

Let $x_0 \prec \ldots \prec x_k$ be a cell of ηC. Then

(19)
$$i f_\eta(x_0 \prec \ldots \prec x_k) = \begin{cases} f(x_0) \prec \ldots \prec f(x_k) & |f(x_0) \prec \ldots \prec f(x_k)| = |x_0 \prec \ldots \prec x_k| \\ 0 & |f(x_0) \prec \ldots \prec f(x_k)| < |x_0 \prec \ldots \prec x_k| \end{cases}$$

Now, the simplicial map $\mathcal{G}(f) : \mathcal{G}(C) \to \mathcal{G}(D)$ induces a chain map $\mathcal{SG}(f) : \mathcal{SG}(C) \to \mathcal{SG}(D)$ in the same way that the poset map of a cellular map of regular chain complexes induces a chain map. That is to say,

(20)
$$\mathcal{SG}i(f)(x_0 \prec \ldots \prec x_k) = \begin{cases} f(x_0) \prec \ldots \prec f(x_k) & |f(x_0) \prec \ldots \prec f(x_k)| = |x_0 \prec \ldots \prec x_k| \\ 0 & |f(x_0) \prec \ldots \prec f(x_k)| < |x_0 \prec \ldots \prec x_k| \end{cases}$$

Thus, i is natural with respect to cellular maps. □

We now define the sense in which $\mathcal{SG}(C)$ is naturally equivalent to C.

D 1.41. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a subcategory. Let $\mathcal{F}$ and $\mathcal{G}$ be two functors from $\mathcal{D}$ to $\mathcal{D}$. A *natural transformation under $\mathcal{C}$* between $\mathcal{F}$ and $\mathcal{G}$ is, for each object x of $\mathcal{D}$, a morphism $\phi_x \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{G}(y))$ such that for all $f \in \mathrm{Hom}_{\mathcal{D}}(x, y)$, the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{F}(x) & \xrightarrow{\phi_x} & \mathcal{G}(x) \\
{\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\
\mathcal{F}(y) & \xrightarrow{\phi_y} & \mathcal{G}(y)
\end{array}
$$

R 1.42. A natural transformation under a larger subcategory differs from the usual notion of a natural transformation in that the functors are defined on the smaller subcategory, but the natural transformation lives in the larger category.

T 1.43 (Simplicial Realization Theorem). *Let $\mathcal{G} : CELL \to SIM$ be the geometric realization functor. Let $\mathcal{S} : SIM \to CEL$ be the simplicial chain functor. Let PL be the category of PL spaces and PL maps.*

There is a natural transformation under REG from 1_{CEL} to $\mathcal{SG}$, described in the proof, which is a quasi-isomorphism on objects. There is a natural transformation under PL from 1_{SIM} to $\mathcal{GS}$, described in the proof, which is a PL homeomorphism on objects.

P . To prove the first assertion, we must show that for every cellular map

$$f : (C, B, \partial) \to (D, E, \partial)$$

of regular chain complexes, the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{ibs} & \mathcal{SG}(C) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{SG}(f)} \\
D & \xrightarrow{ibs} & \mathcal{SG}(D)
\end{array}
$$

Here *ibs* denotes the composition of the maps $s : C \to s(C)$ of Definition 1.27, $b : s(C) \to \eta C$ of Proposition 1.34, and $i : \eta C \to \mathcal{SG}(C)$ of Lemma 1.40. By Proposition 1.19, s is a composition of elementary subdivisions and this thus a regular map. By Proposition 1.34, b is an isomorphism and thus is a regular map. By Lemma 1.40, i is an isomorphism and thus is a regular map. Now consider the following diagram:

$$
\begin{array}{ccccccc}
C & \xrightarrow{s} & s(C) & \xrightarrow{b} & \eta(C) & \xrightarrow{i} & \mathcal{SG}(C) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle \mathcal{SG}(f)} \\
D & \xrightarrow{s} & s(D) & \xrightarrow{b} & \eta C & \xrightarrow{i} & \mathcal{SG}(D)
\end{array}
$$

The left-hand square commutes by Proposition 1.32. The middle square commutes by Proposition 1.37. The right-hand square commutes by Lemma 1.40. Thus *ibs* is a natural transformation from 1_{CEL} to $\mathcal{SG}$. Since s is a quasi-isomorphism and b and i are isomorphisms, the natural transformation *ibs* is a quasi-isomorphism on objects.

To prove the second assertion, we observe that the topological identity map from a simplicial complex to its barycentric subdivision is a PL which is a homeomorphism. Moreover, the barycentric subdivision of simplicial complexes is natural with respect to simplicial maps. □

C 1.44. *If $f : C \to D$ is a cellular map which induces an isomorphism on homology, then $\mathcal{G}(f)$ induces an isomorphism on homology.*

P . When we say that a simplicial map

$$\mathcal{G}(f) : \mathcal{G}(C) \to \mathcal{G}(D)$$

induces an isomorphism on homology, we mean that the induced map of simplicial chain complexes

$$\mathcal{SG}(f) : \mathcal{SG}(C) \to \mathcal{SG}(D)$$

induces an isomorphism on homology. Suppose that $f : C \to D$ is a cellular map which induces an isomorphism on homology, and consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{ibs} & \mathcal{SG}(C) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathcal{SG}(f)} \\ D & \xrightarrow{ibs} & \mathcal{SG}(D) \end{array}$$

Since $ibs : C \to \mathcal{SG}(C)$ and $ibs \circ f : C \to \mathcal{SG}(D)$ induces isomorphisms on homology, the map

$$\mathcal{SG}(f) : \mathcal{SG}(C) \to \mathcal{SG}(D)$$

must induce an isomorphism on homology as well. □

C 1.45. *Every regular chain complex is naturally chain equivalent to the simplicial chain complex of a simplicial complex.*

P . For every regular chain complex is naturally chain equivalent to its barycentric subdivision. □

1.4.1. Open stars. We now use geometric realization to prove some properties of regular chain complexes. Let C be a regular chain complex with basis B, and let z be a cell of B. Define the following subsets of $\bar{z}$.

$$\mathrm{St}_z(x) := \{y \in B \mid x \leq y \leq z\}$$

$$\complement_z x := \{y \in B \mid y \leq z, x \not\leq y\}$$

We call these subsets the *open star*, and the *complement* of x in z.

L 1.46. *Let C be a regular chain complex with basis B, and let z be an n-cell of B. Let $x \lneq z$. Then $\complement_z(x)$ is a closed subset of $\bar{z}$, and the augmentation map*

$$\varepsilon : C\big(\complement_z(x)\big) \to \mathbb{Z}$$

is a quasi-isomorphism.

P . Suppose $y \in \complement_z(x)$, and suppose $y' \leq y$. If $x \leq y'$, then $x \leq y$, a contradiction. Thus $y' \in \complement_z(x)$ and so $\complement_z(x)$ is a closed subset of $\bar{z}$. Now consider the commutative diagram

$$\begin{array}{ccc} C\big(\complement_z(x)\big) & \xrightarrow{\varepsilon} & \mathbb{Z} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 1} \\ \eta\Big(C\big(\complement_z(x)\big)\Big) & \xrightarrow{\varepsilon} & \mathbb{Z} \end{array}$$

The lower left-hand chain complex $\eta\Big(C\big(\complement_z(x)\big)\Big)$ is the simplicial chain complex of a triangulation of the complement of the open star of x in $\bar{z}$. The geometric realization of $\bar{z}$ is a homology cell, and the complement of the open star of a proper face x inside a homology cell z is a homology cell. Thus the augmentation map

$$\varepsilon : \eta\Big(C\big(\complement_z(x)\big)\Big) \to \mathbb{Z}$$

is a quasi-isomorphism. Since b is a quasi-isomorphism which commutes with augmentation,

$$\varepsilon : C\big(\complement_z(x)\big) \to \mathbb{Z}$$

is a quasi-isomorphism as well. □

C 1.47. *Let C be a regular chain complex, and let $x, z \in B$ such that $x \leq z$. Then $C\big(\bar{z}, \complement_z(x)\big)$ is acyclic if $x \neq z$ and has a single generator in dimension $|z|$ if $x = z$.*

P . Suppose $x \lneq z$, and consider the commutative diagram

$$\begin{array}{ccc} C\big(\complement_z(x)\big) & \xrightarrow{i} & C(\bar{z}) \\ & {\scriptstyle \varepsilon}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & \mathbb{Z} \end{array}$$

Here i denotes the inclusion map. The first augmentation map is a quasi-isomorphism because $C(\bar{z})$ is regular, and the second is a quasi-isomorphism by Lemma 1.46. Since both augmentation maps are quasi-isomorphisms, the inclusion map i must be a quasi-isomorphism as well. Looking at long exact sequence of the pair $C\big(\bar{z}, \complement_z(x)\big)$, we see that $C\big(\bar{z}, \complement_z(x)\big)$ is acyclic.

Now, suppose $x = z$. Then the chain complex $\mathcal{C}\big(\bar{z}, \complement_z(z)\big)$, has a single generator, namely the cell z in dimension z. □

R 1.48. The generators of the chain complex $\mathcal{C}\big(\bar{z}, \complement_z(z)\big)$ are exactly the elements of the open star

$$\mathrm{St}_z(x) := \{y \in B \mid x \leq y \leq z\}$$

While $\mathrm{St}_z(x)$ is not a subcomplex of $\mathcal{C}(\bar{z})$, we may form a chain complex $\mathcal{C}(\mathrm{St}_z(x))$ by restricting d. This is exactly the chain complex $\mathcal{C}\big(\bar{z}, \complement_z(z)\big)$.

1.4.2. Simply Connected Regular Chain Complexes.

D 1.49. A regular chain complex C is *connected* if the augmentation map

$$\varepsilon : C \to \mathbb{Z}$$

is an isomorphism on H_0.

L 1.50. *A regular chain complex is connected if and only if its geometric realization is connected.*

P . A triangulated space X is connected if and only if the augmentation map

$$\varepsilon \mathcal{C}(X) \to \mathbb{Z}$$

is an isomorphism on H_0. □

D 1.51. A regular chain complex C is *simply connected* if the geometric realization $\mathcal{G}(C)$ is connected and simply connected. Let $\widetilde{REG}$ and $\widetilde{CEL}$ denote the full subcategories of REG and CEL consisting of the simply connected regular chain complexes.

C 1.52. *The quasi-isomorphism type of a simply connected regular chain complex in $\widetilde{CEL}$ determines a homotopy type of simply connected topological spaces.*

P . A regular chain complex C determines a simplicial complex $\mathcal{G}(C)$. If

$$f : C \to D$$

is a quasi-isomorphism in $\widetilde{CEL}$, then by Corollary 1.44,

$$\mathcal{G}(f) : \mathcal{G}(C) \to \mathcal{G}(D)$$

induces an isomorphism on homology. Since $\mathcal{G}(C)$ and $\mathcal{G}(D)$ are simply connected, $\mathcal{G}(f)$ is a homotopy equivalence. □

R 1.53. Corollary 1.52 is false without the simply connectivity assumption. Let X be the Poincaré dodecahedral space with an open ball removed, so that X has no homology and a nonzero fundamental group. Then chains on some triangulation of X give a regular chain complex C. The augmentation map $\varepsilon : C \to \mathbb{Z}$ is a cellular chain map to the regular chain complex $(\mathbb{Z}, *, 0)$, the based chain complex with the basis a one point set. The cellular map ε is a quasi-isomorphism, but $\mathcal{G}(\varepsilon)$ is not a homotopy equivalence.

CHAPTER 2

Categories of Local Chain Complexes

We will now associate to each regular chain complex C a category of chain which are "local with respect to C". We know from the results of Chapter 1 that the basis B of a regular chain complex is a homology cell decomposition of a space. The differential of a B-local chain complex should be "local" with respect to this decomposition. We the define additional structure on our category of local chain complexes so that we can say what it means for such chain complexes to satisfy local Poincaré duality.

The definitions in this chapter are inspired by Ranicki's "categories over simplicial complexes" [**Ran92**, Chapters 4 and 5]; indeed they specialize to those definitions in the case where B is the poset of a simplicial complex. Related definitions are developed in [**RW90**] and [**RW09**].

2.1. *B*-local chain complexes

D   2.1 (The B-local Category). Given a regular based chain complex (C, B, ∂), we define the category B-LOC of *B-local chain complexes*.

(1) A B-local module is a finitely generated free $\mathbb{Z}$-module D with the additional structure of a decomposition

$$D = \sum_{x \in B} D(x)$$

(2) A B-local map is a module map $f : D \longrightarrow E$ such that for all $x \in B$,

$$f(D(x)) \subset \sum_{x \leq y} E(y)$$

(3) A B-local chain complex is a chain complex of free $\mathbb{Z}$-modules (D, d) such that
   (a) D is *bounded*. That is to say, there exist $m, n \in \mathbb{Z}$ such that $D_k = 0$ for $k < m$ and $k > n$.
   (b) Each D_k is B-local
   (c) $d : D_k \longrightarrow D_{k-1}$ is a B-local map

(4) A map of $f : D \to E$ of B-local chain complexes is a map of the underlying chain complexes satisfying (2).

(5) A *local chain equivalence* of B-local chain complexes is a chain equivalence which has a B-local chain homotopy inverse.

We think of a local chain complex D as one which has been decomposed into pieces $D(x)$, with each piece lying over a cell x of B. The image of $D(x)$ under the differential d must be contained in the pieces of D which lie over the open star of x. In most interesting examples, the chain complex D will be related to the chain complex C and the geometry of its geometric realization. Note however, that while D is a free chain complex, B need not be a basis for D.

R 2.2. Given a regular chain complex B, the category of B-local modules is an additive category. If D and E are B-local, then $D \oplus E$ has an obvious direct sum decomposition

$$(D \oplus E)(x) := D(x) \oplus E(x)$$

The trivial module is the zero object.

Recall that given any additive category $\mathbb{A}$, there is an additive category $\mathbb{B}(\mathbb{A})$ of chain complexes in $\mathbb{A}$. In particular, the expression $\partial\partial = 0$ makes sense in any additive category, as the set of morphisms between any two objects is an abelian group. The category B-LOC of local chain complexes is equivalent to the category of chain complexes in the additive category of local modules.

E 2.3. Let $B = \{*\}$ be a one point set in degree 0. The $(\mathbb{Z}, *, 0)$ is a regular chain complex, and the category of B-local chain complexes is equivalent to the ordinary category of chain complexes.

The most important examples of B-local chain complexes are those that capture local information about the geometric realization of the regular chain complex C. We discuss a central example in two guises.

E 2.4. Let C be a regular chain complex with basis B. Then the barycentric subdivision C' has the following B-local structure. The summand $C'(x)$ is generated by

$$\{x_0 \prec \ldots \prec x_k \in B' \quad | \quad x_0 = x\}$$

Observe that

$$d'(x_0 \prec \ldots \prec x_k) = \sum_{i=0}^{k} (-1)^{i+1} x_0 \prec \ldots \hat{x}_i \ldots \prec x_k \subset \sum_{x \leq y} C'(y)$$

Thus the differential d' of the barycentric subdivision satisfies property (2) of Definition 2.1 As a chain complex, $C'(x)$ is the simplicial chain complex of the open dual cone of the cell x.

We now describe another example of a B-local chain complex which is locally chain equivalent to Example 2.4.

D 2.5 (Pair Subdivision). Given a regular chain complex (C, B, ∂), we define a new based chain complex (P, E, d) called the *pair subdivision* of C. The basis E is given by

$$E_k := \{(y, x) \in B \times B \;\mid\; x \leq y, |y| - |x| = k\}$$

The differential d is given by

$$d(y, x) = (\partial y, x) + (1)^{|y|-|x|}(y, \delta x)$$

where

(1) δ is the adjoint of ∂ defined by $\langle \delta y, x \rangle = \langle y, \partial x \rangle$
(2) If $\partial y = \sum \alpha^i z_i$, then the expression $(\partial y, x)$ is to be interpreted as the sum $\sum \alpha^i (z_i, x)$, where pairs (z, x) are declared to be zero unless $x \leq z$.

We will have much more to say about the pair subdivision in Chapter 3; for now we merely mention it as an example of a B-local chain complex.

E 2.6. The pair subdivision (P, d) of a regular chain complex C has the following B-local structure.

$$P = \sum_{x \in B} P(x)$$

where $P(x)$ is generated by pairs

$$\{(y, x) \in B \times B \;\mid\; x \leq y\}$$

If $(y, x) \in P(x)$, then

$$d(y, x) = (\partial y, x) + (-1)^{|y|-|x|}(x, \delta y) \in P(x)$$

Thus (P, d) satisfies condition 2 of Definition 2.1.

2.1.1. Mapping Cones. Let $\mathbb{A}$ be any additive category, and let $\mathbb{B}$ be the additive category of bounded chain complexes in $\mathbb{A}$. Recall that a chain complex D in $\mathbb{B}$ is *contractible* if there a degree $+1$ map $h : D \to D$ such that $dh + hd = 1_D$. A chain complex is *acyclic* if $H_k(D) = 0$ for all k.

D 2.7 (Mapping Cone). Let $f : D \to E$ be a chain map in $\mathbb{B}$. Then the *mapping cone of f* is the chain complex $\mathcal{C}(f)$ defined as follows.

$$\mathcal{C}(f)_k := D_{k-1} \oplus E_k$$

The differential

$$(d_C)_k : D_{k-1} \oplus E_k \to D_{k-2} \oplus E_{k-1}$$

is given by the matrix

$$\begin{pmatrix} d_D & 0 \\ (-1)^k f & d_E \end{pmatrix}$$

A B-local map of B-local chain complexes has a well-defined mapping cone. To be explicit, if $f : D \to E$ is a B-local map, then the mapping cone $C(f)$ has the following B local decomposition.

$$C(f)(x)_k = D(x)_{k-1} \oplus E(x)_k$$

The differential d_C is a B-local map because d_D, d_E, and f are all B-local.

P 2.8. *Let $\mathbb{A}$ be an additive category and $\mathbb{B}$ be the additive category of bounded chain complexes in $\mathbb{A}$. Then a chain map $f : D \to E$ in $\mathbb{B}$ is a chain equivalence — i.e. has a chain homotopy inverse — if and only if the mapping cone $C(f)$ is contractible.*

P . This proposition is a generalization of the standard fact about the category of chain complexes in the additive category of $\mathbb{Z}$-modules. See [**Ran85**, Proposition 1.1] for an explicit proof. □

The proof of the following proposition is routine.

P 2.9. *Let $f : D \to E$ be a map of chain complexes in some additive category $\mathbb{A}$. Then inclusion and projection define chain maps*

$$E_k \xrightarrow{i} C(f)_k$$

$$C(f)_k \xrightarrow{\pi} D_{k-1}$$

These maps can be chained together to give a long exact sequence on homology

$$\ldots H_k(D) \xrightarrow{f_*} H_k(E) \xrightarrow{i_*} H_k(C(f)) \xrightarrow{\pi_*} H_{k-1}(D) \ldots$$

P 2.10. *If (D, d) is a B-local chain complex, then each $D(x)$ forms a chain complex by restricting d and projecting.*

P . The map

$$d_x : D(x) \xrightarrow{d} \sum_{x \leq y} D(y) \twoheadrightarrow D(x)$$

squares to 0 because $d^2 = 0$ and d satisfies property (2) of Definition 2.1. Thus $(D(x), d_x)$ is a chain complex. (Here the two-headed arrow $\twoheadrightarrow$ denotes the projection map from $\sum_{x \leq y} D(y)$ to $D(x)$. We will use this notation in the sequel.) □

P 2.11. *(Compare* [**RW90**, Proposition 2.7]*.) A B-local chain complex D is contractible in the B-local category if and only if for all $x \in B$, the chain complex $D(x)$ is acyclic.*

P . Suppose that (D, d) is a B-local chain complex which is B-locally chain contractible. that is to say, suppose there is a degree 1 B-local map $h : D \to D$ such that

$$dh + hd = 1_D$$

By definition of being a B-local map, for all $x \in B$,

$$h(D(x)) \subset \sum_{x \leq y} D(y)$$

Let h_x denote the component of h which maps $D(x)$ into $D(x)$:

$$h_x : D(x) \xrightarrow{h} \sum_{x \leq y} D(y) \twoheadrightarrow D(x)$$

Then the restrictions of h and d satisfy

$$d_x h_x + h_x d_x = 1_{D_x}$$

Thus each local chain complex $D(x)$ is chain contractible, and thus *a fortiori* acyclic.

Suppose conversely that each $D(x)$ is acyclic. We will constructed a B-local chain homotopy $h : D \to D$ such that

$$dh + hd = 1_D$$

We argue using a double induction, over the dimensions of the cells of B and over the degrees of the chains of D. Each D_k is a finitely generated free $\mathbb{Z}$-module with a decomposition

$$D_k = \sum_{x \in B} D_k(x)$$

Choose a basis $\{e^i_{j,k}\}$ for each D_k, where j denotes the dimension of the cell $x \in B$ such that $e^i_{j,k} \in D(x)$. The based chain complex C is finite, so there exists some n such that $B_k = 0$ for $k < 0$ and $k > n$. Furthermore D is bounded, so there exists some r such that $D_k = 0$ for $k < r$. Thus the chain map $d : D_{r+1} \to D_r$ is surjective. Let $e^i_{n,r}$ be a generator of $D_r(x)$ for some n-cell $x \in B$. By hypothesis, $H_r(D(x)) = 0$. Since $d_x(e^i_{n,r}) = 0$, there is some chain $\alpha \in D_{r+1}(x)$ such that $d_x(\alpha) = e^i_{n,r}$. Since x is a top-dimensional cell of C, x is a maximal element of the poset B, and so

$$\sum_{x \leq y} D(y) = D(x)$$

Thus $d_x = d : D(x) \to D(x)$. We define $h(e^i_{n,r}) \coloneqq \alpha$, so that

$$\begin{aligned} dh + hd(e^i_{n,r}) &= d\alpha + h(0) \\ &= d_x \alpha \\ &= e^i_{n,r} \end{aligned}$$

Since D_r is a free $\mathbb{Z}$-module, we may make similar choices for each generator of each $D(x)$ with $|x| = n$ and then extend linearly to define a map

$$h : \sum_{|x|=n} D_r(x) \to \sum_{|x|=n} D_{r+1}(x)$$

such that

$$dh + hd = 1 : D_r \to D_r \tag{21}$$

$$h(D(x)) \subset \sum_{x \leq y} D(y) \tag{22}$$

Now suppose we have defined a map h satisfying (21) and (22) on $D(x)_r$ for all $x \in B$ with $|x| > m$. Let $x \in B$ be an m-cell and let $e^i_{m,r}$ be a generator of $D_r(x)$. Once again, since $D_{r-1} = 0$, $d(e^i_{m,r}) = 0$. Since $H_r(D(x)) = 0$, there exists some $\alpha \in D_{r+1}(x)$ such that $d_x(\alpha) = e^i_{m,r}$. However, it is no longer the case that $d = d_x$, since x is not a top dimensional cell. Rather,

$$d(\alpha) = e^i_{m,r} + \beta$$

where

$$\beta \in \sum_{x \lneq y} D_r(y)$$

By our inductive hypothesis, $h\beta$ is defined and satisfies (21) and (22). We define

$$h(e^i_{m,r}) \coloneqq \alpha - h\beta$$

Now we compute

$$\begin{aligned} dh + hd(e^i_{m,r}) &= d(\alpha - h\beta) + h(0) \\ &= e^i_{m,r} + \beta - dh(\beta) \\ &= e^i_{m,r} + \beta - (1 - hd)(\beta) \\ &= e^i_{m,r} + \beta - \beta \\ &= e^i_{m,r} \end{aligned}$$

Thus by induction we have defined a map $h : D_r \to D_{r+1}$ satisfying (21) and (22)

Now suppose that a map h satisfying (21) and (22) has been defined on D_l for $l < k$, and also for $D_k(x)$ with $|x| > m$. Let x be an m-cell of B, and let $e^i_{m,k}$ be a generator of $D_k(x)$. We compute

$$\begin{aligned} d\left(e^i_{m,k} - hd(e^i_{m,k})\right) &= de^i_{m,k} - dh(de^i_{m,k}) \\ &= de^i_{m,k} - (1 - hd)(de^i_{m,k}) \\ &= de^i_{m,k} - de^i_{m,k} + hdde^i_{m,k} \\ &= 0 \end{aligned}$$

Since $d(e^i_{m,k} - hd(e^i_{m,k})) = 0$, $d_x(e^i_{m,k}) = 0$ as well. By hypothesis, $H_k(D(x)) = 0$, so there must be some $\alpha \in D_{k+1}(x)$ such that $d_x(\alpha) = 0$. Then

$$d(\alpha) = e^i_{m,r} + \beta$$

where

$$\beta \in \sum_{x \lneq y} D_k(y)$$

Once again we define

$$h(e^i_{m,r}) := \alpha - h\beta$$

By the exact same computation as in the previous paragraph,

$$dh + hd(e^i_{m,r}) = e^i_{m,r}$$

By making similar choices for each generator $e^i_{m,r}$ and each $x \in B$ with $|x| = m$, we may extend h to $\sum_{|x|=m} D_k(x)$. Thus by induction we define a degree $+1$ map $h : D \to D$ satisfying (21) and (22), that is, a B-local chain contraction of D. □

If $f : D \to E$ is a B-local chain map, let $f(x)$ denote the restriction

$$f(x) : D(x) \xrightarrow{f} \sum_{x \leq y} E(y) \twoheadrightarrow E(x)$$

C 2.12. *Let $f : D \to E$ be a B-local chain map. Then f has a B-local chain homotopy inverse if and only if $f(x)$ is a quasi-isomorphism for each x in B.*

P . By Proposition 2.8, $f : D \to E$ has has a B-local chain homotopy inverse if and only if its mapping cone $C(f)$ is contractible. By Proposition 2.11, $C(f)$ is contractible if and only if each $C(f)(x)$ is a contractible chain complex. As chain complexes,

$$C(f)(x) = C(f(x))$$

By Proposition 2.8, each $C(f(x))$ is contractible if and only if each $f(x)$ is has a chain homotopy inverse. Since each $f(x) : D(x) \to E(x)$ is a map of free chain complexes over $\mathbb{Z}$, each $f(x)$ has a chain homotopy inverse if and only if each $f(x)$ is a quasi-isomorphism. □

2.2. A duality functor on B-LOC

We wish to describe Poincaré duality objects in the B-local category, so we need a notion of the dual of a local chain complex. A B-local chain complex (D, d) has a decomposition

$$D = \sum_{x \in B} D(x)$$

Let D^{-*} denote the hom dual chain complex defined by

$$D^{-*}_k := \mathrm{Hom}(D_{-k}, \mathbb{Z})$$

with differential

$$d^*(\alpha)(x) := \alpha(dx)$$

Then the B-local structure on D induces a decomposition of D^{-*} as follows:

$$\left(D^{-*}\right)(x) := D(x)^{-*} \tag{23}$$

However, the differential d^* of D^{-*} need not satisfy property 2 of Definition 2.1, so (23) does not in general give D^{-*} the structure of a B-local chain complex.

We need a more refined notion of dual on the B-local category; the structure we require a chain duality in the sense of Ranicki. Fist let us observe the following.

R 2.13. Let $\mathbb{A}$ be an additive category and let $\mathbb{B}$ be the additive category of bounded chain complexes in $\mathbb{A}$. Then, as Ranicki observes, any contravariant additive functor

$$T : \mathbb{A} \to \mathbb{B}$$

extends to a contravariant additive functor

$$T : \mathbb{B} \to \mathbb{B}$$

as follows. Given a chain complex C in $\mathbb{B}$, define a chain complex TC as follows.

$$(TC)_r := \sum_{p+q=r} T(C_{-p})_q$$

Observe that for each map

$$d_C : C_p \to C_{p-1}$$

the functor T defines a chain map

$$T(d_C) : T(C_{p-1})_* \to T(C_p)_*$$

Thus the following map is a degree -1 chain map on TC:

$$d_{TC} = \sum_{p+q=r} (-1)^p \left(d_{T(C_{-p})} + T(d_C) \right) : (TC)_r \to (TC)_{r-1}$$

Given a chain map $f : C \to D$, we define a chain map $T(f)$ in the same fashion:

$$T(f)_r = \sum_{p+q=r} (f_{-p})_q : (TD)_r \to (TC)_r$$

Similarly, a natural transformation $e : F \to G$ between two functors $\mathbb{A} \to \mathbb{B}$, can be extended to a natural transformation between the extensions of F and G.

D 2.14. [**Ran92**, Definition 1.1] Let $\mathbb{A}$ be an additive category, and let $\mathbb{B}$ be the additive category of bounded chain complexes in $\mathbb{A}$. A *chain duality* (T, e) on $\mathbb{A}$ is

(1) A contravariant functor $T : \mathbb{A} \to \mathbb{B}$
(2) A natural transformation $e : T^2 \to 1$

Here T^2 denotes the composition of the functor T with the extension of T to $\mathbb{B}$. The functor $1 : \mathbb{A} \hookrightarrow \mathbb{B}$ is the inclusion functor which maps each object A of $\mathbb{A}$ to the chain complex which is equal to A in dimension 0 and is 0 in all other dimensions.

These data satisfy:

(1) The natural transformation e is a quasi-isomorphism

$$e_A : T^2(A) \longrightarrow A$$

for each object A of $\mathbb{A}$.

(2) For each object A of $\mathbb{A}$, the following diagram commutes:

$$\begin{array}{ccc} T(A) & \xrightarrow{T(e_A)} & T^3(A) \\ & \searrow^{1} & \downarrow{\scriptstyle e_{T(A)}} \\ & & T(A) \end{array}$$

The most familiar example of such a structure is the usual hom dual functor on the ordinary category of chain complexes.

E 2.15. Let $\mathbb{A}$ be the category of finitely generated $\mathbb{Z}$-modules, and let $T : \mathbb{A} \to \mathbb{A}$ be defined by

$$T(M) := \mathrm{Hom}(M, \mathbb{Z})$$

Then

$$T^2(M) = \mathrm{Hom}(\mathrm{Hom}(M, \mathbb{Z}))$$

For each $x \in M$, let eval_x denote the map

$$\begin{aligned} \mathrm{eval}_x : TM = \mathrm{Hom}(M, \mathbb{Z}) &\to \mathbb{Z} \\ \alpha &\mapsto \alpha(x) \end{aligned}$$

Then there is a natural isomorphism $e : 1 \to T^2$ given by

$$\begin{aligned} e(M) : M &\to T^2 M \\ x &\mapsto \mathrm{eval}_x \end{aligned}$$

Then (T, e^{-1}) is a chain duality on $\mathbb{A}$; that is to say, these data satisfy the conditions of Definition 2.14. We will denote this chain duality (T_C, e_C) to indicate that it is the standard hom duality on the category C of chain complexes.

2.2.1. The Functor $T : B$-LOC $\to B$-LOC. We now define a chain duality on B-LOC.

D 2.16. Let (C, B, ∂) be a regular chain complex. Let $\mathbb{A}$ denote the additive category of of B-local modules. We define a contravariant functor

$$T : \mathbb{A} \longrightarrow B\text{-LOC}$$

as follows. Let M be B-local module in $\mathbb{A}$ with decomposition

$$M = \sum_{x \in B} M(x)$$

We define a chain complex TM in B-LOC as follows:

$$TM_k := \sum_{\substack{x\in B\\ |x|=-k}} \sum_{x\leq y} \mathrm{Hom}(M(y), \mathbb{Z})$$

This chain complex is defined so as to have a natural B-local decomposition:

$$TM(x) := \sum_{x\leq y} \mathrm{Hom}(M(y), \mathbb{Z})$$

For every pair of generators (x, y) with $x \leq y$ there is an inclusion map:

$$\sum_{y\leq z} M(z) \overset{i_{x,y}}{\hookrightarrow} \sum_{x\leq z} M(z)$$

Let $\delta_{x,y}$ denote the hom dual restriction map:

$$TM(x) = \sum_{x\leq y} \mathrm{Hom}(M(z), \mathbb{Z}) \xrightarrow{\delta_{x,y}} \sum_{y\leq z} \mathrm{Hom}(M(z), \mathbb{Z}) = TM(y)$$

We define a differential $\delta : TM \longrightarrow TM$ by

$$\delta = \sum_{\substack{x,y\in B\\ |y|=|x|+1}} \langle \partial y, x\rangle \left(\delta_{x,y} : TM(x) \to TM(y)\right)$$

R 2.17. Observe that

$$TM_k := \sum_{\substack{x\leq y\\ |x|=-k}} \mathrm{Hom}(M(y), \mathbb{Z})$$

Thus for each $x \leq y$, the module $\mathrm{Hom}(M(y), \mathbb{Z})$ is contained in $TM(x)_{-|x|}$. Since $M(y)^{-*}$ is standard notation for $\mathrm{Hom}(M(y), \mathbb{Z})$, it makes sense to write

$$TM(x) = \sum_{x\leq y} M(y)^{-|x|-*}$$

R 2.18. As a chain complex,

$$\begin{aligned} TM &= \sum_{x\in B} TM(x) \\ &= \sum_{x\in B} \sum_{x\leq y} M(y)^{-|x|-*} \\ &= \sum_{y\in B} \sum_{x\leq y} M(y)^{-|x|-*} \\ &= \sum_{y\in B} (C(\overline{y}) \otimes M(y))^{-*} \end{aligned}$$

The B-local decomposition of TM is somewhat obscured by this presentation of the chain complex. However we see that

$$TM(x) \cong \sum_{x \leq y} (x \otimes M(y))^{-*}$$

where x denotes the copy of $\mathbb{Z}$ in degree $|x|$ generated by x.

L 2.19. *(TM, δ) is a B-local chain complex. That is to say δ is a degree -1 map such that*

$$\delta(TM(x)) \subset \sum_{x \leq y} TM(y) \tag{24}$$

$$\delta^2 = 0 \tag{25}$$

P . Since M is a module concentrated in degree 0,

$$TM(x)_k := \begin{cases} \sum_{x \leq y} \mathrm{Hom}(M(y), \mathbb{Z}) & |x| = -k \\ 0 & |x| \neq -k \end{cases}$$

If $|y| = |x| + 1$, then $TM(x)$ is concentrated in degree $-|x|$ and $TM(y)$ is concentrated in degree $-|x| - 1$, so $\delta_{x,y} : TM(x) \to TM(y)$ is a degree -1 map. Thus

$$\delta = \sum_{\substack{x < y \\ |y| = |x|+1}} \langle \partial y, x \rangle \delta_{x,y}$$

is a degree -1 map.

The map δ is defined to satisfy (25).

It remains to check that $\delta^2 = 0$. Since $TM := \sum_{x \in B} TM(x)$, $\delta^2 = 0$ if and only if

$$TM(x) \hookrightarrow TM \twoheadrightarrow TM(y)$$

for every x and y.

$$TM(x) \xrightarrow{\delta^2} TM(y) = \sum_{\substack{x < z < y \\ |y| = |z|+1 = |x|+2}} \langle \partial y, z \rangle \langle \partial z, x \rangle \left(TM(x) \xrightarrow{\delta x, z} TM(z) \xrightarrow{\delta z, y} TM(y) \right)$$

The sum

$$\sum_{\substack{x < z < y \\ |y| = |z|+1 = |x|+2}} \langle \partial y, z \rangle \langle \partial z, x \rangle$$

is the coefficient of x in the formal expression $\partial^2(y)$. Since $\partial^2 = 0$, this coefficient must be 0, and so

$$\delta^2 : TM(x) \longrightarrow TM(y)$$

is identically 0. □

We have defined an local chain complex (TM, δ) for each local module M. We now extend this correspondence of objects to a contravariant functor.

D 2.20. If $f : M \to N$ is a morphism of B-local modules, then by definition

$$M(x) \hookrightarrow M \xrightarrow{f} N \twoheadrightarrow N(y)$$

is 0 unless $x \leq y$. Thus restricting f to $M(x)$ gives a map

$$f|_{M(x)} : M(x) \longrightarrow \sum_{x \leq y} N(y)$$

For each $x, z \in B$ with $x \leq z$, there is a map

$$f_{x,z} : M(z) \xrightarrow{f|_{M(z)}} \sum_{z \leq y} N(y) \hookrightarrow \sum_{x \leq y} N(y)$$

Thus for each $x \in B$ there is a map

$$f_x := \sum_{x \leq z} f_{x,z} : \sum_{x \leq z} M(z) \longrightarrow \sum_{x \leq y} N(y)$$

If $f : M \to N$ is a morphism of B-local modules, we define

$$f_x^{-*-|x|} : TN(x) = \sum_{x \leq y} N(y)^{-*-|x|} \longrightarrow \sum_{x \leq z} M(z)^{-*-|x|} = TM(x)$$

where $f_x^{-*-|x|} = S^{-|x|} T_C(f_x)$ is the hom dual of f_x, shifted down in degree by $|x|$. Finally, we define:

$$T(f) = \sum_{x \in B} f_x^{-*-|x|} : TN \longrightarrow TM$$

L 2.21. *If $f \in Hom_{B\text{-}LOC}(M, N)$, then $T(f) \in Hom_{B\text{-}LOC}(TN, TM)$. That is to say, $T(f)$ is a degree 0 B-local chain map.*

P . First observe that $T(f)$ maps $TN(x)$ into $TM(x)$, so *a fortiori* $T(f)$ satisfies the condition

$$T(f)(TN(x)) \subset \sum_{x \leq y} TM(y)$$

Thus $T(f)$ is a B-local map. Since $TN(x)$ and $TM(x)$ are both concentrated in degree $-|x|$, $T(f)$ is a degree 0 map.

Next we must check that $T(f)$ is a chain map, that is, that the following diagram commutes.

$$\begin{array}{ccc} TN & \xrightarrow{T(f)} & TM \\ \downarrow{\scriptstyle \delta_{TN}} & & \downarrow{\scriptstyle \delta_{TM}} \\ TN & \xrightarrow{T(f)} & TM \end{array}$$

Since

$$TN = \sum_{x \in B} TN(x) = \sum_{x \in B} \sum_{x \leq y} N(y)^{-*-|x|}$$

it suffice to check that the diagram commutes on $N(y)^{-*-|x|}$ for each $x \preceq y \in B$.

Let $x \prec y \in B$, and let $\alpha \in N(y)^{-*-|x|}$. Then

$$\alpha : S^{-|x|}N(y) \to \mathbb{Z}$$

where $S^{-|x|}$ denotes desuspension by $|x|$. Then $T(f)(\alpha) \in TM(x)$ is the composition

$$\sum_{x \leq v} S^{-|x|}M(v) \xrightarrow{\sum_{x \leq v} S^{-|x|} f_{x,v}} \sum_{x \leq z} S^{-|x|}N(z) \xrightarrow{\alpha} \mathbb{Z} \tag{26}$$

Since $\alpha : N(y) \to \mathbb{Z}$, this composition reduces to:

$$\sum_{x \leq v} S^{-|x|}M(v) \xrightarrow{\sum_{x \leq v} S^{-|x|} f_{x,v,y}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z}$$

where $f_{x,v,y}$ is the composition

$$M(v) \xrightarrow{f_{x,v}} \sum_{x \leq z} N(z) \twoheadrightarrow N(y)$$

Since f is B-local, the composition

$$S^{-|x|}M(v) \xrightarrow{S^{-|x|} f_{x,v,y}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z}$$

is zero unless $v \preceq y$. Thus (26) can be written as:

$$\sum_{x \leq v \leq y} S^{-|x|}\left(M(v) \xrightarrow{f_{x,v,y}} N(y) \xrightarrow{\alpha} \mathbb{Z}\right) \tag{27}$$

Recall that $\delta_{TM} : TM(x) \to \sum_{w \leq x} TM(w)$ is defined to be

$$\sum_{\substack{x \leq w \\ |w|=|x|+1}} \langle \partial w, x \rangle \delta_{x,w}$$

where $\delta_{x,w}$ is the degree -1 map

$$\sum_{x \leq z} M(z)^{-*-|x|} \longrightarrow \sum_{w \leq z} M(z)^{-*-|w|}$$

hom dual to the degree $+1$ inclusion map

$$\sum_{w \leq z} S^{-|w|}M(z) \xrightarrow{i_{x,w}} \sum_{x \leq z} S^{-|x|}M(z)$$

Thus $\delta_{x,w}T(f)(\alpha)$ is the composition:

$$\sum_{w \leq z} S^{-|w|}M(z) \xrightarrow{i_{x,w}} \sum_{x \leq v \leq y} S^{-|x|}M(v) \xrightarrow{f_{x,v,y}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z} \tag{28}$$

and this composition is 0 unless $y \preceq w$, and so can be written more simply as:

$$\sum_{w \preceq v \preceq y} \left(S^{-|w|}M(v) \xrightarrow{i_{x,w}} S^{-|x|}M(v) \xrightarrow{f_{x,v,y}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z} \right) \tag{29}$$

Note that

$$i_{x,w} : S^{-|w|}M(v) \longrightarrow S^{-|x|}M(v)$$

is simply the suspension map which shifts degree by $+1$. Thus

$$\delta_{TM}T(f)(\alpha) \in \sum_{\substack{x \preceq w \\ |w|=|x|+1}} TM(w)$$

is given by:

$$\sum_{\substack{w \preceq x \\ |w|=|x|+1}} \langle \partial w, x \rangle \sum_{y \preceq v \preceq w} \left(S^{-|w|}M(v) \xrightarrow{i_{x,w}} S^{-|x|}M(v) \xrightarrow{f_{x,v,y}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z} \right) \tag{30}$$

We check that going around the diagram the other way gives the same result. Start with the same $\alpha : S^{-|x|}N(y) \to \mathbb{Z}$. Then $\delta_{TN}(\alpha) \in \sum_{x \preceq w} TN(w)$ is given by the composition:

$$\sum_{\substack{x \preceq w \\ |w|=|x|+1}} \langle \partial w, x \rangle \left(\sum_{w \preceq z} S^{-|w|}N(z) \xrightarrow{i_{x,w}} \sum_{x \preceq z} S^{-|x|}N(z) \xrightarrow{\alpha} \mathbb{Z} \right)$$

Since α is nonzero only on $N(y)$, this reduces to:

$$\sum_{\substack{x \preceq w \\ |w|=|x|+1}} \langle \partial w, x \rangle \left(\sum_{w \preceq z} S^{-|w|}N(z) \xrightarrow{i_{x,w}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z} \right)$$

This composition is 0 unless $w \preceq y$, and so can be written as:

$$\sum_{\substack{x \preceq w \\ |w|=|x|+1}} \langle \partial w, x \rangle \sum_{w \preceq z \preceq y} \left(S^{-|w|}N(z) \xrightarrow{i_{x,w}} S^{-|x|}N(y) \xrightarrow{\alpha} R \right) \tag{31}$$

Composing with $T(f)$ gives

$$T(f)(\delta_{TN})(\alpha) \in \sum_{\substack{x \preceq w \\ |w|=|x|+1}} TM(w)$$

which is the composition:

$$\sum_{\substack{x \preceq w \\ |w|=|x|+1}} \langle \partial w, x \rangle \left(\sum_{w \preceq v} S^{-|w|}M(v) \xrightarrow{\sum_{w \preceq v} S^{-|w|} f_{w,v}} \sum_{w \preceq z \preceq y} S^{-|w|}N(z) \xrightarrow{i_{x,w}} S^{-|x|}N(y) \xrightarrow{\alpha} \mathbb{Z} \right) \tag{32}$$

Since f is B-local, this composition is 0 unless $v \le y$. Thus (32) can be written more simply as:

$$(33) \qquad \sum_{\substack{x \le w \\ |w|=|x|+1}} \langle \partial w, x \rangle \sum_{w \le v \le y} \left(S^{-|w|} M(v) \xrightarrow{f_{w,v,y}} S^{-|w|} N(y) \xrightarrow{i_{x,w}} S^{-|x|} N(y) \xrightarrow{\alpha} \mathbb{Z} \right)$$

We see that (30) and (33) are the same map, because

$$i_{x,w} \circ f_{w,v,y} = f_{x,v,y} \circ i_{x,w} : S^{-|w|} M(v) \to S^{-|x|} N(y)$$

That is to say, f restricted to $M(v)$ commutes with the suspension map $i_{x,w}$. □

E 2.22. Let $(\mathbb{Z}, B, 0)$ be the trivial regular chain complex where the basis consists of a single point. Then a B-local chain complex D is just an ordinary chain complex with no additional structure. In this case $TD = D^{-*}$; that is to say, the local duality functor T coincides with the "global" hom duality functor.

2.2.2. The Natural Transformation between T^2 and the Identity. Throughout this section (C, B, ∂) is a regular chain complex. We now show that the functor T of Definition 2.16 extends to a chain duality in the sense of Definition 2.14. Let M be a B-local module, and let T_C denote the hom duality functor of Example 2.15. We compute $T^2 M$.

$$\begin{aligned} T^2 M(x) &= \sum_{x \le y} S^{-|x|} T_C \left(T M(y) \right) \\ &= \sum_{x \le y} S^{-|x|} T_C \left(\sum_{y \le z} S^{-|y|} T_C M(z) \right) \\ &= \sum_{x \le y \le z} S^{|y|-|x|} T_C^2 M(z) \end{aligned}$$

Now we introduce some notation to simplify computing with $T^2 M$. Let $\alpha_{x,y,z}$ denote an element of $S^{|y|-|x|} T_C^2 M(z)$. Then

$$d_{TM} \alpha_{x,y,z} = (-1)^{|y|} \alpha_{\delta x,y,z} + \alpha_{x,\partial y,z}$$

where

$$\alpha_{\delta x,y,z} := \sum_{\substack{x \le w \\ |w|=|x|+1}} \langle \partial w, x \rangle \delta_{w,x}(\alpha_{x,y,z})$$

Note that $\delta_{w,x}(\alpha_{x,y,z}) = 0$ unless $w \le y$. Similarly,

$$\alpha_{x,\partial y,z} := \sum_{\substack{w \le y \\ |w|=|y|-1}} \langle \partial y, w \rangle i_{w,y} \alpha_{x,y,z}$$

Note that $i_{w,y}(\alpha_{x,y,z}) = 0$ unless $z \le w$.

D 2.23. Let M be a B-local module We define a B-local map as follows.

$$e(M) : T^2M(x) = \sum_{x\le y\le z} S^{|y|-|x|}T_C^2M(z) \to \sum_{x\le z} M(z)$$

$$\alpha \in S^{|y|-|x|}T_C^2M(z) \mapsto \begin{cases} e_C(M(z))\alpha & x = y \\ 0 & x \ne y \end{cases}$$

Using the notation established above,

$$e(M)(\alpha_{x,y,z}) = \begin{cases} (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z))(\alpha) & x = y \\ 0 & x \ne y \end{cases}$$

L 2.24. *If M is a B-local module, then*

$$e(M) : T^2M \to M$$

is a B-local chain equivalence.

P . First we check that e is a chain map. Observe that M is concentrated in degree 0 and that $e(M)$ is nonzero only on the degree 0 part of T^2M. Since M is concentrated in degree 0, we need only check that for any 1-chain α in T^2M, $e(M)d_{TM}\alpha = 0$. Let $\alpha_{x,y,z}$ be a 1-chain in $S^{|y|-|x|}T_C^2M(z)$. Then $|y| - |x| = 1$ and x is a codimension 1 face of y. We compute:

$$\begin{aligned} e(M)d_{TM}\alpha_{x,y,z} &= e(M)\left((-1)^{|y|}\alpha_{\delta x,y,z} + \alpha_{x,\partial y,z}\right) \\ &= e(M)\left((-1)^{|y|}\langle \partial y, x\rangle \alpha_{y,y,z} + \langle \partial y, x\rangle \alpha_{x,x,z}\right) \\ &= \langle \partial y, x\rangle (-1)^{|y|+\frac{y(y+1)}{2}} e_C(M(z))(\alpha) + \langle \partial y, x\rangle (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z))(\alpha) \\ &= \langle \partial y, x\rangle e_C(M(z))(\alpha)\left((-1)^{|y|+\frac{|y|(|y|+1)}{2}} + (-1)^{\frac{(|y|-1)|y|}{2}}\right) \\ &= 0 \end{aligned}$$

To see that the final equation of this computation is true, observe that

$$|y| + \frac{|y|(|y|+1)}{2} = \frac{y^2+3y}{2}$$

$$\frac{(|y|-1)|y|}{2} = \frac{y^2-y}{2}$$

Since $y^2 + 3y = y^2 - y \mod 4$,

$$\frac{y^2+3y}{2} = \frac{y^2-y}{2} \mod 2$$

Thus $e(M)$ is a chain map.

Next we show that $e(M)$ is a B-local chain equivalence. By Corollary 2.12, it suffices to show that each

$$e(M)(x) : T^2M(x) \to M(x)$$

is a quasi-isomorphism. Let $x \in B$. As previously observed,

$$T^2M(x)_k = \{\alpha_{x,y,z} \mid x \le y \le z, |y| - |x| = k, \alpha \in T^2_C(M(z))\}$$

The restriction $d_{TM}(x)$ is the map

$$d_{TM}(x) : \alpha_{x,y,z} \mapsto \alpha_{x,\partial y,z}$$

Thus $T^2M(x)$ splits as a direct sum of chain complexes:

$$\begin{aligned} T^2M(x) &= \sum_{x \le z} \{\alpha_{x,y,z} \mid x \le y \le z, \alpha \in T^2_C(M(z))\} \\ &\cong \sum_{x \le z} \left(S^{-|x|}C(\mathrm{St}_z(x)) \otimes T^2_C(M(z))\right) \end{aligned}$$

Here we have observed that the collection

$$\{\alpha_{x,y,z} \mid x \le y \le z\}$$

is precisely the basis for the open star of x in z, and the differential

$$\alpha_{x,y,z} \mapsto \alpha_{x,\partial y,z}$$

is the differential of the chain complex $C(\mathrm{St}_z(x))$. However there is a shift in dimension because the generator corresponding to $\{x, y, z\}$ has dimension $|y|$ in $C(\mathrm{St}_z(x))$ and dimension $|y| - |x|$ in $T^2M(x)$.

Now, by Corollary 1.47 and Remark 1.48, $C(\mathrm{St}_z(x))$ is acyclic if $x \ne z$ and has a single generator in degree x if $x = z$. Thus all of the summands of $T^2M(x)$ are acyclic except for the summand corresponding to $x = z$. Thus $H_*(T^2M(x))$ is concentrated in degree 0, and is generated by

$$S^{-|x|}C(\mathrm{St}_x(x)) \otimes T^2_C(M(x))$$

where $S^{-|x|}C(\mathrm{St}_x(x))$ is a has one generator in degree 0 corresponding to x. Thus the map

$$e_M(x) : T^2M(x) = \sum_{x \le z} \left(S^{-|x|}C(\mathrm{St}_z(x)) \otimes T^2_C(M(z))\right) \to M(x)$$

$$\alpha_{x,y,z} \mapsto \begin{cases} (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z))\alpha & x = y = z \\ 0 & x \ne y \text{ or } y \ne z \end{cases}$$

induces an isomorphism on homology, because $e_C(M(z)) : T^2_C M(z) \to M(z)$ is an isomorphism. We have shown that $e(TM)(x)$ is a quasi-isomorphism for each x, so $e(TM)$ is a B-local equivalence. □

L 2.25. *The map*

$$e : T^2 \to 1$$

is a natural transformation between T^2 and the inclusion functor

$$1 : \mathbb{A} \to B\text{-}LOC$$

That is to say, for each B-local map $f : M \to N$ of B-local modules, the following diagram commutes:

(34)
$$\begin{array}{ccc} T^2 M & \xrightarrow{e(M)} & M \\ {\scriptstyle T^2 f}\downarrow & & \downarrow{\scriptstyle f} \\ T^2 N & \xrightarrow{e(N)} & N \end{array}$$

P . Let $x \in B$. Since

$$T(f)\,(TN(x)) \subset TM(x)$$

it follows that

$$T^2(f)\left(T^2 M(x)\right) \subset T^2 N(x)$$

Let $T^2 f(x)$ denote the map

$$T^2(f)|_{T^2 M(x)} : T^2 M(x) \to T^2 N(x)$$

Let (T_C, e_C) denote the chain duality on the category of chain complexes of Example 2.15. As in Definition 2.16, let f_x denote the map

$$f_y := f|_{\sum_{y \le z} M(z)} : \sum_{y \le z} M(z) \longrightarrow \sum_{y \le z} N(z)$$

Then we have

$$\begin{aligned} (T^2 f)(x) &:= S^{-|x|} T_C((Tf)_x) \\ &= S^{-|x|} T_C \left(\sum_{x \le y} S^{-|y|} T_C f_y \right) \\ &= \sum_{x \le y} S^{|y|-|x|} \left(T_C^2(f_y) : T_C^2 \left(\sum_{y \le z} M(z) \right) \longrightarrow T_C^2 \left(\sum_{y \le z} N(z) \right) \right) \end{aligned}$$

Thus

$$\begin{aligned} e(N) T^2 f(x) &= e(N) \left(\sum_{x \le y} S^{|y|-|x|} \left(T_C^2(f_y) : T_C^2 \left(\sum_{y \le z} M(z) \right) \longrightarrow T_C^2 \left(\sum_{y \le z} N(z) \right) \right) \right) \\ &= \begin{cases} (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z)) \left(T_C^2 \left(f_x : \left(\sum_{x \le z} M(z) \right) \longrightarrow \left(\sum_{x \le z} N(z) \right) \right) \right) & x = y \\ 0 & x \neq y \end{cases} \\ &= \begin{cases} f_x (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z)) T_C^2 \left(\sum_{x \le z} M(z) \right) \longrightarrow T_C^2 \left(\sum_{x \le z} N(z) \right) & x = y \\ 0 & x \neq y \end{cases} \\ &= f_x \left(e(M)(x) \right) \\ &= (f e(M))(x) \end{aligned}$$

Thus Diagram 34 commutes as claimed. □

L 2.26. *The functor T and natural transformation e satisfy the following coherence condition. For any B-local module M, the following diagram commutes.*

$$\begin{array}{ccc} T(M) & \xrightarrow{T(e(M))} & T^3(M) \\ & \searrow^{1} & \downarrow e_{T(M)} \\ & & T(M) \end{array}$$

P . Let $x \in B$. Then

$$TM(x) = \sum_{x \leq y} S^{-|x|} T_C M(y)$$

By definition,

$$e_M : T^2 M(x) = \sum_{x \leq y \leq z} S^{|y|-|x|} T_C^2(M(z)) \to M(z)$$

$$\alpha_{x,y,z} \mapsto \begin{cases} (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z))\alpha & x = y \\ 0 & x \neq y \end{cases}$$

Thus

$$T(e_M) : TM(x) = \sum_{x \leq y} S^{-|x|} T_C M(y) \xrightarrow{S^{-|x|} T_C \left(e(M)|_{\sum_{x \leq y} T^2 M(y)}\right)} \sum_{x \leq y} S^{-|x|} T_C T^2 M(y) = T^3 M(x)$$

Now, the map

$$e(M) : \sum_{x \leq y} T^2 M(y) = \sum_{x \leq y \leq z} S^{|y|-|x|} T_C^2 M(z) \to \sum_{x \leq z} M(z) \twoheadrightarrow M(w)$$

is nonzero only on the summand where $x = y$ and $z = w$. Thus

$$\begin{aligned} T(e(M))\left(S^{-|x|} T_C M(w)\right) &= (-1)^{\frac{|x|(|x|+1)}{2}} T_C(e_C(M(w))\left(S^{-|x|} T_C M(w)\right) \\ &= (-1)^{\frac{|x|(|x|+1)}{2}} S^{-|x|} T_C T_C^2 M(w) \end{aligned}$$

By definition

$$e(TM) : T^2(TM)(x) = \sum_{x \leq y \leq z} S^{|y|-|x|} T_C^2(TM(z)) \to TM(z)$$

$$\alpha_{x,y,z} \mapsto \begin{cases} (-1)^{\frac{|x|(|x|+1)}{2}} e_C(M(z))\alpha & x = y \\ 0 & x \neq y \end{cases}$$

Thus the composition

$$\begin{aligned} e(TM)\circ T(e(M))\left(S^{-|x|}T_C M(w)\right) &= e(TM)(-1)^{\frac{|x|(|x|+1)}{2}} S^{-|x|}T_C T_C^2 M(w)\\ &= e(TM)(-1)^{\frac{|x|(|x|+1)}{2}} S^{-|x|}T_C^2 T_C M(w)\\ &= (-1)^{\frac{|x|(|x|+1)}{2}}(-1)^{\frac{|x|(|x|+1)}{2}} S^{-|x|}e_C(T_C M(w))T_C^2 T_C M(w)\\ &= S^{-|x|}T_C M(w) \end{aligned}$$

We have shown that $e(TM)\circ T(e(M))$ maps $S^{-|x|}T_C M(w)$ to $S^{-|x|}T_C M(w)$ via the map

$$T_C(e_C(M(w))\circ e_C(T_C M(w))$$

Since (T_C, e_C) is a chain duality, this map is the identity. Thus

$$e(TM)\circ T(e(M)) : TM \to TM$$

is the identity map, as desired. □

P 2.27. *The functor $T : \mathbb{A} \to B$-LOC and the natural transformation $e : T^2 \to 1$ of Definition 2.23 are a chain duality on the additive category $\mathbb{A}$ of B-local modules.*

P . Lemmas 2.24, 2.25, and 2.26 show that the data (T, e) satisfy the conditions of Definition 2.14. □

2.3. Algebraic Bordism Categories

Once again, our goal in the is chapter is to sufficiently enrich the category of B-local chain complexes so that we can define what it means for such chain complexes to satisfy local Poincaré duality. Roughly, a n-dimensional Poincaré duality object in a category should something which is equivalent to its dual shifted in dimension by n. Thus, to define algebraic Poincaré duality, we will need to consider categories equipped with notions of duality, shift in dimension, and equivalence. We now introduce the definition, due to Ranicki, of such categories.

D 2.28. [**Ran92**, Definition 3.2] An *algebraic bordism category* $(\mathbb{A}, \mathbb{B}, \mathbb{C})$ is

(1) An additive category $\mathbb{A}$ with chain duality (T, e)
(2) A full subcategory $\mathbb{B}$ of the additive category of bounded chain complexes in $\mathbb{A}$
(3) A full subcategory $\mathbb{C}$ of $\mathbb{B}$

These data satisfy

(1) The categories $\mathbb{B}$ and $\mathbb{C}$ are closed under the operation of taking mapping cones.
(2) For each object B of $\mathbb{B}$, the mapping cone of the identity morphism $1 : B \to B$ is an object of $\mathbb{C}$.

(3) For each object B of $\mathbb{B}$, the mapping cone of the morphism

$$e_B : T^2(B) \longrightarrow B$$

is an object of $\mathbb{C}$.

E 2.29. Let $\mathbb{A}$ be the category of free $\mathbb{Z}$-modules, and let (T_C, e_C) be the usual hom duality on this category. Let $\mathbb{B}$ be the category of bounded chain complexes of free $\mathbb{Z}$-modules, and let $\mathbb{C}$ be the subcategory of contractible chain complexes. Then $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ form an algebraic bordism category. We call Λ the *global* algebraic bordism category, as its objects are chain complexes that do not possess and B-local structure.

D 2.30. Let (C, B, ∂) be a regular based chain complex. We define two algebraic bordism categories associated to C. Let $\mathbb{A}(B)$ be the category of B-local modules, and let (T, e) be the chain duality on $\mathbb{A}$ of Proposition 2.27. Let $\mathbb{B}(B)$ be the full subcategory of B-LOC consisting of those chain complex which have no local homology in negative degrees. That is to say D is an object in $\mathbb{B}$ if D is B-local and for each $x \in B$

$$H_k(D(x)) = 0 \text{ for } k < 0$$

Let $\mathbb{C}_s(B)$ be the full subcategory of $\mathbb{B}$ consisting of locally contractible chain complexes. That is to say, a B-local chain complex D is in $\mathbb{C}_s(B)$ if the 0 map D to the 0 chain complex has a B-local chain homotopy inverse. We define

$$\Lambda_s(B) \coloneqq (\mathbb{A}(B), \mathbb{B}(B), \mathbb{C}_s(B))$$

to be the *strong local* algebraic bordism category of (C, B, ∂).

Let $\mathbb{C}_w(B)$ be the full subcategory of $\mathbb{B}$ consisting of chain complexes D which satisfy

(1) For all $x \in B$,

$$H_k(D(x)) = 0 \text{ for } k < 1$$

(2) The chain complex D is globally contractible, that is to say it is chain contractible after forgetting the B-local structure.

Observe that $\mathbb{C}_s(B)$ is a subcategory of $\mathbb{C}_w(B)$. We define

$$\Lambda_w(B) \coloneqq (\mathbb{A}(B), \mathbb{B}(B), \mathbb{C}_w(B))$$

to be the *weak local* algebraic bordism category of (C, B, ∂).

P 2.31. *If (C, B, ∂) is a regular chain complex, then $\Lambda_s(B)$ and $\Lambda_w(B)$ are algebraic bordism categories.*

P . We have given the data required for an algebraic bordism category; we check that this data satisfies the necessary conditions. First let us check that $\mathbb{B}(B)$,

$\mathbb{C}_w(B)$, and $\mathbb{C}_s(B)$ are closed under the operation of taking mapping cones. If $f : D \to E$ is a map in $\mathbb{B}(B)$ then for each $x \in B$,

$$H_k(D(x)) = H_k(E(x)) = 0$$

for $k < 0$. It follows immediately from looking at the long exact sequence of Proposition 2.9 that $H_k(C(f)(x)) = 0$ for $k < 0$ as well. If $f : D \to E$ is a map in $\mathbb{C}_s(B)$, then D and E are locally contractible, that is to say, contractible as chain complexes in B-LOC. Thus f must induce an isomorphism on homology. Since a map of free chain complexes which induces an isomorphism on homology is a chain equivalence, $C(f)$ is (locally) contractible by Proposition 2.8. A combination of the two previous arguments shows that $\mathbb{C}_w(B)$ is closed under mapping cones as well.

The identity map is a B-local chain equivalence, so the cone of the identity map is a chain complex in $\mathbb{C}_s(B)$ and thus in $\mathbb{C}_w(B)$ as well. Finally, it follows from the proof of Proposition 2.27 that the natural transformation e_B is a local chain equivalence for each object B in $\mathbb{B}(B)$. This its mapping cone is an object in $\mathbb{C}_s(B)$ and thus also an object in $\mathbb{C}_w(B)$. □

2.4. Algebraic Poincaré Complexes

The extra structure that an algebraic bordism category possesses is exactly the structure needed to define what it means for a chain complex in an additive category to satisfy Poincaré duality. The idea is that a chain complex satisfying n-dimensional Poincaré duality should be

(1) A chain complex D in $\mathbb{B}$
(2) A map $\phi : \Sigma^n TD \to D$, where Σ^n denotes n-fold suspension, such that the cone of ϕ is a chain complex in $\mathbb{C}$

Let us make this idea precise.

D 2.32. Let D be a bounded chain complex in any additive category $\mathbb{A}$. Then let $\Sigma^n D$ denote the chain complex defined as follows:

$$(\Sigma^n D)_k = D_{k-n}$$
$$d_{\Sigma^n D} = (-1)^k d_D : (\Sigma^n D)_k \to (\Sigma^n D)_{k-1}$$

We introduce the sign so that the abelian group $H_n(\mathrm{Hom}_{\mathbb{A}}(D, E))$ is the group of chain homotopy classes of chain maps $\Sigma^n D \to E$.

D 2.33. Let $\mathbb{A}$ be an additive category with chain duality (T, e), and let D be a chain complex in $\mathbb{A}$. Let $\mathrm{Hom}(TD, D)$ denote the chain complex where

$$\mathrm{Hom}(TD, D)_k = \sum_{q-p=k} \mathrm{Hom}(TD_p, D_q)$$
$$d_{\mathrm{Hom}}(f) = d_D f + (-1)^q f d_{TD}$$

Then let $\mathcal{T}(D)$ denote the chain map

$$\mathcal{T}(D) : \mathrm{Hom}(TD, D)_k = \sum_{q-p=k} \mathrm{Hom}(TD_p, D_q) \to \sum_{q-p=k} \mathrm{Hom}(TD_p, D_q)$$
$$f \mapsto (-1)^{pq} e(D) T(f)$$

D 2.34 (Ranicki). An *n-dimensional Poincaré complex* in an algebraic bordism category $(\mathbb{A}, \mathbb{B}, \mathbb{C})$ is

(1) A chain complex D in $\mathbb{B}$
(2) A sequences of maps $\{\phi_s : \Sigma^{n+s} TD \to D\}_{s \geq 0}$ in $\mathbb{B}$

These data satisfy

(1) For each ϕ_s with $s > 1$,
$$d_{\mathrm{Hom}(TD,D)} \phi_s = (-1)^{n+s} \left(\phi_{s-1} + (-1)^s \mathcal{T}_D(\phi_{s-1})\right)$$
(2) The map ϕ_0 is a chain map such that the desuspension of the mapping cone $S^{-1}C(\phi_0 : \Sigma^n TD \to D)$ a chain complex in $\mathbb{C}$

We should caution the reader that what we have defined here are what Ranicki calls *symmetric* Poincaré complexes. We will not discuss the relation notions of *quadratic* Poincaré complexes.

R 2.35. Let C be an ordinary chain complex over $\mathbb{Z}$. Then a chain map $\phi : \Sigma^n C^{-*} \to C$ is equivalent to a n-cycle in $C \otimes C$. Thus, specifying a Poincaré duality map $\phi : \Sigma^n C^{-*} \to C$ is equivalent to specifying a co-inner product $\eta : \mathbb{Z} \to C \otimes C$. A co-inner product product on C is the action of a particular properad on the chain complex C. One can work out the definition of a "coherent homotopy co-inner product" or "infinity co-inner product". The higher terms ϕ_s in our definition of an algebraic Poincaré are exactly an extension of ϕ_0 to be an infinity co-inner product. This interpretation of the higher terms is not important to us here, so we will not develop it further.

E 2.36. Let M be a simplicial complex which is a closed oriented n-dimensional manifold, or more generally an n-dimensional Poincaré duality space. Let $C_*(M)$ denote the simplicial chain complex of M, and let

$$D_M : C^{n-*}(M) \longrightarrow C_*(M)$$

denote the cap product with the fundamental class of M. Then $(C_*(M), D_M)$ determine a Poincaré complex in the algebraic bordism category of Example 2.29. The chain equivalence $\phi_0 = D_M$ is given by the Poincaré duality map. The higher homotopies ϕ_s for $s > 0$ come from the symmetries of the diagonal map, exactly as in Steenrod's construction of the Steenrod squares. See Proposition 4.10 for a construction of the higher ϕ_s using Steenrod's method of acyclic carriers.

Just as one can define bordisms between geometric Poincaré complexes, one can define bordisms between algebraic Poincare complexes.

D 2.37 (Ranicki). Let (D, ϕ) and (D', ϕ') be Poincaré complexes in an algebraic bordism category $(\mathbb{A}, \mathbb{B}, \mathbb{C})$. A *cobordism* between (D, ϕ) and (D', ϕ') is

(1) A map in $\mathbb{B}$

$$f \oplus f' : D \oplus D' \longrightarrow E$$

(2) A collection of maps in $\mathbb{B}$

$$\{\theta_s : \Sigma^{n+1+s} TE \longrightarrow E\}_{s \geq 0}$$

These data satisfy

(1) The map θ_0 satisfies

$$d_{\mathrm{Hom}(TE,E)}\theta_0 = (-1)^n \left(f\phi_0 T(f) - f'\phi_0' T(f') \right)$$

(2) For each θ_s with $s > 1$

$$d_{\mathrm{Hom}(TE,E)}\theta_s = (-1)^{n+s-1} \left(\theta_{s-1} + (-1)^s \mathcal{T}(E)(\theta_{s-1})\right) + (-1)^n \left(f\phi_0 T(f) - f'\phi_0' T(f') \right)$$

(3) The mapping cone of the following map is a chain complex in $\mathbb{C}$

$$\theta_0 \oplus \phi_0 T(f) \oplus \phi_0' T(f') : \Sigma^{n+1} TE \longrightarrow C(f \oplus f')$$

where C denotes the mapping cone.

E 2.38. Let (W, M, M') be a geometric cobordism of manifolds. Then $(C_*(M), D_M)$ and $(C_*(M'), D_{M'})$ are Poincaré complexes, as discussed in Example 2.36 Let

$$i : M \hookrightarrow W$$
$$i' : M' \hookrightarrow W$$

denote the inclusion maps. Let

$$D_W : C^{n+1-*}(W) \to C_*(W)$$

denote the cap product map with the relative fundamental class of the manifold with boundary W. This map is not a chain map. However, the relative fundamental class $[W]$ is a homology between $[M]$ and $[M']$. Thus:

$$dD_W + D_W d^* = iD_M i^* - i' D_{M'} (i')^*$$

That is to say, capping with the relative fundamental class $[W]$ is a chain homotopy between capping with $[M]$ and capping with $[M']$. Furthermore, the Lefschetz duality map

$$D_W : C^{n+1-*} \longrightarrow C_*(W, M \amalg M') = C(i \oplus i')$$

is a quasi-isomorphism. The data

$$i \oplus i' : C_*(M) \oplus C_*(M') \longrightarrow W$$

$$D_W : C^{n+1-*}(W) \to C_*(W)$$

determine a cobordism between the Poincaré complexes $(C_*(M), D_M)$ and $(C_*(M'), D_{M'})$.

We have now given examples of Poincaré complexes in the "global" algebraic bordism category of chain complexes and hom duality. We have yet to give examples of Poincaré complexes in the "local" algebraic bordism categories of Definition 2.30; such examples are the subject of the next chapter.

P 2.39 (Ranicki). *Let Λ be an algebraic bordism category. Cobordism is an equivalence relation on n-dimensional Poincaré complexes in Λ. The cobordism classes of n-dimensional Poincaré complexes form an abelian group under direct sum, with the 0 chain complex and 0 duality map as the identity. These groups are known as the* symmetric L-groups *of* Λ.

P . This is Proposition 3.2 of [**Ran80a**] stated in the language of algebraic bordism categories. □

P 2.40. *Let (D,ϕ) and (D',ϕ') be Poincaré complexes in an algebraic bordism category $\Lambda = (\mathbb{A},\mathbb{B},\mathbb{C})$. Let $f : D \to D'$ be a chain map in $\mathbb{B}$. Then for each ϕ_s, let $f^{\%}\phi_s$ denote the map*

$$f^{\%}\phi_s : TD' \xrightarrow{Tf} TD \xrightarrow{\phi_s} D \xrightarrow{f} D'$$

Then $(D', f^{\%}\phi)$ is a Poincaré complex in Λ. We say f is a homotopy equivalence *between (D,ϕ) and (D',ϕ') if*

(1) *f is a chain equivalence in $\mathbb{B}$*
(2) *For each $s \geq 0$, there exists a map $\theta_s : \Sigma^{n+s}TD' \to D'$ such that*

$$d_{Hom(TD,D)}\theta_s + (-1)^{n+s+1}(\theta_{s-1} + (-1)^s\mathcal{T}(D')\theta_{s-1}) = \phi'_s - f^{\%}\phi_s$$

Homotopy equivalent complexes are cobordant.

P . [**Ran80a**, Proposition 3.2] □

2.5. Functors of Algebraic Bordism categories

Suppose that $\mathbb{A}$ and $\mathbb{A}'$ are two additive categories with chain dualities T and T' which have been given the structure of algebraic bordism categories. A functor $F : \mathbb{A} \to \mathbb{A}'$ wil not not necessarily induce a map from Poincaré complexes in $\mathbb{A}$ to Poincaré complexes in $\mathbb{A}'$. For suppose (D,ϕ) is a n-dimensional Poincaré complex in $\mathbb{A}$. Then

$$\phi : \Sigma^n TD \to D$$

so

$$F(\phi) : \Sigma^n F(TD) \to F(D)$$

But this is not the data of a Poincaré complex in $\mathbb{A}'$; we need a map

$$\Sigma^n T'(FD) \to F(D)$$

Thus we need the additional data of a natural map

$$T'(FD) \to F(TD)$$

in order to define a Poincaré complex in $\mathbb{A}'$. This need motivates the following definition.

D 2.41. [**Ran92**, Definition 3.7] A functor of algebraic bordism categories

$$F : (\mathbb{A}, \mathbb{B}, \mathbb{C}) \longrightarrow (\mathbb{A}', \mathbb{B}', \mathbb{C}')$$

is a covariant additive functor $F : \mathbb{A} \to \mathbb{A}'$ such that

(1) For each chain complex D of $\mathbb{B}$, $F(D)$ is a chain complex in $\mathbb{B}'$
(2) For each chain complex D of C, $F(D)$ is a chain complex in C'
(3) The chain dualities T and T' are related in the following way. For each object A in $\mathbb{A}$, there is a natural transformation

$$G_A : T'FA \longrightarrow FTA$$

such that
 (a) The mapping cone $C(G_A)$ is a chain complex in $\mathbb{C}'$
 (b) The following diagram commutes:

$$\begin{array}{ccc} T'FT(A) & \xrightarrow{G_{TA}} & FT^2(A) \\ \downarrow{\scriptstyle T'G_A} & & \downarrow{\scriptstyle Fe_A} \\ (T')^2FA & \xrightarrow[e'_{FA}]{} & FA \end{array}$$

P 2.42. [**Ran92**, Proposition 3.8] *A functor of algebraic bordism categories induces a morphism of cobordism groups.*

P 2.43. *Let C be a regular chain complex. Then the augmentation map $\varepsilon : C \to \mathbb{Z}$ is a cellular map from the regular chain complex (C, B, ∂) to the regular chain complex $(\mathbb{Z}, *, 0)$ where $*$ is the one point set. The augmentation map induces an* assembly *functor*

$$A : B\text{-}LOC \to C$$

where C the category of ordinary chain complexes with no local structure. For any B-local chain complex D, AD is simply the chain complex D with the B-local structure forgotten. The functor A extends to functors of algebraic bordism categories:

$$A : \Lambda_s(B) \longrightarrow \Lambda$$

$$A : \Lambda_w(B) \longrightarrow \Lambda$$

where Λ is the global algebraic bordism category of Example 2.29

P . The assembly of B-local chain complex is a chain complex. The assembly of a B-locally contractible chain complex is globally contractible, since the assembly of a B-local chain contraction is a global chain contraction. To extend A to a functor of algebraic bordism categories, we must define for each B-local module M, a natural equivalence

$$G_M : T_C(AM) \longrightarrow A(TM)$$

As we observed in Remark 2.18

$$ATM = \sum_{x \in B} T_C\left(C(\overline{x}) \otimes M(x)\right)$$

For each x, let

$$\varepsilon_x : C(\overline{x}) \to \mathbb{Z}$$

denote the augmentation map. Then we define:

$$\varepsilon_x \otimes 1 : C(\overline{x}) \otimes M(x) \to M(x)$$
$$y \otimes \alpha \mapsto \varepsilon_x(y)\alpha$$

Then

$$G_M := T_C\left(\varepsilon_x \otimes 1\right) : T_C AM = T_C\left(\sum_{x \in B} M(x)\right)$$
$$\to T_C\left(\sum_{x \in B} C(\overline{x}) \otimes M(x)\right) = ATM$$

is a natural chain equivalence. □

P 2.44. *Let C be a regular chain complex There is a forgetful functor of algebraic bordism categories*

$$F : \Lambda_s(B) \longrightarrow \Lambda_w(B)$$

inducing a morphism of cobordism groups

$$F : L^n(\Lambda_s(B)) \longrightarrow L^n(\Lambda_w(B))$$

P . Recall that

$$\Lambda_s(B) := (\mathbb{A}(B), \mathbb{B}(B), \mathbb{C}_s(B))$$
$$\Lambda_w(B) := (\mathbb{A}(B), \mathbb{B}(B), \mathbb{C}_w(B))$$

where $\mathbb{A}(B)$ is the additive category of B-local modules and $\mathbb{B}(B)$ is the full additive subcategory of B-LOC consisting B-local chain complexes such that

$$H_k(D(x)) = 0 \text{ for } k < 0$$

The additive category $\mathbb{A}(B)$ has the chain duality (T, e) of Proposition 2.27. Thus we define $F : \mathbb{A}(B) \to \mathbb{A}(B)$ to be the identity functor, and

$$G : TF \longrightarrow FT$$

to be the identity natural transformation. The only thing we need to check is that for each chain complex D in $\mathbb{C}_s(B)$, $F(D) = D$ is a chain complex in $\mathbb{C}_w(B)$. If D is a chain complex in $\mathbb{C}_s(B)$, then D by definition is B-locally chain contractible. Then by Proposition 2.11, $H_k(D(x)) = 0$ for all k and x. Furthermore, a B-local chain contraction of D is, by forgetting the B-local structure, a global chain contraction of D. Thus D is globally chain contractible and $H_k(D(x)) = 0$ for all $k < 1$, so D is a chain complex in $\mathbb{C}_w(B)$. □

D 2.45. Given two Poincaré complexes in $\Lambda_w(B)$ we will say that they are *weakly locally cobordant* if there is a cobordism between them in $\Lambda_w(B)$. Given two Poincaré complexes in $\Lambda_s(B)$ we will say that they are *strongly locally cobordant* if there is a cobordism between them in $\Lambda_s(B)$, and *weakly locally cobordant* if there is a cobordism between their images under F in $\Lambda_w(B)$.

We now show that algebraic bordism categories we have defined are equivalent to certain categories defined by Ranicki.

P 2.46. *Let (C, B, ∂) be a simply connected regular chain complex, and let (C', B', ∂') be its barycentric subdivision. Let $K = \mathcal{G}(C)$, so that $\mathcal{S}_*(\mathcal{G}(C)) = C'$. Then the simplicial chain functor*

$$\mathcal{S} : SIM \to CEL$$

induces an equivalence of categories

$$K\text{−}LOC \to B'\text{-}LOC$$

This equivalence extends to equivalences of algebraic bordism categories

$$\Lambda_w(K) \xrightarrow{\sim} \Lambda_w(B')$$
$$\Lambda_s(K) \xrightarrow{\sim} \Lambda_s(B')$$

This in turn in induces isomorphisms of groups such that the following diagram commutes and the vertical arrows are isomorphisms.

(35)
$$\begin{array}{ccc} H_n(K; \mathbb{L}^\bullet) & \xrightarrow{A} & VL^n(K) \\ \downarrow & & \downarrow \\ L^n(\Lambda_s(B')) & \xrightarrow{F} & L^n(\Lambda_w(B')) \end{array}$$

P . First we explain some of the notation in the statement of the theorem, which refers to algebraic bordism categories and groups defined by Ranicki. The category $K\text{−}LOC$ is the additive category of chain complexes which are local over a finite simplicial complex K [**Ran92**, Definition 4.1]. Ranicki denotes this category $\mathbb{B}(\mathbb{Z}, K)$. In the case where B' is the set of simplicies of a simplicial complex, then the categories $K\text{−}LOC$ and B'-LOC are the same. Indeed, all of the definitions in this chapter are equivalent to Ranicki's definitions in Chapters 4 and 5 of [**Ran92**] in the case where B is the poset of a simplicial complex. Since every regular chain complex (C, B, ∂) is equivalent to the simplicial chain complex $\mathcal{S}_*(\mathcal{G}(C))$, we may associate to every regular chain complex an algebraic bordism categories $\Lambda_w(B')$ and $\Lambda_s(B')$ which are equivalent to Ranicki's categories associated to $K = \mathcal{G}(C)$. Our $\Lambda_s(B')$ is equivalent to Ranicki's

$$\Lambda(\mathbb{A}(\mathbb{Z}, K), \mathbb{B}\langle 0\rangle(\mathbb{Z}, K), \mathbb{C}\langle 0\rangle(\mathbb{Z})_*(K))$$

Our $\Lambda_w(B')$ is equivalent to Ranicki's

$$\Lambda(\mathbb{A}(\mathbb{Z}, K), \mathbb{B}\langle 0\rangle(\mathbb{Z}, K), \mathbb{C}\langle 1\rangle(\mathbb{Z}, K))$$

See [**Ran92**, Chapter 15] for Ranicki's definitions of these algebraic bordism categories.

The (symmetric) L-groups of the algebraic bordism category $L^n(\Lambda_s(K))$ are a generalized homology theory, called symmetric $\mathbb{L}$-theory [**Ran92**, Proposition 13.7]. Hence the groups $L^n(\Lambda_s(K))$ are equal to the homology groups $H_n(K; \mathbb{L}^\bullet)$ for a spectrum $\mathbb{L}^\bullet$ described in [**Ran92**, Chapter 13]. The group $VL^n(K)$ is the bordism group of n-dimensional algebraic normal complexes in $\Lambda_w(K)$. However, by [**Ran92**, Remark 9.8] and [**Wei92**], $VL^n(K)$ is isomorphic to the symmetric L-group of n-dimensional algebraic Poincaré complexes $L^n(\Lambda_w(K))$. (This follows because every chain complex in $\mathbb{C}_w$ is globally contractible.) Furthermore, Ranicki's assembly functor

$$A : H_n(K; \mathbb{L}^\bullet) \to VL^n(K)$$

is simply the forgetful functor

$$F : L^n(\Lambda_s(K)) \to L^n(\Lambda_w(K))$$

so Diagram 35 commutes. □

CHAPTER 3

The Pair Subdivision of a Regular Chain Complex

In this chapter we discuss the pair subdivision of a regular chain complex. As we will see, the pair subdivision is a B-local chain complex. If regular chain chain complex C satisfies Poincaré duality, then the pair subdivision of C determines an Poincaré complex in $\Lambda_w(B)$. In the next chapter, we will relate lifts of this Poincaré complex to one in $\Lambda_s(B)$ to topological manifold structures in the homotopy type determined by C.

3.1. The Pair Subdivision

D 3.1. Let (C, B, ∂) be a regular chain complex, and let $y \in B$. Let $C^*(\overline{y})$ denote the hom dual cochain complex of $C(\overline{y})$.

$$C^*(\overline{y})_k := \mathrm{Hom}(C(\overline{y})_k, \mathbb{Z})$$

For a face x of y, let $x^* \in C^*(\overline{y})$ denote the cocycle defined on generators $z \in B$ by the relations:

$$\langle x^*, z\rangle = \begin{cases} 1 & x = z \\ 0 & x \neq z \end{cases}$$

R 3.2. The augmentation map

$$\varepsilon_x : C^*(\overline{x}) \longrightarrow \mathbb{Z}$$

is a element of $C^*(\overline{x})$. Indeed:

$$\varepsilon_x = \sum_{\substack{v<x \\ |v|=0}} v^*$$

P 3.3. *Let (C, B, ∂) be a regular based chain complex, and let $x \in B$. Then the map*

$$\begin{aligned} \overline{\varepsilon} : \mathbb{Z} &\to C^*(\overline{x}) \\ 1 &\mapsto \varepsilon \end{aligned}$$

which sends $1 \in \mathbb{Z}$ to the augmentation map ε_x is a quasi-isomorphism.

P . Since

$$H_q(C(\overline{x})) = \begin{cases} 0 & q > 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

the Universal Coefficient Theorem implies that

$$H^q(C^*(\overline{x})) = \begin{cases} 0 & q > 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

Thus it suffices to show that the cochain given by the augmentation map

$$\varepsilon_x : C^*(\overline{x}) \longrightarrow \mathbb{Z}$$

a cocycle and a generator of $H^0(C^*(\overline{x}))$. We compute

$$\begin{aligned} \delta\varepsilon_x &= \delta \sum_{\substack{v \le x \\ |v|=0}} v^* \\ &= \sum_{\substack{v \le x \\ |v|=0}} \delta v^* \\ &= \sum_{\substack{v \le x \\ |v|=0}} \sum_{\substack{v \le e \\ |e|=1}} \langle \partial e, v \rangle e^* \\ &= \sum_{\substack{e \le x \\ |v|=1}} \sum_{\substack{v \le e \\ |e|=0}} \langle \partial e, v \rangle e^* \end{aligned}$$

By Proposition 1.11, for each 1-cell $e \in B$,

$$\partial e = v_1 - v_0$$

for some $v_1, v_0 \in B$. Thus we have

$$\begin{aligned} \sum_{\substack{v \le e \\ |e|=0}} \langle \partial e, v \rangle e^* &= \langle \partial e, v_1 \rangle e^* + \langle \partial e, v_0 \rangle e^* \\ &= \langle v_1 - v_0, v_1 \rangle e^* + \langle v_1 - v_0, v_0 \rangle e^* \\ &= 0 \end{aligned}$$

Furthermore, the cocycle ε_x is a generator of the cyclic group $H_0(\mathrm{Hom}(C(\overline{x}), \mathbb{Z}))$, and thus a generator of $H^0(C^*(\overline{x}))$. □

Recall Definition 2.5 of the pair subdivision of a regular chain complex. Given a based chain complex C with basis B, the pair subdivision P is the based chain complex with basis

$$E_k := \{(y, x) \in B \times B \mid x \le y, |y| - |x| = k\}$$

and differential

$$d(y, x) = (\partial y, x) + (1)^{|y|-|x|}(y, \delta x)$$

R 3.4. Equivalently, we may think of P as chain complex generated by pairs (y, x), with $y \in B$ and $x \in C^*(\overline{y})$ a cochain supported in the closure of y.

If C is the regular chain complex of simplical chains on a simplicial complex, then P geometric pair subdivision studied by Zeeman [**Zee63**] and McCrory [**McC79**] in analyzing the failure of polyhedra to satisfy Poincaré duality. The chain complex P is bigraded, and filtering this chain complex with respect to δ gives rise to the Zeeman dihomology spectral sequence. More recently, the pair subdivision has arisen in the bivariant chains of Chataur [**Cha10**].

P 3.5. *If C is a regular chain complex, then the map $s : C \to P$ given on generators by the formula:*

$$x \mapsto (x, \varepsilon_x) = \sum_{\substack{v \leq x \\ |v|=0}} (x, v)$$

is a chain map from C to the associated pair complex P.

P . We must check that for each $x \in B$, $s(\partial x) = ds(x)$. By definition this equation says:

$$\sum_{\substack{v \leq \partial x \\ |v|=0}} (\partial x, v) = \sum_{\substack{v \leq x \\ |v|=0}} (\partial x, v) + (-1)^{|x|-|v|}(x, \delta v)$$

Because illegal pairs are defined to be 0,

$$\sum_{\substack{v \leq \partial x \\ |v|=0}} (\partial x, v) = \sum_{\substack{v \leq x \\ |v|=0}} (\partial x, v)$$

and by Proposition 3.3

$$\sum_{\substack{v \leq x \\ |v|=0}} \delta v = 0$$

□

R 3.6. The map ε_x is the unit cochain of the cup product on the cellular cochain complex of the cell x. The natural map

$$F : C \otimes C^{-*} \longrightarrow P$$

which maps $y \otimes x$ to the pair (y, x) gives the cap product on homology. (This map is called the Flexnor cap product [**McC79**].) Then the following diagram commutes:

$$\begin{array}{ccc} C \otimes C^{-*} & \xrightarrow{F} & P \\ \uparrow{\scriptstyle \bar{\varepsilon}} & \nearrow{\scriptstyle s} & \\ C & & \end{array}$$

Here F is the Flexnor cap product, s is the pair subdivision map, and $\bar{\varepsilon}^* : C \longrightarrow C \otimes C^{-*}$ is the coaugmentation map which sends x to $x \otimes \varepsilon$. Thus, the subdivision map s corresponds to capping with the unit cochain.

T 3.7. *If (C, B, ∂) is a regular based chain complex, then $s : C \longrightarrow P$ is a quasi-isomorphism.*

P . Observe that P is bigraded chain complex, with P_p^q generated by

$$\{(y, x) \in B_p \times B_q \mid x \leq y\}$$

with horizontal differential $\partial : P_p^q \longrightarrow P_{p-1}^q$ and vertical differential $\delta : P_p^q \longrightarrow P_p^{q+1}$. Since the basis B is finite, there is an n such that B_k is empty for $k > n$.

Furthermore, the chain complex C has a tautologous bigrading:

$$C_p^q = \begin{cases} 0 & q > 0 \\ C_p & q = 0 \end{cases}$$

with horizontal differential $\partial : C_p^q \longrightarrow C_{p-1}^q$ and vertical differential 0.

Filtering with respect to p gives

(1) a spectral sequence E associated to P
(2) a spectral sequence $\overline{E}$ associated to C

As the subdivision map $s : C \to P$ maps C_p^q into P_p^q, it induces a map $s_* : \overline{E} \longrightarrow E$ of spectral sequences. To show that s is a quasi-isomorphism, it suffices to show that:

L 3.8. *The induced map $s_1 : \overline{E}_1 \longrightarrow E_1$ is an isomorphism at the E_1 page.*

The page $E_{0,p}^q$ is generated by

$$\{(y, x) \in B_p \times B_q \mid x \leq y\}$$

The differential d_0 of this page is δ. As a chain complex,

$$\begin{aligned} (E_0, d_0) &= \sum_{y \in B} \left(\sum_{x \leq y} (y, x), \delta \right) \\ &= \sum_{y \in B} C^*(\overline{y}) \end{aligned}$$

The homology of this chain complex is

$$E_{1,p}^q = H_q(E_0, d_0) = \sum_{\substack{y \in B \\ |y| = p}} H_q(C^*(\overline{y}))$$

Now let us consider the spectral sequence $\overline{E}$ arising from the tautologous bigrading on C. The first differential d_0 is identically 0, so

$$\overline{E}_{1,p}^q = \overline{E}_{0,p}^q = \begin{cases} C_p & q = 0 \\ 0 & q \neq 0 \end{cases}$$

The subdivision map $s : C \longrightarrow P$ maps a generator y to the pair (y, ε_y). s induces a map

$$s_0 : \overline{E}^{\,q}_{0,p} \longrightarrow E^{q}_{0,p}$$

which takes $y \in \overline{E}^{\,0}_{0,p}$ to $\varepsilon_y \in E^0_{0,p}$. Each $\overline{E}^{\,0}_{0,p}$ is a free $\mathbb{Z}$-module which basis y, so restricting to a single generator y give the coaugmentation map

$$\overline{e}_y : \mathbb{Z} \longrightarrow C^*(\overline{y})$$

If (C, B, ∂) is regular, then by Corollary 3.3 this map is a quasi-isomorphism. Thus the induced map

$$s_1 : \overline{E}^{\,0}_{1,p} = C_p \longrightarrow H^0(C^*(\overline{y})) = E^{q}_{1,p}$$

is an isomorphism, and

$$\overline{E}^{\,q}_{1,p} = E^{q}_{1,p} = 0$$

for $q > 0$. Explicitly, s_1 is given on generators by the formula

$$s_1(y) = [\varepsilon_y]$$

where $[\varepsilon_y]$ is the cohomology class of the cochain ε_y in $H^0(C^*(\overline{y}))$.

Since both $\overline{E}$ and E are concentrated on the horizontal line $q = 0$, both spectral sequences collapse at the E_2 page and

$$s_2 : H_p(C) = \overline{E}^{\,0}_{2,p} \longrightarrow E^0_{2,p} = H_p(P)$$

is an isomorphism. □

R 3.9. This argument depends crucially on the fact that C is regular. If the closure of any cell $x \in B$ has homology in positive degrees, then E_1 will not be concentrated in the line $q = 0$ and s_1 need not be an isomorphism. For example the chain complex of Example 1.10 does not have the same homology as its pair subdivision.

R 3.10. Zeeman considered based chain complexes in [**Zee62**], though he did not define regular based chain complexes. Nevertheless, our proof of Theorem 3.7 is essentially the same as Zeemans Theorem 1 in [**Zee62**], even though the statements are different. (The author discovered Zeeman's proof after writing this one.)

P 3.11. *Let x be a cell of a regular chain complex C, and let $P(C(\overline{x}))$ denote the pair subdivision of the chain complex $C(\overline{x})$. Then the augmentation map*

$$P(C(\overline{x})) \to \mathbb{Z}$$

is a quasi-isomorphism.

P . Consider the commutative diagram:

$$\begin{array}{ccc} C(\overline{x}) & \xrightarrow{s} & P(C(\overline{x})) \\ & \searrow^{\varepsilon} & \downarrow \varepsilon \\ & & \mathbb{Z} \end{array}$$

The chain complex $C(\overline{x})$ is regular, so the following two statements are true.

(1) s is a quasi-isomorphism by Theorem 3.7.
(2) The augmentation map $\varepsilon : C(\overline{x}) \to \mathbb{Z}$ is a quasi-isomorphism.

Since the diagram commutes,

$$\varepsilon : P(C(\overline{x})) \to \mathbb{Z}$$

is a quasi-isomorphism as well. □

3.2. A Geometric Picture of the Pair Subdivision

Given a regular chain complex C we have described two functorially related chain equivalent chain complexes, the barycentric subdivision C' and the pair subdivision P. The barycentric subdivision has a geometric description as the simplicial complex of the nerve of the poset of C. In this section, we give a geometric description of the pair subdivision. More precisely, we shall prove the following theorem.

T 3.12. *Let $\partial\Delta^{n+1}$ be the simplicial complex which is the boundary standard $n+1$-simplex in $\mathbb{R}^{n+2}$. Then $\partial\Delta^{n+1}$ has a regular cell decomposition such that:*

(1) *Each k-dimensional cell is labeled by a pair (y, x), where y and x are faces of Δ^{n+1} such that $x \leq y$ and $|y| - |x| = k$*
(2) *The closure of the cell (y, x) is the amalgamation of the simplices of the barycentric subdivision of $\partial\Delta^{n+1}$ of the form $x \prec \ldots \prec y$*
(3) *The vertices of the cell (y, x) are the barycenters of all simplices z such that $x \leq z \leq y$*
(4) *The codimension 1 faces of the cell (y, x) are all cells of the form (z, x) where z a codimension 1 face of y and all cells of the form (y, z), where x is a codimension 1 face of z.*

P . We describe a second triangulation of $\partial\Delta^{n+1}$, called the *dual triangulation*. For each k-face σ of $\partial\Delta^{n+1}$, we define the *dual cell* σ^* of to be the following subcomplex of the barycentric subdivision of $\partial\Delta^{n+1}$

$$\sigma^* := \{x_0 \prec \ldots \prec x_k \in b(\partial\Delta^{n+1}) \mid x_0 = \sigma\}$$

If σ is a k-simplex, then a simplex in the barycentric subdivision of the form

$$\sigma \prec x_1 \cdots \prec x_l$$

is of dimension at most $n - k$. Thus σ^* is an $(n-k)$-dimensional subcomplex of $b(\partial\Delta^{n+1})$. Since $\partial\Delta^{n+1}$ is a closed triangulated manifold, each σ^* is a PL ball, and the collection of dual cells form a regular cell decomposition of $\partial\Delta^{n+1}$. (See [**RS72**, p. 27].) The poset of cells of the dual decomposition of $\partial\Delta^{n+1}$ is the opposite poset of the standard triangulation of $\partial\Delta^{n+1}$. That is to say, for simplices σ and τ of $\partial\Delta^{n+1}$.

$$\sigma^* \leq \tau^* \quad \Leftrightarrow \quad \tau \leq \sigma$$

We claim that the dual decomposition of $\partial\Delta^{n+1}$ is in fact a triangulation. Given a k-simplex σ in $\partial\Delta^{n+1}$, σ^* is a $(n-k)$-cell with one vertex for τ^* for each n-simplex τ such that $\sigma \leq \tau$. A k-simplex in $\partial\Delta^{n+1}$ is determined by choosing $k+1$ vertices. Since $\partial\Delta^{n+1}$ has $n+2$ vertices, each k-simplex σ is a face of

$$n + 2 - (k + 1) = n - k + 1$$

n-dimensional simplices. Thus σ^* is an $(n-k)$-cell with $n-k+1$ vertices. Furthermore every subset of those $n-k+1$ vertices determines a cofaces of σ and thus a face of σ^*. Thus σ^* is an $(n-k)$-dimensional simplex.

We have described two triangulations of the n-sphere $\partial\Delta^{n+1}$. The intersection of two triangulations is a regular cell complex. (See [**RS72**, p. 15].) Let P denote the cell complex formed by intersecting the standard triangulation and the dual triangulation of $\partial\Delta^{n+1}$. The cells of P are the set theoretic intersections

$$\sigma \cap \tau^*$$

where σ is a simplex of the standard triangulation and τ^* is a simplex of the dual triangulation. The simplex σ is made of of all simplices in the barycentric subdivision $b(\partial\Delta^{n+1})$ of the form

$$x_0 \prec \ldots \prec \sigma$$

The simplex τ^* is made of of all simplices in the barycentric subdivision $b(\partial\Delta^{n+1})$ of the form

$$\tau \prec \ldots \prec x_k$$

We conclude that

$$\sigma \cap \tau^* \neq \emptyset \quad \Leftrightarrow \quad \tau \leq \sigma \tag{36}$$

Each nonempty intersection $\sigma \cap \tau^*$ is a subcomplex of the barycentric subdivision of $\partial\Delta^{n+1}$). If σ is a k-simplex and τ is an l-simplex, then σ has codimension $n-k$ and τ^* has codimension l. Thus the codimension of the intersection $\sigma \cap \tau^*$ is $n-k+l$, and so the dimension of the intersection is $k-l$. Let $|\sigma|$ denote the dimension of σ. Then we have:

$$|\sigma \cap \tau^*| = |\sigma| - |\tau| \tag{37}$$

A face of $\sigma \cap \tau^*$ is of the form $\eta \cap \nu^*$, where η is a face of σ and ν is a coface of τ.

We claim that the cell complex P satisfies all of the conditions listed in Theorem 3.12. We show how each condition is satisfied.

(1) We define the pair (σ, τ) to be the cell $\sigma \cap \tau^*$. Then we have:
   (a) $\sigma \leq \tau$

(b) $|\sigma \cap \tau^*| = |\sigma| - |\tau|$

(2) Recall that σ is the subcomplex of the barycentric subdivision of $\partial\Delta^{n+1}$ consisting of all simplices of the form

$$x_0 \prec \cdots \prec \sigma$$

and τ^* is the subcomplex of the barycentric subdivision of $\partial\Delta^{n+1}$ consisting of all simplices of the form

$$\tau \prec \ldots \prec x_k$$

Thus $(\sigma, \tau) := \sigma \cap \tau^*$ is the subcomplex of the barycentric subdivision consisting of all simplices of the form

$$\tau \prec \ldots \prec \sigma$$

(3) The faces of (σ, τ) are the pairs (ν, η) such that $\nu \leq \eta$, $\nu \leq \sigma$, and $\tau \leq \eta$. The vertices are those pairs (ν, η) such that $|\nu| - |\eta| = 0$. If $\nu \leq \eta$ and $|\nu| - |\eta| = 0$, we must have $\nu = \eta$. The cell $\nu \cap \nu^*$ is exactly the barycenter of ν. Thus, the vertices of of (σ, τ) are precisely the barycenters of the simplices ν such that $\tau \leq \nu \leq \sigma$.

(4) A codimension 1 face of $\sigma \cap \tau^*$ is one of the following two types.
   (a) A cell of the form $\partial_i\sigma \cap \tau^*$, where $\partial_i\sigma$ is a codimension 1 face of σ
   (b) A cell of the form $\sigma \cap \partial_i\tau^*$, where $\partial_i\tau^*$ is a codimension 1 face of τ^*, or equivalently τ is a codimension 1 face of $\delta_i\tau = \partial_i\tau^*$.

Thus the codimension 1 faces of (σ, τ) are exactly the cells $(\partial_i\sigma, \tau)$, where $\partial_i\sigma$ is a codimension 1 face of σ, and the cells $(\sigma, \delta_i\tau)$, where τ is a codimension 1 face of $\delta_i\tau$.

□

C 3.13. *Let K be a finite simplicial complex. The K has a cellular subdivision P satisfying all of the conditions of Theorem 3.12.*

P . The set of vertices of K is finite; suppose K has n vertices. A choice of ordering of the vertices of K defines a simplicial embedding of K into $\partial\Sigma^n$ where we map the i-th vertex of K to the i-th vertex of of $\partial\Sigma^n$. We restrict the pair subdivision of $\partial\Sigma^n$ constructed in the proof of Theorem 3.12 to the image of K. We define this restriction to be the pair subdivision of K. □

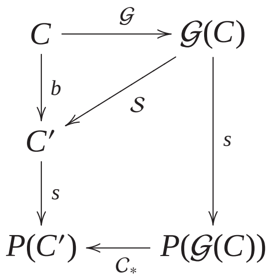
R 3.14. If x is an n-simplex, then each n-cell (x, v) of the pair subdivision of x is an amalgamation of $n!$ simplices of the barycentric subdivision of x.

D 3.15. If K is a simplicial complex, we define the *cellular chain complex of the pair subdivision of K* as follows:

$$C_*(P(K)) := P(\mathcal{S}_*(K))$$

That is to say, we define the cellular chain complex of geometric the pair subdivision $P(K)$ to be the algebraic pair subdivision of the simplicial chains of the simplicial complex K.

R 3.16. Let C be a regular chain complex with basis B. We have defined the cellular chain complex of $(P(K)$ so that the following diagram commutes.



$P(K)$ is a cell complex with one cell for each generator of the based chain complex $P(\mathcal{S}_*(K))$. To be more explicit, the set of cells of $P(K)$ and the set of generators of $P(\mathcal{S}_*(K))$ are both isomorphic to the following set

$$\{(y, x) \in B' \times B' \mid x \leq y\}$$

Furthermore, the poset structure on the set of generators for $P(\mathcal{S}_*(K))$ is the same as the poset structure on the cell complex $P(K)$. Thus the only freedom we have in defining the chain complex $C_*(P(K))$ is choosing the signs of the boundary map. By Theorem 3.7, the map

$$s : \mathcal{S}_*(K) \to P(\mathcal{S}_*(K))$$

is a quasi-isomorphism. Thus, the chain complex $C_*(P(K))$ as we have defined it computes the homology of $P(K)$.

R 3.17. As shown in Figure 3.1, the pair subdivision of a combinatorial manifold M is a decomposition of M that is the coarsest common subdivision of both the original cell decomposition and the dual cell decomposition.

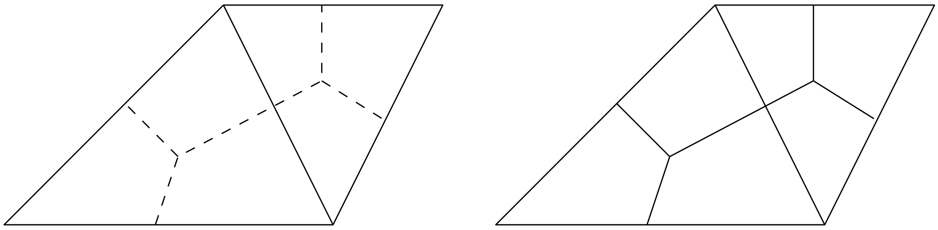
D 3.18. If K is a finite cell complex and x is a cell of K, then the *dual cone* of x, denoted Dx, is the contractable subcomplex of the barycentric subdivision of K' consisting of all cells of the form

$$\{x_0 \prec \ldots \prec x_k \in K' \mid x \preceq x_0\}$$

The boundary of the dual cone of x, denoted ∂Dx, is the subcomplex of the barycentric subdivision consisting of all cells of the form

$$\{x_0 \prec \ldots \prec x_k \in K' \mid x \precneqq x_0\}$$

Dual cones have a simple combinatorial description in the pair subdivision.



F 3.1. The left-hand picture shows a piece of a cell decomposition of a surface, with the dual cell decomposition shown with dashed lines. The right hand picture shows the coarsest common subdivision of the original decomposition and its dual. The pair complex is a combinatorial description of this common subdivision.

P 3.19. *Let K be a simplicial complex, and let P be its pair subdivision Let x be a cell of K. Then*

$$Dx = \{(y, x') \in P \mid x \leq x'\}$$
$$\partial Dx = \{(y, x') \in P \mid x \lneqq x'\}$$

P . The topological identity map from K' to P is a PL map which sends

$$\{x_0 \prec \ldots \prec x_k \in K' \mid x \leq x_0\} \mapsto \{(y, x') \in P \mid x \leq x'\}$$
$$\{x_0 \prec \ldots \prec x_k \in K' \mid x \lneqq x_0\} \mapsto \{(y, x') \in P \mid x \lneqq x'\}$$

□

3.3. Algebraic Structure on the Pair Subdivision

In this section we describe some algebraic structure on the pair subdivision of a regular chain complex. All of the algebraic structures we describe are well-defined for the pair subdivision of any regular chain complex. However, we wish to make use of the geometric picture of the pair subdivision given in Section 3.2. Thus throughout this section we assume that C is the simplicial chain complex of a simplicial complex. By Theorem 1.43, this is no loss of generality, as we may always replace C by its barycentric subdivision C'.

The pair subdivision has been studied previously, as noted in Remark 3.4. To the author's knowledge, however, the formulas in this section have not appeared in the literature. The author learned the formulas for the diagonal map and cup product from Dennis Sullivan.

P 3.20. *Let (C, B, ∂) be a regular based chain complex, and let P be its pair subdivision. The map $\Delta : P \longrightarrow P \otimes P$ given on generators by the formula*

$$\Delta(y, x) \coloneqq \sum_{x \leq z \leq y} (y, z) \otimes (z, x)$$

gives P the structure of a differential graded coassociative coalgebra. The augmentation map $\varepsilon : P \longrightarrow \mathbb{Z}$ given on generators by the formula:

$$\varepsilon(y, x) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

is a conuit for this coalgebra.

P . First we check that formula for Δ defines a differential graded coalgebra. We need the following diagram to commute:

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \otimes P \\ {\scriptstyle d_P}\downarrow & & \downarrow{\scriptstyle d_{P\otimes P}} \\ P & \xrightarrow{\Delta} & P \otimes P \end{array}$$

Let (y, x) be a generator of P.

$$\begin{aligned} \Delta d_P(y, x) &= \Delta\Big((\partial y, x) + (-1)^{|y|-|x|}(y, \delta x)\Big) \\ &= \sum_{x \leq z \leq y} \Big((\partial y, z) \otimes (z, x) + (-1)^{|y|-|x|}(y, z) \otimes (z, \delta x)\Big) \end{aligned}$$

We compute going around the diagram the other way:

$$\begin{aligned} d_{P\otimes P}\Delta(y, x) &= d_{P\otimes P} \sum_{x \leq z \leq y} (y, z) \otimes (z, x) \\ &= \sum_{x \leq z \leq y} (\partial y, z) \otimes (z, x) + (-1)^{|y|-|z|}(y, \delta z) \otimes (z, x) \\ &\quad + (-1)^{|y|-|z|}(y, z) \otimes (\delta z, x) + (-1)^{|y|-|z|}(-1)^{|z|-|x|}(y, z) \otimes (z, \delta x) \end{aligned}$$

Now, each term

$$(-1)^{|y|-|z|}(y, \delta z) \otimes (z, x)$$

in the above expansion represents a sum of terms

$$(-1)^{|y|-|z|}\langle \partial w, z \rangle (y, w) \otimes (z, x)$$

where z is a codimension 1 face of w. Similarly, each term

$$(-1)^{|y|-|z|}(y, z) \otimes (\delta z, x)$$

represents a sum of terms

$$(-1)^{|y|-|w|}\langle \partial w, z \rangle (y, w) \otimes (z, x)$$

where z is a codimension 1 face of w. Since $|w| = |z| + 1$, these terms are equal but with opposite signs. Thus all of these cross terms cancel, and

$$\Delta d_P(y, x) = d_{P\otimes P}\Delta(y, x)$$

Next we check that Δ coassociative. We need the following diagram to commute.

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \otimes P \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ P \otimes P & \xrightarrow{\Delta\otimes 1} & P \otimes P \otimes P \end{array}$$

We chase an generator around the diagram:

$$\begin{array}{ccc} (y,x) & \xmapsto{\Delta} & \sum_{x \leq w \leq y} (y,w) \otimes (w,x) \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ \sum_{x \leq z \leq y} (y,z) \otimes (z,x) & \xmapsto{\Delta\otimes 1} & \sum_{x \leq w \leq z \leq y} (y,w) \otimes (w,z) \otimes (z,x) \end{array}$$

Finally, we check that the augmentation map ε is a conunit. We need the following diagram to commute.

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \otimes P \\ {\scriptstyle \Delta}\downarrow & \searrow & \downarrow{\scriptstyle 1\otimes\varepsilon} \\ P \otimes P & \xrightarrow{\varepsilon\otimes 1} & \mathbb{Z} \otimes P \cong P \otimes \mathbb{Z} \end{array}$$

Let (y, x) be a generator. Then

$$1 \otimes \varepsilon \left(\sum_{x \leq z \leq y} (y,z) \otimes (z,x) \right)$$

is nonzero only on the term

$$(y,x) \otimes (x,x)$$

Thus

$$(1 \otimes \varepsilon)\Delta(y,x) = (y,x)$$

Similarly

$$\varepsilon \otimes 1 \left(\sum_{x \leq z \leq y} (y,z) \otimes (z,x) \right)$$

is nonzero only on the term

$$(y,y) \otimes (y,x)$$

and so

$$(\varepsilon \otimes 1)\Delta(y,x) = (y,x)$$

□

The pair subdivision P of a regular chain complex (C, B, ∂) is itself a based chain complex, with basis

$$E = \{(y, x) \in B \times B \ \mid \ x \prec y\}$$

Let P^{-*} denote the chain complex with

$$(P^{-*})_k \coloneqq \mathrm{Hom}(P_{-k}, \mathbb{Z})$$

The set E also gives a set of generators for P^{-*}, where we identify the pair $(y, x) \in E$ with the map in $\mathrm{Hom}(P^{-k}, \mathbb{Z})$ which is given on generators by the formula:

$$(w, z) \mapsto \begin{cases} 1 & (w, z) = (y, x) \\ 0 & (w, z) \neq (y, x) \end{cases}$$

We abuse notation and let (y, x) denote both an element of P and its dual in P^{-*}. The differential of the chain complex P^{-*} is given by the formula[1]

$$d(y, x) = (\delta y, x) - (-1)^{|y|-|x|}(y, \partial x) \tag{38}$$

P 3.21. *The coalgebra structure on P induces a unital differential graded associative algebra structure on P^{-*}, with the multiplication $\cup : P^{-*} \otimes P^{-*} \longrightarrow P^{-*}$ given on generators by the formula:*

$$(y, x) \cup (w, z) = \begin{cases} (y, z) & x = w \\ 0 & x \neq w \end{cases}$$

The coaugmentation map

$$\begin{aligned} \overline{\varepsilon} : \mathbb{Z} &\to P^{-*} \\ 1 &\mapsto \varepsilon \end{aligned}$$

is the unit. The unit can be written as a sum of generators

$$\overline{\varepsilon}(1) = \varepsilon = \sum_{x \in B} (x, x)$$

This in turn induces a left action $P^{-*} \otimes P \longrightarrow P$ given on generators by the formula:

$$(y, x) \cap (w, z) = \begin{cases} (w, y) & x = z \\ 0 & x \neq z \end{cases}$$

which gives P the structure of a left differential graded P^{-*}-module. Note that the expression (w, y) is defined to be 0 if $y \npreceq w$.

P . Both of the statements of this Proposition are formal consequences of Proposition 3.20. The hom dual of any differential graded coalgebra P is a differential graded algebra P^{-*}. Right multiplication in an associated differential graded algebra A induces a left action of any hom dual A^{-*}, giving A^{-*} the structure of a

[1] The minus sign is correct!

differential graded right module over A. To be more explicit, we use the following to define an action of A on A^{-*}:

$$\begin{aligned}\mathrm{Hom}(A, \mathrm{Hom}(A, A)) &\to \mathrm{Hom}(A, \mathrm{Hom}(A^{-*}, A^{-*}))\\ &\cong \mathrm{Hom}(A \otimes A^{-*}, A^{-*})\end{aligned}$$

To get a left action of P^{-*} on P, we identify P with its double dual $(P^{-*})^{-*}$ in the canonical way. It is elementary to verify that the provided formulas are the ones which result from applying these formal constructions to $\Delta : P \otimes P \to P$.

We check that d is a derivation of $\cup$. That is to say, let us check that

$$d\,((y, x) \cup (z, w)) = d(y, x) \cup (z, w) + (-1)^{|y|-|x|}(y, x) \cup d(z, w)$$

First we compute

$$\begin{aligned}d\,((y, x) \cup (z, w)) &= \begin{cases} d(y, w) & x = z \\ 0 & x \neq z \end{cases}\\ &= \begin{cases} (\delta y, w) - (-1)^{|y|-|w|}(y, \partial w) & x = z \\ 0 & x \neq z \end{cases}\end{aligned}$$

Next we compute

$$\begin{aligned}&d(y, x) \cup (z, w) + (-1)^{|y|-|x|}(y, x) \cup d(z, w)\\ &= (\delta y, x) \cup (z, w) - (-1)^{|y|-|x|}(y, \partial x) \cup (z, w)\\ &+ (-1)^{|y|-|x|}(y, x) \cup (\delta z, w) - (-1)^{|y|-|x|+|z|-|w|}(y, x) \cup (z, \partial w)\end{aligned} \tag{39}$$

If $x = z$, then the only the first and the last term are nonzero, and

$$d(y, x) \cup (z, w) + (-1)^{|y|-|x|}(y, w) \cup d(z, w) = (\delta y, w) + (-1)^{|y|-|w|}(y, \partial w)$$

If $x \neq w$ then all of the terms of (39) are 0 except possibly when z is a codimension 1 face of x. In that case, the nonzero terms of (39) are

$$\begin{aligned}&-(-1)^{|y|-|x|}(y, \partial x) \cup (z, w) + (-1)^{|y|-|x|}(y, x) \cup (\delta z, w)\\ &= -(-1)^{|y|-|x|}\langle \partial x, w\rangle(y, w) + (-1)^{|y|-|x|}\langle \partial x, w\rangle(y, w)\\ &= 0\end{aligned}$$

Let us check that $\varepsilon = \overline{\varepsilon}(1)$ is a unit for $\cup$. If (y, x) is a generator of P^{-*}, then

$$\begin{aligned}(y, x) \cup \varepsilon &= (y, x) \cup \sum_{z \in B}(z, z)\\ &= (y, x) \cup (x, x)\\ &= (y, x)\end{aligned}$$

This shows ε is a right unit; an analogous computation shows it is a left unit as well.

Now let us check that $\cap$ gives P the structure of a differential graded left P^{-*} module. That is to say, we check that

$$d((y,x)\cap(z,w)) = (-1)^{|y|-|x|+|z|-|w|}d(y,x)\cap(z,w)+(y,x)\cap d(z,w) \tag{40}$$

First we compute

$$\begin{aligned} d((y,x)\cap(z,w)) &= \begin{cases} d(z,y) & x=w \\ 0 & x\neq w \end{cases} \\ &= \begin{cases} (\partial z,y)+(-1)^{|z|-|y|}(z,\delta y) & x=w \\ 0 & x\neq w \end{cases} \end{aligned}$$

Next we compute

$$\begin{aligned} &(-1)^{|y|-|x|+|z|-|w|}d(y,x)\cap(z,w)+(y,x)\cap d(z,w) \\ &= (-1)^{|y|-|x|+|z|-|w|}(\delta y,x)\cap(z,w)-(-1)^{|z|-|w|}(y,\partial x)\cap(z,w) \\ &+ (y,x)\cap(\partial z,w)+(-1)^{|z|-|w|}(y,x)\cap(z,\delta w) \end{aligned} \tag{41}$$

If $x=w$, then the only the first and the third terms of (41) are nonzero, and

$$(-1)^{|y|-|x|+|z|-|w|}d(y,x)\cap(z,w)+(y,x)\cap d(z,w) = (-1)^{|z|-|y|}(z,\delta y)+(\partial z,y)$$

as desired. If $x\neq w$, then the only way that any of the terms of (41) can be nonzero is if w is a codimension 1 face of x. In that case

$$\begin{aligned} &(-1)^{|y|-|x|+|z|-|w|}d(y,x)\cap(z,w)+(y,x)\cap d(z,w) \\ &= -(-1)^{|z|-|w|}(y,\partial x)\cap(z,w)+(-1)^{|z|-|w|}(y,x)\cap(z,\delta w) \\ &= -(-1)^{|z|-|w|}\langle\partial x,w\rangle(y,w)\cap(z,w)+(-1)^{|z|-|w|}\langle\partial x,w\rangle(y,x)\cap(z,x) \\ &= -(-1)^{|z|-|w|}\langle\partial x,w\rangle(z,y)+(-1)^{|z|-|w|}(z,y) \\ &= 0 \end{aligned}$$

Thus $\cap$ satisfies (40).

Now we check that ε acts by the identity on P. If (y,x) is a generator of P, then

$$\begin{aligned} (y,x)\cap\varepsilon &= (y,x)\cup\sum_{z\in B}(z,z) \\ &= (y,x)\cup(y,y) \\ &= (y,x) \end{aligned}$$

□

R 3.22. We used right multiplication in P^{-*} to get a left action of P^{-*} on P. We could have also used left multiplication in P^{-*} to get a right action of P^{-*} on

P. Since $\cup$ is commutative on homology (as we shall show), these two $\cap$ products induce the same map on homology. Let

$$T : P \otimes P \to P \otimes P$$

denote the transposition operator which maps

$$(y, x) \otimes (z, w) \mapsto (-1)^{(|y|-|x|)(|z|-|w|)}(z, w) \otimes (y, x)$$

A chain homotopy between Δ and $T\Delta$ will induce a chain homotopy between the two cap products coming from left and right multiplication in P^{-*}. We will develop this idea further in Chapter 4.

To characterize the algebraic structure we have defined on the pair subdivision of a based chain complex, we introduce some terminology due to Whitney [**Whi38**]. We will give the definition as Whitney gave it in 1938.

D 3.23. (Whitney) A *complex admitting a product theory* is the following data.

(1) A finite, nonnegatively graded poset $(B, \leq)$ of *cells*
(2) For each k, a function $i : B_k \times B_{k-1} \to \mathbb{Z}$ called the *incidence number*

Here B_k denotes the degree k part of the graded poset B. The *closure* of a cell x is the poset

$$\{y \in B \mid y \leq x\}$$

Given this data, let C_k denote the free abelian group generated by B_k, and define a map

$$\partial : C_k \to C_{k-1}$$
$$\sum \alpha^i x_i \mapsto \sum \alpha^i \left(\sum_{y \in B_{k-1}} i(x_i, y) y \right)$$

A chain in $\alpha \in C$ is *boundary-like* if either

(1) α is a k-chain and $k > 0$
(2) $\alpha = \sum \alpha^i v_i$ is a 0-chain and $\sum \alpha^i = 0$

This data must satisfy

(1) If x is a k cell and y is a $(k-1)$ cell such that $i(x, y) \neq 0$, then $y \leq x$.
(2) $\partial^2 = 0$
(3) If α is a boundary-like cycle in the closure of a cell, then α is a boundary

P 3.24. *Let (C, B, ∂) be a regular based chain complex in our sense. Then $(B, \leq)$ is a complex admitting a product theory in Whitney's sense.*

P . To get a complex in Whitney's sense, we define the incidence number $i(x, y)$ to be $\langle \partial x, y \rangle$. Then for a k-cell x and $(k-1)$-cell y, $y \preceq x$ if and only if $i(x, y) \neq 0$. If C is regular, than for each cell x,

$$\varepsilon : C(\overline{x}) \to \mathbb{Z}$$

is a quasi-isomorphism. Thus, every positive dimensional cycle in $C(\overline{x})$ bounds. A 0-dimensional $\sum \alpha^i v_i$ bounds exactly if

$$\varepsilon\left(\sum \alpha^i v_i\right) = \sum \alpha^i = 0$$

Thus our regularity axiom implies that every boundary-like cycle in the closure of a cell bounds. □

As the name suggests, complexes admitting a product theory admit additional algebraic structure. First observe that if B is a complex admitting a product theory, and C is the associated chain complex, then the hom dual complex C^{-*} is a free chain complex with the same basis B. As usual, if $x \in B$ is an element of poset, let

$$\mathrm{St}(x) := \{y \in B \mid x \preceq y\}$$

Let ε denote the cochain

$$\sum_{\substack{x \in B \\ |x|=0}} x$$

and by abuse of notation let ε denote the chain

$$\sum_{\substack{x \in B \\ |x|=0}} x$$

as well. If E and F are to subsets of B, $E \sqcap F$ denotes their set-theoretic intersection.

T 3.25. *(Whitney [**Whi38**]) Let B be a complex admitting a product structure. Then there exists a degree 0 product*

$$C^{-*} \otimes C^{-*} \to C^{-*}$$

satisfying the following 3 axioms. For any cells x and y in B.

$$\delta(x \cup y) = \delta(x) \cup y + (-1)^{|x|} x \cup \delta(y) \tag{42}$$

$$\varepsilon \cup x = x \tag{43}$$

$$\textit{If } x \cup y \neq 0, x \cup y \in St(x) \sqcap St(y) \tag{44}$$

There exists a degree 0 product

$$C^{-*} \otimes C \to C$$

satisfying the following 3 axioms.

$$\partial(x \cap y) = (-1)^{|x|+|y|}\delta(x) \cap y + x \cap \partial(y) \tag{45}$$

$$\langle \varepsilon, x \cap x \rangle = 1 \tag{46}$$

$$\textit{If } x \cap y \neq 0, x \cap y \in \overline{St(x) \sqcap y} \tag{47}$$

Any two products satisfying the axioms for $\cup$ *give the same product on cohomology. Any two products satisfying the axioms for* $\cap$ *give the same product on homology and cohomology.*

C 3.26. *Let* (C, B, ∂) *be a regular chain complex, and let* $(C', B'\partial)$ *be its barycentric subdivision. Let* P *denote the pair subdivision of* C'. *Then by Proposition 3.12,* P *is a regular chain complex, and thus a complex admitting a product structure in Whitney's sense. The products* $\cup$ *and* $\cap$ *of Proposition 3.21 satisfy Whitney's axioms.*

P . We checked in the proof of Proposition 3.21 that (42), (43), and (45) are satisfied. To check (46), we observe that for any cell (y, x),

$$\begin{aligned}
\langle \varepsilon, (y, x) \cap (y, x) \rangle &= \langle \varepsilon, (y, y) \rangle \\
&= \langle \sum_{\substack{(z,w)\in P \\ |(z,w)|=0}} (z, w), (y, y) \rangle \\
&= \langle \sum_{x \in B'} (x, x), (y, y) \rangle \\
&= \langle (y, y), (y, y) \rangle \\
&= 1
\end{aligned}$$

Next we check (44) and (47), which assert in a precise way that $\cup$ and $\cap$ are local products. First observe that in the pair complex,

$$(x, y) \leq (z, w) \quad \Leftrightarrow \quad x \leq z \text{ and } w \leq y$$

Suppose $(x, y) \cup (z, w)$ is nonzero. Then

$$(x, y) \cup (z, w) = (x, w)$$

and $y = z$. We have $w \leq y$, so (x, y) is a face of (x, w). Similarly, $z \leq x$, so (z, w) is a face of (x, w). Thus if $(x, y) \cup (z, w) \neq 0$,

$$(x, y) \cup (z, w) \in \mathrm{St}(x, y) \sqcap \mathrm{St}(z, w)$$

Next we must show that if $(x, y) \cap (z, w) \neq 0$,

$$(x, y) \cap (z, w) \in \overline{\mathrm{St}(x, y) \sqcap (z, w)}$$

Observe that

$$\mathrm{St}(x,y) \sqcap (z,w) = \begin{cases} (z,w) & (x,y) \le (z,w) \\ 0 & (x,y) \not\le (z,w) \end{cases}$$

$$\overline{\mathrm{St}(x,y) \sqcap (z,w)} = \begin{cases} \overline{(z,w)} & (x,y) \le (z,w) \\ 0 & (x,y) \not\le (z,w) \end{cases}$$

Suppose $(x,y) \cap (z,w)$ is nonzero. Then

$$(x,y) \cap (z,w) = (z,x)$$

and $y = w$. Since $x \le z$ and $y = w$, $(x,y) \le (z,w)$. Thus

$$\overline{\mathrm{St}(x,y) \sqcap (z,w)} = \overline{(z,w)}$$

Since $w = y \le x$, $(z,x) \le (z,w)$. Thus $(z,x) \in \overline{(z,w)}$. We have shown that if $(x,y) \cap (z,w) \neq 0$,

$$(x,y) \cap (z,w) \in \overline{\mathrm{St}(x,y) \sqcap (z,w)}$$

□

R 3.27. In particular, Corollary 3.26 implies that the ∪ and ∩ products we have defined on the pair subdivision of a simplicial complex agree with the Alexander-Whitney ∪ and ∩ products on homology.

CHAPTER 4

Local Poincaré Complexes and Topological Manifolds

In this chapter, we show how the algebraic structure on the pair subdivision of a regular based chain complex (C, B, ∂) can be used to describe B-local Poincaré complexes. As we will see, topological manifolds structures in the homotopy type determined by a regular chain complex satisfying Poincaré duality correspond to choices of local inverse to the Poincaré duality map.

Throughout this chapter, we work with the category $\widetilde{REG}$ of simply connected regular chain complexes; that is to say, regular chain complexes whose geometric realization is connected and simply connected. If C is a regular chain complex, C' denotes the barycentric subdivision of C, and P denotes the pair subdivision of C'. Furthermore, let s denote the composition

$$s : C \to C' \to P = P(C')$$

Thus, in this Chapter, P always denotes the pair subdivision of the barycentric subdivision of C. Since P is the pair subdivision of a simplicial complex, we may use the geometric properties of P developed in section 3.2.

4.1. Acyclic Carriers

We now briefly recall Steenrod's method of acyclic carriers, as we will use it to construct algebraic Poincaré complexes. Let K be a connected finite regular cell complex, in the sense of [**Ste52**]. In particular, the cellular chain complex of K is a regular chain complex in our sense. Then the usual face relation gives the set of cells of K the structure of a poset. Let $\mathcal{P}(K)$ denote the set of all nonempty subcomplexes of K. Then inclusion of subcomplexes gives $\mathcal{P}(K)$ the structure of a poset.

D 4.1. (Steenrod) Let K and L be cell complexes. A *carrier* from K to L is is a poset map

$$\Gamma : K \to \mathcal{P}(L)$$

such that each The carrier is *acyclic* if for each $\sigma \in K$, the kernel of the augmentation map

$$\varepsilon : C(\Gamma(\sigma)) \to \mathbb{Z}$$

is acyclic; that is to say the kernel of the augmentation map has no homology.

Let $\mathrm{Hom}(C(K), C(L))$ denote the chain complex where

$$\mathrm{Hom}(C(K), C(L))_i = \mathrm{Hom}(C_*(K), C_{*+i}(L))$$

Let $f \in \mathrm{Hom}(C(K), C(L)$. We say f is carried by $\Gamma : K \to L$ if for each $\sigma \in K$

$$f(\sigma) \in C(\Gamma(\sigma))$$

D 4.2. (Steenrod) Let $\Gamma : K \to L$ an acyclic carrier. Then the *operator complex* $\mathcal{O}(\Gamma)$ is the subcomplex of $\mathrm{Hom}(C(K), C(L))$ consisting of all maps $\phi_i : C_*(K) \to C_{*+i}(L)$ such that

(1) Γ is a carrier for ϕ_i
(2) If $i = 0$, then
  (a) ϕ_0 is a cycle
  (b) There exists a integer k, called the *index* of ϕ_0, such that for each vertex 0-chain c of $C_0(K)$

$$\varepsilon(\phi_0(c)) = k\varepsilon(c)$$

  where ε denotes augmentation in K and L.
(3) If $i < 0$ then $\phi_i = 0$

Note that a chain map $f : C(K) \to C(L)$ which commutes with augmentation is exactly a 0-cycle of index 1 in $\mathrm{Hom}(C(K), C(L)$.

T 4.3. *(Steenrod) Let* $\Gamma : K \to L$ *be an acyclic carrier. Then the following augmentation map is a quasi-isomorphism.*

$$\mathcal{O}(\Gamma) \to \mathbb{Z}$$
$$\phi_i \mapsto \begin{cases} Index(\phi) & i = 0 \\ 0 & i > 0 \end{cases}$$

In particular this theorem implies:

(1) There exists an augmented chain map $f : K \to L$ carried by Γ.
(2) Any two chain maps carried by Γ are homotopic via a homotopy carried by Γ.
(3) Any two homotopies between chain maps are themselves homotopic via a homotopy carried by Γ, and so forth.

P 4.4. *Let* (C, B, ∂) *be a regular chain complex. Let* $P = P(\mathcal{G}(C))$ *be the pair subdivision of its geometric realization, and by abuse of notation, let* P *also denote the cellular chain complex of* $P(\mathcal{G}(C))$*. Then*

$$\Gamma_\Delta : P \longrightarrow P \times P$$
$$(y, x) \mapsto \overline{(y, x)} \times \overline{(y, x)}$$

is an acyclic carrier for the chain map

$$\Delta : P \to P \otimes P$$
$$(y, x) \mapsto \sum_{x \le z \le y} (y, z) \otimes (z, x)$$

Let T denote the transposition operator

$$T : P \otimes P \longrightarrow P \otimes P$$
$$(y, x) \otimes (w, z) \mapsto (-1)^{(|w|-|z|)(|y|-|x|)}(w, z) \otimes (y, x)$$

Then Γ_Δ is an acyclic carrier for the chain map $T\Delta$.

P . P is a regular chain complex, so each augmentation map

$$\varepsilon : C\left(\overline{(y, x)} \times \overline{(y, x)}\right) \to \mathbb{Z}$$

is a quasi-isomorphism. Thus Γ_Δ is an acyclic carrier. If $x \le z \le y$, then both (y, z) and (z, x) are faces of (y, x), so Δ is carried by Γ_Δ. Since $\Gamma_\Delta(y, x)$ is invariant under the transposition operator T, Γ_Δ is a carrier for $T\Delta$ as well. □

Note that $T^2 = 1$.

C 4.5. *Define a chain map $\Delta_0 : P \to P \otimes P$ as follows:*

$$\Delta_0 = T\Delta$$

Then for each $s > 0$ there exists a degree s map

$$\Delta_s : P \longrightarrow P \otimes P$$

such that

$$\Delta_s \in \mathcal{O}(\Gamma_\Delta) \tag{48}$$

$$d_{\mathcal{O}(\Gamma_\Delta)}\Delta_s = (-1)^s \Delta_{s-1} + T\Delta_{s-1} \tag{49}$$

The reason for choosing $\Delta_0 = \Delta$ will become clear in the proof of Proposition 4.10

P . By Proposition 3.20, $\Delta : P \to P \otimes P$ is a chain map. If $(x, x) \in P$ is vertex, then $\Delta(x, x) = (x, x) \otimes (x, x)$. We have

$$\varepsilon(x, x) = \varepsilon\left((x, x) \otimes (x, x)\right) = 1$$

so Δ is a chain map of index 1 in Steenrod's sense. Thus Δ is a degree 0 cycle in $\mathcal{O}(\Gamma_\Delta)$, the operator complex of the acyclic carrier Γ_Δ. Furthermore, Since T is an isomorphism, $\Delta_0 = T\Delta$ is a degree 0 cycle as well. Since the augmentation map

$$\varepsilon : \mathcal{O}(\Gamma_\Delta) \to \mathbb{Z}$$

is a quasi-isomorphism, $\Delta_0 - T\Delta_0$ must be a boundary in $\mathcal{O}(\Gamma_\Delta)$. Thus there is some $\Delta_1 \in \mathcal{O}(\Gamma_\Delta)$ such that

$$d_{\mathcal{O}(\Gamma_\Delta)}\Delta_1 = \Delta_0 - T\Delta_0$$

Now suppose inductively that maps Δ_k have been constructed for $k < s$ satisfying the stated conditions. Then

$$\begin{aligned} d_{\mathcal{O}(\Gamma_\Delta)}((-1)^s\Delta_{s-1} + T\Delta_{s-1}) &= (-1)^s\left((-1)^{s-1}\Delta_{s-2} + T\Delta_{s-2}\right) + (-1)^{s-1}\Delta_{s-2} + T\Delta_{s-2} \\ &= -\Delta_{s-2} + (-1)^s T\Delta_{s-2} + (-1)^{s-1}T\Delta_{s-2} + T^2\Delta_{s-2} \\ &= 0 \end{aligned}$$

Thus

$$(-1)^s\Delta_{s-1} + T\Delta_{s-1}$$

is an $(s-1)$-cycle in $\mathcal{O}(\Gamma_\Delta)$. Since $\mathcal{O}(\Gamma_\Delta)$ has no homology above dimension 0, there must be some $\Delta_s \in \mathcal{O}(\Gamma_\Delta)_s$ such that

$$d_{\mathcal{O}(\Gamma_\Delta)}\Delta_s = (-1)^s\Delta_{s-1} + T\Delta_{s-1}$$

□

C 4.6. *Let c be an n-cycle of P. Then*

$$d_{P\otimes P}(\Delta_s(c)) = (-1)^{n+s}\,(\Delta_{s-1}(c) + (-1)^s T\Delta_{s-1}(c))$$

P . The operator complex $\mathcal{O}(\Gamma_\Delta)$ is a subcomplex of the hom complex

$$\mathrm{Hom}(P, P\otimes P)$$

We compute:

$$\begin{aligned} (d_{\mathrm{Hom}(P,P\otimes P)}\Delta_s)\,(c) &= \Delta_s d_P(c) + (-1)^n d_{P\otimes P}(\Delta_s(c)) \\ &= (-1)^n d_{P\otimes P}(\Delta_s(c)) \end{aligned}$$

By Corollary 4.5

$$d_{\mathrm{Hom}(P,P\otimes P)}\Delta_s = (-1)^s\Delta_{s-1} + T\Delta_{s-1}$$

Thus,

$$d_{P\otimes P}(\Delta_s(c)) = (-1)^{n+s}(\Delta_{s-1}(c) + (-1)^s T\Delta_{s-1}(c)$$

□

4.2. Poincaré Duality Spaces

D 4.7. Let X be a compact topological space which has the homotopy type of a simplicial complex. We say that X is an *n-dimensional Poincaré Duality space* if there is a cycle $[X] \in C_n(X)$ such that the cap product map

$$\begin{aligned} \_\cap [X] : C^{n-*}(X) &\longrightarrow C_*(X) \\ \alpha &\mapsto \alpha \cap [X] \end{aligned}$$

is a quasi-isomorphism.

P 4.8. *Let C be a regular chain complex, and let $\mu \in C_n$ such that*

$$\begin{aligned} \_ \cap s(\mu) : P^{n-*} &\longrightarrow P \\ (y,x) &\mapsto (y,x) \cap s(\mu) \end{aligned}$$

is a quasi-isomorphism. Then the geometric realization $\mathcal{G}(C)$ is a Poincaré duality space.

P . The simplicial complex $\mathcal{G}(C)$ has a subdivision which satisfies Poincaré duality with respect to the cap product on the pair complex, so the result follows from Proposition 3.26. □

D 4.9. If C is a regular chain complex an μ is an n-cycle satisfying the hypotheses of Proposition 4.8, we say that μ is a *fundamental cycle* for C.

P 4.10. *(Compare* [**Ran80b**, Proposition 1.1].*) Let C be a regular chain complex with n-dimensional fundamental cycle μ. Then the data $(P, \_ \cap s(\mu))$ can be extended to an global algebraic Poincaré complex, that is to say, a Poincaré complex in sense of Definition 2.34 in the "global" algebraic bordism category of Example 2.29.*

P . The cap product map $\phi_0 := \_ \cap s(\mu)$ provides the chain equivalence

$$P^{n-*} \longrightarrow P$$

We must construct the higher terms $\phi_s : \Sigma^{n+s} T_C P \to P$ such that

(50) $$d_{\mathrm{Hom}(T_C P, P)} \phi_s = (-1)^{n+s} \left(\phi_{s-1} + (-1)^s \mathcal{T}_C(P)(\phi_{s-1})\right)$$

We will construct these maps using the higher diagonal maps of Corollary 4.5. First we define the following chain map:

$$\begin{aligned} \diagdown : P \otimes P &\longrightarrow \mathrm{Hom}(T_C P, P) \\ (y,x) \otimes (w,z) &\mapsto \quad (\beta,\alpha) \mapsto \langle (\beta,\alpha), (y,x) \rangle (w,z) \end{aligned}$$

Here $\langle (\beta,\alpha), (y,x) \rangle$ means evaluate the cochain (β,α) on the chain (y,x). Let us abuse notation and let the symbol (y,x) denote both the cell of P and the generator of P^{-*} which takes the value 1 on (y,x) and 0 on all other generators. Then we may write $\diagdown((y,x) \otimes (w,z))$ as follows

$$(\beta,\alpha) \mapsto \begin{cases} (w,z) & y = \beta, x = \alpha \\ 0 & y \neq \beta \text{ or } x \neq \beta \end{cases}$$

Let $T : P \otimes P \to P \otimes P$ denote the transposition operator. Then the following diagram commutes:

(51) $$\begin{array}{ccc} P \otimes P & \xrightarrow{\diagdown} & \mathrm{Hom}(T_C P, P) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathcal{T}_C(P)} \\ P \otimes P & \xrightarrow{\diagdown} & \mathrm{Hom}(T_C P, P) \end{array}$$

Let $(y,x) \otimes (w,z) \in P \otimes P$. Then

$$\diagdown T\,((y,x) \otimes (w,z))$$

is the map

$$(\beta,\alpha) \mapsto \begin{cases} (-1)^{(|w|-|z|)(|y|-|x|)}(y,x) & w=\beta, z=\alpha \\ 0 & w \neq \beta \text{ or } z \neq \beta \end{cases} \tag{52}$$

The map $\mathcal{T}_C(P)$ is defined in Definition 2.33. For $f : T_C P_r \to P_s$,

$$\mathcal{T}_C(P)(f) \coloneqq (-1)^{rs} e_C(P) T_C(f)$$

$$\begin{aligned} \mathcal{T}_C(P)\diagdown((y,x)\otimes(w,z)) &= \mathcal{T}_C(P)\left((\beta,\alpha) \mapsto \begin{cases} (w,z) & y=\beta, x=\alpha \\ 0 & y \neq \beta \text{ or } x \neq \beta \end{cases}\right) \\ &= (\beta,\alpha) \mapsto \begin{cases} (-1)^{(|\alpha|-|\beta|)(|w|-|z|)}(w,z) & y=\beta, x=\alpha \\ 0 & y \neq \beta \text{ or } x \neq \beta \end{cases} \end{aligned} \tag{53}$$

Comparing (52) and (53), we see that (51) commutes.

Now we inductively construct the maps ϕ_s for $s > 0$. Let $\psi : C \to D$ be a degree n map of chain complexes. The we define $\epsilon\psi$ to be the degree n map with components:

$$\begin{aligned} (\epsilon\psi)_{r,s} : C_r &\to D_{s+n} \\ c &\mapsto (-1)^{rs}\psi(c) \end{aligned}$$

We claim that

$$\phi_0 : (y,x) \mapsto (y,x) \cap s(\mu)$$

is exactly the map $\epsilon\diagdown T\Delta(s(\mu))$. The subdivision of the fundamental cycle μ is the cycle (μ,ε), where

$$\varepsilon = \sum_{\substack{x \in B \\ |x|=0}} x$$

Thus

$$\begin{aligned} (y,x) \cap s(\mu) &= (y,x) \cap (\mu,\varepsilon) \\ &= \begin{cases} (\mu,y) & |x|=0 \\ 0 & |x| \neq 0 \end{cases} \end{aligned}$$

We compute

$$
\begin{aligned}
\backslash T\Delta(s(\mu)) &= \backslash T\left(\sum_{z\in B'}(\mu,z)\otimes(x,\varepsilon)\right)\\
&= \backslash\left(\sum_{z\in B'}(-1)^{|(|\mu|-|z|)(|z|-|\varepsilon|}(z,\varepsilon)\otimes(\mu,z)\right)\\
&= \left((y,x)\mapsto\begin{cases}(-1)^{|(|\mu|-|z|)(|z|-|\varepsilon|}(\mu,y) & |x|=0\\ 0 & |x|\neq 0\end{cases}\right)\\
&= \epsilon\phi_0(y,x)
\end{aligned}
$$

Thus $\phi_0 = \epsilon\backslash T\Delta(\mu,\varepsilon) = \epsilon\Delta_0(\mu,\varepsilon)$. We define

$$\phi_s \coloneqq \epsilon\backslash\Delta_s(\mu,\varepsilon)$$

Using the computation of Corollary 4.6 and the fact that Diagram 51 commutes, we see that for $s > 0$,

$$
\begin{aligned}
d_{\mathrm{Hom}(TP,P)}(\phi_s) &= d_{\mathrm{Hom}(TP,P)}\epsilon\backslash\Delta_s(\mu,\varepsilon)\\
&= \epsilon\backslash d_{P\otimes P}\Delta_s(\mu,\varepsilon)\\
&= \epsilon\backslash(-1)^{n+s}\left(\Delta_{s-1}(\mu,\varepsilon)+(-1)^s T\Delta_{s-1}(\mu,\varepsilon)\right)\\
&= (-1)^{n+s}\left(\epsilon\backslash\Delta_{s-1}(\mu,\varepsilon)+(-1)^s\epsilon\backslash T\Delta_{s-1}(\mu,\varepsilon)\right)\\
&= (-1)^{n+s}\left(\phi_{s-1}+(-1)^s\mathcal{T}_C(P)\epsilon\backslash\Delta_{s-1}(\mu,\varepsilon)\right)\\
&= (-1)^{n+s}\left(\phi_{s-1}+(-1)^s\mathcal{T}_C(P)\phi_{s-1}\right)
\end{aligned}
$$

Thus the maps ϕ_s satisfy (50). □

T 4.11. *Given a regular chain complex (C, B, ∂) and a fundamental cycle $\mu \in C_n$, there is a B'-local map $\phi_\mu : \Sigma^n TP \longrightarrow P$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\Sigma^n TP & \xrightarrow{\phi_\mu} & P\\
\uparrow{\scriptstyle \varepsilon} & \nearrow{\scriptstyle \cap s(\mu)} & \\
\Sigma^n P^{-*} & &
\end{array}
$$

where $\varepsilon : \Sigma^n P^{-*} \longrightarrow \Sigma^n TP$ is the natural quasi-isomorphism of Proposition 2.43 and $\cap s(\mu)$ is the cap product with the pair subdivision of μ. The B-local map ϕ_μ a global quasi-isomorphism, but not necessarily a B-local quasi-isomorphism.

The geometric idea of Theorem 4.11 is as follows. A fundamental cycle μ determines a relative fundamental class of each dual cone Dx. To be more explicit, if μ is a fundamental cycle for triangulated geometric Poincaré duality space, then

(μ, x) is the relative fundamental class of the dual cone Dx. There is a local cap product map

$$\cap[Dx] : C^{n-|x|-*}(Dx) \to C\Big(Dx, \partial\overline{Dx}\Big) \tag{54}$$

Recall from Proposition 3.19 that these chain complexes have simple descriptions in terms of the pair subdivision. $C^{n-|x|-*}(Dx)$ is generated by pairs of the form

$$\{(y, x') \in P \mid x \preceq x' \precneqq y\}$$

and $C\Big(\overline{Dx}, \partial\overline{Dx}\Big)$ is generated by pairs of the form

$$\{(y, x) \in P \mid x \precneqq y\}$$

Thus (54) is given the following formula

$$\begin{aligned} \cap[Dx] : C^{n-|x|-*}(Dx) &\to C\Big(Dx, \partial\overline{Dx}\Big) \\ (z, y) &\mapsto (z, y) \cap (\mu, x) \end{aligned}$$

This formula and its geometric meaning motivate the definition of ϕ_μ.

P T 4.11. First we describe the B-local chain complex $\Sigma^n TP$. Recall the local structure on P. $P(x)$ is generated by all pairs of the form (y, x). By definition,

$$TP(x) = \sum_{x \preceq y} P(y)^{-|x|-*}$$

Thus $TP(x)$ is generated by

$$\{(z, y) \in P^{-*} \mid x \preceq y\}$$

The generator (z, y) has degree $-|z| + |y| - |x|$. By definition,

$$TP = \sum_{x \in B} TP(x)$$

so the chain complex TP is generated by triples

$$\{(z, y, x) \in B \times B \times B \mid x \preceq y \preceq z\}$$

and each generator has degree $-|z|+|y|-|x|$. From the formula (38) for the differential of P^{-*} and Remark 2.13, we see that the differential d_{TP} is as follows.

$$d_{TP}(z, y, x) = (\delta z, y, x) - (-1)^{|z|-|y|}(z, \partial y, x) + (-1)^{|z|-|y|}(z, y, \delta x)$$

Thus the chain complex $\Sigma^n TP$ is generated by triples

$$\{(z, y, x) \in B \times B \times B \mid x \preceq y \preceq z\}$$

and each generator has degree $n - |z| + |y| - |x|$. Recall from Definition 2.32 that for a chain complex D,

$$d_{\Sigma^n D} = (-1)^k d_D : (\Sigma^n D)_k \to (\Sigma^n D)_{k-1}$$

Thus the differential $d_{\Sigma^n TP}$ is as follows.

$$d_{\Sigma^n TP}(z,y,x) = (-1)^{n-|z|+|y|-|x|}(\delta z,y,x) - (-1)^{n-|x|}(z,\partial y,x) + (-1)^{n-|x|}(z,y,\delta x) \tag{55}$$

Now we define the map ϕ_μ.

$$\begin{aligned} \phi_\mu : \Sigma^n TP &\to P \\ (z,y,x) &\mapsto (z,y) \cap (\mu,x) \end{aligned}$$

This formula makes sense, as $(z,y) \in P^{-*}$, and $(\mu,x) \in P$. Thus we may multiply (z,y) and (μ,x) using the cap product of Proposition 3.21. First observe that if $\phi_\mu(z,y,x) \neq 0$, then $y = x$ and

$$\begin{aligned} |\phi_\mu(z,y,x)| &= |(\mu,z)| \\ &= n - |z| \\ &= n - |z| + |y| - |x| \\ &= |(z,y,x)| \end{aligned}$$

thus ϕ_μ is a degree 0 map. We claim that for any n-cycle $\mu \in C_n$, ϕ_μ is a chain map.

$$\begin{aligned} d_P\phi_\mu(z,y,x) &= d_P\left((z,y) \cap (\mu,x)\right) \\ &= \begin{cases} d_P(\mu,z) & x = y \\ 0 & x \neq y \end{cases} \\ &= \begin{cases} (\partial\mu,z) + (-1)^{n-|z|}(\mu,\delta z) & x = y \\ 0 & x \neq y \end{cases} \\ &= \begin{cases} (-1)^{n-|z|}(\mu,\delta z) & x = y \\ 0 & x \neq y \end{cases} \end{aligned}$$

Next we compute:

$$\begin{aligned} \phi_\mu d_{\Sigma^n TP}(z,y,x) &= \phi_\mu\left((-1)^{n-|z|+|y|-|x|}(\delta z,y,x) - (-1)^{n-|x|}(z,\partial y,x) + (-1)^{n-|x|}(z,y,\delta x)\right) \\ &= (-1)^{n-|z|+|y|-|x|}(\delta z,y) \cap (\mu,x) \\ &\quad - (-1)^{n-|x|}(z,\partial y) \cap (\mu,x) + (-1)^{n-|x|}(z,y) \cap (\mu,\delta x) \end{aligned}$$

If $x = y$, then only the first term is nonzero, and

$$\begin{aligned} \phi_\mu d_{\Sigma^n TP}(z,y,x) &= (-1)^{n-|z|+|y|-|x|}(\delta z,y) \cap (\mu,x) \\ &= (-1)^{n-|z|}(\mu,\delta z) \end{aligned}$$

If $x \neq y$, then all of the terms of $\phi_\mu d_{\Sigma^n TP}(z, y, x)$ are zero unless x is a codimension 1 face of y. In that case,

$$\begin{aligned}\phi_\mu d_{\Sigma^n TP}(z, y, x) &= -(-1)^{n-|x|}(z, \partial y) \cap (\mu, x) + (-1)^{n-|x|}(z, y) \cap (\mu, \delta x) \\ &= -(-1)^{n-|x|}\langle \partial y, x\rangle(\mu, z) + (-1)^{n-|x|}\langle \partial y, x\rangle(\mu, z) \\ &= 0\end{aligned}$$

Thus ϕ_μ is a chain map.

Next we claim that ϕ_μ is a B-local chain map.

$$\Sigma^n TP(x) = \{(z, y, x) \mid x \preceq y \preceq z$$

If $\phi_\mu(z, y, x) \neq 0$, then $\phi_\mu(z, y, x) = (\mu, z)$. Recall that

$$P(x) = \{(w, x) \mid x \preceq w\}$$

Thus

$$\phi_\mu\left(P(x)\right) \subset \sum_{x \preceq z} P(z)$$

so ϕ_μ is a B-local chain map.

Let us consider the chain complex $\Sigma^n P^{-*}$ and the map ε^*. First observe that P^{-*} is generated by pairs (y, x) of degree $-|y| + |x|$. The differential of P^{-*} is given by

$$d(y, x) = (\delta y, x) - (-1)^{|y|-|x|}(y, \delta x)$$

Thus $\Sigma^n P^{-*}$ is generated by pairs (y, x) of degree $n - |y| + |x|$, and the differential $d_{\Sigma^n P^{-*}}$ is given by

$$d(y, x) = (-1)^{n-|y|+|x|} - (-1)^n(y, \delta x)$$

By Proposition 2.43, there is a natural chain equivalence

$$e^* : P^{-*} \to TP$$

for any B-local chain complex P. This is a map of global chain complexes that does not consider B-local structure. We have the following explicit formula for ε^*.

$$\begin{aligned}\Sigma^n P^{-*} &\to \Sigma^n TP \\ (z, y) &\mapsto (z, y, \varepsilon)\end{aligned}$$

Here ε denotes the cochain $\varepsilon : C \to \mathbb{Z}$ given by augmentation. Explicitly,

$$\varepsilon = \sum_{v \in B |v|=0} v$$

Recall from the proof of Proposition 3.3 that $\delta\varepsilon = 0$. We check that ε^* gives a chain map from $\Sigma^n P^{-*}$ to $\Sigma^n TP$.

$$\begin{aligned} d_{\Sigma^n TP} e^*(z, y) &= d_{\Sigma^n TP}(z, y, \varepsilon) \\ &= (-1)^{n-|z|+|y|-|\varepsilon|}(\delta z, y, \varepsilon) - (-1)^{n-|e|}(z, \partial y, \varepsilon) + (-1)^{n-|x|}(z, y, \delta\varepsilon) \\ &= (-1)^{n-|z|+|y|}(\delta z, y, \varepsilon) - (-1)^n(z, \partial y, \varepsilon) \\ &= e^*((-1)^{n-|z|+|y|}(\delta z, y) - (-1)^n(z, \partial y) \\ &= e^* d_{\Sigma^n P^{-*}}(z, y) \end{aligned}$$

The subdivision map $s : C \to P$ is given by the formula

$$s(x) = (x, \varepsilon)$$

Thus the map $\cap s(\mu)$ is given by

$$\begin{aligned} \cap s(\mu) : \Sigma^n P^{-*} &\to P \\ (y, x) &\mapsto (y, x) \cap (\mu, \varepsilon) \end{aligned}$$

Note that this map sends the pair subdivision (x, ε) of a cell x to (μ, x), the fundamental class of the dual cone of x.

We check that the diagram commutes. Let $(z, y) \in \Sigma^n P^{-*}$. Then

$$\begin{aligned} \phi_\mu \varepsilon^*(z, y) &= \phi_\mu(z, y, \varepsilon) \\ &= (z, y) \cap (\mu, \varepsilon) \\ &= (z, y) \cap s(\mu) \end{aligned}$$

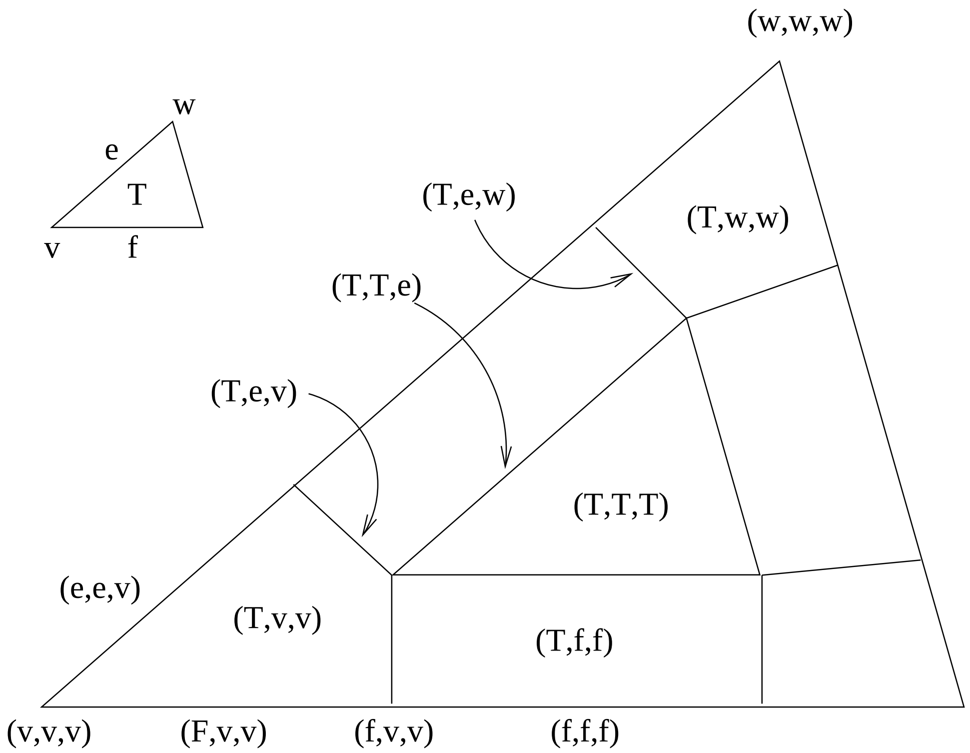
The coaugmentation map e^* is always a quasi-isomorphism. If $\cap s(\mu)$ is an isomorphism, then ϕ_μ must be a (global) quasi-isomorphism as well. □

R 4.12. If P is the pair subdivision of a simplicial complex K, the chain complex TP is the chain complex of a cellular subdivision of K, called the triples subdivision. Just as the cells of P are labeled by pairs, the cells of TP are labeled by triples. Figure 4.1 shows the triples subdivision of a triangle.

We might now expect an analogue of Proposition 4.10. That is to say, a global Poincaré duality map gives rise to a global algebraic Poincaré complex, so we expect that the local data (P, ϕ_μ) should extend to a local algebraic Poincaré complex. And indeed this is almost true. The local map ϕ_μ can be extend to a series of higher maps ϕ_s satisfying the conditions of Definition 2.34. Since ϕ_μ is a global equivalence, $S^{-1}C(\phi_\mu)$ is globally contractible, so (P, ϕ_μ) satisfies condition (2) of Definition 2.30. However, the fact that ϕ_μ is a global quasi-isomorphism does not guarantee that for each $x \in B'$,

$$H_k\left(S^{-1}C(\phi_\mu)(x) = 0\right) \text{ for } k < 1$$

Thus (P, ϕ_μ) need not satisfy condition (1) of Definition 2.30. We need an additional hypothesis on C in order for ϕ_μ to determine a Poincaré complex in $\Lambda_w(B')$.



F 4.1. A triangle with some of its faces labeled, and the triples subdivision of the triangle.

D 4.13. Let C be a regular chain complex with fundamental cycle $\mu \in C_n$. Let (P, ϕ_μ) be as defined in Theorem 4.11. We say that (C, μ) is *codimension two Euclidean* if

(1) Each cell $x \in B'$ is a face of some n-cell
(2) The cycle μ is the sum of all the n-cells:

$$\mu = \sum_{\substack{x \in B' \\ |x|=n}} x$$

(3) for all $y \in B'_{n-1}$, $\langle \partial x, y \rangle$ is nonzero for exactly two $x \in B'_n$
(4) for all$x \in B'$, the map $F : C' \longrightarrow P(x)$ given on generators by the formula

$$F(y) = \begin{cases} (y, x) & |x| = n \\ 0 & |x| \neq n \end{cases}$$

induces an isomorphism from $H_n(C) \longrightarrow H_{n-|x|}(P(x))$.

R 4.14. If (C, μ) is a codimension two Euclidean regular cell complex, then the geometric realization of C is n-circuit in the sense of [**McC77**] and a normal pseudomanifold in the sense of [**GM80**].

P 4.15. *Let C be a regular chain complex with fundamental cycle $\mu \in C_n$. Then (P, ϕ_μ) extends to algebraic Poincaré complex in $\Lambda_w(B')$.*

P . Recall that B' denotes the basis for chains on the barycentric subdivision of C, and P denotes the pair subdivision of P. For each $s \geq 0$, we must give a B'-local map $\phi_s : \Sigma^{n+s} TP \to P$ such that

(1) For each $s > 1$,

$$d_{\mathrm{Hom}(TP,P)} \phi_s = (-1)^{n+s} \left(\phi_{s-1} + (-1)^s \mathcal{T}(P)(\phi_{s-1})\right)$$

(2) ϕ_0 is a B'-local chain map such that
   (a) The desuspension of the mapping cone of ϕ_0 is globally contractible. That is to say, the B'-local chain complex $S^{-1}C(\phi_0 : \Sigma^n TD \to D)$ is contractible after forgetting the B'-local structure.
   (b) For each $x \in B'$,

$$H_k\left(S^{-1}C(\phi_0)(x)\right) = 0 \quad \text{for } k < 1$$

We define $\phi_0 \coloneqq \phi_\mu$. By Proposition 4.11, $\phi_0 : \Sigma^n TP \to P$ is a B'-local chain map which is a global quasi-isomorphism. Thus the mapping cone

$$C(\phi_0 : \Sigma^n TD \to D)$$

is globally contractible. We must check that

$$H_k\left(S^{-1}C(\phi_0 : \Sigma^n TP \to P)(x)\right) = 0$$

for $k < 1$ and $x \in B'$. (This is condition (1) of Definition 2.30.) Since $\Sigma^n TP$ and P are concentrated in nonnegative degrees, the exact sequence of Proposition 2.9 implies that this condition is equivalent to the following two conditions

(1) The chain map $\phi_0(x)$ induces an isomorphism

$$\Sigma^n TP_0(x) \to P_0(x)$$

(2) The chain map $\phi_0(x)$ induces a surjection

$$\Sigma^n TP_1(x) \to P_1(x)$$

In order to check these two conditions and construct the higher terms ϕ_s, we give a geometric description of the local chain complexes $\Sigma^n TP(x)$ and $P(x)$.

Recall that $\Sigma^n TP(x)$ is generated by triples (z, y, x) where $x \leq y \leq z$, and each triple has dimension $n - |z| + |y| - |x|$. By (55), the differential

$$d_{\Sigma^n TP(x)}(x) : \Sigma^n TP(x) \to \Sigma^n TP(x)$$

is given by

$$d(z, y, x) = (-1)^{n-|x|}\left((-1)^{|y|-|z|}(\delta z, y) - (z, \partial y)\right)$$

Let

$$C^{n-|x|-*}(\overline{Dx}) \coloneqq \Sigma^{n-|x|} T_C P(\overline{Dx})$$

denote the chain complex of cochains on the closed dual cone of x. Then $C^{n-|x|-*}(\overline{Dx})$ is generated by pairs (z, y) such that $x \le y \le z$, and each pair has dimension $n - |x| - (|z| - |y|)$. The differential of this chain complex is given by

$$d_{C^{n-|x|-*}(\overline{Dx})}(z, y) = (-1)^{n-|x|}\left((-1)^{|y|-|z|}(\delta z, y) - (z, \partial y)\right)$$

Thus the following map is an isomorphism of chain complexes.

$$\begin{aligned} C^{n-|x|-*}(\overline{Dx}) &\longrightarrow \Sigma^n T P(x) \\ (z, y) &\mapsto (z, y, x) \end{aligned} \tag{56}$$

Similarly, the chain complex $P(x)$ by pairs (y, x) with $x \le y$, and the dimension of each pair is $|y| - |x|$. The differential of the chain complex $(P(x), d_P(x))$ is

$$d_P(x)(y, x) = (\partial y, x)$$

Let $C_*(\overline{Dx}, \partial\overline{Dx})$ denote the relative chain complex of chains on the closed dual cone of x relative to the boundary of the dual cone of x. The chain complex $C_*(\overline{Dx})$ is exactly the subcomplex of the pair subdivision of C' generated by cells (y, x') with $x \le x'$. The differential of this complex is

$$d_{C_*(\overline{Dx})}(y, x') = (\partial y, x') + (-1)^{|y|-|x'|}(y, \delta x')$$

All of the $(y, \delta x')$ terms lie in the boundary $\partial\overline{Dx}$. Thus the differential of $C_*(\overline{Dx}, \partial\overline{Dx})$ is the map

$$d_{C_*(\overline{Dx}, \partial\overline{Dx})}(y, x) = (\partial y, x)$$

and the following map is an isomorphism of chain complexes

$$\begin{aligned} C_*(\overline{Dx}, \partial\overline{Dx}) &\longrightarrow P(x) \\ (y, x) &\mapsto (y, x) \end{aligned} \tag{57}$$

Recall that $\phi_0 = \phi_\mu$ is the map

$$\begin{aligned} (z, y, x) &\mapsto (z, y) \cap (\mu, x) \\ &= \begin{cases} (\mu, z) & x = y \\ 0 & x \ne y \end{cases} \end{aligned}$$

Recall that $\phi_0(x)$ is defined to be the composition

$$\Sigma^n T P(x) \xrightarrow{\phi_0} \sum_{x \le y} P(y) \twoheadrightarrow P(x)$$

Thus,

$$\phi_0(x)(z, y, x) = \begin{cases} (\mu, x) & x = y = z \\ 0 & x \neq y \text{ or } y \neq z \end{cases}$$

Using the isomorphisms (56) and (57), we see that

$$\phi_0(x) \cong {}_- \cap [Dx] : C^{n-|x|-*}(\overline{Dx}) \longrightarrow C_*(\overline{Dx}, \partial\overline{Dx})(z, y) \qquad \mapsto (z, y) \cap [Dx]$$

Here $[Dx]$ denotes the fundamental chain (μ, x) of the closed dual cone $\overline{Dx}$.

We will now use this description of the local map $\phi_0(x)$ to show that for each $x \in B'$,

(1) The chain map $\phi_0(x)$ induces an isomorphism

$$\Sigma^n TP_0(x) \to P_0(x)$$

(2) The chain map $\phi_0(x)$ induces a surjection

$$\Sigma^n TP_1(x) \to P_1(x)$$

Let x in B'. Observe that $C^{n-|x|}(\overline{Dx})$ is generated by cells of the form (c, x), where $|c| = n$. (Recall that $\mu = \sum_{|c|=n} c$.)

Suppose $|x| < n$. Then the map

$$\phi_0(x) : C^{n-|x|}(\overline{Dx}) \to C_0(\overline{Dx}, \partial\overline{Dx})$$

is the 0 map, since

$$\phi_0(x)(c, x) = (c, x) \cap (\mu, x) = (c, c)$$

and $(c, c) = 0$ in $C_0(\overline{Dx}, \partial\overline{Dx})$ since $c \neq x$. Thus we must show that

$$H_0(\overline{Dx}, \partial\overline{Dx}) = 0$$

This chain complex has a single 0 cell, namely (x, x), and $d(x, x) = 0$. Since every cell of C' is the face of a n-cell by hypothesis, x is a codimension 1 face of some cell y. Then

$$\begin{aligned} d\left((-1)^{\langle \partial y, x\rangle}(y, x)\right) &= (-1)^{\langle \partial y, x\rangle}\langle \partial y, x\rangle(x, x) \\ &= (x, x) \end{aligned}$$

Thus (x, x) is a boundary and

$$H_0(\overline{Dx}, \partial\overline{Dx}) = 0$$

Now suppose that $|x| = n$. Then $C^{n-|x|}(\overline{Dx})$ has a single generator (x, x), $C_0(\overline{Dx}, \partial\overline{Dx})$ has a single generator (x, x), and $\phi_0(x)$ is the map

$$(x, x) \mapsto (x, x) \cap (x, x) = (x, x)$$

Thus $\phi_0(x)$ is an isomorphism.

Next we show that for each x,

$$\phi_0(x) : C^{n-|x|-1}(\overline{Dx}) \to C_1(\overline{Dx}, \partial\overline{Dx})$$

is a surjection. If $|x| = n$, then

$$C^{n-|x|-1}(\overline{Dx}) = C_1(\overline{Dx}, \partial\overline{Dx}) = 0$$

If $|x| = n - 1$, then by hypothesis x is a face of exactly two n-cells, call them c_1 and c_2. Thus $C^{n-|x|-1}(\overline{Dx})$ is generated by the 3 cells (x, x), (c_1, c_1), and (c_2, c_2), and $C_1(\overline{Dx}, \partial\overline{Dx})$ is generated by the 2 cells (c_1, x) and (c_2, x). The chain complex $C^{-*}(\overline{Dx})$ has the homology of a point, because the cone $\overline{Dx}$ is contractible, and the generator of the single homology class in degree 0 is the sum of all the degree 0 cells. Thus $C^{n-|x|-1}(\overline{Dx})$ is cyclic with generator

$$(x, x) + (c_1, c_1) + (c_2, c_2)$$

The map $\phi_0(x)$ sends

$$\begin{aligned}(x, x) + (c_1, c_1) + (c_2, c_2) &\mapsto (x, x) \cap (\mu, x) \\ &= (c_1, x) + (c_2, x)\end{aligned}$$

We must show that $\phi_0(x)$ is a surjection on H_1. Let

$$\epsilon_i := \langle \partial c_i, x \rangle$$

By hypothesis,

$$\partial\mu = \partial \sum_{|c|=n} c = 0$$

and x is a coface of only the two n-cells c_1 and c_2 So we must have

$$\epsilon_1 = -\epsilon_2$$

Thus

$$\begin{aligned}d_P(x)\left((c_1, x) + (c_2, x)\right) &= \epsilon_1(x, x) + \epsilon_2(x, x) \\ &= 0\end{aligned}$$

Thus $(c_1, x) + (c_2, x)$ generates the group of 0-cycles of $C_1(\overline{Dx}, \partial\overline{Dx})$ and $\phi_0(x)$ is surjective on homology.

If $|x| < n - 1$, then

$$\phi_0(x) : C^{n-|x|-1}(\overline{Dx}) \to C_1(\overline{Dx}, \partial\overline{Dx})$$

is the 0 map. For $\phi_0(x)(z, y) = 0$ unless $|z| = n$ and $y = x$, and thus $\phi_0(x)(z, y) = 0$ for all (z, y) in $C^{n-|x|-1}(\overline{Dx})$. Thus we must show that

$$H_1(\overline{Dx}, \partial\overline{Dx}) = 0$$

Recall that $\overline{Dx}$ has the homology of a point. Looking at the long exact sequence of the pair $(\overline{Dx}, \partial\overline{Dx})$, we see that

$$H_1(\overline{Dx}, \partial\overline{Dx}) = 0$$

if and only if

$$H_0(\partial\overline{Dx}) = \mathbb{Z}$$

By hypothesis, the map

$$\begin{aligned} C' &\to P(x) \\ y &\mapsto (y, x) \end{aligned}$$

induces an isomorphism

$$\begin{aligned} H_n(C') &\to H_{n-|x|}(P(x)) \\ [\mu] &\mapsto [(\mu, x)] \end{aligned} \tag{58}$$

Let $X = \mathcal{G}(C)$ denote the geometric realization of C, so that $C_*(\mathcal{G}(C)) = C'$. Let c_x denote the barycenter of a cell x of $\mathcal{G}(C)$. Then (58) implies for each cell x, the natural map

$$H_n(X) \longrightarrow H_n(X, X \setminus c_x) \tag{59}$$

is an isomorphism. By the Proposition of [**GM80**, p. 151], (59) holds for a pseudomanifold X if and only if for each cell x of X with $|x| < n-1$, the link of x is connected. The link of a cell x of $\mathcal{G}(C)$ is connected if and only if

$$H_0(\partial\overline{Dx}) = \mathbb{Z}$$

Thus

$$H_1(\overline{Dx}, \partial\overline{Dx}) = 0$$

if $|x| < n-1$.

The higher terms ϕ_s can be constructed in a manner analogous to the construction of the global ϕ_s maps of Proposition 4.10, using the method of acyclic carriers and the diagonal maps

$$\Delta : C_*(\overline{Dx}) \to C_*(\overline{Dx}) \otimes C_*(\overline{Dx})$$

□

Let C be codimension two Euclidean with fundamental cycle μ. Then we have a B-local map

$$\phi_\mu : \Sigma^n TP \longrightarrow P$$

which is a global chain equivalence. But ϕ_μ need not be a B'-local equivalence — the map ϕ_μ need not have a B'-local chain homotopy inverse. As we shall see, the question of whether or not such a local inverse exists is the crucial one for detecting manifold structures in the homotopy type determined by C.

L 4.16. *Let C be codimension two Euclidean with n-dimensional fundamental cycle μ. The isomorphism*

$$VL^n(\mathcal{G}(C)) \to L^n(\Lambda_w(B'))$$

of Proposition 2.46 maps $\sigma^*(\mathcal{G}(C))$, Ranicki's "1/2-connective visible symmetric signature"$\sigma(\Gamma(C))$ to the bordism class represented by the algebraic Poincaré complex (P, ϕ_μ).

P . Since C is codimension two Euclidean, the geometric realization $\mathcal{G}(C)$ is a normal pseudomanifold in the sense of [**GM80**]. Thus the 1/2-connective visible symmetric signature of $\sigma^*(\mathcal{G}(C))$ is equal to the visible symmetric signature (C', ϕ') [**Ran92**, Remark 16.8]. By Proposition 2.46,

$$VL^n(\mathcal{G}(C)) \cong L^n(\Lambda_w(B'))$$

The visible symmetric signature $\sigma(\Gamma(C))(C'', \phi')$ is a B'-local chain Poincaré complex, where B' is the basis for the barycentric subdivision of B. Let x be a cell of C'. Then $C''(x)$ is the chain complex generated by all cells of the form $x \prec \ldots \prec x_k$. That is to say $C''(x)$ is the chain complex of the relative dual cone $C_*(\overline{Dx}, \partial Dx)$ in barycentric subdivision of B'. $TC''(x)$ is the chain complex of the closed dual cone of x in the barycentric subdivision of B'.

$$C^{n-|x|-*}(\overline{Dx})$$

The map $\phi' : \Sigma^n TC'' \to C''$ is locally given by the cap product map

$$[Dx] \cap \_ : \Sigma^n TC'' \to C''$$

We claim that (P, ϕ_μ) and (C'', ϕ') are homotopy equivalent in the sense of of Definition 2.40. Let $s : P \to C''$ be a chain equivalence from the chain complex P, which is the pair subdivision of the barycentric subdivision of C, to the chain complex C'', which is the second barycentric subdivision of C. Consider the diagram

(60)
$$\begin{array}{ccc} TP & \xleftarrow{\;Ts\;} & TC'' \\ \big\downarrow{\scriptstyle \phi_{\mu_s}} & & \big\downarrow{\scriptstyle \phi'_s} \\ P & \xrightarrow{\;s\;} & C'' \end{array}$$

We must show that this diagram commutes for all s up to chain homotopies θ_s in the sense of Definition 2.40. First observe that ϕ'_0 and $s\mathcal{T}_P\phi_{\mu_0}T(s)$ are both local cap product maps

$$[Dx] \cap \_ : \Sigma^{n-|x|} C^{n-|x|-*}(\overline{Dx}) C_*(\overline{Dx}, \partial Dx)$$

The difference is that ϕ'_0 is the Alexander-Whitney cap product, and $s\mathcal{T}_P\phi_{\mu_0}T(s)$ is the cap product of Proposition 3.21. Since both of these cap products satisfy Whitney's axioms, they are chain homotopic. The higher terms ϕ'_s and ϕ_{μ_s} are each constructed using the symmetry of the diagonal map and Streenrod's method of acyclic carriers. The difference is that the ϕ'_s are constructed using Alexander-Whitney diagonal, and the ϕ_{μ_s} are constructed using the diagonal map of Proposition 3.20. However, both of these diagonal maps are carried by the acyclic carrier Γ_Δ of Proposition 4.4. Thus the chain homotopies θ_s between ϕ'_s and $s\phi_{\mu_0}T(s)$ can be constructed using the method of acyclic carriers. □

Now we are ready to state our main result.

T 4.17. *Let (C, B, ∂) be a simply connected regular chain complex. Let $\mu \in C_n$ be a fundamental cycle such that $n > 4$ and (C, μ), is codimension two Euclidean. Then topological manifold structures in the homotopy type determined by C are in one-to-one correspondence with the set of (P', μ') such that*

(1) *(P', μ') is weakly B'-local cobordant to (P, ϕ_μ)*
(2) *μ' has a B-local chain homotopy inverse*

up to strong B'-local bordism.

P . Let $X = \mathcal{G}(C)$ denote the geometric realization of C. Observe that if (P, ϕ_μ) is weakly B'-local cobordant to some (P', μ') such that (P', μ') has a B-local inverse, then (P', μ') determines an algebraic Poincaré complex in $\Lambda_s(B')$. Furthermore, since (P', μ') is weakly B'-local cobordant to (P, ϕ_μ), (P', μ') is a lift of (P, ϕ_μ) to $\Lambda_s(B')$ under the map

$$F : \Lambda_s(B') \longrightarrow \Lambda_w(B')$$

Thus, the theorem states that topological manifold structures in the homotopy type of C are in one-to-one correspondence with $(P', \mu') \in \Lambda_s(B')$ such that

$$F((P', \mu')) = (P, \phi_\mu)$$

By Proposition 2.46 and Lemma 4.16, such lifts are in one-to-one correspondence with $(P', \mu') \in H_n(X; \mathbb{L}^\bullet)$ such that $A((P', \mu')) = \sigma^*(X)$, where $\sigma^*(X)$ denotes the 1/2-connective visible symmetric signature of X. By Ranicki's theory of the total surgery obstruction [**Ran92**, Chapter 17], lifts of $\sigma^*(X)$ to $H_n(X; \mathbb{L}^\bullet)$ correspond to lifts of the Spivak normal fibration of X to a topological normal bundle with 0 surgery obstruction. By Browder-Novikov-Sullivan-Wall surgery theory, such lifts are in one-to-one correspondence with topological manifolds structures in the homotopy type of X. □

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