Block bundles: III. Homotopy theory

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Introduction

This paper is a continuation of 'Block bundles: I and II (Transversality)' [37, 38]. In these papers we established the theory of block bundles as a tool in the PL category, drawing strong analogies with the usage of vector bundles in the differential category. Our methods were geometrical wherever possible.

In this paper our aim is to use the theory in order to reduce geometrical problems to homotopy theory. This aim is achieved in several cases through studying the homotopy of the various semi-simplicial groups and complexes involved. Our methods will be algebraic as far as possible, that is to say, we will start from the homotopy theory and deduce the geometry. It is worth noting, however, that all our geometrical techniques, apart from the triangulation theorems of Whitehead [46] and the Cairns-Hirsch smoothing theorem [12], belong to the PL category, whereas we recover several known results and some new results for the differential category.

The plan of the paper is the following. In § 0 we define all the complexes and groups that we will use (including those defined in previous papers) and then in subsequent sections, we interpret geometrically exact sequences, braids, etc. derived from these complexes.

Section 1 is concerned with Levine's braid [28; 2.2] which we prove to be isomorphic with a homotopy braid. We deduce one of the main results of Haefliger's paper [5; 3.4].

Section 2 is concerned with various types of solid torus knots and Stiefel manifolds (PL, C^∞, Γ- cf. § 0); we have braids displaying the obstructions to both types of smoothing, and we deduce results in the metastable range. Also in this section, there is a geometrical version of Haefliger's suspension sequence [5; 6.4] and two geometrical interpretations of the groups $\pi_n(G_{q+1}, G_q)$ and $\pi_n(F_q, G_q)$ for $q > 2$ (see § 0 for definitions). The first is in terms of isotopy classes of knotted "ribbons" (embeddings of $\Sigma^n \times I$ in Σ^{n+q+1}) and the second in terms of knots of I^n in $I^n \times \Sigma^q$. This enables us to give a new proof that $\pi_n(F_q, G_q) = 0$ for $2q > n + 2$ (cf. James [20], Haefliger [5]), which is, in some sense, the key metastable result (cf. 2.9–2.11).

In § 3 we derive an exact PL suspension sequence

$$E_n^q(R) \longrightarrow E_n^{q+1}(I) \longrightarrow \pi_n(F_q, G_q) \longrightarrow E_{n-1}^q(R) \longrightarrow$$

where $E_n^q(R)$ is the set of concordance classes of open tubes on Σ^n in Σ^{n+p}, and $E_n^q(I)$ is the set of concordance classes of closed tubes. This implies that metastably $E_n^q(R) \cong E_n^{q+1}(I)$ and that unstably there are many examples of non-zero groups $E_n^q(R)$ or $E_n^q(I)$. The sequence forms part of a diagram of suspension sequences (3.8) which reduces results of Hirsch [11] to diagram chasing, and also implies that $E_n^3(I) \cong \Gamma_n^3 \oplus E_n^2(R)$. We use the main result of [39] to prove that there is a differentiable framed embedding of S^{18} in S^{27} whose normal bundle is non-standard as a topological microbundle, but such that, on suspension to S^{28}, both the embedding and the framing become standard!

In § 4 we establish a theory of q-block bundles with a (never-zero) section. The obstructions for the existence of such a section fall into two sets, *homotopy* and *geometrical* obstructions. The homotopy obstructions have coefficients in $\pi_i(S^{q-1})$ and correspond to finding a section of the associated Serre fibration, while the geometric obstructions have coefficients in certain groups of links of spheres. This situation contrasts with the theory of sections of a PL micro- or fibre bundle (fibre R^q, Σ^{q-1}, or I^q) in which there are only the homotopy obstructions, and furnishes further evidence for the divergence of the theories of block bundles and microbundles. We give examples of block bundles with section which do not split a (corresponding) line subbundle, and we prove that metastably the geometrical obstructions vanish and a section always splits. We use the theory of sections to give an obstruction theory for (instantly) isotoping a PL submanifold off itself (Theorem 4.21), and we conclude the section with a short proof that either $E_{2s}^{s+1}(R)$ or $E_{2s+1}^{s+2}(R)$ is non-zero for s even > 3 (recall that we showed $E_{18}^9(R) \neq 0$ in [39]).

Section 5 on immersions of spheres in spheres is a complement to Haefliger's work [7], and we link our definitions with those of Haefliger. We give a classification of PL immersions with an (open or closed) normal bundle, and we derive braids which theoretically show which immerisons (of spheres in spheres) possess open or closed normal bundles. We give *suspension* sequences for all types of immersions (framed, C^∞, Γ-, PL) and these fit together with diagram (3.8) into a large "ladder" diagram (5.15), which simultaneously displays the precise obstructions to compression for all types of immersion and differentiable embedding of spheres in spheres, and the precise obstructions to (both types of) smoothing of a PL immersion, and for a PL immersion to possess an open or closed normal bundle. As an example of the use of this diagram, we prove that all non-trivial PL-immersions of Σ^4 in Σ^7 have no PL normal disc bundle (and hence *a fortiori* are unsmoothable).

Section 6 is an appendix in which we display on one diagram (6.2) a large number of the results proved in the paper. Using this diagram, the reader may construct diagrams of exact sequences *ad libitum*.

Some of the results of this paper (notably from § 1) have been announced by Haefliger [8] and Morlet [32].

0. Notation and main tools

We collect here all the notations, definitions, and major theorems that will be used.

Notation. We work principally in the *smooth category* C of C^∞-manifolds (abbreviated C-manifold) and C-maps, and in the *piecewise-linear category* PL of PL-spaces and PL-maps. Thus a C-isomorphism is what in more conventional terminology would be termed a diffeomorphism of class C^∞. Definitions and basic properties of the categories C and PL may be found in [33] and [49].

A *PD-map* $f: X \to M$, where X is a PL-space and M is a C-manifold, is a map which is C on each simplex of some triangulation K of X. If for each point $b \in K$ the differential of f (see [33; § 8]) df_b is injective, then f is a *PD-immersion*. If, in addition, f is a homeomorphism onto $f(X)$, then f is a *PD-embedding*. A *compatible PL*-structure on a C-manifold M is a PL-structure determined by a *PD-homeomorphism* (PD-embedding onto M) $h: X \to M$, where X is a PL-manifold. Similarly h determines a *compatible* C-structure on X.

Let X and M be manifolds as above. Then a PD-map $g: X \to M$ is a Γ-*map* (resp. Γ-*embedding*, Γ-*immersion*, Γ-*homeomorphism*) if there is a PD-homeomorphism $h: X \to N$ such that $g \circ h^{-1}$ is a C-map (resp. C-embedding, C-immersion, C-homeomorphism). Note that a PD-embedding g is a Γ-embedding if and only if $f(X)$ is a smooth submanifold of M, and that a PD-homeomorphism is always a Γ-homeomorphism, (n.b., Haefliger [7] calls Γ-maps "lisse").

As standard objects in the categories C and PL, we have: R^n is the subspace of Hilbert space with coordinates $x_i = 0$ for $i > n$, $R^n_+ \subset R^n$ is defined by $x_n \geqq 0$, and $R^n_- \subset R^n$ by $x_n \leqq 0$. R^n is also identified with the n-fold cartesian product of the real line R. $D^n \subset R^n$ is defined by $\|x\| \leqq 1$ and $S^{n-1} = \partial D^n$ is its boundary. S^n, D^n, and R^n all have natural C-structures. $S^n_\pm = S^n \cap R^{n+1}_\pm$. $I^n = [-1, +1]^n \subset R^n$ and $\Sigma^n = \partial I^{n+1}$. I^n, Σ^n and R^n have natural PL-structures. $\Sigma_\pm^n = \Sigma^n \cap R_\pm^{n+1}$. $I = [0, 1] \subset R$ is not to be confused with $I^1 = [-1, +1]$. The standard n-simplex $\Delta^n \subset R^n$ has vertices v_i $(i = 0, \cdots, n)$ where v_i has coordinates $x_j = 1, j \leqq i, x_j = 0$ otherwise. $\partial_i^n: \Delta^{n-1} \to \Delta^n$ is the order preserving simplicial map which fails to cover v_i. We shall often abbreviate ∂_i^n to ∂_i. $\Lambda_{n,i} = \mathrm{cl}\,(\dot{\Delta}^n - \partial_i \Delta^{n-1})$.

We denote by $t: \Sigma^n \to S^n$ the radial projection map from the origin and also use $t: I^{n+1} \to D^{n+1}$ for a choice of PD-extension. t furnishes S^n and D^n with (standard) compatible PL-structures, and for convenience of notation, we will often speak of PD-, Γ-, or even PL-maps S^n or $D^n \to S^{n+q}$ or D^{n+q}, meaning that S^n and D^n are to be interpreted as PL-spaces where necessary *via* t. [The reader is warned, however, that the inclusion of S^{n-1} or D^n in R^n is not a PL-map under this convention.]

From now on, when no prefix is used, *all maps are assumed to be PL.* Thus "homeomorphism" will mean PL-isomorphism.

The inclusion $\Sigma^n \times I^q \subset \Sigma^{n+q}$ comes directly from the definitions, as does the identification $\Sigma^{n+q-1} = \Sigma^{n-1} \times I^q \cup I^n \times \Sigma^{q-1}$. We pick a "standard" PL-homeomorphism $s_1: \mathrm{int}\, I^n \to R^n$, and this gives us an inclusion $s_2: \Sigma^n \times R^q \to \Sigma^{n+q}$. We also use a standard C-isomorphism $s_3: \mathrm{int}\, D^n \to R^n$, and a standard C-inclusion $s_4: S^n \times R^q \to S^{n+q}$ such that $s_4 \mid S^n \times \{0\} = \mathrm{id}$. The *standard cell structure* on Σ^n has i-cells $\Sigma_\pm^i$, $0 \leqq i \leqq n$.

Δ-sets and Δ-groups. Δ is the category with objects Δ^n, $n = 0, 1, \cdots$ and morphisms generated by the ∂_i defined above (note that $\partial_j^{n+1}\partial_i^n = \partial_i^{n+1}\partial_{j-1}^n$ if $i < j$). A *Δ-set* (resp. *pointed Δ-set*, *Δ-group*) or simply a *complex* is a contravariant functor X from Δ to the category of sets (resp. pointed sets, groups). The function $X(\partial_i^n): X(\Delta^n) \to X(\Delta^{n-1})$ is called a *face map* and an element $\sigma \in X(\Delta^n)$ is called an *n-simplex* of X. A natural transformation $f: X \to Y$ is called a *Δ-map*. Notice that a Δ-set differs from a (complete) semi-simplicial complex in that it does not have "degeneracy" functions. These have proved to be a nuisance in previous papers [37, 38] and, since by [40] they are irrelevant to our present purposes, we discard them completely.

We now define the various (pointed) Δ-sets that we will use. We attempt to give a uniform definition for the Δ-sets $X(Y)$ and $\widetilde{X}(Y)$ where X denotes one of PL, PD, G, or O and Y denotes one of R^q, I^q, D^q, Σ^{q-1}, or S^{q-1}:

A k-simplex of $\widetilde{X}(Y)$ is an X-isomorphism $\sigma: \Delta^k \times Y \to \Delta^k \times Y$ which satisfies

(i) (block-preserving) for each subcomplex $K \subset \Delta^k$, $\sigma^{-1}(K \times Y) = K \times Y$,

(ii) (zero-preserving) if $Y = R^q$, I^q, or D^q, then $\sigma \mid \Delta^k \times \{0\} = \mathrm{id}$.

A k-simplex of $X(Y)$ is a k-simplex of $\widetilde{X}(Y)$ which satisfies the stronger condition (i′) (fibre-preserving) σ commutes with projection on Δ^k.

Face maps are defined by restriction. In practise we write $X_q(Z)$ rather than $X(Z^q)$ (resp. $X(Z^{q-1})$) when $Z = R$, D or I (resp. $Z = S$ or Σ). A similar remark applies to $\tilde{X}_q(Z)$.

A simplex of $X_q(\mu)$ or $\tilde{X}_q(\mu)$ is a germ of an X-isomorphism $\Delta^k \times R^q \circlearrowleft$ defined in a neighbourhood of $\Delta^k \times \{0\}$ and satisfying conditions analogous to the ones given above.

We now have to interpret these definitions in the various cases. If $X = PL$, then the definitions make sense.

If $X = PD$ we must then regard Y as both a C- and a PL- object (this can be done for $Y = I^q$, D^q, S^{q-1}, or Σ^{q-1} by making use of $t: I^n \to D^n$, and interpret PD-isomorphism to mean PD-homeomorphism. If $X = G$, and $Y = S^{q-1}$, or Σ^{q-1} (resp. $Y = R^q$, D^q, or I^q) we interpret *σ is a G-isomorphism* to mean σ (resp: $\sigma \mid \Delta^k \times (Y - \{0\})$) is a (topological) homotopy equivalence.

A simplex $\sigma \in O(Y)$ is a C-isomorphism such that $\sigma \mid \{x\} \times Y: \{x\} \times Y \circlearrowleft$ is orthogonal, $Y = R^q$, D^q or S^{q-1} (if $Y = I^q$ or Σ^{q-1}, one must use t). $\tilde{O}(Y)$ is never used.

We identify $Z_q(\Sigma)$ with $Z_q(S)$, and $Z_q(D)$ with $Z_q(I)$ for all Z, *via* t.

Subcomplexes $X_q(Y_s) \subset X_q(Y)$ and $\tilde{X}_q(Y_s) \subset \tilde{X}_q(Y)$ ($Y = \Sigma$, I, or R) are defined by $\sigma^k \in X_q(Y_s)$ or $\tilde{X}_q(Y_s)$ if and only if $\sigma \mid \Delta^k \times \{*\} = \mathrm{id}$, where $* = (1, 0, \cdots, 0)$ if $Y = \Sigma$ and $* = (1/2, 0, \cdots, 0)$ if $Y = I$, or R.

All the complexes satisfy the extension condition (Kan [21]). For $Z_q(Y)$, $Z \neq O$, this follows from the fact that $\Delta^k \cong \Lambda_k \times I$ and Whitehead's results [46], cf. [37; 6.1]. For $Z = O$, one uses an argument similar to [37; 6.2]. Since this has essentially been done elsewhere [27, 37] we omit details. For $X = PL$ or O, the Δ-sets are in fact Δ-groups.

We have the following standard abbreviations.

$O_q(Y)$, $Y = D$, S, R, or μ are all called O_q (since they may be naturally identified).

$\widetilde{PL}_q(I) \equiv \widetilde{PL}_q$ (since this is the natural block bundle group) and $\widetilde{PL}_q(I_s) \equiv \widetilde{P_sL}_q$.

$\widetilde{PD}_q(R) \equiv \widetilde{PD}_q$ (since this is the most natural complex for comparison with vector bundles, see [37, §6]).

$PL_q(\mu)$ and $PD_q(\mu) \equiv PL_q$ and PD_q, as this is well established usage now, cf. [27, 31].

$G_{q+1}(\Sigma_s)$, $G_q(\Sigma)$, and $\tilde{G}_q(\Sigma) \equiv F_q$, G_q and $\tilde{G}_q$; F_q and G_q are well established too.

Inclusions of the complexes. There are the following inclusions:

$$
\begin{array}{lcccc}
 & & G_q(Y) & \subset & \widetilde{G}_q(Y) \\
 & & \cup & & \cup \\
(0.1) \qquad O_q(Y) & \subset & PD_q(Y) & \subset & \widetilde{PD}_q(Y) \\
 & & \cup & & \cup \\
 & & PL_q(Y) & \subset & \widetilde{PL}_q(Y)
\end{array}
$$

$O_q(Y)$ acts on $PD_q(Y)$ and $\widetilde{PD}_q(Y)$ on the left by composion, and $PL_q(Y)$, $\widetilde{PL}_q(Y)$ act on the right. We have the composite theorem.

THEOREM 0.2. *$PL_q(Y) \subset PD_q(Y)$ and $\widetilde{PL}_q(Y) \subset \widetilde{PD}_q(Y)$ are both homotopy equivalences, all Y.*

(See [40] for notions of homotopy equivalence, weak homotopy equivalence, etc. in Δ-sets, and also an analogue of Whitehead's theorem). Theorem 0.2 has been proved elsewhere in several of the cases, e.g. [37; 6.1], [12; 3.4]. As these special cases essentially contain the general proof, we will omit it.

Let Z denote any one of PL, PD, G, $\widetilde{PL}$, $\widetilde{PD}$, $\widetilde{G}$, then there are the following Δ-maps.

Cone: $Z_q(\Sigma) \to Z_q(I)$ by inductive conical extension, cf. [37; 5.1].

Bdry: $Z_q(I) \to Z_q(\Sigma)$ $(Z \neq G, \widetilde{G})$ by restriction.

Int: $Z_q(I) \to Z_q(R)$ by the identification s_1 when $Z = PL, G, \widetilde{PL}$, or $\widetilde{G}$, and by s_1 on the right and s_3 on the left when $Z = PD$ or $\widetilde{PD}$.

Collar: $Z_q(I) \to Z_q(R)$ $(Z \neq G, \widetilde{G})$ by extension *via* suitable open PL- or C-collars.

Germ: $Z_q(R) \to Z_q(\mu)$, by taking germs.

Warning. Because the identifications s_1 and s_3 are not equal on the domain of s_3, int and collar do not commute with the inclusions (0.1). However, it is easily proved that they commute up to homotopy, which is all that we require.

Int and collar are easily proved to be homotopic maps, cf. remarks in [37; § 5], and the following results indicate when these maps are known to be homotopy equivalences.

THEOREM 0.3. (i) *For $Z = \widetilde{PL}$ or $\widetilde{PD}$, all the above maps are homotopy equivalences* (cone *and* bdry *being inverse*).

(ii) *For $Z = PL$ or PD,* cone, bdry, *and* germ *are homotopy equivalences* (cone *and* bdry *being inverse*).

(iii) *For $Z = G, \widetilde{G}$,* cone, int, *and* germ *are homotopy equivalences as is the inclusion $G_q(Y) \subset \widetilde{G}_q(Y)$, $Y = \Sigma, I, R$, or μ.*

PROOF. The $\widetilde{PL}$ case is [37; Remark 5.5]. The $\widetilde{PD}$ case now follows, using 0.2. For the PL case, the result about cone and bdry is well-known and easily proved by the "Alexander trick" (see e.g. Browder [2]); that germ is a homotopy equivalence is proved in Kuiper-Lashof [25] (also Hirsch). The PD case follows using 0.2. The G and $\widetilde{G}$ cases are well-known and easy to prove by suitable homotopies, and will be left to the reader. $G_q \subset \widetilde{G}_q$ is a homotopy equivalence by [37; 5.8], and the rest now follows.

Remark. All the inclusions (0.1) and the above maps not covered by 0.2 and 0.3 are known not to be homotopy equivalences. These are consequences of Browder [2], Hirsch [11], and ourselves [39], see also §§1, 3, and 4.

Inclusions $Z_q(Y) \subset Z_{q+1}(Y) \subset \cdots$ are defined as follows.

For $Y = R$ or μ, identify $\sigma: \Delta^k \times R^q \circlearrowleft$ with $\sigma \times \mathrm{id}: \Delta^k \times R^q \times R \circlearrowleft$.

For $Y = I$ and $Z = PL$, $\widetilde{PL}$, G, or $\widetilde{G}$ identify $\sigma: \Delta^k \times I^q \circlearrowleft$ with $\sigma \times \mathrm{id}$: $\Delta^k \times I^q \times I^1 \circlearrowleft$.

For $Y = I$ and $Z = PD$ or $\widetilde{PD}$, we need to be rather more careful because strictly (i.e., without using the conventional identification of I^n and D^n) a simplex of $\widetilde{PD}_q(I)$ is a PD-homeomorphism $\sigma: \Delta^k \times I^q \to \Delta^k \times D^q$. $\sigma \times \mathrm{id}$ maps $\Delta^k \times I^q \times I^1 \to \Delta^k \times D^q \times D^1$, and there is a homeomorphism analogous to t, $t_1: D^q \times D^1 \to D^{q+1}$. Composing $\sigma \times \mathrm{id}$ with $\mathrm{id} \times t_1$ gives the required simplex of $\widetilde{PD}_{q+1}(I)$.

For $Y = \Sigma$, the composion $Z_q(\Sigma) \xrightarrow{\text{cone}} Z_q(I) \subset Z_{q+1}(I) \xrightarrow{\text{bdry}} Z_{q+1}(\Sigma)$ gives the required inclusion (this is essentially suspension).

These inclusions commute with (0.1), and cone, bdry, and germ by definition. But as before, in the PD and $\widetilde{PD}$ cases, the commutativity with int and collar is only up to homotopy.

We denote the direct limit of $Z_q(Y) \subset Z_{q+1}(Y) \subset \cdots$ by $Z(Y)$.

Subcomplexes $Z_{q+r,r}(Y) \subset Z_{q+r}(Y)$ are defined as follows:

$Z_{q+r,r}(R)$ consists of simplexes which preserve $\{0\} \times R^r \subset R^q \times R^r = R^{q+r}$ (i.e., $\sigma \in Z_{q+r,r}(R)$ if and only if $\sigma \in Z_{q+r}(R)$, and $\sigma \mid \Delta^k \times \{0\} \times R^r = \mathrm{id}.$).

$Z_{q+r,r}(I) \subset Z_{q+r}(I)$ consists of simplexes which preserve I^r, and $Z_{q+r,r}(\Sigma) \subset Z_{q+r}(\Sigma)$ consists of simplexes which preserve Σ^{r-1}.

We now have inclusions

$$Z_q(Y) \subset Z_{q+r,r}(Y) \subset Z_{q+r}(Y) \quad \text{and} \quad Z_{q+1,1}(Y) \subset Z_{q+1}(Y_s) \subset Z_{q+1}(Y) .$$

For information about when these inclusions are homotopy equivalences or not, see later sections, in particular 2.2, 3.2, and §4. Note that $O_{q,1}$ and $O_q(Y_s)$ are all essentially O_{q-1}. Note also that G is the direct limit of $G_q \subset F_q \subset G_{q+1} \subset F_{q+1} \subset \cdots$.

Homotopy groups. We always denote by $* \subset Z_q(Y)$ the subcomplex consisting of identity simplexes (thus $Z_q(Y)$ is a pointed Δ-set). In [40] we showed how to define the homotopy groups of (pairs of) pointed Δ-sets;

we recall the definition. An ordered simplicial complex K may be regarded as a Δ-set with typical k-simplex an order preserving simplicial embedding $\Delta^k \to K$. Thus we may speak of Δ-maps $K \to X$, where X is any Δ-set. A *map* $f: P \to X$ where P is a compact PL-space is a triangulation of P by an ordered simplicial complex K and a Δ-map $f: K \to X$. f_0, f_1 are *homotopic* if there is a map $F: P \times I \to X$ such that $F | P \times \{i\} = f_i$, $i = 0, 1$. We denote the set of homotopy classes by $[P; X]$. Now let $Y \subset X$ be pointed Δ-sets. Then by definition $\pi_n(X, Y) = [I^n, \Sigma_+^{n-1}, \Sigma_-^{n-1}; X, Y, *]$.

We will need the following result from [40].

PROPOSITION 0.4. *Any homotopy class in $\pi_n(X, Y)$ is represented by a Δ-map $K, K_+, K_- \to X, Y, *$, where K, K_+, K_- is any ordered triangulation of $I^n, \Sigma_+^{n-1}, \Sigma_-^{n-1}$. Any homotopy may be replaced by a homotopy over J, J_+, J_-, where J, J_+, J_- is any ordered triangulation of $I^n \times I$,* etc., *extending $K_0 \times \{0\} \cup K_1 \times \{1\}$.*

Remarks. (1) 0.4 gives us great freedom in choice of representatives for homotopy elements. For example, let K, K_+, K_- be complexes which triangulate $\Sigma_+^n, \Sigma_+^{n-1}, \Sigma_-^{n-1}$ (linearly), (then t gives a corresponding smooth triangulation of S_+^n etc.). An element of $\pi_n(\widetilde{PD}_q, O_q)$ for example, will (by 0.4) be represented by a block and zero preserving PD-homeomorphism $K \times R^q \circlearrowleft$ which is orthogonal and fibre-preserving over K_+ (and C over simplexes), and the identity over K_-. In practice we will omit the *block-preserving* condition in both representatives and homotopies. This is allowable, since the results of [37; § 4] (and Whitehead [46] for the PD-case) show that any homeomorphism is isotopic to a block-preserving one. Thus we will say that a representative of $\pi_n(\widetilde{PD}_q, O_q)$ is a zero-preserving PD-homeomorphism $\Sigma_+^n \times R^q \circlearrowleft$ which is an orthogonal isomorphism over Σ_+^n, and the identity over Σ_-^n.

(2) A representative for $\pi_n(O_q)$ is an orthogonal and fibre-preserving homeomorphism $\sigma: S_+^n \times R^q \circlearrowleft$ (identity over S^{n-1}) which is C over the simplexes of some smooth triangulation of S_+^n. However, by standard approximation theorems, we may always assume that σ is in fact a C-isomorphism, and by uniqueness of C-collars [33], that σ is the identity in a neighbourhood of $S^{n-1} \times R^q$.

We will also use the homotopy groups of a square

$$\square \equiv \begin{matrix} A & \supset & C \\ \cup & & \cup \\ B & \supset & D \end{matrix}$$

of pointed Δ-sets. $\pi_n(\square)$ is by definition the set $[I^n, \Sigma_+^{n-1}, \Sigma_-^{n-1}, \Sigma_+^{n-2}, \Sigma_-^{n-2}; A, B, C, D, *]$ and its properties are proved in an analogous way to those of a square of spaces, cf. remarks in [40], in particular it is a group if $n > 2$, and abelian if $n > 3$.

There are two exact sequences including $\square$; one is, for example,

$$\pi_n(B, D) \longrightarrow \pi_n(A, C) \longrightarrow \pi_n(\square) \xrightarrow{\partial} \pi_{n-1}(B, D) ,$$

and, together with the exact sequences of the triples (A, B, D) and (A, C, D), they form a commutative braid (at least up to sign). We write $\pi_n(\square)$ as $\pi_n(A; B, C; D)$, and in the case $D = B \cap C$, we have the homotopy group of a triad and write $\pi_n(A; B, C)$. (The exact sequences and braid of a square are reasonably well-known, and are derived in a similar way to those of a triad, cf. [1].)

There is an exact sequence associated with a diagram

$$\begin{array}{ccccc} A & \supset & C & \supset & E \\ \cup & \square_1 & \cup & \square_2 & \cup \\ B & \supset & D & \supset & F \end{array}$$

namely

$$\pi_n(\square_2) \longrightarrow \pi_n(\square_3) \longrightarrow \pi_n(\square_1) \xrightarrow{\partial} \pi_{n-1}(\square_2) \tag{0.5}$$

where $\square_3$ is the outside square. This can easily be proved directly (for an direct proof for triads, see Haefliger [5; 6.2]).

Main tools. We now state the main tools that we will use. By a *concordance* between embeddings $f_0, f_1: M \to Q$, we mean an embedding $F: M \times I \to Q \times I$ such that $F^{-1}(Q \times \{i\}) = M \times \{i\}$, and $F | M \times \{i\} = f_i$, $i = 0, 1$.

THEOREM 0.6. Hudson and Lickorish [16A]. Concordance extension theorem. *If $q > 2$ and M is compact, then any proper concordance F of M^n in Q^{n+q} extends to concordance of Q,* i.e., *there exists a homeomorphism $H: Q \times I \circlearrowleft$ such that $H^{-1}(Q \times \{i\}) = Q \times \{i\}$, $i = 0, 1$, $H | Q \times \{0\} =$ id, and $H \circ (f_0 \times \mathrm{id}) = F$. Further, if G is a concordance of ∂Q extending $F | \partial M \times I$, then it may be assumed that H extends G.*

(By a *proper* embedding of manifolds $f: M \to Q$, we mean, as usual, $f^{-1}(\partial Q) = \partial M$ and f^{-1} (compact) $=$ compact).

An *isotopy* is a level-preserving concordance. It is *locally unknotted* if, for each subinterval $J \subset I$, $F | M \times J$ is locally flat.

THEOREM 0.7. Hudson and Zeeman [19]. Isotopy extension theorem. *Any locally unknotted isotopy with M compact extends to an isotopy of Q.*

THEOREM 0.8. Zeeman. Unknotting balls and spheres. *Any proper PL ball or sphere pair $B^n \subset B^{n+q}$ or $S^n \subset S^{n+q}$, $q > 2$, is PL homeomorphic to the standard pair.*

THEOREM 0.9. Hirsch [12]. Product smoothing theorem. *Suppose M is a PL-manifold, and Q is a C-manifold, and that $f\colon M^n \times R^q \to Q^{n+q}$ is a PD-embedding such that $f|M$ is proper and $f(\partial M \times R^q) \subset \partial Q$. Then there is a PD-isotopy of f carrying f to f_1 (via similar embeddings), so that $f_1|M$ is a Γ-embedding, and $f_1'\colon f_1(M) \times R^q \to Q$ defined by $f_1'(f_1x, y) = f_1(x, y)$ is a C-embedding. If, in addition, K is a subcomplex of a triangulation of M, N an open neighbourhood of K such that $f'|f(N) \times R^q$ is a C-embedding, then the isotopy may be assumed to be fixed on $N' \times R^q$ where $N' \subset N$ is a closed neighbourhood.*

THEOREM 0.10. Hirsch [11]. *Any orientation-preserving PL-homeomorphism $f\colon \Sigma^{n+q} \circlearrowleft$ such that $f|\Sigma^n =$ id is isotopic to the identity via homeomorphisms which are the identity on Σ^n.*

THEOREM 0.11. Hirsch [11]. Knob theorem. *Let $M^n \subset Q^{n+q}$ be a PL submanifold, and let $B^n \subset$ int M^n be an n-disc. Then any two orientation preserving embeddings $f_i\colon B^n \times I^q \to Q^{n+q}$, $i = 0, 1$, such that $f_i | B^n \times \{0\} =$ id and $f_i^{-1}(M) = B^n$ are isotopic through an isotopy which is fixed on M^n.*

THEOREM 0.12. Kuiper-Lashof [25] and Hirsch [13]. *There is a homotopy equivalence $h\colon PL_{q+1,1}(I) \to PL_q(R)$ such that*

$$\begin{array}{ccc} PL_{q+1,1}(I) & \xrightarrow{h} & PL_q(R) \\ \cap & & \cap \\ \widetilde{PL}_{q+1,1} & \xleftarrow{q} & \widetilde{PL}_q(R) \end{array}$$

commutes up to homotopy, where q is the (homotopy) inverse of int *composed with the standard inclusion.* q itself is a homotopy equivalence, see 2.2. The homotopy equivalence h is given in [25], and the commutativity follows from the geometrical interpretation given in [13].

We also make crucial use of the relative regular neighbourhood theorems of Hudson and Zeeman [17, 18]. The statements are rather technical and so we refer to [17] for details. [17] has a flaw in it, see [43], but this will not affect our usage, see [18] for the corrected theorems. See also Cohen [2B] for a general theory of relative regular neighbourhoods.

1. Some results of Levine and Haefliger

In this section we obtain a geometric interpretation of the braid of the triple $O_q \subset \widetilde{PD}_q \subset \widetilde{G}_q(R)$, and we prove that it is isomorphic with the braid described by Levine [28]. We deduce results of Haefliger [5].

We display the braid under consideration

(1.1)
$$\begin{tikzcd}
\pi_n(O_q) \arrow[dr] \arrow[rr, bend left] & & \pi_n(\widetilde{G}_q(R)) \arrow[dr] \arrow[rr, bend left] & & \pi_n(\widetilde{G}_q(R), \widetilde{PD}_q) \\
& \pi_n(\widetilde{PD}_q) \arrow[ur] \arrow[dr] & & \pi_n(\widetilde{G}_q(R), O_q) \arrow[ur] \arrow[dr] & \\
\pi_{n+1}(\widetilde{G}_q(R), \widetilde{PD}_q) \arrow[ur] \arrow[rr, bend right] & & \pi_n(\widetilde{PD}_q, O_q) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(O_q)
\end{tikzcd}$$

For the geometric interpretation of (1.1) we will need some definitions.

Definition of $FE_n^q(I)$. An orientation preserving embedding $f\colon \Sigma^n \times I^q \to \Sigma^{n+q}$ such that $f \mid \Sigma^n \times \{0\}$ is the standard inclusion is called a *framing* of Σ^n in Σ^{n+q}. Two such framings f_0, f_1 are *concordant* if there is a concordance between the embeddings f_0, f_1; $F\colon \Sigma^n \times I^q \times I \to \Sigma^{n+q} \times I$ such that $F \mid \Sigma^n \times \{0\} \times I$ is the standard inclusion. Concordance is an equivalence relation and the set of equivalence classes is $FE_n^q(I)$.

Remark 1.2. A set $FE_n^q(R)$ is defined by replacing I^q by R^q in the above definition. The inclusion $I^q \subset R^q$ and the standard homeomorphism $s_1\colon \operatorname{int} I^q \to R^q$ are readily seen to induce inverse bijections between $FE_n^q(R)$ and $FE_n^q(I)$.

Now let $\sigma\colon \Sigma_+^n \times I^q \circlearrowleft$ represent an element of $\pi_n(\widetilde{PL}_q)$, so that

$$\sigma \mid \Sigma^{n-1} \times I^q = \mathrm{id}\ .$$

Define $\Psi[\sigma]$ to be the element of $FE_n^q(I)$ represented by the embedding $f\colon \Sigma^n \times I^q \to \Sigma^{n+q}$ defined by

(i) $f \mid \Sigma_+^n \times I^q = \sigma$, and

(ii) $f \mid \Sigma_-^n \times I_q = \mathrm{id}$.

Clearly Ψ is well-defined.

LEMMA 1.3. *The function* $\Psi\colon \pi_n(\widetilde{PL}_q) \to FE_n^q(I)$ *is a bijection,* $n > 0$.

PROOF. It follows from 0.11 that a class in $FE_n^q(I)$ has a representative $f\colon \Sigma^n \times I^q \to \Sigma^{n+q}$ such that $f \mid \Sigma_-^n \times I^q = \mathrm{id}$, and it follows from the regular neighbourhood theorem of [18] that f is isotopic mod $\Sigma_-^n \times I^q$ to an embedding f' satisfying (i) and (ii) above for some representative σ of $\pi_n(\widetilde{PL}_q)$ (note the remarks in § 0 on defining the homotopy of $\widetilde{PL}_q$, etc.). Hence Ψ is onto.

Now suppose $\Psi[\sigma_0] = \Psi[\sigma_1]$, and let f_i be the embeddings defined by replacing σ by σ_i in (i) and (ii) above for $i = 0, 1$. Let F be a concordance between f_0 and f_1. That Ψ is injective follows from the existence of a homeomorphism $G: \Sigma^{n+q} \times I \circlearrowleft$ such that $G \mid \Sigma^{n+q} \times \partial I \cup \Sigma^n \times \{0\} = \mathrm{id}$, and $G \circ F \mid \Sigma^n_- \times I^q \times I$ is standard. An application of the regular neighbourhood theorem of [18] to the concordance $G \circ F$ will then show that $[\sigma_0] = [\sigma_1]$. Now it is clear how to define G on $F(\Sigma^n_- \times I^q \times I \cup \Sigma^n_+ \times \{0\} \times I) \cup \Sigma^{n+q} \times \partial I$. [49; Lem. 18] provides the required extension of G.

Definition of $F\Gamma^q_n$. Let $f: S^n \times R^q \to S^{n+q}$ be an orientation preserving PD-embedding such that

(i) $f \mid S^n \times \{0\}$ is a Γ-embedding

(ii) $f': f(S^n \times \{0\}) \times R^q \to S^{n+q}$, defined by $f\big(f(x, 0), y\big) = f(x, y)$, is a C-embedding, and

(iii) $f \mid S^n \times \{0\}$ is PD-isotopic to the identity.

Two such framed embeddings f_0, f_1 are concordant if there is a PD-embedding $F: S^n \times R^q \times I \to S^{n+q} \times I$ such that

(i) $F \mid S^n \times \{0\} \times I \to S^{n+q} \times I$ is a Γ-embedding,

(ii) F' (defined as in (ii) above) is a C-embedding,

(iii) $F \mid S^n \times \{0\} \times I$ is PD-isotopic to the identity, and

(iv) $F \mid S^n \times R^q \times \{i\} = f_i$, $i = 0, 1$.

The set of concordance classes is denoted $F\Gamma^q_n$.

It follows easily from 0.9 and 0.10 that there is a bijection $FE^q_n(R) \to F\Gamma^q_n$. Combining this with the bijections of Remark 1.2 and Lemma 1.3, and the isomorphism $\pi_n(\widetilde{PL}_q) \to \pi_n(\widetilde{PD}_q)$ (given by 0.2) we have

PROPOSITION 1.4. *There is a bijection* $\Psi_1: \pi_n(\widetilde{PD}_q) \to F\Gamma^q_n$, $n > 0$.

We now redefine the set Γ^q_n which was defined in [37; §6] (as the set of concordance classes of smoothings of $\Sigma^n \subset \Sigma^{n+q}$ such that Σ^{n+q} is smoothed by a C-structure concordant to the standard one). We prove that the two sets are isomorphic by showing that the new set is also isomorphic with $\pi_n(\widetilde{PD}_q, O_q)$.

Definition of Γ^q_n. Let $f: S^n \to S^{n+q}$ be a Γ-embedding such that f is PD-isotopic to the identity. Two such embeddings are concordant if there is a Γ-concordance $F: S^n \times I \to S^{n+q} \times I$ connecting them, such that F is PD-isotopic to the identity. The set of concordance classes is denoted Γ^q_n.

Now let $\sigma: S^n_+ \times R^q \circlearrowleft$ represent an element of $\pi_n(\widetilde{PD}_q, O_q)$, (cf. remarks in §0, we may assume that $\sigma \mid S^{n-1} \times R^q$ is a C-homeomorphism). Define the PD-embedding $\sigma_1: S^n_+ \times R^q \to S^{n+q}_+$ to be $s_4 \mid \circ \sigma$. By uniqueness of PD-collars (which follows from [46] and [19]) σ_1 is PD-isotopic to σ_2 where σ_2 preserves a standard C-collar of S^{n+q-1} in S^{n+q}_+. By applying 0.9, we have σ_2 PD-isotopic, keeping a smaller collar fixed, to σ_3 where $\sigma_3 \mid S^n_+ \times \{0\}$ is a Γ-embedding. $\sigma_4: S^n \to S^{n+q}$ defined by $\sigma_4 \mid S^n_+ = \sigma_3$ and $\sigma_4 \mid S^n_- = \mathrm{id}$ is now a representative of an element $\Psi_2[\sigma] \in \Gamma^q_n$. A similar construction shows that Ψ_2 is well-defined.

PROPOSITION 1.5. *The function* $\Psi_2: \pi_n(\widetilde{PD}_q, O_q) \to \Gamma^q_n$ *is a bijection.*

The proof of this (which follows from considerations similar to those in the proof of Lemma 1.3) will be left to the reader.

Definition of P^q_n, $n > 0$ (not to be confused with the P^q_n defined in [5; p. 424]). We consider framed submanifolds (M^n, f) where M^n is an oriented (PL-) submanifold of Σ^{n+q-1} such that ∂M is homeomorphic to Σ^{n-1}, and f is an orientation preserving embedding $f: M^n \times I^{q-1} \to \Sigma^{n+q-1}$ such that $f(x, 0) = x$ and $f \mid \partial M \times \{0\}$ is isotopic to standard position.

Two such framed submanifolds (M_0, f_0) and (M_1, f_1) are framed cobordant if there is submanifold $N^{n+1} \subset \Sigma^{n+q-1} \times I$ and an embedding $F: N \times I^{q-1} \to \Sigma^{n+q-1} \times I$ such that

(i) $F(x, 0) = x$,

(ii) $F \mid (N \cap \Sigma^{n+q-1} \times \{i\}) = f_i$, $i = 0, 1$, and

(iii) $F(\mathrm{cl}(\partial N - N \cap \Sigma^{n+q-1} \times \partial I))$ is isotopic to $\Sigma^{n-1} \times I \subset \Sigma^{n+q-1} \times I$.

P^q_n denotes the set of framed cobordism classes.

Now let $\sigma: I^n \times \Sigma^{q-1} \circlearrowleft$ represent an element of $\pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma))$ so that $\sigma \mid \Sigma^{n-1}_+ \times \Sigma^{q-1}$ is a (PL-) homeomorphism, and $\sigma \mid \Sigma^{n-1}_- \times \Sigma^{q-1} = \mathrm{id}$ (σ itself being a topological map). By [48] we may assume that σ is PL, and by [38; Th. 3.1] that σ is block transverse regular (see for definition) to $M' = I^n \times \{(1, 0, \cdots, 0)\}$. Then $M = \sigma^{-1}(M')$ is a submanifold of $I^n \times \Sigma^{q-1}$ and M receives a framing *via* σ and a standard framing of M' in $I^n \times \Sigma^{q-1}$. Regarding M as a framed submanifold of $I^n \times \Sigma^{q-1} \cup \Sigma^{n-1} \times I^q = \Sigma^{n+q-1}$, we have an element $\Phi[\sigma]$ of P^q_n.

By further applications of [48] and [38; Th. 3.1] Φ is seen to be well-defined.

LEMMA 1.6. *The function* $\Phi: \pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma)) \to P^q_n$ *is a bijection,* $n > 0$.

PROOF. We first show that Φ is onto. Let (M_1, f_1) represent an element of P^q_n. Let $c: \partial M_1 \times I \to M_1$ be a collar of M_1. By definition of P^q_n and 0.7, we may assume that $c(\partial M_1 \times \{1/2\}) = \Sigma^{n-1}$. As in the proof of 1.3, we may assume that the induced framing on Σ^{n-1}_- is standard, and that

$$f_1(c(\partial M_1 \times I) \times I^{q-1}) = \Sigma^{n-1} \times I^q .$$

We may replace (M_1, f_1) by (M, f) where $M = M_1$ minus the collar and $f = f_1 \mid M$, then we have $f(M \times I^{q-1}) \subset I^n \times \Sigma^{q-1}$.

We now define a map $\sigma: I^n \times \Sigma^{q-1} \circlearrowleft$ representing an element

$[\sigma] \in \pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma))$, and such that $\Phi[\sigma] = [(M, f)]$, which will prove what we want. Define $\sigma \mid \Sigma^{n-1} \times \Sigma^{q-1} \to \Sigma^{n-1} \times \Sigma^{q-1}$ to be the inverse of $f_1 \mid$, so that $\sigma \mid \Sigma_-^{n-1} \times I^{q-1} = \mathrm{id}$, and $\sigma^{-1}(\partial M') = \partial M$. Extend σ in stages as follows:

Stage 1. Extend σ to M such that $\sigma(M) = M'$ by mapping a collar of ∂M in M pseudo-radially to $M' = I^n \times \{(1, 0, \cdots, 0)\}$, see [49], and the rest of M to $\{0\} \times \{(1, 0, \cdots, 0)\}$.

Stage 2. Extend fibrewise to $f(M \times I^{q-1})$ using the framing f and the framing $I^n \times I_+^{q-1}$ of M' in $I^n \times \Sigma^{q-1}$. (Where $I_+^{q-1} = I^{q-1} \times \{(0, \cdots, 0, 1)\} \subset \Sigma^{q-1}$).

Stage 3. Extend to the rest by the same construction as Stage 1 (note that $\mathrm{cl}\,(I^n \times \Sigma^{q-1} - I^n \times I_+^{q-1})$ is a ball).

σ now satisfies the requirements, and an analogous proof shows that Φ is $1:1$.

Now by 0.3 and the 5-lemma, there is an isomorphism $\pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma)) \to \pi_n(\widetilde{G}_q(R), \widetilde{PD}_q)$; combining this with Lemma 1.6 we have.

PROPOSITION 1.7. *There is a bijection* $\Psi_3 \colon \pi_n(\widetilde{G}_q(R), \widetilde{PD}_q) \to P_n^q$.

The sets P_n^q, $F\Gamma_n^q$, Γ_n^q inherit abelian group structures from the bijections of Propositions 1.4, 1.5, and 1.7 (the sum operations are easily described geometrically by various connected sum operations). Using these bijections together with the isomorphisms $\pi_n(G_q) \to \pi_n(\widetilde{G}_q(R))$ and $\pi_n(G_q, O_q) \to \pi_n(\widetilde{G}_q(R), O_q)$ from 0.3, we have

THEOREM 1.8. *There is a braid,*

$$\begin{tikzcd}
\pi_n(O_q) \arrow[rr, bend left] \arrow[dr] & & \pi_n(G_q) \arrow[rr, bend left] \arrow[dr] & & P_n^q \\
& F\Gamma_n^q \arrow[ur] \arrow[dr, "\varphi_4"] & & \pi_n(G_q, O_q) \arrow[ur] \arrow[dr] & \\
P_{n+1}^q \arrow[ur, "\partial_1"] \arrow[rr, bend right] & & \Gamma_n^q \arrow[ur] \arrow[rr, bend right, "\partial_4"'] & & \pi_{n-1}(O_q)
\end{tikzcd}$$

Remarks 1.9. (1) By Smale [41], Γ_n^q coincides with $\theta^{n+q,n}$ for $n > 4$, $q > 2$ (see Levine for definition); and in this range (1.8) coincides with Levine's braid (it is quite easy to verify that the homomorphisms are the same).

(2) Stably (1.8) is the Kervaire-Milnor braid [23; II], and in that case has been shown to be isomorpic with (1.1) by Williamson (unpublished).

(3) By the product smoothing theorem we see that P_n^q can be defined in the smooth category—cave—∂M may be an exotic sphere.

(4) The homomorphisms in (1.8) can be described geometrically; e.g.,

(i) $\partial_1[(M, f)]$ is the framed sphere obtained by restricting f to $\partial M \times$ (collar).

(ii) φ_4 ignores framing, and

(iii) $\partial_4[f]$ is the classifying element for the normal bundle. (For more detail see Levine [28]).

The standard framing of the inclusion $\Sigma^{n+q-1} \subset \Sigma^{n+q+r-1}$ induces a function $s_r\colon P_n^q \longrightarrow P_n^{q+r}$ which coincides (using the isomorphism of 1.6) with the homomorphism $i_r\colon \pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma)) \longrightarrow \pi_n(\widetilde{G}_{q+r}, \widetilde{PL}_{q+r}(\Sigma))$ induced by inclusion. From the work of Levine, s_r is an isomorphism for $q > 2$ so that we have

THEOREM 1.10. *For $q > 2$, $i_r\colon \pi_n(\widetilde{G}_q, \widetilde{PL}_q(\Sigma)) \longrightarrow \pi_n(\widetilde{G}_{q+r}, \widetilde{PL}_{q+r}(\Sigma))$ is an isomorphism.*

From the exact sequences of the triad $(\widetilde{G}_{q+r}; \widetilde{PL}_{q+r}(\Sigma), \widetilde{G}_q)$ and Theorem 1.10 we have.

THEOREM 1.11. *$i_*\colon \pi_n(\widetilde{PL}_{q+r}(\Sigma), \widetilde{PL}_q(\Sigma)) \longrightarrow \pi_n(\widetilde{G}_{q+r}, \widetilde{G}_q)$ is an isomorphism for $q > 2$.*

Remarks. (1) Levine's work was only for the range $n \geqq 5$; for $n < 5$ one needs special arguments: Surjectivity of $P_n^q \rightarrow P_n$ for $n < 5, q > 2$ follows from the fact that the generators of P_n, $n < 5$, can be constructed in a similar way to the higher dimensional ones, see the next remark. For injectivity, regard a cobordism between M_0 and M_1 as a cobordism between $M_0 \cup M_1 \cup \Sigma_-^{q-1} \times I$ and $\Sigma_+^{q-1} \times I$. Use surgery to remove index 4 and 5 handles if $n = 4$ or index 4 handles if $n = 3$, then Levine's methods work to compress the cobordism into codimension 2. For $n < 3$ injectivity is trivial. See also Haefliger [8], Sullivan [42A] and Remark (1) at the end of § 2.

(2) The groups P_n have been computed (for $n > 5$ see Kervaire-Milnor [23], for $n \leqq 5$ use the stable version of (1.8) and known homotopy groups) so that for $q > 2$ we have

$$
\begin{aligned}
P_n^q &\cong Z && n \equiv 0\ (4) \quad \text{(classified by index)} \\
&\cong Z_2 && n \equiv 2\ (4) \quad \text{(classified by Kervaire-Arf invariant)} \\
&\cong 0 && n \text{ odd}\ .
\end{aligned}
$$

N.B. The generator of P_4^q has index 16 (Rohlin [35A]), whereas that of P_{4k}^q, $k > 1$, has index 8, cf. Milnor [31A] for construction. Note also that $\Gamma_4 = 0$ (Cerf [2A]) is used to prove $P_5 = 0$.

(3) 1.10 and 1.11 are false for $q = 2$ since $\widetilde{PL}_2(\Sigma) \subset \widetilde{G}_2$ is a homotopy

equivalence (cf. Wall [45]).

Definition of C_n^q. Consider C-embeddings $f: S^n \to S^{n+q}$ such that f is PD-isotopic to the identity. f_0, f_1 are concordant if there is a C-concordance $F: S^n \times I \to S^{n+q} \times I$ between them, and such that F is PD-isotopic to the identity. C_n^q denotes the set of concordance classes.

Now let $f: S_+^{n+1} \times R^N \circlearrowleft$ represent an element of $\pi_{n+1}(\widetilde{PD}; O, \widetilde{PD}_q)$ i.e., (see § 0) f is a zero-preserving PD-homeomorphism such that $f \mid S_+^n \times R^N$ is orthogonal, $f \mid S_-^n \times R^N = f_1 \times \mathrm{id}$, where $f_1: S_-^n \times R^q \circlearrowleft$ is a PD-homeomorphism, and $f \mid S_-^{n-1} \times R^N = \mathrm{id}$. We may assume that $f \mid S_+^n \times R^N$ is a C-homeomorphism which preserves a standard C-collar of $S^{n-1} \times R^N$ in $S_+^n \times R^N$. As in the proof of 1.5 we may assume that f_1 preserves a standard C-collar of $S^{n-1} \times R^q$ in $S_-^n \times R^q$, and then (as in 1.5) an application of 0.9 yields a PD-isotopy of $f_1 \circ s_4$ in S_-^{n+q} ending with $f_1 \circ s_4(S_-^n \times \{0\}) \cup S_+^n$ a C-submanifold $M^n \subset S^{n+q}$. A further application of 0.9 (after fixing, as usual, a collar of the boundary) shows that M^n bounds a C-submanifold M^{n+1} of S_+^{n+N+1} which is PD-homeomorphic with D^{n+1} (and hence C-homeomorphic by Munkres [34]). So up to isotopy there is a well-defined C-homeomorphism $f_2: S^n \to M^n$ which extends to a C-homeomorphism of D^{n+1} with M^{n+1}. This determines an element $\theta[f] \in C_n^q$. A similar argument shows that θ is well-defined. Again, considerations analogous to Lemma 1.3 show that θ is a bijection. We have proved

PROPOSITION 1.12. *There is a bijection*

$$\theta: \pi_{n+1}(\widetilde{PD}; O, \widetilde{PD}_q) \longrightarrow C_n^q .$$

Combining this with Theorem 1.11, 0.3, and the 5-lemma applied to

$$\begin{array}{ccccc} \pi_{n+1}(\widetilde{PD}; O, \widetilde{PD}_q) & \longrightarrow & \pi_n(O, O_q) & \longrightarrow & \pi_n(\widetilde{PD}, \widetilde{PD}_q) \\ \big\downarrow & & = \big\downarrow & & \big\downarrow \cong \ q > 2 \\ \pi_{n+1}(G(R); O, G_q(R)) & \longrightarrow & \pi_n(O, O_q) & \longrightarrow & \pi_n(G(R), G_q(R)) \end{array} ,$$

we have the theorem of Haefliger [5; 3.4].

THEOREM 1.13. *For* $q > 2$, $C_n^q \cong \pi_{n+1}(G; O, G_q)$.

Definition of FC_n^q, $n > 0$. Consider orientation preserving C-embeddings $f: S^n \times R^q \to S^{n+q}$ such that $f \mid S^n \times \{0\}$ is PD-isotopic to the identity. f_0, f_1 are concordant if there is a C-concordance $F: S^n \times R^q \times I \to S^{n+q} \times I$ between them, such that $F \mid S^n \times \{0\} \times I$ is PD-isotopic to the identity. The set of concordance classes is denoted FC_n^q.

By a very similar argument to that given above, one can prove

PROPOSITION 1.14. *There is a bijection*

$$\theta_1\colon \pi_{n+1}(\widetilde{PD};\, O,\, \widetilde{PD}_q;\, *) \longrightarrow FC_n^q\ .$$

Then by 0.3 and 5-lemma applied to

$$\begin{tikzcd}
\pi_{n+1}(\widetilde{PD};\, O,\, \widetilde{PD}_q;\, *) \arrow[r] \arrow[d] & \pi_n(O) \arrow[r] \arrow[d, "="] & \pi_n(PD,\, PD_q) \arrow[d, "\cong\ q>2"] \\
\pi_{n+1}(G(R);\, O,\, G_q(R);\, *) \arrow[r] & \pi_n(O) \arrow[r] & \pi_n(G(R),\, G_q(R))\ ,
\end{tikzcd}$$

we have

THEOREM 1.15. Haefliger [5; 5.7]. *For* $q > 2$

$$FC_n^q \cong \pi_{n+1}(G;\, O,\, G_q;\, *)\ .$$

Remarks. (1) Using results of this section, one can recover homotopically all the exact sequences given in Haefliger [5].

(2) The suspension functions $\Gamma_n^q \to \Gamma_n^{q+1}$, etc. induced by inclusion of S^{n+q} in S^{n+q+1} are easily seen to correspond to the obvious homotopy maps. We denote the stable limit by Γ_n, etc., and, for example, we recover the result of Hirsch [14; § 0], $\Gamma_n \cong \pi_n(\widetilde{PD}, O)$, and that of Haefliger-Wall [10; § 3], $F\Gamma_n \cong \pi_n(\widetilde{PL})$ (one also needs to know that $\widetilde{PL}$ and $\widetilde{PD}$ are homotopy equivalent to PL and PD, see § 5).

2. Stiefel manifolds and solid torus knots

This section is devoted to interpretations (2.13) and (2.14) of the braid (2.1) of the square $\square_q^s \equiv (\widetilde{PD}_{q+s};\, \widetilde{PD}_q,\, O_{q+s};\, O_q)$. The interpretations are in terms of various types of solid torus knots and also groups defined in § 1. The $\widetilde{PL}$, Γ and ordinary (C) Stiefel manifolds come naturally into this program. We use a result of § 1 to prove that, for $q > 2$, $\pi_n(\square_q^1) \cong \pi_n(F_q, G_q)$, and from this and a result of James, we deduce some well-known results and some relations between the Stiefel manifolds. The geometric interpretations promised in the introduction come easily from these results.

(2.1)
$$\begin{tikzcd}
\pi_n(\widetilde{PD}_q,\, O_q) \arrow[rr, bend left] \arrow[dr] & & \pi_n(\widetilde{PD}_{q+s},\, O_{q+s}) \arrow[rr, bend left, "1"] \arrow[dr] & & \pi_{n-1}(O_{q+s},\, O_q) \\
& \pi_n(\widetilde{PD}_{q+s},\, O_q) \arrow[ur] \arrow[dr] & & \pi_n(\square_q^s) \arrow[ur, "2"] \arrow[dr, "3"] & \\
\pi_n(O_{q+s},\, O_q) \arrow[ur] \arrow[rr, bend right] & & \pi_n(\widetilde{PD}_{q+s},\, \widetilde{PD}_q) \arrow[ur] \arrow[rr, bend right, "4"] & & \pi_{n-1}(\widetilde{PD}_q,\, O_q)
\end{tikzcd}$$

Our first aim is to prove that the pair $(\widetilde{PD}_{q+s}, \widetilde{PD}_q)$ is homotopy equivalent to the $\widetilde{PL}$-Stiefel manifold $\widetilde{V}_{q+s,s}$ defined in [38; § 5] (and redefined below). This is achieved in 2.2–2.4.

THEOREM 2.2. *$\widetilde{PL}_q \subset \widetilde{PL}_{q+s,s}$ is a homotopy equivalence.*

PROOF. First identify $\pi_n(\widetilde{PL}_q)$ as isomorphism classes of q-block bundles over $\partial\Delta^{n+2}$: Let $\xi^q/\partial\Delta^{n+2}$ be a block bundle. Trivialise $\xi \mid \Lambda_{n+2}$ by [37; 1.1] and extend the trivialisation of Λ_{n+1} over Δ^{n+1} by [37; 1.3]. This determines another trivialisation of $\xi \mid \Delta^n$ which agrees with the first on $\partial\Delta^n$, and hence determines an element of $\pi_n(\widetilde{PL}_q)$ which is easily seen to depend only on the isomorphism class of ξ^q. The function so defined is clearly surjective, and is proved to be injective by an application of [37; 1.11].

The group $\pi_n(\widetilde{PL}_{q+s,s})$ may be identified with isomorphism classes of pairs (cf. [38; § 4]) $\varepsilon^s \subset \xi^{q+s}$ of block bundles over $\partial\Delta^{n+2}$, where ε^s is the trivial s-block bundle and the isomorphisms involved are required to restrict to the identity on $\partial\Delta^{n+2} \times I^s$. The proof of this is a generalisation of the above proof using the analogues of [37; 1.1, 1.6, 1.11] for pairs (see [38; §§ 4, 5]).

Now $i_*: \pi_n(\widetilde{PL}_q) \to \pi_n(\widetilde{PL}_{q+s,s})$ can be interpreted as the function which assigns to each isomorphism class $\xi^q/\partial\Delta^{n+2}$ the isomorphism class of the pair $\varepsilon^s \subset \xi^q \times I^s$, where $\xi^q \times I^s$ denotes the block bundle over $\partial\Delta^{n+2}$ total space $E(\xi) \times I^s$ and typical block $\beta_i(\xi) \times I^s$. It follows from [38; 4.5, 5.1] that i_* is an isomorphism.

Let $\widetilde{V}_{q+s,s}$ be the Δ-set whose k-simplexes are isomorphisms of ε^s/Δ^k onto a subbundle of $\varepsilon^{q+s}/\Delta^k$. It is easily shown that $\widetilde{V}_{q+s,s}$ satisfies the extension condition. Let $\pi: \widetilde{PL}_{q+s} \to \widetilde{V}_{q+s,s}$ be the function induced by restriction.

THEOREM 2.3. *π is a fibration with fibre $\widetilde{PL}_{q+s,s}$.*

PROOF. We first show that π is a fibration. Let $f: \Lambda_n \times I^{q+s} \circlearrowleft$ be a block and zero preserving homeomorphism which extends the restriction of $\tau: \Delta^n \times I^s \to \Delta^n \times I^{q+s} \in \widetilde{V}_{q+s,s}$. $\operatorname{Im}(\tau)$ is a subbundle of $\varepsilon^{q+s}/\Delta^n$, and so by [38; 5.1], there is a block bundle ξ^q/K, $|K| = \operatorname{im}(\tau)$, such that blocks of $\varepsilon^{q+s}/\Delta^n$ are unions of blocks of τ. Moreover, by the uniqueness part of this result, we may assume that $\xi \mid \tau(\Lambda_n \times I^s)$ is the block bundle given by f (with block $f(f^{-1}(\sigma_i) \times I^q)$ over the cell $\sigma_i \in K$). ξ and [37; 1.6] allow us to extend $f \cup \tau$ to an n-simplex in $\widetilde{PL}_{q+s}$.

It is clear that the fibre is as stated.

COROLLARY 2.4. $\pi_n(\widetilde{V}_{q+s,s}) \cong \pi_n(\widetilde{PL}_{q+s}, \widetilde{PL}_q) \cong \pi_n(\widetilde{PD}_{q+s}, \widetilde{PD}_q)$.

PROOF. The first isomorphism follows from 2.2 and 2.3, the second from 0.2.

We now give a similar interpretation of the pair $(\widetilde{PD}_{q+s}, O_q)$. $V^\Gamma_{q+s,s}$ is the Δ-set of which a typical k-simplex is a proper block-preserving Γ-embedding $\sigma: \Delta^k \times R^s \to \Delta^k \times R^{q+s}$, such that $\sigma \mid \Delta^k \times \{0\}$ is PD-isotopic (through block-preserving embeddings) to the identity.

THEOREM 2.5. *There is an isomorphism* $\mu: \pi_n(V^\Gamma_{q+s,s}) \to \pi_n(\widetilde{PD}_{q+s}, O_q)$.

PROOF. Let $\sigma: \Delta^n \times R^s \to \Delta^n \times R^{q+s}$ represent an element of $\pi_n(V^\Gamma_{q+s,s})$. Since $\sigma(\Delta^n \times R^s)$ is a smooth submanifold, and $\Delta^n \times R^s$ deformation retracts to $\Delta^n \times \{0\} \cup \dot{\Delta} \times R^s$, which is contractible, it follows from the smooth tubular neighbourhood theorem that σ extends to a PD-embedding $\sigma_1: \Delta^n \times R^{q+s} \circlearrowleft$ such that $\sigma_1 \mid \Lambda_n \times R^{q+s} = \text{id}$, and $\sigma_1 \mid \Delta^{n-1} \times R^q \times R^s = \beta \times \text{id}$, where β: $\Delta^{n-1} \times R^q \circlearrowleft$ represents an element of $\pi_{n-1}(O_q)$. By definition of $V^\Gamma_{q+s,s}$, we may PD-isotope $\sigma_1 \bmod \dot{\Delta}^n \times R^{q+s}$ so that $\sigma_1 \mid \Delta^n \times \{0\} = \text{id}$. We now show that we may further PD-isotope $\sigma_1 \bmod \dot{\Delta}^n \times R^{q+s} \cup \Delta^n \times \{0\}$ until it is a homeomorphism and thus represents an element of $\pi_n(\widetilde{PD}_{q+s}, O_q)$. First, by uniqueness of PD-collars, we may assume that σ_1 is a homeomorphism on $Q \times R^{q+s}$ where Q is a collar of $\dot{\Delta}^n$ in Δ^n. Next by [46], we may assume that $\sigma_1 \mid (\Delta^n - Q) \times R^{q+s}$ is PL (use the collar $Q \times R^{q+s}$ to keep $\dot{\Delta}^n \times R^{q+s}$ fixed). By the regular neighbourhood theorem [17], we may assume that σ_1 preserves $(\Delta^n - Q) \times I^{q+s}$. Now $R^{q+s} - I^{q+s}$ is an (open) collar or $\dot{I}^{q+s}$, so we can complete the isotopy by "combing" the collar to infinity (cf. [12]).

A similar construction shows that this gives a well-defined function $\mu: \pi_n(V^\Gamma_{q+s,s}) \to \pi_n(\widetilde{PD}_{q+s}, O_q)$, and two applications of 0.9 show that μ is a bijection. Finally it is easy to check that the group structures are the same.

THEOREM 2.6. *For $q > 2$ there is an isomorphism* $\Phi: \pi_n(\square^1_q) \to \pi_n(F_q, G_q)$.

PROOF. First we show that, when $s = 1$, the sequence 2 of (2.1) falls into split short exact sequences. To see this consider the homomorphism $\beta: \pi_n(\tilde{V}_{q+1,1}) \to \pi_n(S^q)$, where $\beta[\sigma]$ is the class obtained by composing $\sigma \mid \Delta^n \times \{+1\}$ with the projection onto Σ^q. Let $i_1: \pi_n(S^q) \to \pi_n(O_{q+1}, O_q)$ be the standard isomorphism and $i_2: \pi_n(\widetilde{PD}_{q+1}, \widetilde{PD}_q) \to \pi_n(\tilde{V}_{q+1,1})$ be the isomorphism of Corollary 2.4. One easily checks that $\beta i_2 \alpha i_1 = 1$, where $\alpha: \pi_n(O_{q+1}, O_q) \to \pi_n(\widetilde{PD}_{q+1}, \widetilde{PD}_q)$ is the map in (2.1), and this gives the result.

From the fibration $F_q \subset G_{q+1} \to S(S^q)$, where $S(S^q)$ denotes the singular complex, we have an isomorphism $\pi_n(S^q) \cong \pi_n(G_{q+1}, F_q)$. Using this together with

$$(2.7)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \pi_n(S^q) & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & \pi_n(\widetilde{PD}_{q+1}, \widetilde{PD}_q) & \longrightarrow & \pi_n(\square^1_q) & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\cong} & & \Big\downarrow{\Phi} & & \\ 0 & \longleftarrow & \pi_n(S^q) & \longleftarrow & \pi_n(G_{q+1}, G_q) & \longleftarrow & \pi_n(F_q, G_q) & \longleftarrow & 0 \end{array}$$

1.11, 0.2, and the exact sequence of the triple $G_q \subset F_q \subset G_{q+1}$, we have for $q>2$; Here $\alpha' = \alpha i_1$ and $\beta' = \beta i_2$. It is easily verified that the left hand square commutes and hence the lower sequence splits (a well-known fact). The splitting induces the isomorphism Φ.

COROLLARY 2.8. *For* $q > 2$,

(i) $\pi_n(\widetilde{V}_{q+1,1}) \cong \pi_n(S^q) \oplus \pi_n(F_q, G_q)$

(ii) $\pi_n(V^{\Gamma}_{q+1,1}) \cong \pi_n(S^q) \oplus \Gamma^{q+1}_n$.

PROOF. (i) follows directly from 2.4 and 2.7, for (ii) notice that the splitting of sequence 2 of (2.1) implies the splitting of sequence 1.

COROLLARY 2.9. (i) $\pi_n(\square^s_q) = 0$, *for* $2q > n + 2$.

(ii) $C^q_{n-1} \cong \pi_n(\square^s_q)$ *for* $2(q + s) > n + 3$.

PROOF. Part (i) follows from the result of James [20], (cf. also Remark (2) below 2.18) that $\pi_n(F_q, G_q) = 0$ for $2q > n + 2$, and an induction on s using the exact sequence

$$\pi_n(\square^1_q) \longrightarrow \pi_n(\square^s_q) \longrightarrow \pi_n(\square^{s-1}_{q+1}) \longrightarrow \pi_{n-1}(\square^1_q) .$$

For part (ii) notice that, by part (i), $\pi_n(\square_{q+s}) = \pi_{n+1}(\square_{q+s}) = 0$, and so the result follows from the exact sequence below and 1.12.

$$\pi_{n+1}(\square_{q+s}) \longrightarrow \pi_n(\square^s_q) \longrightarrow \pi_n(\square_q) \longrightarrow \pi_n(\square_{q+s}) .$$

We deduce the well-known result:

COROLLARY 2.10. *For* $2q > n + 3$,

(i) $\Gamma^q_n \cong \Gamma_n$, *and*

(ii) $C^q_n = 0$.

PROOF. Part (i) follows from 1.5 and 2.9 (i) applied to sequence 3 of (2.1). Part (ii) follows from 2.9.

Now using the standard isomorphism $\pi_n(V_{q+s,s}) \cong \pi_n(O_{q+s}, O_q)$ (where $V_{q+s,s}$ denotes the classical Stiefel manifold), we have homomorphisms $\alpha_1: \pi_n(V_{q+s,s}) \to \pi_n(\widetilde{V}_{q+s,s})$ and $\alpha_2: \pi_n(V_{q+s,s}) \to \pi_n(V^{\Gamma}_{q+s,s})$.

COROLLARY 2.11. *For* $2q > n + 3$, α_1 *is an isomorphism and* α_2 *is the inclusion of* $\pi_n(V_{q+s,s})$ *in* $\pi_n(V^{\Gamma}_{q+s,s}) \cong \pi_n(V_{q+s,s}) \oplus \Gamma_n$.

PROOF. This follows from diagram (2.1), 1.5, 2.9, and 2.10.

Geometrical interpretations of (2.1). We first interpret the Stiefel manifolds. Define $T^q_{n,s}(C)$ and $T^q_{n,s}$ to be the set of equivalence classes of C- (resp. PL-) embeddings $f: S^n \times D^s \to S^{n+q+s}$ such that $f \mid S^n \times \{0\}$ is standard, under the equivalence of C- (resp. PL-) concordance which is standard on $S^n \times \{0\} \times I$. Define $T^q_{n,s}(\Gamma)$ to be the set of equivalence classes of Γ-embeddings $f: S^n \times D^s \to S^{n+q+s}$, such that $f \mid S^n \times \{0\}$ is PD-isotopic to the identity, under the equivalence of Γ-concordance whose restriction to $S^n \times \{0\} \times I$ is PD-isotopic to the identity.

Note that, if $q > 2$, by Zeeman 0.8, Hudson [16], and 0.10, the sets $T^q_{n,s}$ and $T^q_{n,s}(\Gamma)$ are simply PL- and Γ-isotopy classes of PL- and Γ-embeddings.

Now given a representative $f\colon S^n_+ \times D^s \to S^n_+ \times D^{q+s}$ of $\pi_n(V_{q+s,s})$, one defines $\Phi_C[f] \in T^q_{n,s}(C)$ to be represented by the embedding $f_1\colon S^n \times D^s \to S^{n+q+s}$ given by $f_1 \mid S^n_- \times D^s = s_4 \mid$ and $f_1 \mid S^n_+ \times D^s = s_4 \mid \circ f$. Φ_C is clearly well-defined, and there are similar functions Φ_{PL} and Φ_Γ.

PROPOSITION 2.12. *Φ_C, Φ_{PL}, and Φ_Γ are bijections $\pi_n(V_{q+s,s})$, $\pi_n(\tilde{V}_{q+s,s})$, $\pi_n(V^\Gamma_{q+s,s}) \to T^q_{n,s}(C)$, $T^q_{n,s}$, $T^q_{n,s}(\Gamma)$.*

PROOF. The C-case, which is well-known, follows from the smooth tubular neighbourhood theorem and the fact that the action of $\pi_1(V_{q+s,s})$ on $\pi_n(V_{q+s,s})$ is always trivial. The PL- and Γ-cases follow from the block neighbourhood theory [37, § 4] and arguments similar to 1.3.

Combining results from this section and 1.5 we have

THEOREM 2.13. *For $2(q + s) > n + 3$, there is a braid*

$$\begin{tikzcd}
\Gamma^q_n \arrow[rr, bend left, "\varphi_3"] \arrow[dr, "\omega_4"'] & & \Gamma_n \arrow[rr, bend left, "\partial_1"] \arrow[dr, "\partial_3"] & & T^q_{n-1,s}(C) \\
& T^q_{n,s}(\Gamma) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_4"] & & C^q_{n-1} \arrow[ur, "\omega_2"'] \arrow[dr, "\omega_3"] & \\
T^q_{n,s}(C) \arrow[ur, "\omega_1"] \arrow[rr, bend right, "\varphi_2"'] & & T^q_{n,s} \arrow[ur, "\partial_2"] \arrow[rr, bend right, "\partial_4"'] & & \Gamma^q_{n-1}
\end{tikzcd}$$

We now give direct descriptions of some of the homomorphisms in 2.13. The reader may easily verify that they agree with the homotopy definitions.

Sequence 3 is essentially Haefliger [5; 1.9] where the homomorphisms were defined directly ($\omega_3[f] = [f]$ and φ_3 is induced by $S^{n+q} \subset S^{n+q+1} \subset \cdots$).

ω_4: Let $f\colon S^n \to S^{n+q}$ be a Γ-embedding, then $s_4 \circ (f \times \mathrm{id})\colon S^n \times D^s \to S^{n+q+s}$ represents $\omega_4[f]$.

ω_1: $\omega_1[f] = [f]$.

φ_1: $\varphi_1[f] = [f \mid S^n \times \{0\}]$.

φ_4: Let $[f_0] \in T^q_{n,s}(\Gamma)$; f_0 is PD-isotopic to a PL-embedding by Whitehead [46]. $\varphi_4[f_0] = [f_1]$.

ω_2: Same formula as ω_4.

∂_4: Let $[f] \in T^q_{n,s}$; the obstuction to trivialising the normal block bundle of f is an element of $\pi_{n-1}(\widetilde{PL}_q) \cong F\Gamma^q_{n-1}$, which determines $\partial_4[f] \in \Gamma^q_{n-1}$.

We have neglected to describe ∂_2: $T^q_{n,s} \to C^q_{n-1}$ since it is complicated; but the reader may do so by studying the composition

$$T^q_{n,s} \cong \pi_n(\tilde{V}_{q+s,s}) \cong \pi_n(\widetilde{PD}_{q+s}, \widetilde{PD}_q) \longrightarrow \pi_n(\square_q) \cong C^q_{n-1} .$$

Remarks. (1) In § 5 for $q + s > n + 1$, we will give another interpretation of (2.1).

(2) From 2.13 and 2.10, we have $T^q_{n,s} \cong T^q_{n,s}(C)$ and $T^q_{n,s}(\Gamma) \cong T^q_{n,s} \oplus \Gamma_*$ for $2q > n + 3$.

The case $s = 1$. Let $A^q_n \subset T^q_{n,1}$ denote the subgroup whose elements are represented by (PL-) embeddings f: $S^n \times D^1 \to S^{n+q+1}$ such that

$$f| : S^n \times \{+1\} \longrightarrow S^{n+q+1} - f(S^n \times \{-1\})$$

is null-homotopic (i.e., the second linking class of $f \mid S^n \times S^0$ is zero cf. [39]).

THEOREM 2.14. *There is a braid*

$$\begin{tikzcd}
\Gamma^q_n \arrow[rr, bend left, "\omega_3"] \arrow[dr, "\omega_4"'] & & \Gamma^{q+1}_n \arrow[rr, bend left, "\partial_1"] \arrow[dr, "\varphi_3"'] & & \pi_{n-1}(S^q) \\
& T^q_{n,1}(\Gamma) \arrow[ur, "\varphi_1"'] \arrow[dr, "\varphi_4"] & & A^q_n \arrow[ur, "\partial_2"'] \arrow[dr, "\partial_3"] & \\
\pi_n(S^q) \arrow[ur, "\omega_1"] \arrow[rr, bend right, "\omega_2"'] & & T^q_{n,1} \arrow[ll, dashed, "\beta_1"'] \arrow[ur, "\varphi_2"'] \arrow[rr, bend right, "\partial_4"'] & & \Gamma^q_{n-1}
\end{tikzcd}$$

PROOF. It suffices to show that $A^q_n \cong \pi_n(\square^1_q)$. But clearly $A^q_n = \ker \beta_1$ where β_1: $T^q_{n,1} \to \pi_n(S^q)$ is the homomorphism given by β (in the proof of 2.6) and 2.12, so the result follows from 2.6 and 2.7. (Notice that sequence 2 is isomorphic with the top sequence of 2.7.)

We now give a direct description of the homomorphisms of 2.14.

ω_4, φ_4, and ∂_4 were described in 2.13, and φ_1 has the same formula (so $\omega_3 = \varphi_1\omega_4$ is 'suspension' induced by the inclusion $S^{n+q} \subset S^{n+q+1}$). Using the isomorphisms $\pi_n(S^q) \cong \pi_n(V_{q+1,1}) \cong T^q_{n,1}(C)$, ω_1 and ω_2 were described also in 2.13. φ_2 is the projection of $T^q_{n,1} \cong \pi_n(S^q) \oplus A^q_n$ on A^q_n. This is described by altering a representative f: $S^n \times D^1 \to S^{n+q+1}$ by introducing a 'local twist' by an element $\alpha \in \pi_n(S^q)$ (i.e., replace $[f]$ by $[f] + \omega_2(\alpha)$) so that the second linking class of $f_1 | S^n \times S^0$ is zero.

Since $A^q_n \subset T^q_{n,1}$, ∂_4 defines ∂_3. Also, since sequences 1 and 2 split, $\partial_1 = \partial_2 = 0$. So it remains to describe φ_3. Notice that, by the above discussion, A^q_n can also be defined as equivalence classes $[f]/\sim$ of $[f] \in T^q_{n,1}$ where $[f_1] \sim [f_2]$ if they differ by a local twist. Now since the normal bundle of h ($[h] \in \Gamma^{q+1}_n$) is easily seen to be fibre homotopy trivial (cf. [3]), it admits a never-zero section, and this determines a line subbundle and hence, on triangulating, an equivalence class in A^q_n. Moreover any two sections differ, up to homotopy of sections, by a local twist, and hence this is well-defined.

Remark. Sequence 3 is a geometrical version of Haefliger's suspension sequence $\Gamma^q_n \to \Gamma^{q+1}_n \to \pi_n(F_q, G_q) \to$, defined in [5] for $q > 2$.

Geometrical interpretations of $\pi_n(G_{q+1}, G_q)$ and $\pi_n(F_q, G_q)$ for $q > 2$. By combining several results of this section we have

COROLLARY 2.15. *For $q > 2$, there are isomorphisms*

$$\pi_n(G_{q+1}, G_q) \cong T^q_{n,1}$$
$$\pi_n(F_q, G_q) \cong A^q_n\ .$$

We now give another geometrical interpretation of these groups. Denote by $\mathrm{Emb}\,(I^n, I^n \times \Sigma^q)$ the set of equivalence classes of proper embeddings $f: I^n \to I^n \times \Sigma^q$ such that $f | \dot{I}^n =$ identification: $\dot{I}^n \to \dot{I}^n \times \{(1, 0, \cdots, 0)\}$ under the equivalence of isotopy mod $\dot{I}^n$. Now let $\sigma: I^n \times I^1 \to I^n \times I^{q+1}$ represent an element of $\pi_n(\widetilde{V}_{q+1,1})$, $\sigma_1 = \sigma | I^n \times \{(1, 0, \cdots, 0)\}$ determines an element $\Phi[\sigma] \in \mathrm{Emb}\,(I^n, I^n \times \Sigma^q)$. Moreover, if τ represents the same homotopy element, then σ_1 and τ_1 are concordant mod $\dot{I}^n$, and hence isotopic mod $\dot{I}^n$ by Hudson [16]. So Φ is well-defined.

PROPOSITION 2.16. *$\Phi: \pi_n(\widetilde{V}_{q+1,1}) \to \mathrm{Emb}\,(I^n, I^n \times \Sigma^q)$ is a bijection for $q > 2$.*

PROOF. Consider the set $\mathrm{Emb}\,(I^n \times \Sigma^0, I^n \times \Sigma^q)$ of isotopy classes of proper embeddings mod $\dot{I}^n \times \Sigma^0$. Restriction again gives a well-defined function $\Phi_1: \pi_n(\widetilde{V}_{q+1,1}) \to \mathrm{Emb}\,(I^n \times \Sigma^0, I^n \times \Sigma^q)$ which is a bijection by the Lickorish cone unknotting theorem [29]. Restriction gives a function $\Phi_2: \mathrm{Emb}\,(I^n \times \Sigma^0, I^n \times \Sigma^q) \to \mathrm{Emb}\,(I^n, I^n \times \Sigma^q)$ and, since $\Phi = \Phi_2\Phi_1$, we have to prove that Φ_2 is a bijection. This follows at once from the following lemma and Zeeman's unknotting theorem 0.8.

LEMMA 2.17. *Let $f: I^n \to I^n \times \Sigma^q$ represent an element of $\mathrm{Emb}\,(I^n, I^n \times \Sigma^q)$, and let N be a regular neighbourhood of $f(I^n)$ in $I^n \times \Sigma^q$ which meets the boundary regularly. Then $\mathrm{cl}\,(I^n \times \Sigma^q - N)$ is an $(n + q)$-ball.*

PROOF. It is easily proved that $Q = (N \cup \Sigma^{n-1} \times I^{q+1})$ is a regular neighbourhood of $f(I^n)$ in Σ^{n+q}, and hence is an $(n + q)$-ball by [49; Th. 5]. But $\mathrm{cl}\,(I^n \times \Sigma^q - N) = \mathrm{cl}\,(\Sigma^{n+q} - Q)$, and so the result is just [49; Th. 2].

COROLLARY 2.18. *For $q > 2$, there are bijections*

$$\Psi_1\colon \mathrm{Emb}\,(I^n, I^n \times \Sigma^q) \longrightarrow \pi_n(G_{q+1}, G_q)$$
$$\Psi_2\colon \mathrm{SEmb}\,(I^n, I^n \times \Sigma^q) \longrightarrow \pi_n(F_q, G_q)\ ,$$

where SEmb ( ) *denotes the subset of embeddings, homotopic* mod $\dot{I}^n$ *to the standard embedding.*

PROOF. Ψ_1 is the composition of 2.16, 2.4 and 1.11. Ψ_2 is the restriction of Ψ_1 on noticing that SEmb ( ) is precisely the subset corresponding to the kernel of β' in (2.7).

Remarks. (1) There is a cobordism interpretation of $\pi_n(G_{q+1}, G_q)$ proved in an analogous way to 1.6; namely, as the set of framed cobordism classes of framed proper n-submanifolds of $I^n \times \Sigma^q$ with boundary in $\Sigma^{n-1} \times \Sigma^{q-1}$. Now a representative $f\colon I^n \to I^n \times \Sigma^q$ of an element of Emb $(I^n, I^n \times \Sigma^q)$ gives an essentially unique framed submanifold $(f(I^n), F)$ of $I^n \times \Sigma^q$ such that the framing $F \mid \Sigma^{n-1}_+$ is standard. This determines a function Emb $(I^n, I^n \times \Sigma^q) \to \pi_n(G_{q+1}, G_q)$ which can be verified to coincide with Ψ_1. In fact, using this interpretation of Ψ_1, one can prove directly that it is a bijection using framed surgery, *à la* Levine [28], Haefliger [5]. This proves directly Theorem 1.11 for $r = 1$ (from which the general case follows), implying the stability of P^q_n, and thus gives a different approach to §§ 1 and 2.

(2) Using 2.18, it is a direct corollary of Zeeman's unknotting theorem [49, Ch. 8] that $\pi_n(F_q, G_q) = 0$ for $2q > n + 2$. This gives an alternative geometric proof of the result of James used in 2.9.

(3) Composition of the isomorphism of 2.15 with the boundary map $T^q_{n,1} \to \pi_n(G_{q+1}, G_q) \to \pi_{n-1}(G_q)$ measures the precise obstruction to fibre homotopy trivialization of the normal block bundle of an element of $T^q_{n,1}$ (cf. [3] and [37; 5.8] for the associated fibre space to a block bundle). This follows from the naturality of the isomorphism 1.11, [37; 5.8] and the interpretation of (2.14) ∂_4. A similar remark applies to $T^q_{n,s}$.

3. The suspension sequences

In this section we study a PL analogue of (2.1) when $s = 1$, namely the braid of the square $(*_q) \equiv (\widetilde{PL}_{q+1}; \widetilde{PL}_{q+1,1}, PL_{q+1}(I); PL_{q+1,1}(I))$.

PROPOSITION 3.2. *The sequence* 2 *of* (3.1) *is isomorphic with sequence* 2 *of* (2.1), *and hence with sequence* 2 *of* (2.14).

$$
(3.1)\qquad
\begin{tikzcd}[column sep=small]
\pi_n(\widetilde{PL}_{q+1,1}, PL_{q+1,1}(I)) \arrow[rr, bend left] \arrow[dr] & & \pi_n(\widetilde{PL}_{q+1}, PL_{q+1}(I)) \arrow[rr, bend left, "1"] \arrow[dr] & & \pi_{n-1}(PL_{q+1}(I), PL_{q+1,1}(I)) \\
& \pi_n(\widetilde{PL}_{q+1}, PL_{q+1,1}(I)) \arrow[ur] \arrow[dr] & & \pi_n(*_q) \arrow[ur, "2"'] \arrow[dr, "3"] & \\
\pi_n(PL_{q+1}(I), PL_{q+1,1}(I)) \arrow[ur] \arrow[rr, bend right] & & \pi_n(\widetilde{PL}_{q+1}, \widetilde{PL}_{q+1,1}) \arrow[ur] \arrow[rr, bend right, "4"'] & & \pi_{n-1}(\widetilde{PL}_{q+1,1}, PL_{q+1,1}(I))
\end{tikzcd}
$$

PROOF. Consider the commutative diagram

$$
(3.3)\qquad
\begin{tikzcd}
\pi_n(PD_{q+1}(I), PD_{q+1,1}(I)) \arrow[r] & \pi_n(\widetilde{PD}_{q+1}, \widetilde{PD}_{q+1,1}) \arrow[r] & \pi_n(*'_q) \arrow[r] & {} \\
\pi_n(O_{q+1}, O_q) \arrow[r] \arrow[u, "j_1"] & \pi_n(\widetilde{PD}_{q+1}, \widetilde{PD}_q) \arrow[r] \arrow[u, "j_2"] & \pi_n(\square^1_q) \arrow[r] \arrow[u, "j_3"] & {}
\end{tikzcd}
$$

where $(*'_q)$ denotes the PD analogue of $(*_q)$, and the vertical maps are all induced by inclusion. It follows from 0.2 that the top sequence of (3.3) is isomorphic with 2 of (3.1) in a natural way. By 2.2 and 0.2, j_2 is an isomorphism. Now let $\alpha: PL_{q+1}(I) \to S(S^q)$ be defined by restricting $\sigma^k \in PL_{q+1}(I)$ to $\Delta^k \times \{(1, 0, \cdots, 0)\}$, and projecting on S_q; $\big(S(S^q)$ denotes the (PL-) singular complex of $S^q\big)$. One easily checks that α is a fibration with fibre homotopy equivalent to $PL_{q+1,1}(I)$ (use the homotopy equivalences 'cone' and 'bdry' of 0.3). Hence there is an isomorphism $\pi_n(PL_{q+1}(I), PL_{q+1,1}(I)) \to \pi_n(S^q)$, and one can check that the composition

$$
\begin{aligned}
\pi_n(S^q) \cong \pi_n(O_{q+1}, O_q) \xrightarrow[j_1]{} & \pi_n(PD_{q+1}(I), PD_{q+1,1}(I)) \\
& \cong \pi_n(PL_{q+1}(I), PL_{q+1,1}(I)) \cong \pi_n(S^q)
\end{aligned}
$$

is the identity, and consequently j_1 is an isomorphism. The proposition now follows by the 5-lemma.

COROLLARY 3.4. *There are isomorphisms*

(i) $\pi_n(\widetilde{PL}_{q+1}(R), PL_q(R)) \cong \pi_n(\widetilde{PL}_{q+1}, PL_{q+1}(I)) \oplus \pi_n(S^q)$

(ii) $\pi_n(PD_q(R), O_q) \cong \pi_n(PD_{q+1}(I), O_{q+1})$

(iii) $\pi_n(PD_{q+1}(I), O_q) \cong \pi_n(PD_q(R), O_q) \oplus \pi_n(S^q)$

PROOF. (i) From the homotopy equivalence $h: PL_{q+1,1}(I) \longrightarrow PL_q(R)$ (see 0.12) we have $\pi_n(\widetilde{PL}_{q+1}(R), PL_q(R)) \cong \pi_n(\widetilde{PL}_{q+1}, PL_{q+1,1}(I))$, and the result now follows from the fact that sequence 1 in (3.1) splits (since sequence 2 does by 3.2). Then (ii) and (iii) both follow from h, the isomorphism j_1 (in 3.3) and the braid of the square $(PD_{q+1}(I); PD_{q+1,1}(I), O_{q+1}; O_q)$.

Remark. From 3.4 (ii) one can recover easily part of Browder's result [2] namely that $\pi_n(PL_q(R), PL_q(I)) \neq 0$ for some n, q. ($PL_q(I)$ is identified as a subcomplex of $PL_q(R)$ by either 'int' or 'collar', see § 0). For, suppose that this group is always zero, then the exact sequence of the triple $O_q \subset PD_q \subset PD_q(R)$ and 0.2 implies that $\pi_n(PD_q(R), O_q) \cong \pi_n(PD_q(I), O_q)$; but, by 3.4 (ii), this is $\cong \pi_n(PD_{q+1}(I), O_{q+1})$. Hence all these groups must be zero, which is well-known to be false (e.g., by Hirsch [14] $\pi_n(PD(R), O) \cong \Gamma_n$).

This implies either the existence of an R^n-bundle containing no D^n-bundle, or an R^n-bundle containing two inequivalent D^n-bundles, both occur, see Browder [2] and Varadarajan [44]. Varadarajan's work is only for the topological case; Lashof has similar results in the PL case.

Definitions. Denote by $E^q_n(I)$ the set of concordance classes of closed tubes on Σ^n in Σ^{n+q} (i.e., a representative is an embedding $f: E(\xi) \longrightarrow \Sigma^{n+q}$ where ξ^q/Σ^n is an I^q-bundle with zero section Σ^n and $f \mid \Sigma^n = \mathrm{id}$, $f_0 \sim f_1$ if there is $F: E(\eta) \longrightarrow \Sigma^{n+q} \times I$ where $\eta^q/\Sigma^n \times I$ is an $(I^q, 0)$-bundle, $F \mid \Sigma^n \times I = \mathrm{id}$, and $f_i = (F \mid E(\eta \mid \Sigma^n \times \{i\})) \circ g_i$, $i = 0, 1$, where g_i is an $(I^q, 0)$-bundle isomorphism). Similarly denote by $E^q_n(R)$ the set of concordance classes of open tubes on Σ^n in Σ^{n+q}, (Compare also [37, § 5]).

Let $T^q_{n,s}(R)$ denote the set of concordance classes of embeddings of $\Sigma^n \times I^s$ in Σ^{n+q+s} with an open normal tube. More precisely, a representative is an embedding $f: E(\xi) \longrightarrow \Sigma^{n+q+s}$ where $\xi^q/\Sigma^n \times I^s$ is an $(R^q, 0)$-bundle and $f \mid \Sigma^n \times \{0\} = \mathrm{id}$; $f_0 \sim f_1$ if there is an embedding $F: E(\eta) \longrightarrow \Sigma^{n+q+s} \times I$ where $\eta^q/\Sigma^n \times I^s \times I$ is an $(R^q, 0)$-bundle, $F \mid \Sigma^n \times I^s \times \{i\} = f_i \mid i = 0, 1$, $F \mid \Sigma^n \times \{0\} \times I = \mathrm{id}$, and $f_i = (F \mid E(\eta \mid \Sigma^n \times I_s \times \{i\})) \circ g_i$ where g_i is an $(R^q, 0)$-bundle isomorphism, $i = 0, 1$.

PROPOSITION 3.5. *There are bijections*

(i) $E^q_n(R) \cong \pi_n(\widetilde{PL}_q(R), PL_q(R))$

(ii) $E^q_n(I) \cong \pi_n(\widetilde{PL}_q, PL_q(I))$

(iii) $T^q_{n,s}(R) \cong \pi_n(\widetilde{PL}_{q+s}(R), PL_q(R))$.

PROOF. (i) and (ii) follow from [37; 5.6], or can easily be proved directly as in 1.3, etc.

For (iii), define the simplicial set $\widetilde{V}_{q+s,s}(R^q)$ to be the quotient set of $\widetilde{PL}_{q+s}(R)$ under the equivalence $\sigma^k_0 \sim \sigma^k_1$ if

(i) $\sigma_0 \mid \Delta^k \times R^s = \sigma_1 \mid \Delta^k \times R^s$, and

(ii) $\sigma_0^{-1} \circ \sigma_1 \colon (\Delta^k \times R^s) \times R^q \circlearrowleft$ is an R^q-bundle isomorphism (i.e., is fibre preserving).

Now a variant on the usual argument (1.3, etc.) shows that $T^q_{n,s}(R) \cong \pi_n(\widetilde{V}_{q+s,s}(R^q))$, but there is an obvious fibration $\widetilde{PL}_{q+s}(R) \to \widetilde{V}_{q+s,s}(R^q)$, with fibre F having typical k-simplex a self isomorphism of the trivial $(R^q, 0)$-bundle $(\Delta^k \times R^s) \times R^q$. Since Δ^k is a deformation retract of $\Delta^k \times R^s$, F is homotopy equivalent to $PL_q(R)$. The result follows.

Combining (3.1), 3.2, and 3.5, we have

THEOREM 3.6. *There is a braid*

$$\begin{tikzcd}
E^q_n(R) \arrow[rr, bend left, "\omega_3"] \arrow[dr, "\omega_4"'] & & E^{q+1}_n(I) \arrow[rr, bend left, "\partial_1"] \arrow[dr, "\varphi_3"] & & \pi_{n-1}(S^q) \\
& T^q_{n,1}(R) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_4"] & & A^q_n \arrow[ur, "\partial_2"] \arrow[dr, "\partial_3"] & \\
\pi_n(S^q) \arrow[ur, "\omega_1"] \arrow[rr, bend right, "\omega_2"'] & & T^q_{n,1} \arrow[ur, "\varphi_2"] \arrow[ll, dashed, "\beta_1"'] \arrow[rr, bend right, "\partial_4"'] & & E^q_{n-1}(R)
\end{tikzcd}$$

(*in which* β_1 *is the splitting in* 2.14).

COROLLARY 3.7. (i) *there is an isomorphism* $T^q_{n,1}(R) \cong E^{q+1}_n(I) \oplus \pi_n(S^q)$.

(ii) *For* $2q > n + 3$, *there is an isomorphism* $E^q_n(R) \cong E^{q+1}_n(I)$.

(iii) *For* $2q = n + 3$, $\omega_3 \colon E^q_n(R) \to E^{q+1}_n(I)$ *is an epimorphism.*

PROOF. (i) follows from the splitting of sequence 1 in 3.6; (ii) and (iii) follow from exactness of sequence 3 and 2.9 (i).

We now give direct descriptions of those homomorphisms in 3.6 which have not already been described (in 2.14).

ω_4: Let $f \colon E(\xi) \to \Sigma^{n+q}$ represent an element of $E^q_n(R)$, and let

$$c \colon \Sigma^{n+q} \times I^1 \longrightarrow \Sigma^{n+q+1}$$

be the standard collar, then the embedding $f_1 \colon E(\xi) \times I^1 \to \Sigma^{n+q+1}$ defined by $f_1(x, y) = c(f(x), y)$ represents $\omega_4[f] \in T^q_{n,1}(R)$.

φ_4: simply forgets the normal tube.

ω_1: is given by local twisting (cf. remarks in 2.14).

φ_1: Let $f: E(\xi) \to \Sigma^{n+q+1}$ represent an element of $T^q_{n,1}(R)$. Then ξ is equivalent to $(\xi \mid \Sigma^n) \times I^1$ hence ξ is contained fibrewise in $\eta = (\xi \mid \Sigma^n) \oplus \varepsilon^1$. But by the results of Kuiper-Lashof and Hirsch quoted in 0.12, η contains a well-defined disc bundle μ (essentially cone on (ξ + "∞")). $\varphi_1[f] = [\mu]$.

φ_3: This is very similar to φ_3 in 2.14. Let ξ be a disc bundle on Σ^n in Σ^{n+q+1}. ξ is easily proved to be fibre homotopy trivial, and hence possesses a never-zero section. By 0.3 (the homotopy equivalences 'cone' and 'bdry'), this determines a line subbundle and hence an equivalence class in A^q_n. As before, since two sections differ up to homotopy by a local twist, this is well-defined. (Compare also § 4).

∂_3: Since $A^q_n \subset T^q_{n,1}$, we only have to define ∂_4. This is very similar to ∂_4 in 2.13. The normal block bundle determines an element of $FE^q_{n-1}(R)$ by 1.3. Forgetting structure gives the required element of $E^q_{n-1}(R)$.

Remark. By exactness of sequence 4, ∂_3 and ∂_4 measure the obstruction to an element of $T^q_{n,1}$ or A^q_n possessing a normal plane (or micro-) bundle.

THEOREM 3.8. *There is a diagram, commutative up to sign,*

$$\begin{array}{ccccccccc}
T^q_{n+1,1} & \longrightarrow & FC^q_n & \xrightarrow[\omega_1]{} & FC^{q+1}_n & \longrightarrow & T^q_{n,1} & \xrightarrow{1} & \\
\downarrow \cup & & \alpha_1 \downarrow & & \beta_1 \downarrow & & \downarrow \cup & & \\
A^q_{n+1} & \longrightarrow & C^q_n & \xrightarrow[\omega_2]{} & C^{q+1}_n & \longrightarrow & A^q_n & \xrightarrow{2} & \\
\| & & \alpha_2 \downarrow & & \beta_2 \downarrow & & \| & & \\
A^q_{n+1} & \longrightarrow & \Gamma^q_n & \xrightarrow[\omega_3]{} & \Gamma^{q+1}_n & \longrightarrow & A^q_n & \xrightarrow{3} & \\
\| & & \alpha_3 \downarrow & & \beta_3 \downarrow & & \| & & \\
A^q_{n+1} & \longrightarrow & E^q_n(R) & \longrightarrow & E^{q+1}_n(I) & \longrightarrow & A^q_n & \xrightarrow{4} &
\end{array}$$

in which the rows are exact.

PROOF. Sequence 2 is isomorphic with the exact sequence of the diagram

$$\begin{array}{ccccc}
PD & \supset & PD_{q+1} & \supset & PD_q \\
\cup & & \cup & & \cup \\
O & \supset & O_{q+1} & \supset & O_q
\end{array}$$

using 1.12 and 2.14. Sequence 1 comes from the diagram obtained from the above by replacing O_{q+1} and O_q by $*$ and using 1.14 and 2.12. Sequence 3 is 3 of 2.14 and sequence 4 is 3 of 3.6. Vertical maps are all induced on homotopy level by inclusion. Commutativity follows from definition.

The new horizontal homomorphisms in 3.8 have similar interpretations to the old ones (e.g., ω_1 and ω_2 are 'suspension'). The vertical ones are interpreted as follows: $\alpha_1, \alpha_2, \beta_1, \beta_2$ all forget structure. α_3, β_3 forget structure in the normal bundle (after triangulating and isotoping the zero-section to standard position).

COROLLARY 3.9. $E^3_n(I) \cong \Gamma^3_n \oplus E^2_n(R)$

PROOF. This follows from sequences 3 and 4 of 3.8, when $q = 2$, using the fact that $\Gamma^2_n = 0$ (Wall [45]).

Remarks. (1) $\Gamma^3_4 \cong Z_{12}$ (cf. Haefliger [5]), so that $E^3_4(I)$ has non-zero torsion.

(2) The case $q = 1$ of diagram 3.8 is uninteresting since $E^1_n(R)$ and Γ^1_n are both zero by uniqueness of collars, so that $E^2_n(I) \cong \Gamma^2_n = 0$.

COROLLARY 3.10. *Let $f: S^n \to S^{n+q}$ represent an element of C^{q+1}_n or Γ^{q+1}_n, and suppose that f is not compressible* (i.e., $[f] \notin \operatorname{im} \omega_2$ *or* $\operatorname{im} \omega_3$). *Then the normal bundle of f is non-standard as a PL disc bundle.*

PROOF. By exactness of 2 (or 3) the image of $[f]$ in A^q_n is non-zero and by commutativity the image in $E^{q+1}_n(I)$ is non-zero. The result follows from the interpretation of the homomorphisms.

Remark. Corollary 3.10 recovers a result of Hirsch [11]. In fact, we can easily interpret mapping to zero in A^q_n as 'having an engulfable section' (compare interpretation of φ_3 in 3.6), and hence we recover the main result of [11], namely an embedding is compressible if and only if its normal bundle possesses an engulfable section. By chasing diagram 3.8, we also have, (cf. Kervaire [22]).

COROLLARY 3.11. *Suppose $f: S^n \to S^{n+q}$ represents an element of C^q_n or Γ^q_n whose suspension is trivial* (i.e., ω_2 *or* ω_3 $[f] = 0$). *Then f has trivial normal bundle, in particular, by* 2.10, *any element of C^q_n has trivial normal bundle if $2q > n + 1$.*

Remarks. (1) From Corollary 3.9 and 3.11, the generator α of $C^3_3 \cong \Gamma^3_3 \cong Z$ (cf. Haefliger [4]) has trivial normal bundle which is non-standard as a (*PL*) disc bundle, and hence by the usual construction (Hirsch [11]), there is a *PL* embedding $\Sigma^4 \subset M^7$ with no normal *PL* disc bundle. (It is unknown whether the image of α in $E^3_3(R)$ is non-trivial)[1]. See Theorem 5.20 for an immersion $\Sigma^4 \subset \Sigma^7$ with no normal (*PL*) disc bundle.

(2) Any two of the sequences 2, 3, and 4 in 3.8 may be embedded in a braid. Sequences 2 and 3 fit in a braid with sixth term Γ_n. Sequences 3 and

[1] M. W. Hirsch has recently proved that α survives in $E^3_3(R)$, thus $\Sigma^4 \subset \Sigma^7$ and $\Sigma^4 \subseteq \Sigma^7$ (Theorem 5.20) have no normal plane or micro-bundles; it is unknown whether these embeddings have topological normal microbundles. Note that the double of M^7 can be identified with the boundary of a regular neighbourhood of $\Sigma^4 \times I$ in Σ^8, and hence is homeomorphic to $\Sigma^4 \times \Sigma^3$.

4 fit in a braid with sixth term $\pi_n(PD_q(R), O_q) \cong \pi_n(PD_{q+1}(I), O_{q+1})$, and sequences 2 and 4 fit in a braid with sixth term $\pi_n(PD(I); O, PD_{q+1}(I)) \cong \pi_n(PD(R); O, PD_q(R))$. We leave the reader to make geometric interpretations.

In [39] we gave an example of an embedding $f\colon \Sigma^{19} \times I^1 \to \Sigma^{29}$ having no open normal bundle. Thus by exactness of sequence 4 of 3.6, we see that ∂_4, and hence ∂_3, is non-zero for $n = 19$, $q = 9$; and from commutativity of 3.8, we have an element $\alpha \in FC_{18}^9$ with zero image in FC_{18}^{10} and non-zero image in $E_{18}^9(R)$. In fact, using the topological thesis of [39], we can do better than this. Denote by $E_n^q(R, \mathrm{Top})$ the set of topological concordance classes of topological open tubes on Σ^n in Σ^{n+q} (defined in a similar way to $E_n^q(R)$). There is an obvious function $E_n^q(R) \to E_n^q(R, \mathrm{Top})$.

THEOREM 3.12. *There is an element $\alpha \in FC_{18}^9$ which maps trivially to FC_{18}^{10}, but non-trivially to $E_{19}^9(R, \mathrm{Top})$. (Hence using the 'Kister-Mazur' theorem* [24], *we have the result stated in the introduction.)*

PROOF. Let $[f] \in A_{19}^9$ be the element given by [39]. Let α be the image of $[f]$ in FC_{18}^9. We have to show that the image of $[f]$ in $E_{18}^9(R, \mathrm{Top})$ is non-trivial. Now let $f_1\colon \Sigma^{18} \times R^9 \to \Sigma^{27}$ represent the image of $[f]$ in $FE_{18}^9(R)$; (f_1 is obtained from the normal block bundle of f by 1.3 and 1.2). We may assume that $f_1(\Sigma^{18} \times R^9) = \Sigma^{18} \times R^9$ (which, throughout the proof, we identify as a subset of Σ^{27} *via* the standard inclusion), and then $M_1^{28} = \Sigma_+^{19} \times R^9 \cup_{f_1} \Sigma_-^{19} \times R^9$ may be identified with the total space of an (open) normal block bundle on f restricted to $f(\Sigma^{19} \times \{0\})$.

We have to prove that f_1 is not topologically concordant (fixing $\Sigma^{18} \times I$) to an embedding f_2 such that $f_2(\Sigma^{18} \times R^9) = \Sigma^{18} \times R^9$ and f_2 is fibre-preserving. (Note that $M_2^{28} = \Sigma_+^{19} \times R^9 \cup_{f_2} \Sigma_-^{19} \times R^9$ is the total space of an $(R^q, 0)$-bundle over Σ^{19}). Suppose that $F\colon \Sigma^{18} \times R^9 \times I \to \Sigma^{27} \times I$ is such a concordance, then we may assume that $F(\Sigma^{18} \times R^9 \times I) \subset \Sigma^{18} \times R^9 \times I$. By the formula below, there is a topological embedding $e\colon M_2^{28} \to M_1^{28}$ such that $e \mid \Sigma^{19} = \mathrm{id}$, thus e gives a topological normal $(R^q, 0)$-bundle on Σ^{19} in M_1^{28}, contradicting [39]. To define e, choose a collar $\Sigma^{18} \times I$ of Σ_+^{19} and define $e = \mathrm{id}$ on

$$(\Sigma_-^{19} \times R^9) \cup (\Sigma_+^{19} - \text{collar}) \times R^9$$

and $e = (f_1^{-1} \times \mathrm{id}) \circ F$ on $\Sigma^{18} \times R^9 \times I$.

4. Sections of a block bundle and links of spheres

A *never-zero section*, or just "section", of a q-block bundle ξ^q/K (where K is a locally finite PL cell complex, see [37; notation]) is an embedding $s\colon K \to E(\xi)$ satisfying

(i) $s(K) \cap K = \varnothing$, and

(ii) s respects blocks, i.e., for each cell $\sigma_i \in K$, $s^{-1}(\beta_i(\xi)) = \sigma_i$.

Sections s_0, s_1 are *homotopic* if there is a section S of $\xi \times I$ (see [37; § 1]) such that $S \mid K \times \{i\} = s_i$, $i = 0, 1$. There are very similar definitions of sections of block bundles with fibres R^q and Σ^{q-1} (see [37; § 5] for definitions) except that condition (i) is vacuous for sphere block bundles.

A *proper* section also satisfies

(iii) $s(K) \cap E(\dot{\xi}) = \varnothing$,

where $\dot{\xi}$ denotes, as usual, the boundary sphere block bundle. At the other extreme, a *spherical* section is a section of $\dot{\xi}$.

Note that there is nothing to prevent a section from being "knotted" if $q = 2$, so in order to give a sensible theory of sections we will insist throughout this section that $q > 2$ (and for spherical sections that $q > 3$).

We now give a "theory" of proper and spherical sections of a block bundle analogous to [37; § 1]. Block bundles with section (ξ_0, s_0), (ξ_1, s_1) are *isomorphic* if there is an isomorphism $h: E(\xi_0) \longrightarrow E(\xi_1)$ such that $hs_0 = s_1$. The *standard* proper section of ε^q/K, where ε^q denotes as usual the trivial block bundle, is given by id: $K \longrightarrow K \times \{*\}$ where $* \in I^q$ is the point $(1/2, 0, \cdots, 0)$, and the standard spherical section is given by id: $K \longrightarrow K \times *(\Sigma)$ where $*(\Sigma) = (1, 0, \cdots, 0)$. The trivial block bundle with standard sections are denoted $(\varepsilon^q, *)$, $(\varepsilon^q, *(\Sigma))$ respectively. A block bundle with proper (resp. spherical) section is *trivial* if is isomorphic with $(\varepsilon^q, *)$ (resp. $(\varepsilon^q, *(\Sigma))$).

We have the analogue of [37; Prop. 1.3*];

PROPOSITION 4.1. *Suppose* $|K| \cong I^n$ *and* K *has just one* n-*cell* σ^n. *Let* σ^{n-1} *be any* $(n-1)$-*cell in* K, *and let* L *be the subcomplex of* K *consisting of all cells except* σ^n *and* σ^{n-1}. *Suppose given a proper* (*resp. spherical*) *section* s *of* ξ^q/K *and a trivialization* $t: E((\xi^q, s) \mid L) \longrightarrow E((\varepsilon^q, *)/L)$ (*resp.* $E((\varepsilon^q, *(\Sigma))/L)$), *then* t *extends to a trivialization of* (ξ^q, s) *provided that* $q > 2$ (*resp.* $q > 3$).

PROOF. This follows from the concordance extension Theorem 0.6. To see this for the case of a proper section, notice that there is a homeomorphism $g: \sigma^{n-1} \times I \longrightarrow \sigma^n$ such that $g \mid \sigma^{n-1} \times \{1\} = \mathrm{id}$. Using t and [37; 1.3], g extends to a homeomorphism $g': E(\xi \mid \sigma^{n-1}) \times I \longrightarrow E(\xi)$ such that $g' \mid E(\xi \mid \sigma^{n-1}) \times \{1\} = \mathrm{id}$, $g'(E(\xi \mid \dot{\sigma}^{n-1}) \times I \cup E(\xi \mid \sigma^{n-1}) \times \{0\}) = E(\xi \mid L)$, and $(g')^{-1}sg$ is a section of $(\xi \mid \sigma^{n-1}) \times I$. $(g')^{-1}sg$ is now a concordance in $E(\xi \mid \sigma^{n-1}) \times I$. For the case of a spherical section, one uses a similar argument restricting attention to $E(\dot{\xi})$.

Remark 4.2. Using the methods of [37, § 1], we can now prove that any block bundle with proper or spherical section admits charts, that homotopic proper (or spherical) sections are isomorphic, and that we can subdivide or amalgamate block bundles with proper or spherical sections. We also deduce that there is a well-defined bijection between homotopy classes of sections (or spherical sections) of equivalent block bundles. (Note that (as is trivially proved) any section is homotopic to a proper section).

Now let L_q (resp. $L_q(\Sigma)$) denote the Δ-set of which a typical k-simplex is a proper (resp. spherical) section of ε^q/Δ^k. L_q and $L_q(\Sigma)$ are pointed by the standard sections. There are projections $\pi\colon \widetilde{PL}_q \to L_q$ (resp. $\pi(\Sigma)\colon \widetilde{PL}_q(\Sigma) \to L_q(\Sigma)$) got by restricting to $\Delta^k \times \{*\}$ (resp. $\Delta^k \times *(\Sigma)$), which may be proved to be fibrations using 0.6, by a similar proof to 4.1. The fibres are clearly $\widetilde{P_s L}_q$ and $\widetilde{PL}_q(\Sigma_s)$ respectively. We record for future reference

PROPOSITION 4.3. *There are isomorphisms*

$$\pi_n(L_q) \longrightarrow \pi_n(\widetilde{PL}_q, \widetilde{P_s L}_q) \qquad (q > 2)$$
$$\pi_n(L_q(\Sigma)) \longrightarrow \pi_n(\widetilde{PL}_q(\Sigma), \widetilde{PL}_q(\Sigma_s)) \qquad (q > 3)\ .$$

Now let ξ^q/K be a block bundle and K^k an ordered simplicial complex. We form the associated L_q^k (resp. $L_q^k(\Sigma)$)-bundle, denoted $S(\xi)$ (resp. $S(\dot{\xi})$), with base K in an analogous way to [37; § 3]. For example, an r-simplex of $S(\xi)$ is a proper section $s\colon \sigma^r \to E(\xi \mid \sigma^r)$, where σ^r is an r-simplex of K. It is immediate that there is a bijection between proper (resp. spherical) sections of ξ^q/K and cross-sections of $S(\xi)$ (resp. $S(\dot{\xi})$), and hence we have obstruction theories for a block bundle to admit a section or spherical section with coefficients in $\pi_n(L_q)$, $\pi_n(L_q(\Sigma))$ respectively.

Remark 4.4. $L_q(R)$ is defined in an analogous way to L_q, and there is an analogous fibration $\widetilde{PL}_q(R_s) \subset \widetilde{PL}_q(R) \to L_q(R)$ ($q > 2$ as usual). There is an obvious inclusion $L_q \subset L_q(R)$ induced by $I^q \subset R^q$, and this is easily seen to be a homotopy equivalence. It follows that the theories of block bundles with section and open block bundles with section coincide, and that there is a homotopy eqivalence $\widetilde{P_s L}_q \simeq \widetilde{PL}_q(R_s)$ for $q > 2$.

We now seek a criterion for a block bundle with section to "split" a line subbundle. A proper (resp. spherical) section s of ξ^q/K is said to *split* if there is an isomorphism onto a subbundle (see [38, § 4]) $h\colon E(\varepsilon^1/K) \to E(\xi^q/K)$ such that $h \mid K \times \{1/2\}$ (resp. $h \mid K \times \{1\}$) $= s$.

Next note that there are projections $p\colon \tilde{V}_{q,1} \to L_q$, $p(\Sigma)\colon \tilde{V}_{q,1} \to L_q(\Sigma)$ defined by restriction to $\Delta^k \times \{1/2\}$ and $\Delta^k \times \{1\} \subset \Delta^k \times I^1$ respectively. That p and $p(\Sigma)$ are fibrations for $q > 2$, $q > 3$ respectively, follows as usual from 0.6.

PROPOSITION 4.5. *$p(\Sigma)$ is a homotopy equivalence for $q > 3$.*

PROOF. This is just a restatement of 2.16.

COROLLARY 4.6. *$\widetilde{PL}_q(\Sigma_s)$ is homotopy equivalent to $\widetilde{PL}_{q-1}$ for $q > 3$.*

PROOF. This follows from 4.3, 4.5, 2.4, and 5-lemma.

Now let ξ^q/K be a q-block bundle and K an ordered simplicial complex. Let $\varepsilon^1(\xi)$ denote the associated $\widetilde{V}^k_{q,1}$-bundle (see [38; § 5]), then there are projections $p^*: \varepsilon^1(\xi) \to S(\xi)$ and $p(\Sigma)^*: \varepsilon^1(\xi) \to S(\dot{\xi})$ induced by p and $p(\Sigma)$. It is immediate that a proper (resp. spherical) section of ξ splits if and only if the corresponding cross-section of $S(\xi)$ (resp. $S(\dot{\xi})$) lifts to a cross-section of $\varepsilon^1(\xi)$. Hence we have an obstruction theory for a proper (resp. spherical) section to split, with coefficients in the fibre of p (resp. $p(\Sigma)$). We deduce at once from 4.5

COROLLARY 4.7. *A spherical section splits a line subbundle which is unique up to concordance* ($q > 3$).

We also have

COROLLARY 4.8. *A proper section splits a line subbundle if and only if it is homotopic to a spherical section* ($q > 3$).

PROOF. If a section splits, then it is homotopic to a spherical section by sliding along the line subbundle. Conversely, if a proper section is homotopic to a spherical section, then, sliding down the line subbundle given by 4.7, it is homotopic to a proper section which splits. Hence by Remark 4.2, it is isomorphic to a proper section which splits, and hence splits itself.

The rest of the section is devoted to a study of the fibration $p: \widetilde{V}_{q,1} \to L_q$, which, by the above, is crucial to the theory of sections and splittings. We first examine p_* in the metastable range.

THEOREM 4.9. $p_*: \pi_n(\widetilde{V}_{q,1}) \to \pi_n(L_q)$ is

(i) *an epimorphism if* $2q > n + 3$, *and*

(ii) *an isomorphism if* $2q > n + 4$.

COROLLARY 4.10. *If $2q > n + 3$, then any proper section of ξ^q/K, $\dim K = n$, splits a line subbundle, and if $2q > n + 4$, this subbundle is unique up to a concordance which keeps the section fixed.*

PROOF OF 4.9. (i) Let $s: \Delta^n \times \{1/2\} \to \Delta^n \times I^q$ represent an element of $\pi_n(L_q)$, and $t: \dot{\Delta}^n \times I^1 \to \dot{\Delta}^n \times I^q$ be the standard line subbundle. We can extend $s \cup t$ to an embedding t_1 of $\Delta^n \times [0, 1/2]$ such that $t_1 \mid \Delta^n \times \{0\} = \mathrm{id}$ by [36; Th. 2] and (cf. [36; 4(c)]) we may assume that

$$t_1(\operatorname{int}(\Delta^n \times [0, 1/2])) \subset \operatorname{int}(\Delta^n \times I^q) .$$

Now let N be a regular neighbourhood of

$$t_1(\Delta^n \times [0, 1/2]) \bmod t_1(\Delta^n \times (\{0\} \cup \{1/2\})$$

in $\Delta^n \times I^q$ which meets $\partial(\Delta^n \times I^q)$ in some standard neighbourhood. By existence and uniqueness of PL-collars [47, 19], we can prove as in [38; 1.5] that $\mathrm{cl}\,(\Delta^n \times I^q - N) \cong \Delta^n \times \dot{I}^q \times I$ in a block respecting fashion (and standard over $\dot{\Delta}^n$). This collar enables t_1 to be extended to the required line subbundle. Part (ii) is proved in an analogous way.

Remark 4.11. Using the full strength of [36; Th. 2] one can prove in a similar way that any section (not necessarily proper) "splits" for $2q > n + 3$, in the sense that it is a section of a line subbundle (and also a similar uniqueness result using [36; Th. 3]).

Now let ST_{q-1} denote the Δ-set of which a k-simplex is a block-respecting embedding $\sigma\colon \Delta^n \times [0, 1/2] \to \Delta^n \times I^q$ such that $\sigma \mid \Delta^n \times (\{0\} \cup \{1/2\}) = \mathrm{id}$, and σ is disjoint from $\Delta^n \times \dot{I}^q$. By the argument in the second half of the proof of 4.9 (i), the fibre of p is homotopy equivalent to ST_{q-1}, consequently there is an exact sequence

$$\pi_n(ST_{q-1}) \longrightarrow \pi_n(\widetilde{V}_{q,1}) \longrightarrow \pi_n(L_q) \longrightarrow \pi_{n-1}(ST_{q-1})\ . \tag{4.12}$$

We seek to interpret (4.12) geometrically, and we remark that by 4.3, 2.4, and the naturality of all the maps, this sequence is isomorphic with the exact sequence of the triple $\widetilde{PL}_{q-1} \subset \widetilde{P_s L}_q \subset \widetilde{PL}_q$. A general position argument easily proves that $\pi_1(L_q) = 0$ (recall that $q > 2$), hence L_q is n-simple. But $\pi_n(L_q) \cong [\Sigma^n; L_q]$ may be identified with homotopy classes of sections of the trivial block bundle ε^q/Σ^n, where Σ^n has the standard cell structure (see § 0). Given a section s, define an embedding $f\colon \Sigma^n \times \partial I \to \Sigma^{n+q}$ by $f(x, 0) = i(x, 0)$, and $f(x, 1) = i \circ s(x)$ where $i\colon \Sigma^n \times I^q \to \Sigma^{n+q}$ is the standard inclusion. Denote the set of concordance classes of links (PL-embeddings $\Sigma^n \times \partial I \to \Sigma^{n+q}$) by L^q_n, then we have a well-defined function $\Psi\colon \pi_n(L_q) \to L^q_n$.

THEOREM 4.13. *Ψ is a bijection* ($q > 2$).

PROOF. *Ψ is onto.* Let $f\colon \Sigma^n \times \partial I \to \Sigma^{n+q}$ be a link. By Zeeman [47], we may assume that $f \mid \Sigma^n \times \{0\}$ is standard. Since any two embeddings of a disc are isotopic, we may further assume using 0.7 that $f \mid \Sigma^n_- \times \{1\}$ coincides with the standard section over Σ^n_-. It remains to modify $f \mid \Sigma^n_+ \times \{1\}$ by isotoping into $i(\Sigma^n_+ \times I^q)$. But the complement C of $i(\Sigma^n_+ \times I^q)$ is a regular neighbourhood of $X = f(\Sigma^n_- \times \partial I) \cup i(x_0 \times [0, 1/2]) \bmod f(\Sigma^{n-1} \times \partial I)$, where x_0 is an interior point of Σ^n_-, and by general position $f \mid \mathrm{int}\,(\Sigma^n_+ \times \{1\})$ may be assumed not to meet X, and hence it may be assumed not to meet the regular neighbourhood C of X [17].

Ψ is 1–1. We are given a concordance $F\colon \Sigma^n \times \partial I \times I \to \Sigma^{n+q} \times I$ such that $F \mid \Sigma^n \times \{0\} \times \{i\}$ is standard, $i = 0, 1$, and such that $F \mid \Sigma^n \times \{1\} \times \{i\}$, $i = 0, 1$, are sections of $\varepsilon^q/\Sigma^n \times \{i\}$, which are standard over $\Sigma^n_- \times \{i\}$. We

have to show that these sections are homotopic. By 0.6 we can find a homeomorphism $H: \Sigma^{n+q} \times I \circlearrowleft$ such that $HF \mid \Sigma^n \times \{0\} \times I$ is standard, and by 0.10 we may assume that $H \mid \Sigma^{n+q} \times I = \mathrm{id}$. Thus we may assume that $F \mid \Sigma^n \times \{0\} \times I$ is standard. In $\Sigma^{n+q} \times I$ span the circle

$$i(x_0 \times [0, 1/2] \times I) \cup x_0 \times I \cup F(x_0 \times \{1\} \times I)$$

by a 2-disc Q with $\mathrm{int}\, Q \subset \mathrm{int}\, \Sigma^{n+q} \times I$, and disjoint from the image of F (by general position). Define β_1 = regular neighbourhood of

$$Q \cup \Sigma^n_- \times I \cup F(\Sigma^n_- \times \{1\} \times I) \cup i(\Sigma^n_- \times I^q \times \partial I) \bmod (\Sigma^n_+ \times I)$$
$$\cup F(\Sigma^n_+ \times \{1\} \times I) \cup (\mathrm{cl}\,(\Sigma^{n+q} - \Sigma^n_- \times I^q) \times \partial I) ,$$

then this will be an $(n + q)$-ball (see [17]) and will form one block of a block bundle of which F defines a section, namely the block over $\Sigma^n_- \times I$. Now in $\mathrm{cl}\,(\dot{\beta}_1 - \partial(\Sigma^{n+q} \times I))$ we can define a block bundle over $\Sigma^{n-1} \times I$ of which F defines a section by induction on n, since this is (PL) homeomorphic to $\Sigma^{n+q-1} \times I$. Finally we define the block β_2 over $\Sigma^n_+ \times I$ by excluding a suitable relative regular neighbourhood, much as in the first part.

Now by [37; 4.4] we may assume that this block bundle over $\Sigma^n \times I$ is standard, and then the required sections are homotopic, completing the proof.

By general position argument, we again have that $\pi_1(ST_q) = 0$ for $q > 2$, hence $\pi_n(ST_q)$ may be identified (using our previous discussion) with the set of (concordance classes of) splittings of $(\varepsilon^q, *)/\Sigma^n$. Define $ST^q_{n,1}$ to be the set of concordance classes of ribbons $f: \Sigma^n \times I \to \Sigma^{n+q+1}$ such that $f \mid \Sigma^n \times \{0\} = \mathrm{id}$, and $f \mid \Sigma^n \times \{1\} = *$, where a concordance is required to preserve the boundary. The above remarks define (after identifying I with $[0, 1/2]$) a function

$$\Phi: \pi_n(ST_q) \longrightarrow ST^q_{n,1} .$$

PROPOSITION 4.14. *Φ is a bijection* $(q > 2)$.

PROOF. *Φ is onto.* Given a ribbon $f: \Sigma^n \times I \to \Sigma^{n+q+1}$ with standard boundary, we will prove that f is isotopic keeping the boundary fixed to a ribbon contained in a block-wise manner in $\varepsilon^{q+1}/\Delta^n$, and hence is in the image of Φ. $f(\{x_0\} \times I)$ is isotopic to standard position with interior disjoint from $f(\Sigma^n \times \partial I)$ throughout, by general position, and since this may be taken to be an isotopy by moves (see [19]), it can be covered by an ambient isotopy keeping $f(\Sigma^n \times \partial I)$ fixed. Thus we may assume that $f \mid \{x_0\} \times I$ is also standard. Define $\beta_1 = a$ regular neighbourhood of $f(\Sigma^n_- \times I) \bmod f(\Sigma^n_+ \times I)$ in Σ^{n+q+1}, and since β_1 is also a regular neighbourhood of

$$f(\Sigma^n_- \times \partial I \cup \{x_0\} \times I) \bmod f(\Sigma^n_+ \times \partial I)$$

it is isotopic mod $f(\Sigma^n \times \partial I)$ to $i(\Sigma^n_- \times I^{q+1})$. Thus we have $f(\Sigma^n_- \times I)$ contained in $i(\Sigma^n_- \times I^{q+1})$ and $f(\Sigma^n_+ \times I)$ in the closure of the complement. By induction on n, we may assume that $f(\Sigma^{n-1} \times I)$ is contained block-wise in $\varepsilon^{q+1}/\Sigma^{n-1}$, and then if we define β_2 to be the complement of $\beta_1 \cup a$ suitable regular neighbourhood (as in 4.13), then we may assume that $\beta_2 = \Sigma^n_+ \times I^{q+1}$ by uniqueness [17], completing the proof.

Φ *is* 1–1. A very similar argument, the crucial point being that $F(\{x_0\} \times I \times I)$ is isotopic to standard position by general position (with interior disjoint from $F(\Sigma^n \times \partial I \times I)$) which requires $q > 2$.

Combining (4.12), 4.13, and 4.14, together with 2.12, we have

THEOREM 4.15. *There is an exact sequence for* $q > 2$

$$ST^q_{n,1} \xrightarrow{\omega} T^q_{n,1} \xrightarrow{\varphi} L^{q+1}_n \xrightarrow{\partial} ST^q_{n-1,1}\,.$$

The maps in 4.15 may be easily interpreted geometrically as follows.

ω: forgets structure,

φ: restricts to the boundary,

∂: is defined by matching trivialisations of either half of the link (cf. proof of 4.13).

We now observe that both $T^q_{n,1}$ L^{q+1}_n contain the direct summand $\pi_n(S^q)$, on which φ is an isomorphism. The fact that $T^q_{n,1} \cong A^q_n \oplus \pi_n(S^q)$ was proved in § 2. Let β: $L^{q+1}_n \to \pi_n(S^q)$ give the second linking class (see [39]) then $\beta\varphi i$ is the identity on $\pi_n(S^q)$, where i is the inclusion of $\pi_n(S^q)$ in $T^q_{n,1}$, and consequently $L^{q+1}_n \cong N^{q+1}_n \oplus \pi_n(S^q)$, say, where N^{q+1}_n denotes the kernel of β. N^{q+1}_n can be described either as the subgroup of L^{q+1}_n, whose second linking class is zero, or as the equivalence classes of L^{q+1}_n under 'local twisting.'

As a consequence the obstructions for a $(q + 1)$-block bundle to admit a section (resp. spherical section or equivalently a line subbundle) split into two sets of obstructions. The first have coefficients in $\pi_i(S^q)$, and the second in N^{q+1}_i (resp. A^q_i). We will show that the first set of obstructions are the obstructions for the associated fibre space (see [37; § 5]) to admit a cross-section. Thus we may call the obstructions respectively 'homotopy' obstructions and 'geometrical' obstructions.

Remarks. (1) Metastably the geometric obstructions are all zero (by 4.9 and 2.9). Compare also Zeeman [49; Ch. 8] and Haefliger [6] for $N^q_n = 0$ metastably.

(2) N^q_n and A^q_n are not always zero (cf. § 2 and Haefliger [6]) so that the theory of sections of a block bundle differs from the theory of sections of a fibre bundle or fibre space (see below for more details), in which there are

only homotopy obstructions.

Before connecting the homotopy obstructions with the associated Serre fibration, we give an example of a block bundle with section which does not split.

Example 4.16. There is a section of $\varepsilon^{s+1}/\Sigma^{2s-1}$ which does not split for s even and >1.

PROOF. Using Theorem 4.15, and the isomorphism of 4.13, we have to show that $\varphi: T^s_{2s-1,1} \to L^{s+1}_{2s-1}$ is not an epimorphism or equivalently, using the remarks below 4.15, that $\varphi|: A^s_{2s-1} \to N^{s+1}_{2s-1}$ is not epimorphic. But using results of Haefliger [5, 6] and Paechter [35] $A^s_{2s-1} \cong Z_2$ if $s \equiv 2$ (4) and is zero if $s \equiv 0$ (4), whereas $N^{s+1}_{2s-1} \cong Z$, and the result follows.

Remark. The dimensions of these examples are critical for Theorem 4.9. Thus this result is best possible dimensionally.

Sections of the associated Serre fibration. Let ξ^q/K be a block bundle; denote by $\pi: E(\xi) \to K$ the 'projection' given by collapsing (cf. [37; § 4]), and let $E_0(\xi) = E(\xi) - K$. In [37; § 5] we showed that there was a fibre space $p: G(\xi) \to K$ with fibre a homotopy $(q-1)$-sphere and a homotopy equivalence $g: E_0(\xi) \to G(\xi)$ such that

(4.17)
$$\begin{array}{ccc} E_0(\xi) & \xrightarrow{g} & G(\xi) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ K & = & K \end{array}$$

commutes up to homotopy. Now let $s: K \to E_0(\xi)$ be a *map* such that s respects blocks, then it is easily seen that $\pi s \simeq 1$. Diagram (4.17) then shows that $pgs \simeq 1$ and, by the homotopy lifting property of p, gs is homotopic to a cross-section of $G(\xi)$. Conversely, given a cross-section of $G(\xi)$ then the interpretation of $G(\xi)$ given in [37; Remark at end of § 5] gives us a block-respecting map $s: K \to E_0(\xi)$. Thus we see that the obstructions to finding a cross-section of $G(\xi)$ coincide with the obstructions to finding such a block-respecting map. But these latter are trivially verified to coincide with the 'homotopy' obstructions to a section or spherical section of ξ, which gives us the required connection.

Now let η^q/K be a (PL) bundle with fibre Σ^{q-1}, R^q or I^q, and let ξ^q/K be the block bundle obtained from η *via* [37; 5.1 or 5.4]. Then one has a diagram similar to (4.17) in which p is the projection of η. The above discussion then holds in this case also, and we have an isomorphism between the obstructions to finding a (never-zero) section of η and the homotopy obstructions to a section of ξ.

Summarising this discussion, we have

THEOREM 4.18. (i) *The homotopy obstructions to a section of a block bundle ξ^q coincide with the obstructions to a cross-section of the associated Serre fibration.*

(ii) *If ξ comes from a PL-fibre bundle η, then they also coincide with the obstructions to a never-zero section of η.*

Now by the proofs of [37; 5.1, 5.4], $E(\eta)$ may be taken to be contained in a block-respecting fashion in $E(\xi)$, thus a never-zero section of η gives a section of ξ. Moreover if η is a Σ^{q-1}-bundle, then this section will be spherical, and if η is an I^q-bundle, then the Alexander trick (cf. 0.3) shows that the section is homotopic to a spherical section. This proves

COROLLARY 4.19. (i) *If ξ^q contains a PL bundle, then the geometric obstructions to a section of ξ are all zero*;

(ii) *if ξ^q contains a Σ^{q-1} or I^q bundle, then the geometric obstructions to a spherical section also vanish.*

Now let $M^n \subset Q^{n+q}$ be an embedding, and ξ a normal block bundle. Combining 4.19 with the discussion in [37, § 5] we have

COROLLARY 4.20. (i) *If ξ has non-zero geometric obstructions to a spherical section, then M has no normal disc bundle in Q.*

(ii) *If ξ has non-zero geometric obstructions to a section, then M has no normal plane (or micro-) bundle in Q.*

An interpretation of sections. Let $M^n \subset Q^{n+q}$ have normal block bundle ξ. We will interpret the existence of a section or spherical section of ξ or a cross-section of $G(\xi)$.

Definitions. We can *homotope M off itself in Q* if there is a homotopy $H: M \times I \to Q$ such that $H \mid M \times \{0\} = \mathrm{id}$, and $H(M \times \{t\}) \cap M = \varnothing$, $t \neq 0$. If H is also an isotopy, i.e., $H \mid M \times \{t\}$ is an embedding each $t \in I$, then we can *isotope M off itself in Q*. If H is also an embedding, then we can *thicken M in Q*.

THEOREM 4.21. (i) *We can homotope M off itself in Q if and only if $G(\xi)$ has a cross-section.*

(ii) *We can isotope M off itself in Q if and only if ξ has a section.*

(iii) *We can thicken M in Q if and only if ξ has a spherical section.*

PROOF. Since a spherical section splits, (iii) is an easy consequence of the block bundle theorems [37, § 4] and the interpretation of Whitney sums given in [38, § 4]. Let K be the base complex of ξ, and suppose $s: K \to E_0(\xi)$ is a block-respecting map. We will construct a map $H: K \times I \to E(\xi) \times I$ such that H respects blocks of $\xi \times I$, $H \mid K \times \{0\} = \mathrm{id}$, $H \mid K \times \{1\} = s$ and $H(K \times \{t\}) \subset E(\xi) \times I - K \times I$. If s is an embedding, then H will be too. This will prove the 'if' parts of (i) and (ii) simultaneously. H is constructed by induction up the skeleton of K. Let $\sigma^t \in K$ be a cell, and suppose H defined on $\dot{\sigma} \times I$, we define H on $\sigma \times I$. Since we may choose a chart for $\xi \mid \sigma$ [37, 1.4] we may suppose that $\sigma = \Delta^t$ and $\xi \mid \sigma \cong \Delta^t \times I^q$. Let $b \in \operatorname{int} \Delta^t$ denote the barycentre. H is already defined on $\partial(\Delta \times I)$ extend to the interior conewise using $b \times \{0\}$ as vertex.

If we can homotope M off itself in Q, then (4.17) gives a cross-section of $G(\xi)$, so (i) is proved, and it remains to prove the 'only if' part of (ii). Let $H: M \times I \to Q \times I$ be an isotopy which carries M off itself, and let K be the base complex of ξ. We show that $\xi \times \{0\}$ can be extended to a block bundle $\eta/K \times I$ on $M \times I$ in $Q \times I$ such that H is a section of η. [37, 1.10] will then prove that ξ has a section. By Remark 4.2, ξ admits a section if and only if any block bundle equivalent to ξ admits a section. We may therefore suppose that K comes from a handle decomposition of M (as in [37, 4.4]), and by a similar induction, to that used in [37, 4.4], we only have to construct a block of η over a top dimensional cell of $K \times I$, and this is easy by choosing a suitable relative regular neighbourhood.

COROLLARY 4.22. (i) *if we can homotope M off itself in Q and M has a normal plane (or micro-) bundle in Q, then we can isotope M off itself in Q;*

(ii) *if we can homotope M off itself in Q, and M has a normal disc bundle in Q, then we can thicken M in Q.*

Remark. Theorem 4.21 gives an obstruction theory for thickening a submanifold M of Q with coefficients in $\pi_i(\widetilde{V}_{q,1})$. Of course this generalises by a similar proof to give an obstruction theory for extending M to $M \times I^r$ in Q with coefficients in $\pi_i(\widetilde{V}_{q,r})$.

Obstructions to normal plane (or micro-) bundles. To conclude this section, we give a short proof (additional to [39]) that $E_n^q(R) \neq 0$ for some n, q. This means that the obstruction theory given in [37, § 5] for the existence of normal plane and micro-bundles has non-zero coefficients. Let ξ^q/Σ^n be a normal plane bundle on Σ^n in Σ^{n+q}. There are no homotopy obstruction to a section of ξ, so by 4.18 (ii), ξ has a section (alternatively ξ is fibre-homotopy trivial, cf. [3], and so has a section). Now any two sections differ up to homotopy of sections by local twisting (since the obstructions lie in $\pi_n(S^{q-1})$). Thus the section and the zero-section give a link whose class in N_n^q is well-defined. This gives a function $\mu: E_n^q(R) \to N_n^q$. μ fits into a commutative diagram

$$(4.23)\qquad \begin{array}{ccccc} \xrightarrow{\omega_3} E_n^{q+1}(I) & \xrightarrow{\varphi_3} & A_n^q & \xrightarrow{\partial_3} & E_{n-1}^q(R) \\ \Big\downarrow \text{int} & & \Big\downarrow \varphi & & \\ E_n^{q+1}(R) & \xrightarrow{\mu} & N_n^{q+1} & & \end{array}$$

in which the top sequence is sequence 3 of 3.6, φ is the map in 4.15, and int is obtained by forgetting the boundary. Commutativity is easy by the interpretation of φ_3. Now suppose that $E_{n-1}^q(R) = E_n^{q+1}(R) = 0$; then (4.23) implies that $\varphi = 0$. But [39, Th. B] showed that the suspension $\Sigma\colon N_{n-1}^q \to N_n^{q+1}$ factors through φ (cf. [39]), and using results of Haefliger [6] we can easily show that $\Sigma \neq 0$. By [6, 10.7] and the construction of [5, 8.12] Σ is isomorphic with $i_*\colon \pi_{n-q+1}(O, O_{q-1}) \to \pi_{n-q+1}(O, O_q)$ for $3q - 6 \geqq n$, and by calculations of Paechter [35], we have

Example 4.24. $E_{2s}^{s+1}(R)$ or $E_{2s+1}^{s+2}(R)$ is not zero if s is even and >3.

Remark 4.25. μ can be interpreted homotopically as a homomorphism in the exact sequence

$$\longrightarrow \pi_n(\widetilde{PL}_q(R_s), PL_q(R_s)) \longrightarrow \pi_n(\widetilde{PL}_q(R), PL_q(R))$$
$$\xrightarrow{\mu} \pi_n(\widetilde{PL}_q(R); \widetilde{PL}_q(R_s), PL_q(R)) \longrightarrow$$

since $\pi_n(\widetilde{PL}_q(R); \widetilde{PL}_q(R_s), PL_q(R)) \cong N_n^q$ by 4.13, 4.4, and an argument similar to 2.6 or 3.2 (notice that $\pi_n(PL_q(R), PL_q(R_s)) \cong \pi_n(S^{q-1})$ by the obstruction theory for sections of plane bundles, see also Kuiper-Lashof [26]). $\pi_n(\widetilde{PL}_q(R_s), PL_q(R_s))$ can be interpreted as $E_n^q(R_s)$ the set of concordance classes of open tubes with a standard fixed section. Thus there is an exact sequence which fits in (4.23) to form a commuting diagram. The remaining map $E_n^q(R) \to E_n^{q+1}(R_s)$ being interpreted as adding a trivial line bundle. See 5.15 for the complete diagram.

5. Immersions of spheres in spheres

In this section we interpret the braid of the triple $(\widetilde{PL}(R), \widetilde{PL}_q(R), PL_q(R))$ for $q > 1$. Before doing this, we prove in Theorem 5.1 a homotopy equivalence between $\widetilde{PL}_q$ and Haefliger's group $\lim_{n\to\infty}(PL_{n+q,n})$. This implies (see Remark 5.6) that the theory of block bundles is equivalent to Haefliger's theory of microbundle pairs.

THEOREM 5.1. *$i_*\colon \pi_n(PL_{q+s,s}) \to \pi_n(\widetilde{PL}_{q+s,s}(\mu))$ is an isomorphism for $s \geqq n$.*

PROOF. We use the differential (cf. Haefliger-Poenaru [9]) to construct a function d, in (5.2) below, which we will prove commutes.

(5.2)
$$\begin{tikzcd} & \pi_n(PL_{q+n,n}) \arrow[d, "i_*"] \\ \pi_n(\widetilde{PL}_q(\mu)) \arrow[ur, "d"] \arrow[r, "\cong", "j_*"'] & \pi_n(\widetilde{PL}_{q+n,n}(\mu)) \end{tikzcd}$$

j_* is induced by the standard inclusion and is an isomorphism by 2.2 and an analogous proof to 0.3 (i).

We define the differential in the form in which we will use it. Let $\sigma \in \widetilde{PL}_q(\mu)$ represent an element of $\pi_n(\widetilde{PL}_q(\mu))$, and suppose that σ is the identity in some neighbourhood of $\dot{\Delta}^n$. Define $\sigma_1: R^n \times R^q \circlearrowleft$ (in some neighbourhood of $R^n \times \{0\}$) to be σ on $\Delta^n \times R^q$, and id elsewhere. Define $\sigma_2: R^n \times R^q \times R^n \circlearrowleft$ (in some neighbourhood of $R^n \times \{0\} \times R^n$ to be $\sigma_1 \times \text{id}$. Define $e: \Delta^n \times R^q \times R^n \to R^n \times R^q \times R^n$ to be the embedding given by $e(x, y, z) = (x + z, y, x)$. Then $d\sigma = \sigma_2^e$ (i.e., $d\sigma = e^{-1}\sigma_2 e$, where all the maps are restricted to suitable neighbourhoods).

Now any element of $\pi_n(\widetilde{PL}_q(\mu))$ has a representative σ which is the identity in a neighbourhood of $\dot{\Delta}^n$; moreover given a concordance F between representatives σ, τ, we may easily replace F by a concordance which is the identity in a neighbourhood of $\dot{\Delta}^n \times I$, and by the trick of Haefliger [7, 8.11 line 5] modified slightly, we may replace F by an isotopy having this property. Applying the differential to each level of this isotopy, we have an isotopy between $d\sigma$ and $d\tau$. Thus we have a well-defined function d as in (5.2)

Commutativity of (5.2). We now prove that (5.2) commutes. We will show that e is isotopic *via* embeddings e_t to the embedding $e_1: \Delta^n \times R^q \times R^n \circlearrowleft$ defined by $e_1(x, y, z) = (x, y, -z)$; and e_t satisfies $e_t(\Delta^n \times \{0\} \times R^n) \subset R^n \times \{0\} \times R^n$, and $e_t \mid \Delta^n \times \{0\} \times \{0\}$ is a proper embedding in $\Delta^n \times \{0\} \times \Delta^n$. This means that $(\sigma_2)^{e_t}$ defines a homotopy in $PL_{q+n,n}(\mu)$ between $d\sigma$ and $\sigma \times \text{id} \equiv (\sigma_2)^{e_1}$, which proves what we want.

We will write down formulas for e_t which are not PL (since they contain quadratic terms), but may easily be modified to be PL (e.g., by applying Whitehead [46]).

$$\begin{aligned} 0 \leqq t \leqq 1/2 \ , \qquad & e_t(x, y, z) = (x + z(1 - 2t), y, x - z(2t)) \\ 1/2 \leqq t \leqq 1 \ , \qquad & e_t(x, y, z) = (x, y, x(2 - 2t) - z) \ . \end{aligned}$$

Now by the Haefliger-Poenaru theorem [9], any homotopy class in $\pi_n(PL_{q+n,n})$ contains a representative of the form $d\sigma$ for some $\sigma \in \widetilde{PL}_q(\mu)$ consequently d is surjective and, by the commutativity of (5.2), it follows at once that d and i_* are both isomorphisms. This proves the theorem in the case $s = n$. The general case follows by a very similar argument applied to (5.3)

(5.3)
$$\begin{tikzcd} & \pi_n(PL_{q+n+r,n+r}) \arrow[d, "i_*"] \\ \pi_n(\widetilde{PL}_{q+r,r}(\mu)) \arrow[ur, "d"] \arrow[r, "\cong"] & \pi_n(\widetilde{PL}_{q+n+r,n+r}(\mu)) \end{tikzcd}$$

We deduce at once

COROLLARY 5.4. $\pi_n(PL_{q+n,n}) \cong \pi_n(PL_{q+n+r,n+r})$ (This result is also proved in Haefliger [7, 8.6]).

COROLLARY 5.5. *There are homotopy equivalences*

(i) $\widetilde{PL}_q \simeq \lim_{n\to\infty}(PL_{n+q,n})$

(ii) $\widetilde{PL} \simeq PL$.

There are isomorphisms for $q > 1$

(iii) $\pi_n(PL_{n+q}) \cong \pi_n(\widetilde{PL}_{n+q})$

(iv) $\pi_n(PL_{n+q}, PL_{n+q,n}) \cong \pi_n(PL, \lim_{n\to\infty}(PL_{n+q,n})) \cong \pi_n(\widetilde{PL}_{n+q}, \widetilde{PL}_q) \cong \pi_n(\widetilde{PL}, \widetilde{PL}_q)$.

PROOF. (i) is a direct consequence of 5.1 and 0.3. (ii) now follows, since it is easy to prove that $\lim_{n,q\to\infty}(PL_{n+q,n}) = PL$. (iii) follows from (ii), the stability theorem for PL_q [10, Th. 2], and the stability theorem for $\widetilde{PL}_q$ [38, 5.3]. (iv) now follows from (i), (ii), 5.4, and the stability theorems, by a 5-lemma argument.

Remark 5.6. Corollary 5.5 (i) gives a homotopy equivalence between $\widetilde{PL}_q$ and the group for stable microbundle pairs $\varepsilon^N \subset \xi^{q+N}$ (where ε^N denotes the trivial N-microbundle). This does not immediately prove that the theories are the same—one also needs to know that the notions of 'induced bundle' are compatible. However, using the definition of induced block bundle given in [37, § 1], (namely as the restriction of the cartesian product to the graph of the map), one only needs to check that the notions of 'cartesian product' and 'restriction' are compatible and this is easy.

From 5.5 and the classification given in [9, § 4], we deduce

COROLLARY 5.7. *There is a bijection for* $q > 1$, $\pi_n(\widetilde{PL}, \widetilde{PL}_q) \to I_n^q$, *where* I_n^q *denotes the set of regular homotopy classes of immersions of* Σ^n *in* Σ^{n+q}.

Remarks. (1) There is also a bijection for $q > 1$, $\pi_n(\widetilde{PD}, O_q) \to I_n^q(\Gamma)$, where $I_n^q(\Gamma)$ denotes the set of Γ-regular homotopy classes of Γ-immersions of S^n in S^{n+q}, this is proved by a similar argument to 5.11 below. This means that (together with Smale [42]) the braid (2.1) possesses the further geometric interpretation (5.8) below for $q + s > n + 1$ and $q > 1$. As this is proved in detail in Haefliger [7], we omit further proof.

(5.8)
$$\begin{tikzcd}
\Gamma_n^q \arrow[rr, bend left] \arrow[dr] & & \Gamma_n \arrow[rr, bend left, "1"] \arrow[dr] & & I_{n-1}^q(C) \\
& I_n^q(\Gamma) \arrow[ur] \arrow[dr] & & C_{n-1}^q \arrow[ur, "2"] \arrow[dr, "3"] & \\
I_n^q(C) \arrow[ur] \arrow[rr, bend right] & & I_n^q \arrow[ur] \arrow[rr, bend right, "4"'] & & \Gamma_{n-1}^q
\end{tikzcd}$$

(2) for $q > 1$, I_n^q may be equivalently defined using regular concordance, as may the other types of immersion used in this section. This is an easy consequence of stability of the various Stiefel manifolds involved, see Haefliger [7; 9.2].

We now seek to interpret the braid (5.9) of the triple $(\widetilde{PL}(R), \widetilde{PL}_q(R), PL_q(R))$, a precisely similar interpretation holds for the braid of $(\widetilde{PL}, \widetilde{PL}_q, PL_q(I))$ as the reader may construct from results of this section.

(5.9)
$$\begin{tikzcd}
\pi_n(\widetilde{PL}(R)) \arrow[rr, bend left] \arrow[dr] & & \pi_n(\widetilde{PL}(R), \widetilde{PL}_q(R)) \arrow[rr, bend left] \arrow[dr] & & \pi_{n-1}(\widetilde{PL}_q(R), PL_q(R)) \\
& \pi_n(\widetilde{PL}(R), PL_q(R)) \arrow[ur] \arrow[dr] & & \pi_{n-1}(\widetilde{PL}_q(R)) \arrow[ur] \arrow[dr] & \\
\pi_n(\widetilde{PL}_q(R), PL_q(R)) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(PL_q(R)) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(\widetilde{PL}(R))
\end{tikzcd}$$

Our first aim is to interpret $\pi_n(\widetilde{PL}(R), (PL_q(R))$ as immersions of Σ^n in Σ^{n+q} with an open normal bundle. This is achieved in 5.10–5.12.

Classification of immersions with an open (or closed) normal bundle. An R-immersion (resp. I-immersion) of M^n in Q^{n+q} is an immersion $f: M \to Q$

together with an open (resp. closed) normal tube on M in Q. More precisely, we consider immersions $f\colon E(\xi^q) \to Q$ where ξ^q/M is an $(R^q, 0)$-bundle (resp. $(I^q, 0)$-bundle), under the equivalence $f_0 \sim f_1$ if $f_0 \mid M = f_1 \mid M$ and $f_0 = f_1 \circ g$, where g is an $(R^q, 0)$- (resp. $(I^q, 0)$-) bundle isomorphism. There is an obvious notion of regular homotopy of R- and I-immersions, and more generally of simplexes of such immersions. Thus we have Δ-sets denoted $\mathrm{Imm}_R(M, Q)$ and $\mathrm{Imm}_I(M, Q)$.

Now denote by $\tau(M)$, $\tau(Q)$ the tangent microbundles of M and Q (see Milnor [30]). An *R-monomorphism* of $\tau(M)$ in $\tau(Q)$ is an equivalence class of bundle maps $f\colon \tau(E(\xi^q)) \mid M \to \tau(Q)$, where ξ^q/M is an $(R^q, 0)$-bundle, and $f_0 \sim f_1$ if $f_0 = f_1 \circ (dg \mid M)$ where $g\colon E(\xi_0) \to E(\xi_1)$ is an $(R^q, 0)$-bundle isomorphism (over the identity on M). There is a Δ-set $\mathrm{Mono}_R(\tau(M), \tau(Q))$ of R-monomorphisms formed in the obvious way, and there is a precisely analogous Δ-set $\mathrm{Mono}_I(\tau(M), \tau(Q))$.

The following result follows from the main theorem of [9].

THEOREM 5.10. *The differential induces homotopy equivalences*

$$d\colon \mathrm{Imm}_R(M, Q) \longrightarrow \mathrm{Mono}_R(\tau(M), \tau(Q))$$
$$d\colon \mathrm{Imm}_I(M, Q) \longrightarrow \mathrm{Mono}_I(\tau(M), \tau(Q))$$

In analogy to [9, § 4], Theorem 5.10 gives rise to an obstruction theory for the existence of R- and I-immersions with coefficients in $\pi_n(PL_{n+q}, PL_q(R))$ and $\pi_n(PL_{n+q}, PL_q(I))$ respectively (where these homotopy groups must be interpreted as the homotopy groups of the maps of $PL_q(R)$ and $PL_q(I)$ in PL_{n+q} defined in § 0). This implies.

COROLLARY 5.11. *There are bijections*

$$I_n^q(R) \longrightarrow \pi_n(PL_{n+q}, PL_q(R))$$
$$I_n^q(I) \longrightarrow \pi_n(PL_{n+q}, PL_q(I)) \,,$$

where $I_n^q(R)$ denotes the set of R-regular homotopy classes of R-immersions of Σ^n in Σ^{n+q}, and $I_n^q(I)$ is defined similarly.

Combining 5.11, 5.5, and 0.3 we have

COROLLARY 5.12. *For $q > 1$, there are bijections*

$$I_n^q(R) \longrightarrow \pi_n(\widetilde{PL}(R), PL_q(R))$$
$$I_n^q(I) \longrightarrow \pi_n(\widetilde{PL}, PL_q(I)) \,.$$

Now denote by FI_n^q the set of framed regular homotopy classes of framed immersions of Σ^n in Σ^{n+q}. From [9] and 5.5, we have

PROPOSITION 5.13. *For $q > 1$, there are bijections*

$$FI_n^q \longrightarrow \pi_n(\widetilde{PL}) \cong \pi_n(\widetilde{PL}(R)) .$$

Combining results of this section, together with 3.5 and 1.2–3 we have

THEOREM 5.14. *The braid* (5.9) *is isomorphic with the following braid for* $q > 1$,

$$\begin{tikzcd}
FI_n^q \arrow[rr, bend left, "\varphi_3"] \arrow[dr, "\varphi_4"'] & & I_n^q \arrow[rr, bend left, "\partial_1"] \arrow[dr, "\partial_3"'] & & E_{n-1}^q(R) \\
& I_n^q(R) \arrow[ur, "\varphi_1"'] \arrow[dr, "\partial_4"'] & & FE_{n-1}^q(R) \arrow[ur, "\varphi_2"] \arrow[dr, "\omega_3"] & \\
E_n^q(R) \arrow[ur, "\omega_1"'] \arrow[rr, bend right, "\partial_2"'] & & \pi_{n-1}(PL_q(R)) \arrow[ur, "\omega_2"] \arrow[rr, bend right, "\omega_4"'] & & FI_{n-1}^q
\end{tikzcd}$$

We now interpret the homomorphisms in 5.14 geometrically. ω_1, ω_3, and all the φ_i forget structure.

∂_3: Any immersion $f\colon \Sigma^n \longrightarrow \Sigma^{n+q}$ is regularly homotopic to f_1, where $f_1 \mid \Sigma_-^n = \mathrm{id}$, and $f_1(\Sigma_+^n) \subset \Sigma_+^{n+q}$. A trivialisation of the normal block bundle on $f_1 \mid \Sigma_+^n$ determines $\partial_3[f]$.

∂_4 gives the obstruction to trivialising the normal plane bundle.

ω_2 is the construction of 1.3 applied to $PL_q(R)$, and the remaining homomorphisms are given by commutativity.

Remark. (1) By the interpretation of φ_1, ∂_1 measures the precise obstruction for an immersion to possess a normal (plane) bundle.

(2) Metastably, i.e. for $2q > n + 3$, $I_n^q(C) \longrightarrow I_n^q$ is an isomorphism (from 5.8 and 2.10), hence any immersion has a preferred normal vector bundle and (on triangulating) a PL normal plane bundle. Consequently sequence 1 splits and (in this range)

$$I_n^q(R) = I_n^q \oplus E_n^q(R) .$$

The immersion suspension sequences.

THEOREM 5.15. *For* $q > 1$ *there is a commutative diagram with exact rows in which the columns are "multi-exact" at* I_{n+1}^q, *and* I_{n+1}^{q+1}, i.e., *an element survives precisely* t *homomorphisms if and only if it comes from* $4 - t$ *spaces above.*

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & FI^q_{n+1}(C) \arrow[r] \arrow[d] & FI^{q+1}_{n+1}(C) \arrow[r] \arrow[d] & 0 \arrow[r, "1"] \arrow[d] & {} \\
\pi_{n+1}(S^q) \arrow[r] \arrow[d, equal] & I^q_{n+1}(C) \arrow[r] \arrow[d] & I^{q+1}_{n+1}(C) \arrow[r] \arrow[d] & \pi_n(S^q) \arrow[r, "2"] \arrow[d, equal] & {} \\
\pi_{n+1}(S^q) \arrow[r] \arrow[d, equal] & I^q_{n+1}(\Gamma) \arrow[r] \arrow[d] & I^{q+1}_{n+1}(\Gamma) \arrow[r] \arrow[d] & \pi_n(S^q) \arrow[r, "3"] \arrow[d, equal] & {} \\
\pi_{n+1}(S^q) \arrow[r] & I^q_{n+1}(R) \arrow[r] \arrow[d] & I^{q+1}_{n+1}(I) \arrow[r] \arrow[d] & \pi_n(S^q) \arrow[r, "4"] & {} \\
T^q_{n+1,1} \arrow[r] \arrow[u, dotted, "\cap"] \arrow[d, equal] & I^q_{n+1} \arrow[r] \arrow[d] & I^{q+1}_{n+1} \arrow[r] \arrow[d] & T^q_{n,1} \arrow[r, "5"] \arrow[u, dotted, "\cap"] \arrow[d, equal] & {} \\
T^q_{n+1,1} \arrow[r] \arrow[d, "\cup"] & FC^q_n \arrow[r] \arrow[d] & FC^{q+1}_n \arrow[r] \arrow[d] & T^q_{n,1} \arrow[r, "6"] \arrow[d, "\cup"] & {} \\
A^q_{n+1} \arrow[r] \arrow[d, equal] & C^q_n \arrow[r] \arrow[d] & C^{q+1}_n \arrow[r] \arrow[d] & A^q_n \arrow[r, "7"] \arrow[d, equal] & {} \\
A^q_{n+1} \arrow[r] \arrow[d, equal] & \Gamma^q_n \arrow[r] \arrow[d] & \Gamma^{q+1}_n \arrow[r] \arrow[d] & A^q_n \arrow[r, "8"] \arrow[d, equal] & {} \\
A^q_{n+1} \arrow[r] \arrow[d] & E^q_n(R) \arrow[r] \arrow[d] & E^{q+1}_n(I) \arrow[r] \arrow[d] & A^q_n \arrow[r, "9"] \arrow[d] & {} \\
N^{q+1}_{n+1} \arrow[r] & E^{q+1}_n(R_s) \arrow[r] & E^{q+1}_n(R) \arrow[r] & N^{q+1}_n \arrow[r, "10"] & {}
\end{tikzcd}
$$

PROOF. The lower half of the diagram was described in 3.8 and 4.25. $FI^q_n(C) \cong \pi_n(O)$, and sequence 2 is the exact sequence of $O_q \subset O_{q+1} \subset O$ (cf. Smale [42] and Hirsch [15]). Sequence 3 is the exact sequence of $O_q \subset O_{q+1} \subset \widetilde{PD}$ (cf. Remark (1) below 5.7). Sequence 4 is the exact sequence of $PL_{q+1,1}(I) \subset PL_{q+1}(I) \subset \widetilde{PL}$ using the homotopy equivalence of 0.12 and 5.12. Sequence 5 is the exact sequence of $\widetilde{PL}_q \subset \widetilde{PL}_{q+1} \subset \widetilde{PL}$ using 2.12 and 5.7. The vertical maps are all obtained from obvious homotopy maps and 0.2. This proves commutativity and exactness of the rows, it remains to prove "exactness" at I^q_{n+1} and I^{q+1}_{n+1} in the columns. This follows from the fact that the vertical maps (and compositions of them) fit into various exact sequences. For example

$$\longrightarrow FI^q_{n+1}(C) \longrightarrow I^q_{n+1} \longrightarrow FC^q_n \longrightarrow$$

which is one of the exact sequences of the square

$$
\begin{array}{ccc}
\widetilde{PD} & \supset & \widetilde{PD}_q \\
\cup & & \cup \\
O & \supset & * \quad .
\end{array}
$$

The other exact sequences are sequences 2 and 4 of 5.8, sequence 1 of 5.14 and the analogous sequence for I-immersions. (These five exact sequences, which all contain the term I^q_n can also be displayed in the form of a ladder diagram like 5.15, which we leave the reader to construct.)

Remark. Any two of sequences 2, 3, 4, sequences 5 and 6, and any two of sequences 7, 8, 9 of 5.15 can be imbedded in a braid (cf. Remark (2) below 3.11); sequences 2 and 3 have sixth term Γ_n; also any two of the five exact sequences mentioned in the proof of 5.15 can be imbedded in a braid (e.g., 5.8). We leave the reader to construct and interpret these.

Interpretation of 5.15. *Vertical homomorphisms.* The central vertical homomorphisms between sequences 1 to 3 and 4, 5 all forget structure. Between sequences 3 and 4, one first triangulates the normal bundle. $\pi_n(S^q) \to T^q_{n,1}$ was described in 2.14. $I^q_{n+1} \to FC^q_n$ has a complicated interpretation (cf. ∂_2 in 2.13) but the composition $I^q_{n+1} \to \Gamma^q_n$ is obtained from the obstruction to trivializing the normal block bundle of the immersion (cf. ∂_4 in 2.13).

Horizontal homomorphisms. The central horizontal homomorphisms are all 'suspension' of various types. ($I^q_{n+1}(R) \to I^{q+1}_{n+1}(I)$ has a similar interpretation to ω_3 of 3.6.) $T^q_{n,1} \to I^q_n$ was described in [39; remark below Th. B]. The right hand homomorphism in sequences 2 to 5 are all obstructions to splitting a line subbundle of the normal bundle (normal block bundle in the case of sequence 5).

Remarks. (1) By the interpretation given above, the right hand horizontal homomorphisms give the precise obstruction to "compression" of the various types of immersion and imbedding.

(2) By "multi-exactness" and the interpretations of the columns, the (compositions) of vertical homomorphisms from I^q_{n+1} and I^{q+1}_{n+1} measure the precise obstruction to (respectively), C-smoothing with trivial normal bundle, C-smoothing, Γ-smoothing, having a normal plane bundle (I^q_{n+1}) or disc bundle (I^{q+1}_{n+1}).

Diagram 5.15 contains a vast amount of information (some of which has already been given in §§ 3 and 4). We now give a few more corollaries, which are proved by easy diagram chasing.

COROLLARY 5.16. *An element $\alpha \in C^q_n$ (resp. Γ^q_n) with trivial suspension gives rise to an element $\alpha_1 \in I^q_{n+1}$ with trivial suspension, but which is not C-smoothable (resp. Γ-smoothable).*

COROLLARY 5.17. *An element $\alpha \in A^q_{n+1}$ with non-trivial image in $E^q_n(R)$ gives rise to an immersion $\alpha_1 \in I^q_{n+1}$ with trivial suspension but no normal plane (or micro-) bundle.*

Remark. The first half of the last corollary was proved directly in [39], where we deduced the existence of an immersion $\Sigma^{19} \subseteq \Sigma^{28}$ with no normal (topological) plane bundle. Corollary 5.17 shows that this immersion also has trivial suspension.

Now remember that $T^q_{n,1} \cong \pi_n(S^q) \oplus A^q_n$ (cf. § 2), thus the obstruction to compression of a PL-immersion splits into two obstructions; the first in $\pi_n(S^q)$ (the "homotopy" obstruction) being the obstruction to a section of the Serre fibration associated with the normal block bundle. This is interpreted (cf. 4.21 (i)) as the obstruction to locally homotoping the immersion off itself. The second obstruction in A^q_n (the geometric obstruction) is the obstruction to PL-immersing this homotopy (cf. 4.21 (iii)). Similarly $I^{q+1}_{n+1} \to N^{q+1}_n$ gives the geometric obstruction to a section of the normal block bundle (or equivalently to replacing the homotopy mentioned above by a regular homotopy (cf. 4.21 (ii)).

The next corollary (still proved by easy diagarm chasing) is a version of Theorem 4.20 for immersions.

COROLLARY 5.18. (i) *An immersion with non-trivial geometric obstruction to compression (in A^q_n) has no normal disc bundle.*

(ii) *If it also has non-zero image in N^{q+1}_n, then it has no normal plane bundle.*

COROLLARY 5.19. *If an immersion with no normal disc bundle is compressible, then its compression has no normal plane bundle.*

Finally we recall from § 3 that the generator of C^3_3 comes from A^3_4 and maps non-trivially to $E^3_3(I)$, hence there is an element in I^3_4 which comes from A^3_4 and maps non-trivially to $E^3_3(I)$. We have

THEOREM 5.20. *There is a PL-immersion $\Sigma^4 \subseteq \Sigma^7$ with trivial suspension but no normal (PL) disc bundle.*

Remark. From 5.8, one easily proves that $I^3_3 \cong C^3_3$, and so *every* non-trivial immersion of Σ^4 in Σ^7 has no normal disc bundle.

6. Appendix

Here we present on one diagram (6.2) a large number of the results proved in the paper. Each arrow is induced by a combination of the Δ-maps described in § 0. The notation

$$\pi_*(\widetilde{PL}_q) \xleftarrow{\Gamma^q_*} \pi_*(O_q) ,$$

for example, implies the existence of an isomorphism $\Gamma^q_* \cong \pi_*(\widetilde{PL}_q, O_q)$, and the reference (in this case (1.5)) is indicated on (6.3). Here, as elsewhere in this appendix, we shall not worry about writing $\widetilde{PL}_q$ when strictly we should write $\widetilde{PD}_q$, etc.

Notice that each triangle of arrows determines a braid of four exact sequences, which may be written, for example

$$
\begin{tikzcd}
 & & \pi_*(G_{q+1}) \arrow[dl] \arrow[dr] & & \\
 & \pi_*(S^q) \arrow[dl, "\partial"'] \arrow[dr, dashed, "\partial"] \arrow[rr, phantom, "2" near start] & & T^q_{*,1} \arrow[ll, dashed] \arrow[dr, "\partial"] \arrow[rr, phantom, "3" near end] & \\
\pi_*(F_q) \arrow[rr] \arrow[uurr, bend left] & \arrow[r, phantom, "1"] & A^q_* \arrow[ur, dashed] \arrow[rr, "\partial"] & & \pi_*(G_q) \arrow[uull, bend right] \arrow[llll, bend left, "4"]
\end{tikzcd}
$$

The notation below implies an isomorphism

$$A^q_* \cong \pi_*\begin{pmatrix} \widetilde{PL}_{q+1} & O_{q+1} \\ \widetilde{PL}_q & O_q \end{pmatrix}$$

and the reference is again indicated on (6.3).

$$
\begin{tikzcd}
\pi_*(\widetilde{PL}_{q+1}) & & \pi_*(O_{q+1}) \arrow[ll, "\Gamma^{q+1}_*"'] \\
T^q_{*,1} \arrow[r, dashed] & \boxed{A^q_*} \arrow[r, dashed] & \pi_*(S^q) \\
\pi_*(\widetilde{PL}_q) \arrow[uu] & & \pi_*(O_q) \arrow[ll, "\Gamma^q_*"] \arrow[uu]
\end{tikzcd}
$$

The dotted arrows correspond to homomorphisms in the exact sequences of the square. (See § 0). Note that a square determines three braids, two of which are braids of the triangles obtained by filling in the diagonal (in this case, $\pi_*(O_q) \xrightarrow[T^q_{*,1}(\Gamma)]{} \pi_*(\widetilde{PL}_{q+1})$).

The exact sequence (0.5) justifies a process of 'adding' squares. Thus for example adding the two zero squares adjacent to the A^q_* square, we deduce an isomorphism

$$A^q_* \cong \pi_*\begin{pmatrix} G_{q+1} & O_{q+1} \\ G_q & O_q \end{pmatrix} \qquad q > 2 .$$

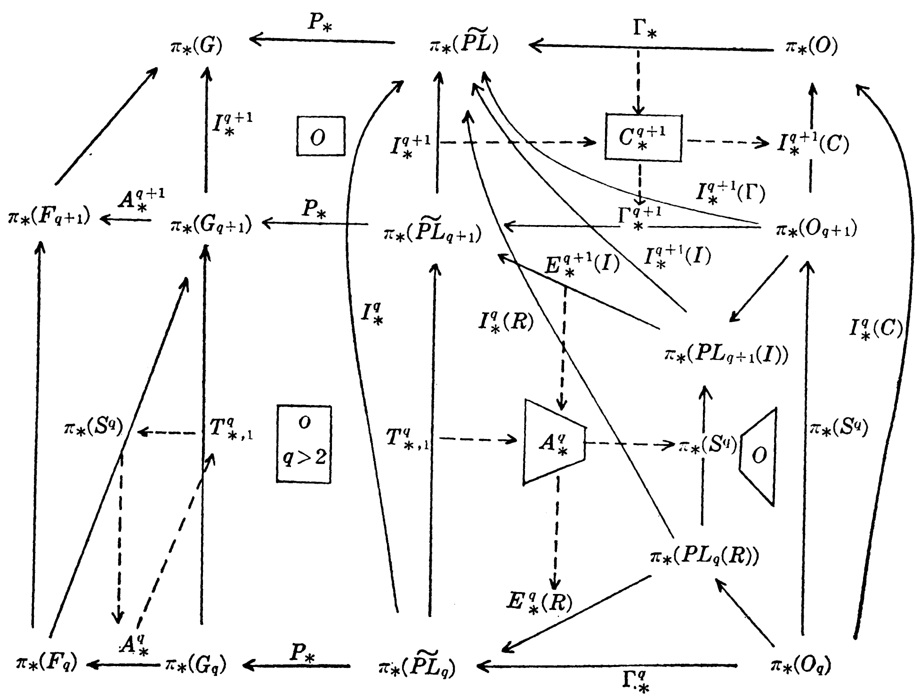
Less obvious is the isomorphism

$$\Gamma^q_* \cong \pi_*\begin{pmatrix} G & \widetilde{PL} \\ G_q & O_q \end{pmatrix} \qquad q > 2 ,$$

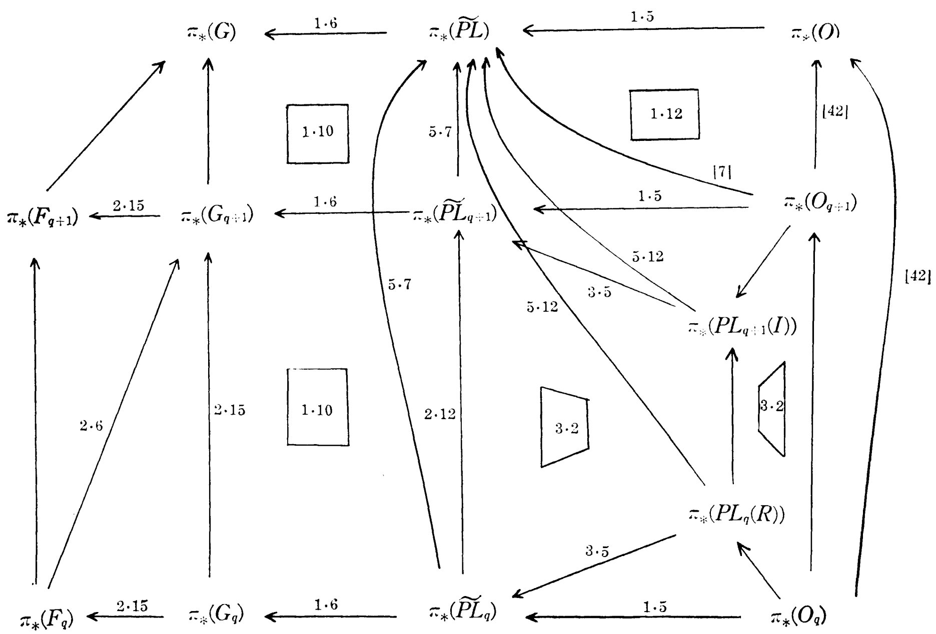
which is obtained by 'adding a triangle' to a zero square

$$\begin{tikzcd}
\pi_*(G) & \pi_*(\widetilde{PL}) \arrow[l] & \pi_*(\widetilde{PL}) \arrow[l, "0"'] \\
\pi_*(G_q) \arrow[u] \arrow[ur, phantom, "\boxed{0}"] & \pi_*(\widetilde{PL}_q) \arrow[l] \arrow[u] \arrow[ur, phantom, "\boxed{\Gamma^q_*}"] & \pi_*(O_q) \arrow[l, "\Gamma^q_*"] \arrow[u]
\end{tikzcd}$$

(6.2)



(6.3)



Collecting these results together we have

Theorem 6.1.

$$I_n^q \cong \pi_n(G, G_q)$$

$$T_{n,1}^q \cong \pi_n(G_{q+1}, G_q)$$

$$A_n^q \cong \pi_n\begin{pmatrix} G_{q+1} & O_{q+1} \\ G_q & O_q \end{pmatrix} \cong \pi_n(F_q, G_q)$$

$$C_n^q \cong \pi_{n+1}\begin{pmatrix} G & O \\ G_q & O_q \end{pmatrix} \qquad FC_n^q \cong \pi_{n+1}\begin{pmatrix} G & O \\ G_q & * \end{pmatrix}$$

$$\Gamma_n^q \cong \pi_{n+1}\begin{pmatrix} G & \widetilde{PL} \\ G_q & O_q \end{pmatrix} \qquad F\Gamma_n^q \cong \pi_{n+1}\begin{pmatrix} G & \widetilde{PL} \\ G_q & * \end{pmatrix} \cong \pi_n(\widetilde{PL}_q)$$

$$P_n \cong \pi_n(G, \widetilde{PL}) \qquad (q > 2 \text{ throughout}) .$$

Remark. The reader may construct further isomorphisms by utilising the isomorphism

$$\pi_*\begin{pmatrix} G_{q+1} & O_{q+1} \\ F_q & O_q \end{pmatrix} \cong 0 .$$

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