Block bundles: II. Transversality

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Block bundles were designed to play the same role in the PL category as vector bundles in the smooth category. This paper is a continuation of *Block bundles*: I [6], where the basic properties (and notation) were established. Here we apply the techniques developed in [6] to the problem of PL transversality, and at the same time establish several properties in continuing analogy with vector bundle theory.

Suppose M^m, $N^n \subset Q^q$ are proper PL submanifolds. We say N is *transverse* to M in Q if locally M and N meet like perpendicular subspaces of R^q. The *absolute* problem is to find an isotopy of Q which carries N transverse to M in Q. This problem has recently been solved by Armstrong and Zeeman [2], and an analogous problem for polyhedra has been solved by Armstrong [1]. In the relative case, they showed that, if $\dot{N}$ is transimplicial (see [2] for definition) to $\dot{M}$ in $\dot{Q}$, then N can be isotoped transverse to M keeping $\dot{Q}$ fixed, but they left the obvious relative problem unsolved; namely, is the same true if $\dot{N}$ is merely transverse to $\dot{M}$ in $\dot{Q}$?

We will work, usually, with a particular normal block bundle on M in Q, and we define transversality of N to M respecting this block structure. Using this strong definition of transversality, we prove both absolute and relative theorems, thus recovering Armstrong and Zeeman's main result (in the case M has a normal microbundle, we also recover Williamson's transversality results [10]). We extend this notion of transversality for embeddings to transverse regularity for maps, and we prove absolute and relative theorems for homotoping a map to be transverse regular. The advantage of our approach lies in its close relationship with the smooth category. Thus, one is now able to develop analogues of the cobordism and surgery techniques already extensively used in the smooth case (see Remark 3.3 and, for example, D. Sullivan [8, 9] and W. Browder [2A]).

By forgetting the given normal block bundle, we define "block transverse", namely, transverse respecting some block structure, and using this definition, we again have absolute and relative theorems. We prove (non-trivially) by a characterization using the Whitney sum bundle defined in [6] that, if N is block transverse to M in Q, then M is block transverse to N in Q. Thus "block transversality" has all the properties one would want of a definition of transversality, and the relative problem would be solved if we could show that "transversality" and "block transversality" are identical.

It is clear that 'block transverse to' implies 'transverse to' but the converse is unknown. The problem is one of deducing a global condition from a local condition, and we give an obstruction theory for solving it with coefficients in a subgroup of $\widetilde{PL}_q$; we also give some equivalent problems. The problem is analogous to that of proving the equivalence of "transversality" and "transimpliciality".

The layout of the paper is as follows. In § 1 we prove the transversality theorems for embeddings. In § 2 we are concerned with the theory of bundle maps which is needed for the treatment of transverse regularity in § 3. In § 4 we examine in detail the Whitney sum bundle, giving a simple alternative construction, and deducing the connection with block transversality. § 5 is pure block bundle theory. We define a (unique) quotient block bundle to a subbundle, and deduce results on stability of Whitney sums. In § 6 we give the obstruction theory and alternative problems outlined above. Throughout the paper we use a purely geometrical approach where there is any choice. In particular, the existence and uniqueness of quotient bundles, which we prove here geometrically, is strongly related to the stability theorem, $\widetilde{PL}_{n+q,n} \simeq \widetilde{PL}_q$. This theorem and other semi-simplicial aspects of the theory will be given in [7], where we will also give applications of the theory to classifying generalized torus knots and various embedding groups of spheres.

Added. Since this paper was written, there have been two developments of relevance. First, W. B. R. Lickorish and one of the authors have found counter-examples to one of the problems stated in § 6; namely, the notorious "three-balls problem" of Armstrong. This means that (see § 6) "block transversality" is strictly stronger than "transversality". This does not necessarily imply the non-existence of a relative transversality theorem. However, the second development is that J. F. P. Hudson, working independently, has in fact proved that relative transversality is false. The outcome of this is that a strong definition of transversality, i.e., "transimpliciality" or "block transversality" according to taste, is necessary for a sensible transversality theory in the PL category. We have not altered the paper in the light of these counter-examples, but the reader should note that the "unsolved problems" of § 6 are now all solved.

1. Transversality for manifolds

Let M be a compact proper submanifold of Q. We shall often find it more convenient to write ξ/M rather than the more cumbersome ξ/K, $|K| = M$, and we assume (throughout the paper) that, if ξ is a normal block bundle on M in Q, then $E(\xi)$ meets $\dot{Q}$ regularly, i.e., in $E(\xi \mid \dot{M})$.

Definitions. Let M, $N \subset Q$ be compact proper submanifolds, and ξ a normal block bundle on M in Q. We say N is *transverse to M with respect to ξ*, and write $N \perp \xi$, if there is a subdivision ξ' of ξ, such that $N \cap E(\xi) = E(\xi' \mid N \cap M)$ (see Figure 1 (a)). We say N is *locally* transverse to M with respect to ξ, and write $N \mathrel{\dot{\perp}} \xi$, if this is true near M, i.e., there exists ξ' a subdivision of ξ and a neighbourhood U of M in Q, such that $N \cap E(\xi) \cap U = E(\xi' \mid N \cap M) \cap U$ (see Figure 1 (b)).

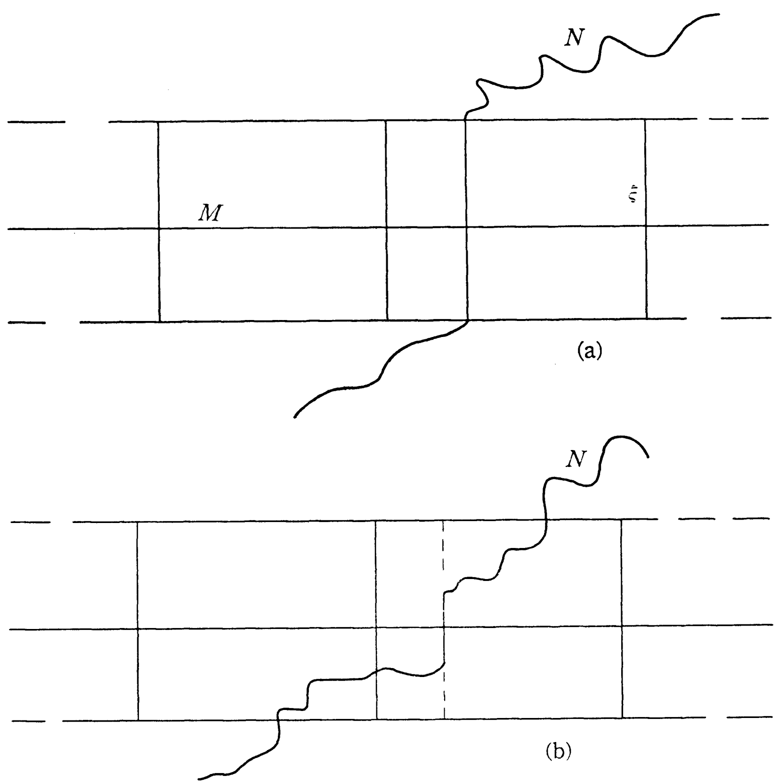

FIGURE 1

We say N is *block transverse* to M in Q if there exists a normal block bundle ξ on M in Q such that $N \perp \xi$. We have analogous definitions of transversality over a subcomplex $L \subset K$, $|K| = M$, by substituting $\xi \mid L$ for ξ in the above definitions.

Remarks. (1) If $N \perp \xi'$ ($N \mathrel{\dot{\perp}} \xi'$) and ξ' is a subdivision of ξ, then $N \perp \xi$ ($N \mathrel{\dot{\perp}} \xi$); but, because of the non-uniqueness of subdivision, the converse is not true.

(2) If $N \pitchfork \xi$ for some normal block bundle ξ on M in Q, then N is block transverse to M in Q. For suppose ξ', U are as in the definition of local transversality; use [11; Th. 1] to triangulate M, U, and all the blocks of ξ'. The second derived neighbourhood P of M in this complex has a block structure given by taking as blocks the intersections of P with blocks of ξ'. Each block is the second derived neighbourhood of the base in a block of ξ', thus condition (1) of a block bundle (see [6; § 1]) is satisfied by the regular neighbourhood theorem [3], and the other conditions are apparent. Thus we have a normal block bundle η on M in Q with $N \perp \eta$.

The main results of this section are:

THEOREM 1.1. (a) *Suppose $N, M \subset Q$ are proper submanifolds, and ξ is a normal block bundle on M in Q. Then there is an ε-isotopy of Q carrying N locally transverse to M with respect to ξ.*

(b) *Suppose that $\dot{N}$ is locally transverse to $\dot{M}$ with respect to $\xi \mid \dot{M}$, then the isotopy of* (a) *may be taken* mod $\dot{Q}$.

THEOREM 1.2. (a) *Let N, M, Q, and ξ be as in Theorem* 1.1. *There is an ambient isotopy of Q carrying N transverse to M with respect to ξ.*

(b) *Suppose further that $\dot{N}$ is transverse with respect to $\xi \mid \dot{M}$. Then the isotopy may be taken* mod $\dot{Q}$.

COROLLARY 1.3. *Suppose $N, M \subset Q$ are proper submanifolds. Then there is an ε-isotopy of Q carrying N block transverse to M in Q and, if $\dot{N}$ is block transverse to $\dot{M}$ in $\dot{Q}$, then the isotopy may be taken* mod $\dot{Q}$.

PROOF. The first part follows easily from Theorem 1.1 and Remark (2) above. For the second part, one needs to know in addition that any normal block bundle on $\dot{M}$ in $\dot{Q}$ is extensible [6; Th. 4.3 (b)].

The proof of the following corollary is postponed (for convenience) until after the proofs of the theorems (cf. Williamson [10]).

COROLLARY 1.4. *Let ξ be a normal plane, disc or micro-bundle on M in Q. Then the analogues of Theorems* 1.1 *and* 1.2 *are true for ξ.*

Remark. The condition that N is a proper submanifold is largely irrelevant, since the theorems can be generalized to the case when N is a compact polyhedron (see [5] for details and compare with [1]).

The proofs of the theorems. We first show that Theorem 1.1 implies Theorem 1.2, and we use the following definition and result.

Definition. Let ξ/K be a block bundle, and K' a subdivision of K. A block bundle η/K' is called a *minidivision* of ξ, if each block of η is contained in a block of ξ (see Figure 2).

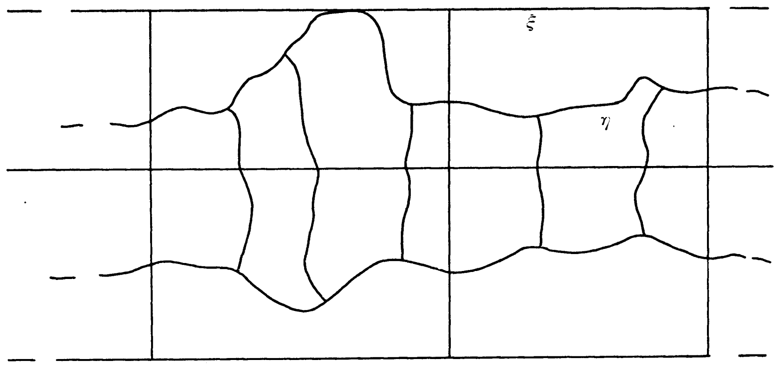

FIGURE 2

η is a *strict* minidivision if their associated (boundary) sphere bundles are disjoint. We write $\mathring{\eta}$ for the boundary sphere bundle of η.

PROPOSITION 1.5. *Suppose η/K is a strict minidivision of ξ/K, then there is a homeomorphism*

$$h\colon E(\mathring{\eta}) \times I \to \mathrm{cl}\,\big(E(\xi) - E(\eta)\big)$$

such that $h \mid E(\mathring{\eta}) \times \{0\} = identity$, and such that h preserves blocks, i.e., *for each $\sigma_i \in K$,*

$$h\big(E(\mathring{\eta} \mid \sigma_i) \times I\big) = \mathrm{cl}\,\big(\beta_i(\xi) - \beta_i(\eta)\big)\ .$$

PROOF. We prove the result by induction on the skeleton of K. For the induction step, suppose $|K| \cong I^n$, K has one n-cell σ_i, and L is its boundary. Suppose $h\colon E(\mathring{\eta} \mid L) \times I \to \mathrm{cl}\,\big(E(\xi \mid L) - E(\eta \mid L)\big)$ satisfies the conclusions of the proposition; we have to extend h over $E(\mathring{\eta}/K) \times I$.

Now by a similar argument to the regular neighbourhood collaring theorem [3], there is a homeomorphism $h'\colon E(\mathring{\eta}) \times I \to \mathrm{cl}\,\big(E(\xi) - E(\eta)\big)$, and since h and h' both give (half) collars of $E(\mathring{\eta} \mid L)$ in $\partial\,\mathrm{cl}\,\big(E(\xi) - E(\eta)\big)$, we may assume by isotopy uniqueness of collars [4] that h' extends h, completing the induction step.

Proof that Theorem 1.1 $\Rightarrow$ *Theorem* 1.2. Since Theorem 1.2 (a) follows by a double application of 1.2 (b), we need only prove 1.2 (b). For brevity, use the notation $N \perp\!\!\!\perp \xi$ if $\xi \mid N \cap M$ is defined (without subdivision) and $N \perp \xi$. Using Remark (1) above, we may assume that $\dot{N} \perp\!\!\!\perp \xi \mid \dot{M}$.

Let K be the base complex of ξ and ∂K the subcomplex corresponding to $\dot{M}$. Pick a collar $h\colon \dot{M} \times I \to M$ of $\dot{M}$ in M, and subdivide K to K' such that each $h(\sigma \times I)$, $\sigma \in \partial K$ is a subcomplex of K'. Pick a subdivision ξ'/K' of ξ. Extend h to a collar of $E(\xi \mid \partial K)$ in $E(\xi)$, such that $h(\beta_i \times I) = E\big(\xi' \mid h(\sigma_i \times I)\big)$ for each $\sigma_i \in \partial K$ (use a similar argument to the proof of Proposition 1.5 above).

Extend h again to a collar of $\dot{Q}$ in Q. Now pick a collar h_1 of $(\dot{Q}, \dot{N})$ in (Q, N) [11]. By [4; Th. 4], there is an isotopy of $Q \bmod \dot{Q}$ carrying h to h_1, so we may assume that $h = h_1$. Note that this implies $h(\dot{N} \times I) \pitchfork \xi' \mid h(\dot{M} \times I)$.

We now work with $Q_1 = \mathrm{cl}\,(Q - h(\dot{Q} \times I))$, $M_1 = \mathrm{cl}\,(M - h(\dot{M} \times I))$, $N_1 = \mathrm{cl}(N - h(\dot{N} \times I))$. By Theorem 1.1 there is an isotopy of $Q_1 \bmod \dot{Q}_1$ carrying $N_1 \mathrel{\underline{\uparrow}} \xi' \mid M_1$. By Remark (2) above, there is a strict minidivision η of $\xi' \mid M_1$ (over the same base complex) such that $N_1 \perp \eta$.

Now by Proposition 1.5, we can expand η blockwise to $\xi' \mid M_1$ keeping a smaller minidivision and the outside of a collar of $E(\dot{\xi}' \mid M_1)$ in $Q_1 - E(\xi' \mid M_1)$ pointwise fixed. Extend this isotopy by the identity to an isotopy of Q_1 and then use the collar to extend to an isotopy of Q mod $\dot{Q}$, to complete the proof.

We need the following result for the proof of Theorem 1.1.

PROPOSITION 1.6. *Let $S^n \subset \Sigma^q$ be an unknotted sphere and ξ a normal block bundle on I^{m+1} in I^{q+1}, such that $S^n \mathrel{\underline{\uparrow}} \xi \mid \Sigma^m$. Then there is an unknotted disc $D^{n+1} \subset I^{q+1}$ spanning S^n and such that $D^{n+1} \mathrel{\underline{\uparrow}} \xi$.*

We prove the implications $1.6_{q-1} \Rightarrow 1.1_q$ and $1.2_q \Rightarrow 1.6_q$. The theorem then follows by induction since we have $1.1_q \Rightarrow 1.2_q$ (above).

$1.6_{q-1} \Rightarrow 1.1_q$. Since (b) implies (a), we only have to prove (b), and as in the proof of 1.2, we may assume $\dot{N} \pitchfork \xi \mid \dot{M}$. Let K be the base complex of ξ.

By Proposition 1.5, we can extend any strict minidivision η of ξ to a subdivision ξ' of ξ. Thus if we have $N \mathrel{\underline{\uparrow}} \eta$, then by Remark (1), we also have $N \mathrel{\underline{\uparrow}} \xi$. We will prove the theorem for a particular choice of η constructed below.

Triangulate N, M, Q, all the blocks of ξ, and cells of K [11; Th. 1] by a complex of mesh less than $\varepsilon/2$, and construct the dual complexes as in [6; proofs of 4.3 and 4.4], call them N^*, M^*, etc. The dual complexes $M^* \subset Q^*$ give rise to a normal block bundle ν on M in Q (see the proof of [6; 4.3]). Now let K' consist of cells $\sigma \cap \tau$ for $\sigma \in K$, $\tau \in M^*$; $\sigma \cap \tau$ is thus a cell of σ^*. And let η/K' consist of blocks $\beta \cap \beta_1$ for $\beta \in \xi$ and $\beta_1 \in \nu$; $\beta \cap \beta_1$ is a disc, since it is a cell of β^*. It is immediate that η is a block bundle, a subdivision of ν, and a strict minidivision of ξ.

Let α_i^t be a simplex in the first derived $(N \cap M)^{(1)}$. Corresponding to α_i^t, there are three dual cells, $\beta_i^{q-t} \in Q^*$, (a block of ν), $\gamma_i^{m-t} \in M^*$, and $\delta_i^{n-t} \in N^*$, and by local flatness (β_i, γ_i) and (β_i, δ_i) are unknotted ball pairs (see e.g. [6; proof of 4.3]). If we assume inductively that there is a subdivision η' of η such that $\dot{\delta}_i \mathrel{\underline{\Uparrow}} \eta' \mid \dot{\gamma}_i$, then by Proposition 1.6, there is a further subdivision η'' of η' and a disc D_i spanning $\dot{\delta}_i$ in β_i, such that $D_i \mathrel{\underline{\Uparrow}} \eta'' \mid \gamma_i$. Now there is an isotopy of β_i mod $\dot{\beta}_i$ carrying δ_i to D_i, since they are both unknotted discs

[11; Remark at end of chapter 4], and this isotopy extends conewise to neighbouring cells of Q^* (each cell has a natural cone structure) by induction on the skeleton of Q^*. If β_i is an interior cell of Q^*, the final isotopy keeps $\dot{Q}$ fixed. Thus working inductively up the skeleton of int M^*, we find an isotopy of $Q \bmod \dot{Q}$ carrying $N \pitchfork \eta_1$ for a subdivision η_1 of η constructed during the induction.

$1.2_q \Rightarrow 1.6_q$. By passing to a minidivision if necessary, we may assume that $S^n \pitchfork \xi \mid \Sigma^m$, and by [6; Th. 4.4] that ξ/K is the trivial bundle $K \times I^{q-m}(\varepsilon)$ for some ε. (Where $I^q(\varepsilon) = [-\varepsilon, +\varepsilon]^q$.)

The following lemma is an easy consequence of the regular neighbourhood theorem and the uniqueness of collars.

LEMMA *There is a homeomorphism of I^{q+1} which keeps $I^{m+1} \times I^{q-m}(\varepsilon/2)$ pointwise fixed and which throws $\Sigma^m \times I^{q-m}(\varepsilon)$ fibrewise onto $\Sigma^m \times I^{q-m}$.*

Thus we may assume that $S^n \cap (\Sigma^m \times I^{q-m}) = (S^n \cap \Sigma^m) \times I^{q-m}$. Denote this manifold by P^n. By Theorem 1.2 applied to $N = \text{cl}\,(S^n - P^n)$ in $(M, Q) = (\text{cl}\,(\Sigma^{m+1} - \Sigma^m \times I^1), \text{cl}\,(\Sigma^q - \Sigma^m \times I^{q-m}))$, and the trivial normal bundle on M in Q, together with isotopy uniqueness of subdivision [6; Th. 4.1], we may

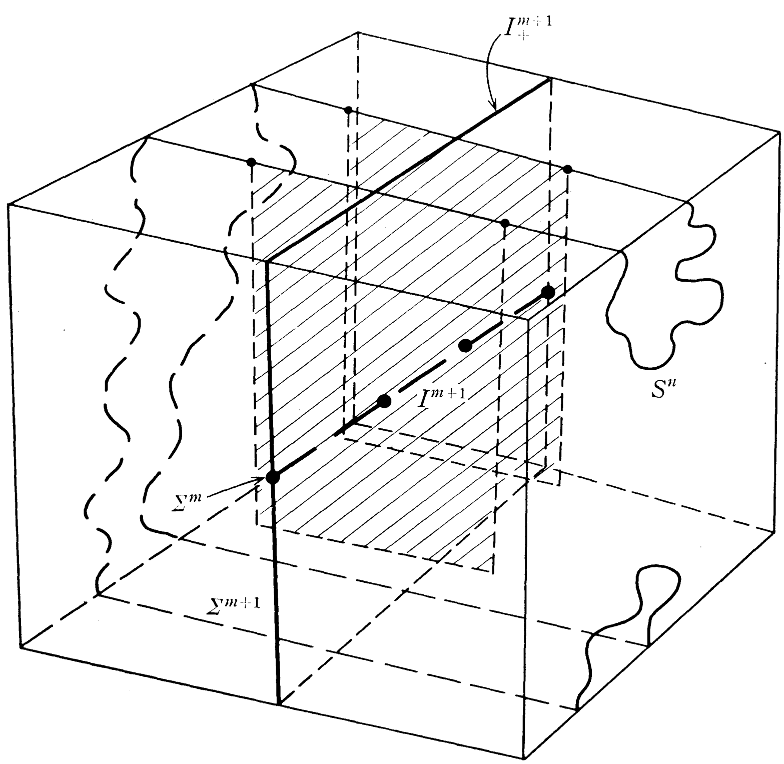

FIGURE 3

assume that $S^n \cap (\Sigma^{m+1} \times I^{q-m-1}) = (S^n \cap \Sigma^{m+1}) \times I^{q-m-1}$, see Figure 3.

Define

$$I_+^{m+1} = I^m \times \{+1\} \subset \Sigma^{m+1}$$
$$B^{q+1} = I^{m+1} \times [-1/2, +1] \times I^{q-m-1}(1/2)$$

and

$$M^{n+1} = (S^n \cap I_+^{m+1}) \times [-1/2, +1] \times I^{q-m-1}(1/2)$$

(the dotted part on the figure). Note that $M^{n+1} \pitchfork \xi$ in B^{q+1}.

Now in the boundary of the $(q+1)$-ball, $D^{q+1} = \mathrm{cl}\,(I^{q+1} - B^{q+1})$ we have the unknotted n-sphere $S_1^n = D^{q+1} \cap (S^n \cup M^{n+1})$. Choose an $(n+1)$-ball B^{n+1} spanning S_1^n in D^{q+1}, and define $D^{n+1} = M^{n+1} \cup B^{n+1}$.

Proof of Corollary 1.4. We prove the analogue of Theorem 1.1, and leave the reader to deduce the analogue of Theorem 1.2 by a similar argument.

(a) Let $|K| = M$. As in [6; § 5] ξ gives rise to a unique (open, closed or micro-) normal block bundle ζ/K on M in Q. Use Theorem 1.1 (a) (or the natural extension to open or micro-block bundles obtained by applying the results of [6; § 5]) to ε-shift $N \pitchfork \zeta$, and let ζ'/K' be the resulting subdivision (as in the definition of $\pitchfork$). Let ζ_1'/K' be the subdivision given by ξ. ζ' and ζ_1' are isotopic by [6; Th. 4.1], and this isotopy may be taken arbitrarily small by making all blocks sufficiently small.

(b) Use the same method as that in the proof of Theorem 1.2 to first make $N \pitchfork \xi$ over some (small) collar of $\dot{M}$ in M, and write, as before, $Q_1 = Q$ minus collar, etc. Apply the proof of (a) above to N_1 in (M_1, Q_1), first choosing K so that $\dot{N}_1 \cap \dot{M}_1$ is a subcomplex, and using 1.1 (b) instead of 1.1 (a). Note that the resulting isotopy keeps a neighbourhood of $\dot{N}_1 \cap \dot{M}_1$ in $\dot{N}_1$ setwise fixed (by the statement of [6; 4.1]), and therefore extends over the collar to an isotopy of Q mod $\dot{Q}$, giving the required result.

2. The theory of bundle maps

In [6] we defined the Δ-group $\widetilde{PL}_q$ and observed that it does not admit degeneracy homomorphisms. This fact complicated the proof of the main result of [6; § 3] (the uniqueness of the associated principal bundle) and also the theory of induced bundles; we were forced to define them geometrically rather than use a ready made semi-simplicial definition. However, as we show below, it is possible to define "admissible" degeneracy functions which are homotopy unique. The existence of these degeneracies explains why block bundles behave like bundles, and using them we can give a sensible account of bundle maps.

Definition. Let $\sigma \in \widetilde{PL}_q^{(k)}$, and let $\lambda : \Delta^l \to \Delta^k$, $l > k$, be a monotone simplicial map. We say $\tau \in \widetilde{PL}_q^{(l)}$ is an *admissible degeneracy of* σ, *associated with* λ, if the following diagram commutes

$$\begin{array}{ccc} \Delta^l \times I^q & \xrightarrow{\tau} & \Delta^l \times I^q \\ \downarrow \lambda \times 1 & & \downarrow \lambda \times 1 \\ \Delta^k \times I^q & \xrightarrow{\sigma} & \Delta^k \times I^q . \end{array}$$

The existence of admissible degeneracies follows (by induction) from

PROPOSITION 2.1. *Let* $\lambda : \Delta^l \to \Delta^k$, $l > k$, *be a monotone simplicial map, and let* $\sigma \in \widetilde{PL}_q^{(k)}$. *Suppose* h: $\dot\Delta^l \times I^q \to \dot\Delta^l \times I^q$ *is a block and zero preserving homeomorphism which satisfies* $\sigma \circ (\lambda \times 1) = (\lambda \times 1) \circ h$. *Then* h *is the restriction of an admissible degeneracy of* σ.

PROOF. Let L, K be triangulations of $\Delta^k \times I^q$ which are linear regarding $\Delta^k \times I^q$ as a subspace of R^{k+q}, and so that $\Delta^k \times \{0\}$ and $\Delta^s \times I^q$ (each face Δ^s of Δ^k) are subcomplexes, and σ: $L \to K$ is simplicial. Linear cell subdivisions L_1, K_1 of $\Delta^l \times I^q$ are obtained by taking inverse images of simplexes of L, K under $\lambda \times 1$, together with their intersections with $\Delta^r \times I^q$ for each face Δ^r of Δ^l. The proposition is now proved by inductively extending h over the skeletons of L_1 first defining $h \mid \Delta^l \times \{0\} = 1$. For the induction step, let α be a cell in L_1, and β the corresponding cell in K_1. Suppose h defined on $\dot\alpha$ but not on int α. Then h can be extended over α by conical extension using linear cone structures on α, β with vertices A, B respectively, chosen so that $\sigma \circ (\lambda \times 1)A = (\lambda \times 1)B$.

We are now able to define block bundle maps and prove several useful properties.

Definition. f: $E(\xi/K) \to E(\eta/L)$ is a *bundle map* if

(1) $f \mid K : K \to L$ is simplicial, and

(2) for each $\sigma \in K$, there exist charts (see [6; § 1]) φ_1: $\sigma \times I^q \to E(\xi)$, and φ_2: $f\sigma \times I^q \to E(\eta)$, such that the following diagram commutes

$$\begin{array}{ccc} \sigma \times I^q & \xrightarrow{f \mid \sigma \times 1} & f\sigma \times I^q \\ \downarrow \varphi_1 & & \downarrow \varphi_2 \\ E(\xi) & \xrightarrow{f} & E(\eta) . \end{array}$$

Remark 2.2. It follows from the existence of admissible degeneracies that the chart φ_2 in the above definition may be chosen arbitrarily. Hence if f: $E(\xi) \to E(\eta)$, g: $E(\eta) \to E(\zeta)$ are bundle maps, then gf: $E(\xi) \to E(\zeta)$ is a bundle map. (See also the Remark 2.5 at the end.)

The next result shows that the domain of a bundle map is isomorphic with the induced bundle from the restriction to the base.

THEOREM 2.3. (a) *Let* ξ/L *be a subbundle of* ε^∞/L (*see* [6; § 1]) *and* $f: K \to L$ *a simplicial map. Then* $f \times 1: E(f^\# \xi) \to E(\xi)$ *is a bundle map.*

(b) *Let* $f_i: E(\xi_i/K) \to E(\eta/L)$ $i = 1, 2$ *be bundle maps such that* $f_1 \mid K = f_2 \mid K$. *Then there exists an isomorphism* $h: E(\xi_1) \to E(\xi_2)$ *such that* $f_2 h = f_1$.

PROOF. (a) Let $\sigma \in K$, $f\sigma \in L$, and choose a chart $\varphi_2: f\sigma \times I^q \to E(\xi \mid f\sigma)$ for ξ. We will show how to define $\varphi_1: \sigma \times I^q \to E(f^\# \xi \mid \sigma)$ so that

$$\begin{array}{ccc} \sigma \times I^q & \xrightarrow{f \mid \sigma \times 1} & f\sigma \times I^q \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ E(f^\# \xi \mid \sigma) & \xrightarrow{f \times 1} & E(\xi \mid f\sigma) \end{array}$$

commutes.

Triangulate φ_2 so that $\varphi_2: J \to P$ is simplicial, and the simplexes of J, P are linear in the (natural) linear structures on $f\sigma \times I^q$, $f\sigma \times R^\infty$. Now we have cell subdivisions of $\sigma \times I^q$, $E(f^\# \xi \mid \sigma)$ by taking $(f \times 1)^{-1}$ of simplexes of J, P intersected with blocks (as in the proof of Proposition 2.1 above), and the existence of φ_1 therefore follows as in 2.1.

(b) This follows directly from Proposition 2.1 and the first part of Remark 2.2.

We now show how to "subdivide" a bundle map.

THEOREM 2.4. *Let* $f: E(\xi/K) \to E(\eta/L)$ *be a bundle map. Suppose* K', L' *are subdivisions of* K, L *so that* $f \mid K$ *is still simplicial, and* η'/L' *is a subdivision of* η/L. *Then there exists a subdivision* ξ'/K' *of* ξ/K *so that* $f: E(\xi') \to E(\eta')$ *is still a bundle map.*

PROOF. It follows from Theorem 2.3 that we need only consider the case $\xi = f^\# \eta$. But this is trivial.

Remark 2.5. If $f: E(\xi) \to E(\eta)$ and $g: E(\eta') \to E(\zeta)$ are bundle maps, where η' is a subdivision of η, then by [11; Lem. 5] and Theorem 2.4, there

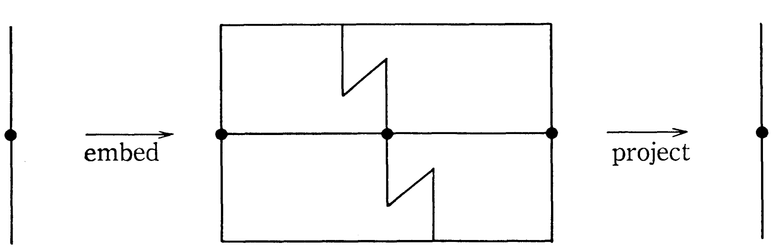

FIGURE 4

exists a subdivision ξ' of ξ so that $gf\colon E(\xi') \to E(\zeta)$ is a bundle map. However there is no hope of composing bundle maps $f\colon E(\xi) \to E(\eta')$ and $g\colon E(\eta) \to E(\zeta)$, η' a subdivision of η, as the example pictured in Figure 4 shows.

3. Transverse regularity

Let $M^m \subset Q^q$ be a compact proper submanifold, and ξ a normal block bundle on M in Q. Let N^n be a compact manifold. A proper map $f\colon N \to Q$ (i.e., $f^{-1}\dot{Q} = \dot{N}$) is said to be *transverse regular with respect to* ξ, written $f \perp \xi$, if the following are true.

(1) $f^{-1}M$ is a (proper) submanifold N_0^{n+m-q} of N.

(2) There is a normal block bundle η on N_0 in N and a subdivision ξ' of ξ such that $f \mid E(\eta)\colon E(\eta) \to E(\xi')$ is a bundle map.

Remarks. (1) This definition is the direct analogue of smooth transverse regularity.

(2) If f is an embedding, then $f \perp \xi$ is equivalent to $fN \perp \xi$, thus *transverse regularity with respect to* ξ extends the notion of *transversality with respect to* ξ given in § 1. There are similar extensions of *local transversality with respect to* ξ and *block transversality*.

THEOREM 3.1. (a) *Let M, N, Q and ξ be as above, and let $f_0\colon N \to Q$ be a proper map. Then there is a homotopy (through proper maps) f_t of N in Q such that f_1 is transverse regular with respect to ξ.*

(b) *Suppose further that $f_0 \mid \dot{N}$ is transverse regular with respect to $\xi \mid \dot{M}$, then we can choose the homotopy so that $f_t \mid \dot{N} = f_0 \mid \dot{N}$.*

Remark 3.2. Theorem 3.1 is the analogue for transverse regularity of Theorem 1.2. There are similar analogues of Theorem 1.1 and Corollaries 1.3 and 1.4. The proofs, similar to that for Theorem 3.1, are left to the reader.

Remark 3.3. Our definitions and results on transverse regularity allow a repetition for the PL category of Thom's classical results on smooth cobordism theory. Suppose $A_q \subset \widetilde{PL}_q$ is a subgroup and let $\Omega_n^q(A)$ denote cobordism classes of closed PL n-submanifolds of Σ^{n+q} with an A_q normal bundle. Then, in analogy with the smooth results, we have $\Omega_n^q(A) \cong \pi_{n+q}(TA_q)$, where TA_q is the Thom space of the classifying bundle (constructed as in [6; § 2]). The proof is analogous to the smooth proof after replacing BA_q by a manifold (embed some skeleton in euclidean space and take a regular neighbourhood). In particular, we recover Williamson's classification [10].

PROOF OF THEOREM 3.1. (a) The idea of the proof is simple. Take the product of Q with a high dimensional cube I^r, and lift f_0 to an embedding q in $Q \times I^r$. Using transversality for manifolds shift N transverse to $\xi \times I^r$

in $Q \times I^r$, and then, by the results on bundle maps in the last section, ensure that q followed by projection on Q is a bundle map.

Choose r so that there exists an embedding $e: N^n \to \operatorname{int} I^r$. $q = f_0 \times e: N \to Q \times I^r$ is thus a proper embedding. Choose a complex J with $|J| = I^r$, and let $\eta = \xi \times J$ (see [6; § 1]). Then η is a normal block bundle on $M \times I^r$ in $Q \times I^r$. Denote by p the projection $Q \times I^r \to Q$, and let $g = p \mid M \times I^r$. Choose subdivisions $(K \times J)'$, K' of $K \times J$, K (K is the base complex of ξ) so that $g: (K \times J)' \to K'$ is simplicial. Choose a subdivision ξ'/K' of ξ. We wish to subdivide η to $\eta'/(K \times J)'$ so that $p \mid E(\eta')$ is a bundle map. But this is easy since, by embedding $E(\xi')$ in the infinite trivial bundle (see [6; § 2]), we may identify $p \mid E(\eta)$ with g $\times$ 1: $E(g^{\#}\xi') \to E(\xi')$.

By Theorem 1.2 (a), there is an isotopy H_t of $Q \times I^r$ such that $H_1 qN \perp \eta'$. Let η''/L be the subdivision given by the definition of $\perp$. Now by Theorem 2.4, there are subdivisions η_1/L_1, ξ_1/K_1 of η' and ξ such that $p \mid E(\xi_1)$ is a bundle map and L_1 is a subdivision of L. Let η_2/L_1 be a subdivision of η''/L; by [6; Th. 4.1] there is a further isotopy H'_t of $Q \times I^r$ carrying η_2 to η_1. Thus by Remark 2.5, $pH'_1H_1q \perp \xi$.

Define $f_t = pH'_tH_tq$, to complete the proof.

(b) We use a similar argument to (a) using Theorem 1.2 (b) instead of 1.2 (a), and we use collars (much as in 1.2 and 1.4) to keep $f_0 \mid \dot{N}$ fixed throughout.

Let ζ/A be the bundle in $\dot{N}$ given by hypothesis. By subdividing ξ if necessary, we may assume that $f \mid E(\zeta): E(\zeta/A) \to E(\xi/B)$ is a bundle map. Let p, J be as in (a). Our first aim is to find q, η as in (a) so that $q \mid E(\zeta)$ is a bundle map.

Choose an embedding $e: |A| \to \operatorname{int} I^r$, and choose simplicial complexes A_1, B_1, L subdivisions of A, B, $B \times J$, respectively, so that the following diagram is simplicial

$$\begin{array}{ccc} & & L \\ & \overset{(f_0 \mid A) \times e}{\nearrow} & \big\downarrow p \mid L \\ A_1 & \xrightarrow[f \mid A_1]{} & B_1 \, . \end{array}$$

Choose a subdivision ξ'/B_1 of ξ, and subdivisions ζ'/A_1 of ζ and η/L of $\xi \times J/B \times J$ so that $f_0 \mid E(\zeta')$ and $p \mid E(\eta)$ are bundle maps (by Theorem 2.4 and the argument used in the proof of (a) respectively). Extend $f_0 \mid A \times e$ to a bundle map $q_1: E(\zeta') \to E(\eta)$, by Theorem 2.3 (b) so that the following diagram commutes.

$$\begin{array}{ccc} & & E(\eta) \\ & \overset{q_1}{\nearrow} & \downarrow {\scriptstyle p \mid E(\eta)} \\ E(\zeta') & \xrightarrow[f \mid E(\zeta')]{} & E(\xi') \end{array} .$$

Now by general position q_1 extends to an embedding $q: N \to Q \times I^r$ such that $pq = f_0$. By the same argument as used in the proof of Theorem 1.2 (b), we may assume that q is transverse regular to η over a collar of $\dot{M} \times I^r$ in $M \times I^r$ (compatible with collars of $\dot{Q} \times I^r$, $\partial(qN)$ in $Q \times I^r$, qN), and by choosing the collar of $\dot{Q} \times I^r$ to be the product of a collar of $\dot{Q}$ in Q with I^r, we have $pq \perp \xi$ over the collar. Now apply the same argument as (a) to $(Q \times I^r)_1 = (Q \times I^r$ minus collar) etc., using the collar to extend the isotopies to $Q \times I^r \bmod \dot{Q} \times I^r$. We have just to check that pq remains transverse regular over the collar, but this follows since H_t keeps $\partial(Q \times I^r)_1$ pointwise fixed, and (by the statement of [6; 4.1]) H'_t keeps $\partial(qN)_1 \cap E(\zeta')$ setwise fixed.

4. Whitney sums and block transversality

In this section we prove that, if N is block transverse to M, then M is block transverse to N. This follows from the fact that the components ξ, η of the Whitney sum bundle defined in [6] are mutually block transverse in their sum $\xi \oplus \eta$. We start by setting up a theory of decompositions.

Definitions. (1) ξ^q/K is said to be a *subbundle* of η^r/K if, for each cell $\sigma_i^t \in K$,

$$(\sigma_i^t, \beta_i(\xi), \beta_i(\eta)) \cong (I^t, I^{t+q}, I^{t+r}) .$$

We write $\xi \subset \eta$.

(2) Subbundles $\xi^q, \zeta^r \subset \eta^{q+r}$ give a *decomposition* of η, if their blocks meet like transverse cubes, i.e.,

$$(\sigma_i^t, \beta_i(\xi), \beta_i(\zeta), \beta_i(\eta)) \cong (I^t, I^t \times I^q \times \{0\}, I^t \times \{0\} \times I^q, I^{t+q+r})$$

for each $\sigma_i^t \in K$.

We say that decompositions $\xi_0, \zeta_0 \subset \eta_0$ and $\xi_1, \zeta_1 \subset \eta_1$ are *isomorphic* if there is an isomorphism of η_0 with η_1 which restricts to isomorphisms of ξ_0 with ξ_1, and ζ_0 with ζ_1. We now develop a theory of decompositions along the lines of [6; § 1]; it is first necessary to prove the following analogue of Proposition 1.3*, under the assumption that a decomposition with base K, $|K| \cong I^{n-1}$ is trivial.

PROPOSITION 4.1*. *Suppose $|K| \cong I^n$ and K has just one n-cell σ_i^n. Let σ_j^{n-1} be any $(n-1)$-cell in K, and let L be the subcomplex consisting of all cells except σ_i and σ_j. Suppose given a decomposition $\xi^q, \zeta^r \subset \eta^{q+r}/K$ and an* *isomorphism* $t: E(\varepsilon_{q,r}^{q+r}/L) \to E(\eta \mid L)$ *of decompositions. Then* t *extends to an isomorphism (of decompositions)* $t': E(\varepsilon_{q,r}^{q+r}/K) \to E(\eta)$.

($\varepsilon_{q,r}^{q+r}/K$ *denotes the trivial decomposition*, $K \times I^r, K \times I^q \subset K \times I^r \times I^q$).

PROOF. The method of extending t is so close to the method used in the proof of [6; 1.3*] that we content ourselves with sketching the more substantial modifications. In place of the result of [6; Lem. 1.2], we need to extend a homeomorphism of $\partial(I^s \times I^q \times I^r) \cup (I^s \times \{0\} \times \{0\})$ with itself, which preserves $\partial(I^s \times \{0\} \times \partial I^r) \cup \partial(I^s \times I^q \times \{0\})$ setwise, to a homeomorphism of $I^s \times I^q \times I^r$ which preserves $I^s \times \{0\} \times I^r \cup I^s \times I^q \times \{0\}$ setwise. The proof of this is an easy adaptation of the proof of [6; Lem. 1.2], see Zeeman [11; Lem. 18].

In place of [3; Cor. 8], we need to know that the complement of an unknotted ball triple (a homeomorph of $(I^s \times I^q \times I^r, I^s \times I^q \times \{0\}, I^s \times \{0\} \times I^r)$) in an unknotted sphere triple is unknotted. We sketch the proof. Let (B, B_1, B_2) denote the ball triple and (S, S_1, S_2) the sphere triple. The method is to find an isotopy of S which preserves S_1, S_2, and $S_1 \cap S_2$ setwise, and which throws the ball triple onto the standard hemisphere triple. First there is an isotopy of $S_1 \cap S_2$ which moves $B_1 \cap B_2$ into standard position. Extend this isotopy to the large sphere S by suspension. Now by a double application of [3; Cor. 8], we find isotopies of S_1 and S_2 keeping $S_1 \cap S_2$ fixed and moving B_1 and B_2 into standard position. Taking the suspension of these isotopies and composing, we have B_1, B_2, and $B_1 \cap B_2$ in standard position. Now B is a regular neighbourhood of $B_1 \cup B_2$ mod cl $(S_1 - B_1) \cup$ cl $(S_2 - B_2)$ in S, and so we can get B into standard position by an application of the relative regular neighbourhood theorem [3].

Now as in the proof of [6; 1.3*], t may be extended to a homeomorphism $t_1: E(\varepsilon_{q,r}^{q+r}/K) \to \dot{\beta}_i(\eta)$ such that $t_1\partial E(\varepsilon^q/K) = \dot{\beta}_i(\xi)$, and similarly ζ. There is an isotopy F of $\dot{\beta}_i(\xi)$ mod $E(\xi \mid L) \cup \sigma_j^{n-1}$ and similarly G for ζ, so that F_1 composed with the restriction of t_1 preserves blocks of ξ over ∂K, and similarly for G_1 (double application of the proof of [6; 1.3*]). Now $(E(\eta \mid L), E(\xi \mid L), E(\zeta \mid L))$ is an unknotted ball triple, and we find an isotopy H of the complementary triple, which extends F and G on the factors and the identity on the boundary, by composing the suspensions of F and G.

Define t_2 to be t_1 on $E(\varepsilon^{q+r} \mid L)$ and $H_1 \circ t_1$ on the complementary triple. Now $t_2\beta_j(\varepsilon^{q+r}) \neq \beta_j(\eta)$, but this can be put right by one application of the relative regular neighbourhood theorem [3]. (Similar to the application in the proof of [6; 1.3*], note that at this point the induction hypothesis is used to ensure that $\eta \mid \sigma_j$ is a trivial decomposition.) The resulting t_3 defined on

$\dot{\beta}_i(\varepsilon^{q+r})$ can be extended to the required t' using the analogue of [6; Lem. 1.2] described above.

We can now repeat [6; § 1] for decompositions; and, in particular, we have (by completely analogous proofs)

THEOREM 4.2. (a) *Any decomposition over a disc is trivial.*

(b) *Given a decomposition $\xi, \zeta \subset \eta/K$ and a subdivision K' of K, then there exist subdivisions $\xi', \zeta' \subset \eta'/K'$ which form a decomposition.*

(c) *If $\xi_1, \zeta_1 \subset \eta_1$, $\xi_2, \zeta_2 \subset \eta_2$ are two such, then there is an isomorphism of decompositions $\eta_1 \cong \eta_2$.*

(More delicate isotopy uniqueness results may be obtained using the cellular shelling techniques of [6; § 4]).

It will be useful to have the following result (which cannot be deduced by analogy to a result in [6]).

PROPOSITION 4.3. *Let $\xi, \zeta \subset \eta/K$ be a decomposition and $\zeta_1, \zeta_2/K'$ two subdivisions of ζ. Then there exists an isomorphism (of decompositions) $h: E(\eta) \to E(\eta)$ such that $h \mid E(\xi) = 1$, and $h \mid E(\zeta_1): E(\zeta_1) \to E(\zeta_2)$ is an isomorphism.*

PROOF. We use the proof of [6; 4.1] (which shows that ζ_1, ζ_2 are isotopic), and extend the isotopy to an isotopy of $E(\eta)$ mod $E(\xi)$. In [6; 4.1] the isotopy was constructed by induction on the skeleton of K. The induction step gave an isotopy of $E(\zeta \mid \sigma_i)$ mod $K \cup E(\zeta \mid \dot{\sigma}_i)$ for some cell $\sigma_i \in K$. Using local triviality (Theorem 4.2 (a)), we can extend this isotopy to an isotopy of $E(\eta \mid \sigma_i)$ mod $E(\xi) \cup E(\zeta \mid \dot{\sigma}_i)$ by taking cross-product with the identity. A similar remark to that made in [6; proof of 4.1] allows us to extend this isotopy to $E(\eta)$ mod $E(\xi)$.

Remark. It follows from 4.3 and 4.2 (b) that given a decomposition $\xi, \zeta \subset \eta/K$ and subdivisions $\xi', \zeta'/K'$ of ξ, ζ, then we can find a subdivision η' of η such that $\xi', \zeta' \subset \eta'$ is a decomposition.

We now define an important subclass of decompositions, *block decompositions*; it is known that there are decompositions which are not block decompositions (see the added remark at the end of the introduction and also § 6).

Definition. Subbundles $\xi^q, \zeta^r \subset \eta^{q+r}/K$ give a *block decomposition* of η if, given any complex J with $|J| = E(\xi)$ such that the blocks of ξ and cells of K are subcomplexes, then there is a bundle ν/J such that

(1) $E(\nu) = E(\eta)$,

(2) ζ is the amalgamation of the restriction of ν to $|K|$,

(3) the blocks of η are unions of blocks of ν.

Remarks. (1) The definition of block decomposition is not apparently symmetric in ξ, ζ, but in fact this will be proved below, where we also show that the definition characterizes the Whitney sum bundle.

(2) Block decompositions also form a *theory* analogous to [6; § 1]. However since a block decomposition is essentially just two block bundles, ξ/K and ν/J, the analogue of the key proposition [6; 1.3*] is trivial by a double application of [6; 1.3*]. In particular, we deduce

PROPOSITION 4.4. (a) *Any block decomposition over a disc is trivial.*

(b) *Given a block decomposition* $\xi, \zeta \subset \eta/K$ *and a subdivision* K' *of* K, *then there exist subdivisions* $\xi', \zeta' \subset \eta'/K'$ *forming a block decomposition.*

(Note that the proposition can also be deduced by double applications of [6; 1.1, 1.5] respectively.)

Remarks. (1) By (a), a block decomposition *is* a decomposition.

(2) By (b) and 4.2 (c), any subdivision as a decomposition of a block decomposition is a block decomposition.

The most useful property of a block decomposition $\xi, \zeta \subset \eta$, is that η is determined by ξ and ζ (see the next theorem). It is precisely this which is not true for decompositions.

THEOREM 4.5. *Suppose* $\xi_1, \zeta_1 \subset \eta_1/K$ *and* $\xi_2, \zeta_2 \subset \eta_2/K$ *are block decompositions, and* $h: E(\xi_1) \rightarrow E(\xi_2)$, $g: E(\zeta_1) \rightarrow E(\zeta_2)$ *are isomorphisms. Then* $h \cup g$ *extends to an isomorphism of decompositions.*

PROOF. Choose J_1, J_2 with $|J_i| = E(\xi_i)$ such that $hJ_1 = J_2$ and the cells of K and blocks of ξ_i are subcomplexes. Let ν_i/J_i be the bundles given by the definition of block decomposition. By Proposition 4.3 we may assume that $g: E(\nu_1 \mid |K|) \rightarrow E(\nu_2 \mid |K|)$ is an isomorphism. This isomorphism extends by [6; Th. 1.6] to an isomorphism of ν_1 with ν_2, which, on considering unions of blocks, is the required isomorphism of decompositions.

We now proceed to the connection with Whitney sum. Recall that the class of $\xi \oplus \eta/K$ is $(\xi \times \eta/K \times K) \mid \Delta_K$.

PROPOSITION 4.6. $\xi \times K$, $K \times \eta \subset \xi \times \eta$, *and* $K \times \eta$, $\xi \times K \subset \xi \times \eta$ *are block decompositions.*

PROOF. Let Q be any complex with $|Q| = E(\xi)$, and such that the cells of K and blocks of ξ are subcomplexes, and let $J = Q \times K$. We show that conditions (1), (2), and (3) hold for J (and hence for any other complex by subdivision and amalgamation). Define $B_{ij}(\nu) = \sigma_i \times \beta_j(\eta)$ where $\sigma_i \in Q$, $\sigma_j \in K$, and the conditions are easily checked. $K \times \eta$, $\xi \times K \subset \xi \times \eta$ is a block decomposition by a similar argument.

THEOREM 4.7. *There exists a block decomposition* $\xi_1, \zeta_1 \subset \xi \oplus \zeta$ *such that*
(1) $\xi_1 \cong \xi$, $\zeta_1 \cong \zeta$, *and*
(2) $\zeta_1, \xi_1 \subset \xi \oplus \zeta$ *is also a block decomposition.*

PROOF. Let $(K \times K)'$ be a subdivision of $K \times K$ such that Δ_K is a subcomplex and a subdivision of K. Choose decomposition subdivisions $(\xi \times K)'$, $(K \times \eta)' \subset (\xi \times \eta)'/(K \times K)'$ by Theorem 4.2 (b), and note that this, and $(K \times \eta)', (\xi \times K)' \subset (\xi \times \eta)'$ are both block decompositions by Proposition 4.6 and Remark (2) below Proposition 4.4.

Since the property of being a block decomposition is evidently invariant under restriction and amalgamation, the theorem follows by taking ξ_1 to be the amalgamation of $(\xi \times K)' \mid \Delta_K$ and ζ_1 similarly.

THEOREM 4.8. *If* $\xi, \zeta \subset \eta$ *is a block decomposition, then*
(1) $\zeta, \xi \subset \eta$ *is a block decomposition, and*
(2) $\eta \cong \xi \oplus \zeta$.

PROOF. Let $\xi_1, \zeta_1 \subset \xi \oplus \zeta$ be the block decomposition provided by Theorem 4.7. Then, from Theorem 4.5, we have an isomorphism $\xi_1, \zeta_1 \subset \xi \oplus \zeta \to \xi, \zeta \subset \eta$, and the first result follows since $\zeta_1, \xi_1 \subset \xi \oplus \zeta$ is also a block-decomposition.

Remark. We now have an alternative construction for $\xi \oplus \zeta$. Choose a complex J with $|J| = E(\xi)$ and the cells and blocks of K, ξ being subcomplexes. Subdivide ζ over K' (the subcomplex corresponding to K), and take ν the induced bundle over J by the collapse $|J| \searrow |K|$. Now take unions of blocks of ν over blocks of ξ to give $\xi \oplus \zeta$.

COROLLARY 4.9. *Suppose* $P \subset M \subset Q$ *are proper compact submanifolds, and let* ξ *be a normal block bundle on* P *in* M, ζ *on* M *in* Q, *and* η *on* P *in* Q. *Then (as classes)* $\xi \oplus \zeta \mid P \sim \eta$.

PROOF. Let K be the base complex of ξ. By subdividing if necessary, we may assume that the base complex of ζ contains J as a subcomplex where $|J| = E(\xi)$, and the cells of K and blocks of ξ are subcomplexes of J. Unions of blocks of ζ over blocks of ξ give a normal bundle on P in Q, which, by uniqueness, we may take to be η [6; Cor. 4.6]. The result now follows by Theorem 4.8.

For the promised connection with block transversality, we need

PROPOSITION 4.10. *Let* $N, M \subset Q$ *be compact proper submanifolds, and suppose* $N \cap M = P$ *is also a proper submanifold. Then* N *is block transverse to* M *in* Q *if and only if there exist normal block bundles* ξ, ζ, η *on* P *in* N, M, Q, *such that* $\xi, \zeta \subset \eta$ *is a block decomposition.*

PROOF. If N is block transverse to M, then ξ, ζ, η exist as in the proof of Corollary 4.9. The converse is obvious. Therefore, using Theorem 4.8, we have;

COROLLARY 4.11. *Block transversality is symmetric.*

5. Quotient bundles and the stability of $\oplus$

This section is concerned with results on block bundles, which, while not strictly relevant to the main theme of the paper, fit well with the preceding discussion of Whitney sums. We define quotient block bundles and the Stiefel manifold, and deduce results on splitting and cancelling trivial bundles.

THEOREM 5.1. *Let ξ/K be a subbundle of η/K.*

(a) *There is a subbundle $\zeta \subset \eta$ such that ξ, ζ give a block decomposition of η.*

(b) *Given any two choices ζ_1, ζ_2 for ζ, then there is an isotopy of $E(\eta)$* mod *$E(\xi)$ carrying ζ_1 to ζ_2 and preserving the blocks of η setwise.*

Remarks. (1) We do not have a relative version of (b), i.e., we cannot keep blocks fixed where ζ_1, ζ_2 agree. This again follows from the new counter-examples mentioned in the introduction.

(2) From Theorem 5.1 and the fact that decompositions form a theory, it follows that we can subdivide pairs $\xi \subset \eta$ (and indeed by Proposition 4.3 we can extend a subdivision of ξ over η), and that any pair over a disc is trivial (and hence we may amalgamate pairs).

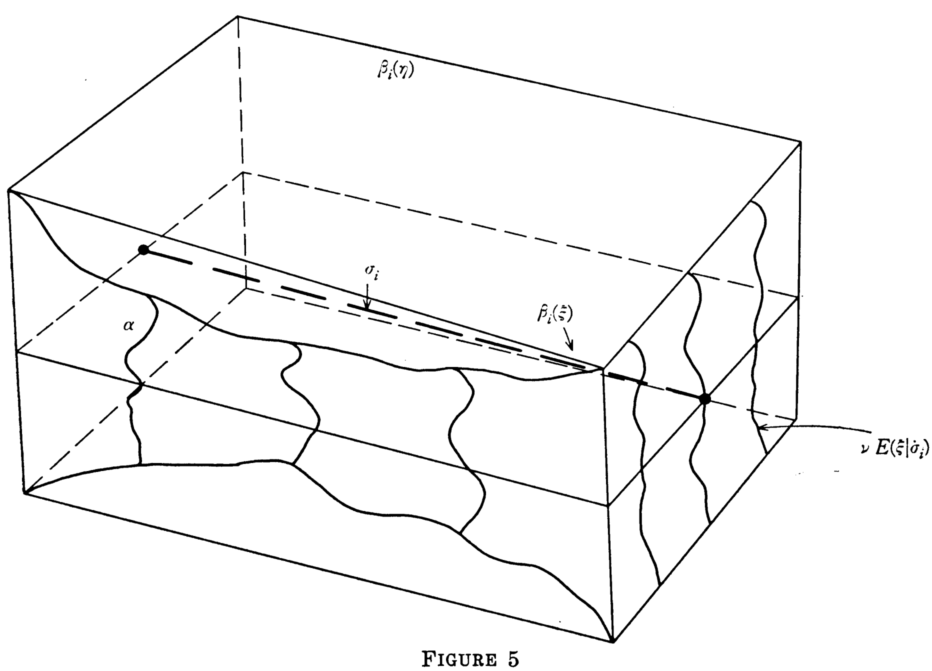

FIGURE 5

PROOF. (a) Choose a complex J with $|J| = E(\xi)$ as for the definition of a block decomposition. We define ν/J inductively over the skeleton of K to satisfy the conditions of a block decomposition, and then $\nu \,|\, |K|$ amalgamated over K gives ζ.

Let $\sigma_i \in K$, and suppose ν defined over $\dot{\sigma}_i$. By [6; Th. 4.3 (b)] we can extend $\nu \,|\, E(\dot{\xi} \,|\, \dot{\sigma}_i)$ to a normal block bundle α/L on $E(\dot{\xi} \,|\, \sigma_i)$ in $\mathrm{cl}\,(\partial\beta_i(\eta) - E(\eta \,|\, \dot{\sigma}_i))$ see Figure 5, where L is the subcomplex of J corresponding to $E(\dot{\xi} \mid \sigma_i)$.

Let L_1 be the subcomplex of J corresponding to $\beta_i(\xi)$. $\gamma = \alpha + \nu \,|\, E(\xi \,|\, \dot{\sigma}_i)$, together with the block $\beta_i(\eta)$ over $\beta_i(\xi)$, is a block bundle over $T = \partial L_1 +$ the cell $\beta_i(\xi)$. γ subdivides over L_1 by the subdivision theorem [6; Th. 1.5], and this defines ν over σ_i to complete the induction step.

(b) Let ν_1, ν_2/J be as in the definition of block decomposition. We define an isotopy inductively over the skeleton of K, which carries ν_1 to ν_2 (and hence ζ_1 to ζ_2) and keeps $E(\xi)$ pointwise fixed, as follows.

Let $\sigma_i \in K$, and suppose ν_1, ν_2 agree over $\dot{\sigma}_i$, define α_1, α_2/L as in (a). By [6; Th. 4.4 (c)] α_1, α_2 are isotopic mod $E(\eta \,|\, \dot{\sigma}_i) \cup E(\xi \mid \sigma_i)$, and then $\nu_1, \nu_2 \,|\, \beta_i(\xi)$ are isotopic by [6; Th. 4.1], the isotopies extend as in [6; proof of 4.1].

The Stiefel manifold. $\widetilde{V}_{n+q,n}$ is the Δ-set of which a k-simplex is an isomorphism onto a subbundle

$$\Delta^k \times I^n \longrightarrow \Delta^k \times I^{n+q} .$$

Given a block bundle ξ^{n+q}/K^k, where K is an ordered simplicial complex, we define the associated $\widetilde{V}^k_{n+q,n}$-bundle to consist of isomorphisms onto subbundles

$$\Delta^t \times I^n \longrightarrow E(\xi \mid \sigma^t) , \qquad \sigma^t \in K .$$

Similarly, we have the associated $\widetilde{V}_{n+q,n}$-bundle base K (cf. [6; § 3]).

One verifies at once that isomorphisms onto subbundles $K \times I^n \longrightarrow E(\xi^{n+q})$ correspond bijectively with cross-sections of the associated $\widetilde{V}_{n+q,n}$-bundle and thus, by Theorem 5.1, we have an obstruction theory for splitting off trivial bundles. From the fact that $\pi_i(\widetilde{V}_{n+q,n}) = 0$ for $i < q$, see [7; 2.11], we deduce

COROLLARY 5.2. *Any ξ^q/K^k splits into $\eta^k \oplus \varepsilon^{q-k}$.*

COROLLARY 5.3. *If $\xi^{k+1} \oplus \varepsilon^t/K^k \cong \eta^{k+1} \oplus \varepsilon^t/K^k$, then $\xi^{k+1} \cong \eta^{k+1}$.*

Now by a general position argument (see [11]) any ξ^q/K^k embeds as a subbundle of ε^r/K^k for large r, thus by Theorem 5.1 and the above corollaries, we have

COROLLARY 5.4. *Given ξ^q/K^k, then there exists η^k/K^k such that $\xi \oplus \eta \cong \varepsilon$.*

Tangent Bundles. Define the tangent class of a manifold M, $t(M)$, to be the class determined by a regular neighbourhood of Δ_M in $M \times M$, (recall [6; Cor. 4.6]).

Let ξ/K be a block bundle, u its equivalence class, where $|K| = M$.

PROPOSITION 5.5. $t(E(\xi)) \mid M = u \oplus t(M)$.

PROOF. Write $N = E(\xi)$ and consider the diagram of embeddings

$$\begin{array}{ccc} M \times M & \subset & N \times N \\ \uparrow{\scriptstyle \Delta_M} & & \uparrow{\scriptstyle \Delta_N} \\ M & \subset & N . \end{array}$$

The block bundle classes given by these embeddings are (reading round clockwise starting at the top) $u \times u$, $t(N)$, u, $t(M)$. Therefore by Corollary 4.9, we have on restricting to M,

$$t(N) \mid M \oplus u = u \oplus u \oplus t(M) .$$

The result follows by stability (Corollaries 5.2–5.5).

COROLLARY 5.6. *Let $M \subset Q$ be a compact submanifold, and u the class of any normal block bundle on M in Q, then $t(M) \oplus u = t(Q) \mid M$.*

PROOF. We show that if $N^q \subset Q^q$ is a submanifold, then $t(N) = t(Q) \mid N$, the result then follows from Proposition 5.5, on taking $N = E(u)$. But to prove this, we only have to note that a regular neighbourhood of N in $N \times N$ is a regular neighbourhood of N mod $\dot{N}$ in $Q \times Q$, and the result therefore follows by the relative regular neighbourhood theorem [3].

6. Unsolved problems and obstructions

The connection between transversality and decompositions is given by the following result, analogous to Proposition 4.10, the proof (similar to that of [6; 4.3]) is omitted.

PROPOSITION 6.1. *Suppose M^m, $N^n \subset Q^q$ are proper submanifolds and $M \cap N = P$ is also a proper submanifold. Then M is transverse to N in Q, if and only if there exist normal block bundles ξ, η, ζ on P in N, M, Q, such that $\xi, \eta \subset \zeta$ is a decomposition.*

Thus the following two problems posed earlier are equivalent.

Problem 1. Does transversality imply block transversality?

Problem 2. Is a decomposition a block decomposition?

However, Problem 2 may be attacked semi-simplicially. Let $\widetilde{PL}^{q,r}_{q+r}$ be the Δ-group of a decomposition, a k-simplex is an isomorphism

$$\sigma \colon E(\varepsilon^{q+r}_{q,r}/\Delta^k) \longrightarrow E(\varepsilon^{q+r}_{q,r}/\Delta^k) ,$$

where $\varepsilon_{q,r}^{q+r}$ denotes the trivial decomposition $\Delta^k \times I^q$, $\Delta^k \times I^r \subset \Delta^k \times I^q \times I^r$. By forgetting ε^{q+r} we have a homomorphism

$$\varphi \colon \widetilde{PL}_{q+r}^{q,r} \rightarrow \widetilde{PL}_q \times \widetilde{PL}_r \ .$$

Let $\widetilde{PL}_{q+r,q,r}$ denote the kernel of φ. A splitting

$$\alpha \colon \widetilde{PL}_q \times \widetilde{PL}_r \rightarrow \widetilde{PL}_{q+r}^{q,r}$$

is given by

$$\alpha(\sigma, \tau) = (\sigma \times 1) \circ (1 \times \tau) \ .$$

Thus $\widetilde{PL}_{q+r}^{q,r} \cong \widetilde{PL}_q \times \widetilde{PL}_r \times \widetilde{PL}_{q+r,q,r}$.

Now by the results of § 4, the subgroup of $\widetilde{PL}_{q+r}^{q,r}$ corresponding to block decompositions is the summand $\widetilde{PL}_q \times \widetilde{PL}_r$, and so we have

THEOREM 6.2. *There is an obstruction theory for a decomposition to be a block decomposition, with coefficients in* $\pi_i(\widetilde{PL}_{q+r,q,r})$.

Thus the following problem is equivalent to Problems 1 and 2.

Problem 3. Is $\widetilde{PL}_{q+r,q,r}$ contractible?

The problem of whether $\pi_0(\widetilde{PL}_{q+r,q,r}) = 0$ is readily seen to be equivalent to the following geometrical problem.

Problem 4. Let $S^{p+q+1} = S^p * S^q$ ($*$ denotes geometric join), and let h be a homeomorphism of S^{p+q+1} keeping $S^p \cup S^q$ pointwise fixed. Is h concordant to the identity keeping $S^p \cup S^q$ pointwise fixed? (S^r denotes a homeomorph of Σ^r).

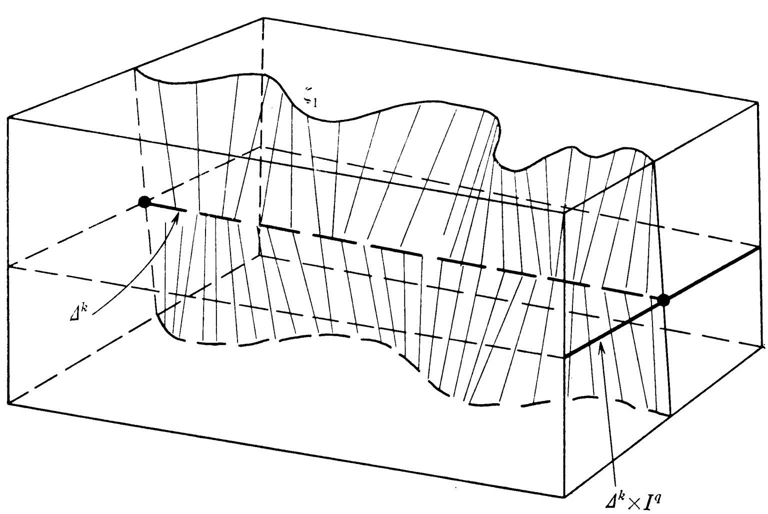

FIGURE 6

We could answer Problems 1 and 2 at once if we had a relative form for Theorem 5.1 (b) (uniqueness of quotient bundles) and the crucial problem here was posed by Armstrong.

Problem 5. (Three balls problem). Let $\varepsilon^q/\Delta^k \subset \varepsilon^{q+r}/\Delta^k$ be the standard inclusion; suppose ζ_1, ζ_2 are quotient bundles which agree over $\dot{\Delta}^k$. Then can we ambient isotope ζ_1 to ζ_2 mod $\Delta^k \times I^q \cup \dot{\Delta}^k \times I^{q+r}$? (See Figure 6).

We can re-express these problems in terms of *transverse neighbourhoods*. Let $P \subset M \subset Q$ be compact proper submanifolds. A *block transverse neighbourhood* on P in (M, Q) is the (total space of) the restriction to P of some normal block bundle on M in Q. A *transverse* (*regular*) *neighbourhood* is defined similarly relaxing *block transverse* to *transverse*. By [6; Th. 4.4 (b)], block transverse neighbourhoods are unique up to ambient isotopy, but because of the lack of a relative version of Theorem 5.1 (b), we do not know

Problem 6. Given two block transverse neighbourhoods on P in (M, Q) which agree over $\dot{P}$, are they ambient isotopic mod $\dot{Q} \cup M$?

And, we also do not know

Problem 7. Given two transverse regular neighbourhoods on P in (M, Q), are they ambient isotopic mod M?

Finally, it is not hard to show that Problems 1 to 7 (excluding 4) are all equivalent. In our treatment of transversality, we have by-passed these problems by sticking to the strong definition, block transversality. It may be that the answer to the problems is, *no*, and *block transversality* is strictly stronger than *transversality*. If this is the case, it is still possible that the unsolved relative transversality theorem stated in the introduction is true.

Added. As noted at the end of the introduction, Problems 1 to 7 (excluding 4) all have counter-examples. The remaining problem (number 4) has also been solved. Lickorish has shown that, in a wide range of dimensions, the answer is in fact *yes*. See W. B. R. Lickorish, "*Homeomorphisms of $S^p * S^q$ keeping $S^p \cup S^q$ pointwise fixed*", J. Lon. Math. Soc. (to appear). More generally the authors have since shown that for q and $r > 2$ $\widetilde{PL}_{q+r,q,r} \simeq \Omega(G/PL)$.

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