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QUADRATIC FORMS OVER FIELDS OF CHARACTERISTIC 2.*

By CHIH-HAN SAH.

In [13], Minkowski solved the problem of rational equivalence for quadratic forms. This was extended by Hasse to algebraic number fields, [10], [11]. Witt [16], and Arf [1], then developed general theories of quadratic forms over fields of characteristic $\neq 2$ and $= 2$ respectively. These works together solved the field equivalence problem of quadratic forms over sufficiently "nice" fields, e.g. global and local fields.

The problem of integral equivalence for quadratic forms over a global field remains open. In a sequence of papers, O'Meara [14], [15], solved the problem of integral equivalence for quadratic forms over a local field of characteristic $\neq 2$. Earlier, Arf [2], solved the same problem for binary and ternary quadratic forms over local fields of characteristic 2. The purpose of the present investigation, which formed the basic part of the author's thesis, is to give a solution to the integral equivalence problem for quadratic forms in any finite number of variables over a local field of characteristic 2. The main results of this paper are contained in Theorem 4.6, for modular lattices, and in Theorem 5.5 for lattices in general.

Familiarity of [1] is assumed. Basic notions on quadratic forms may be found in [4], [5] and [9]. Basic results on fields and algebras may be found in [3], [5] and [17].

I wish to thank Prof. O'Meara for introducing me to the theory of quadratic forms in his seminar given during the spring of 1958 as well as the many interesting conversations on quadratic forms during the preparation of this paper.

1. Notations. Throughout this paper, Ω shall denote a finite field of characteristic 2. $k = \Omega\langle\langle\pi\rangle\rangle$, the formal power-series field over Ω with uniformizer π , $\pi^\infty = 0$. $\mathfrak{o} = \Omega[[\pi]]$, the ring of integral power series. $\Omega[\pi^{-1}]$, the polynomial ring in π^{-1} over Ω . 0 and λ are fixed representatives of $\Omega/\mathfrak{P}\Omega$, where $\mathfrak{P}x = x^2 + x$. ord is the ordinal function in k and Ord is the ordinal function in $k/\mathfrak{P}k$ defined by $\text{Ord}(\Delta) = \max\{\text{ord}(x) \mid x \in \Delta\}$, where $\Delta \in k/\mathfrak{P}k$.

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By a quadratic space (respectively lattice) U over k , we mean a vector space (respectively free module) of finite type over k (respectively \mathfrak{o}), together with a map $Q: U \rightarrow k$ such that $Q(ax) = a^2Q(x)$ for $a \in k$ (respectively $a \in \mathfrak{o}$) and $x \in U$, and such that $\langle x, y \rangle = Q(x+y) - Q(x) - Q(y)$ is bilinear. If U is the direct sum of U_1 and U_2 and $\langle U_1, U_2 \rangle = 0$, then we shall write $U = U_1 \oplus U_2$; such decompositions will be called orthogonal. $\dim U = \text{rank of } U \text{ over } k \text{ (or } \mathfrak{o})$. $R(U) = \{x \in U \mid Q(x) = 0, \langle x, U \rangle = 0\}$ and $K(U) = \{x \in U \mid \langle x, U \rangle = 0\}$ will be called the radical and core of U respectively. It is easily seen that $R(U)$ and $K(U)$ are orthogonal summands of U , and that the structure of U is uniquely determined by $R(U)$ and the natural quadratic structure on $U/R(U)$. Thus, we shall always assume that the given quadratic spaces or quadratic lattices are non-degenerate, i. e. $R(U) = 0$. U is called non-defective when $K(U) = 0$. It is easy to see that $\dim K(U) \leq [k: k^2] = 2$.

If M is a quadratic lattice, then $k \otimes_{\mathfrak{o}} M$ receives a natural quadratic structure, this quadratic space shall be denoted by kM . If $a \neq 0 \in k$, then the space (or lattice) M together with the quadratic map $a \circ Q$ defined by $(a \circ Q)(x) = aQ(x)$ for $x \in M$, shall be denoted by $a \circ M$. This operation is called scaling by a . Homomorphisms which preserve the quadratic structures will be called representations. Isomorphic representations are called isometries and denoted by \cong . From the non-degeneracy hypothesis, it follows easily that representations are always monomorphic.

The quadratic lattice $\mathfrak{o}x$ such that $Q(x) = a \neq 0 \in k$ will be denoted by (a) . The quadratic lattice $\mathfrak{o}x + \mathfrak{o}y$ such that $Q(x) = a, \langle x, y \rangle = b \neq 0$ and $Q(y) = c$, where $a, b, c \in k$, shall be denoted by $\begin{pmatrix} a & b \\ & c \end{pmatrix}$. The subscript k will be used to denote the quadratic spaces spanned by the corresponding lattices. If the basis element x is to be emphasized, we shall write $\mathfrak{o}x = (a)$, otherwise, we shall use $\mathfrak{o}x \cong (a)$, similarly for binary and higher dimensional lattices and spaces.

If V is a quadratic space, then we shall let $C(V)$ denote the Clifford algebra of V , cf. [5; Chap. 9], [9]. If V is non-defective, then $C(V)$ is a k -central simple algebra and $C((1)_k \oplus V)/\mathfrak{R} \cong C(V)$, where \mathfrak{R} is the radical of the algebra $C((1)_k \oplus V)$, cf. [1; p. 151]. In general, $C(U \oplus V)$ is isomorphic to $C(U) \oplus_k C(V)$. If $V = \begin{pmatrix} 1 \\ a & c \end{pmatrix}_k$, then the class of $C(V)$ in the Brauer group of k is denoted by $[a, c]$. It can be shown, [17; p. 135] that, $[a, c] = [c, a]$, $[ad^2, c] = [a, cd^2]$, $[d^2, c] = 1$ and $[a, b+c] = [a, b][a, c]$, where the group operation in the Brauer group is written multiplicatively. Using these, it is easy to see that $[a, c] = 1$ or $[\pi^{-1}, \pi\lambda] (\neq 1)$.

If $V \cong \bigoplus \begin{pmatrix} b_i \\ a_i & c_i \end{pmatrix}_k$ is a non-defective quadratic space, then the Arf invariant is denoted by $\Delta(V) = \sum a_i c_i b_i^{-2} + \mathfrak{P}k \in k/\mathfrak{P}k$, cf. [1], [6], [12] or [18]. We shall need the following fact:

LEMMA 1.1. *Let $\Delta \in k/\mathfrak{P}k$ and $a \in k$, then,*

- 1). $\text{Ord}(\Delta) > 0$ implies that $\Delta = \mathfrak{P}k$ and $\text{Ord}(\Delta) = \infty$;
- 2). $\text{Ord}(\Delta) = 0$ implies that $\Delta = \lambda + \mathfrak{P}k$ and conversely;
- 3). $\text{Ord}(\Delta) < 0$ implies that $\text{Ord}(\Delta) = 2s + 1$, s an integer;
- 4). $a \in \Delta$ and $\text{ord}(a) = 2s + 1 < 0$ implies that $\text{Ord}(\Delta) = 2s + 1$;

5). $a = \pi^{-1}b^2 + D + \mathfrak{P}c$, where $b \in \Omega[\pi^{-1}]$ and $D = 0$ or λ ; b , D and $\mathfrak{P}c$ are uniquely determined by a . In particular, if $\text{ord}(a) = 2s + 1 < 0$, then $\text{ord}(\pi^{-1}b^2 + D) = \text{ord}(\pi^{-1}b^2) = 2s + 1 < \text{ord}(\mathfrak{P}c)$; if $\text{ord}(a) = 2s < 0$, then $\text{ord}(\mathfrak{P}c) = 2s < \text{ord}(\pi^{-1}b^2 + D)$; if $\text{ord}(a) = 0$, then $b = 0$; and, if $\text{ord}(a) > 0$, then $b = D = 0$.

Proof. Applying Hensel's Lemma [7; p. 43] to $X^2 + X + d$, where $d \in \mathfrak{o}$, we get 1) and 2) immediately.

3). Let $a \in \Delta$ such that $\text{ord}(a) = \text{Ord}(\Delta) < 0$. Suppose that $\text{ord}(a) = 2s$ and $a = \pi^{2s}(b^2 + \pi c^2)$, where $s < 0$, $\text{ord}(b) = 0$ and $\text{ord}(c) \geq 0$. Then $a' = a + \mathfrak{P}(\pi^s b) \in \Delta$ and $\text{ord}(a') > 2s$, a contradiction.

4). Let $a' \in \Delta$ such that $\text{ord}(a) = 2s + 1 < \text{ord}(a')$. Then, $a + a' \in \mathfrak{P}k$ and $\text{ord}(a + a') = 2s + 1 < 0$. Thus, $a + a' = b^2 + b$ and $\text{ord}(b) < 0$. Therefore, $\text{ord}(b^2 + b) = 2 \cdot \text{ord}(b) = 2s + 1$, a contradiction.

5) follows easily from the first four results.

Q. E. D.

Let L be a quadratic lattice. $s(L) = \{\langle x, y \rangle \mid x, y \in L\}$ is called the scale of L . $Q(L) = \{Q(x) \mid x \in L\}$ is called the norm of L . $q(L) = \{Q(x) + s(L) \mid x \in L\}$ is called the norm group of L . For $-\infty < i \leq \infty$, $L(i) = \{x \in L \mid \langle x, L \rangle \subseteq \pi^i \mathfrak{o}\}$ is called the i -th invariant sublattice of L and $q_i(L) = \{Q(x) + \pi^i \mathfrak{o} \mid x \in L(i)\}$ is called the i -th norm group of L .

It is easily seen that the scale is an \mathfrak{o} -module of finite type in k and that the norm groups are \mathfrak{o}^2 -modules of finite type in k . If A, B are subsets of k , then let $A + B = \{a + b \mid a \in A, b \in B\}$. If $L = L_1 \oplus L_2$, then $s(a \circ L) = a \cdot s(L_1) + a \cdot s(L_2)$, similarly for $Q(a \circ L)$, $q(a \circ L)$ and $q_i(a \circ L)$. If $s(L) = \pi^i \mathfrak{o}$, then $L = L(i)$. In general $L \supseteq \cdots \supseteq L(i) \supseteq L(i+1) \supseteq \cdots \supseteq L(\infty) = K(L)$; $\pi^j L(i) \subseteq L(i+j)$; $kL(i) = kL$, when $i < \infty$; and $(L_1 \oplus L_2)(i) = L_1(i) \oplus L_2(i)$.

LEMMA 1.2. *Let L be a quadratic lattice. Then there exist $e \in k$ and $u \leq v \leq \infty$ such that $q(L) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$. Moreover, they may be chosen to satisfy the following:*

- 1). $\text{ord}(e) = 0$ and $e\pi^u \in Q(L)$,
- 2). If $q(L) = 0$, then $u = v = \infty$,
- 3). If $\text{rank}_{\mathfrak{o}^2} q(L) = 1$, then $u < v = \infty$, and $s(L) = 0$,
- 4). If $\text{rank}_{\mathfrak{o}^2} q(L) = 2$, then $u < v < \infty$, $u + v \equiv 1 \pmod{2}$ and $v \leq s + 1$, where $s(L) = \pi^s \mathfrak{o}$.

When these conditions hold, then u and v are uniquely determined by L . Such representations of $q(L)$ will be called standard.

Proof. We may assume that $q(L) \neq 0$. Thus, $s(L) = \pi^s \mathfrak{o} \subseteq q(L) \mathfrak{o} = \pi^u \mathfrak{o}$. If $u < s$ ($s < \infty$), then there exists $x \in L$ such that $\text{ord}(Q(x)) = u$. If $u = s$, then there exist $x, y \in L$ such that $\text{ord}(\langle x, y \rangle) = u$ and one of the three elements $Q(x)$, $Q(y)$ and $Q(x + y)$ must have ordinal u . Thus, we can find $x \in L$ such that $Q(x) = e\pi^u$ with $\text{ord}(e) = 0$. By scaling L , we may assume that $e = 1$. Thus, $\pi^u \mathfrak{o}^2 \subseteq q(L)$. In case 3), we see easily that $s(L) = 0$ and $q(L) = \pi^u \mathfrak{o}^2$. In case 4), it is clear that v may be found to satisfy all the conditions. Since \mathfrak{o}^2 -modules of finite type in k have ranks at most 2, e , u and v may always be found to satisfy the conditions stated.

The uniqueness of u follows from the equation $\pi^u \mathfrak{o} = q(L) \mathfrak{o}$. Let $\text{rank}_{\mathfrak{o}^2} q(L) = 2$, then v is the minimal ordinal in the set $\{\text{ord}(a) \mid a \in q(L) \text{ and } \text{ord}(a) \equiv u + 1 \pmod{2}\}$. Thus, v is also uniquely determined, provided that it satisfies the conditions stated.

Q. E. D.

A subset $\{x_i \mid 1 \leq i \leq m\}$ of a free \mathfrak{o} -module M of finite rank is called pure if it generates a direct summand of rank m . Clearly, given a set $\{y_i \mid 1 \leq i \leq m\}$ of k -independent elements in kM , there is a pure subset $\{x_i \mid 1 \leq i \leq m\}$ in M such that for $p = 1, 2, \dots, m$, $\sum_{1 \leq i \leq p} kx_i = \sum_{1 \leq i \leq p} ky_i$.

Let M be a quadratic lattice and $-\infty < i \leq \infty$. Then M is called an i -modular lattice if $\{x\} \subseteq M$ is a pure subset implies that $\langle x, M \rangle = \pi^i \mathfrak{o}$. If M is i -modular, then, $i = \infty$ implies that $M = K(M)$; $i < \infty$ implies that $K(M) = 0$, $s(M) = \pi^i \mathfrak{o}$ and $\{x\} \subseteq M$ is a pure subset if and only if $\langle x, M \rangle = \pi^i \mathfrak{o}$. If $M = M_1 \oplus M_2$, then M is i -modular if and only if M_1 and M_2 are i -modular.

LEMMA 1.3. *Let L be a quadratic lattice and J be a sublattice of L . Then,*

- 1). *If J is i -modular, then J is an orthogonal summand of L if and only*

if, $J \subseteq L(i)$ and J is a direct summand of $L(\infty)$ when $i = \infty$.

2). If L is i -modular with $i < \infty$, then J is an orthogonal summand of L if and only if J is i -modular.

Proof. 1) follows easily as in [15; Proposition 1, p. 160]. 2) follows from 1) and the preceding remarks.

COROLLARY 1.4. Let M be an i -modular lattice, $i < \infty$, then

$$M \cong \bigoplus_j \begin{pmatrix} \pi^i \\ a_j & c_j \end{pmatrix}$$

and conversely.

Proof. By induction on $\dim M$ and Lemma 1.3.

2. Orthogonal decompositions and change of bases.

Definition 2.1. Let L be a quadratic lattice. By an easy induction argument, using Lemma 1.3, we obtain an orthogonal decomposition $L = \bigoplus_{1 \leq i \leq m} L_i \oplus \bigoplus_{1 \leq i \leq p} K_i$, where $\dim L_i = 2$, $\dim K_i = 1$, $0 \leq p \leq 2$, L_i is $s(i)$ -modular with $s(1) \leq \dots \leq s(m) < \infty$, and $K(L) = \bigoplus_{1 \leq i \leq p} K_i$. By Corollary 1.4, we have $L = \bigoplus_{1 \leq i \leq n} M_i \oplus K(L)$, where M_i is $t(i)$ -modular with $t(1) < \dots < t(n) < \infty$. These decompositions will be called complete and canonical respectively.

Let $L_i = \alpha x_i + \alpha y_i$ and $K_i = \alpha z_i$ in the complete decomposition. Then, the following transformations will be called elementary:

$$LT_0: \text{ a). } z'_i = z_{i+1}; z'_{i+1} = z_i,$$

$$\text{ b). } z'_i = z_i + az_{i+1}, \text{ ord}(a) \geq 0,$$

$$\text{ c). } z'_i = az_i, \text{ ord}(a) = 0,$$

$$\text{ d). } x'_m = x_m + az_j, \text{ ord}(a) \geq 0,$$

$$LT_1: \text{ a). } x'_i = y_i, y'_i = x_i,$$

$$\text{ b). } x'_i = x_i + ay_i, \text{ ord}(a) \geq 0,$$

$$\text{ c). } x'_i = ax_i, \text{ ord}(a) = 0,$$

$$LT_2: \text{ a). } x'_i = x_{i+1}, y'_i = y_{i+1}; x'_{i+1} = x_i, y'_{i+1} = y_i, \text{ provided that } s(i) = s(i+1) = \text{ord}(\langle x_i, y_i \rangle) = \text{ord}(\langle x_{i+1}, y_{i+1} \rangle),$$

$$\text{ b). } x'_i = x_i + ax_{i+1}, y'_i = y_i; x'_{i+1} = x_{i+1}, y'_{i+1} = ab_{i+1}b_i^{-1}y_i + y_{i+1}, \text{ where } \text{ord}(a) \geq 0, b_i = \langle x_i, y_i \rangle \text{ and } b_{i+1} = \langle x_{i+1}, y_{i+1} \rangle.$$

It is to be understood that in each of the nine types of transformations, all the remaining basis elements remain unchanged. Transformations obtained by interchanging the roles of x_i and y_i will be called elementary of the same type.

It is easy to see that the new basis elements in the given order will again give a complete decomposition of L .

LEMMA 2.2. *Let L be a quadratic lattice with a complete decomposition as in Definition 2.1. Let $\{x\} \subseteq L$ be a pure subset such that $\langle x, L \rangle = s(L)$. Then there is a sequence of elementary transformations which will lead from the given basis to a new basis such that x is the first basis element.*

Proof. If $m = 0$ or 1 , then the hypothesis that $\langle x, L \rangle = s(L)$ together with transformations of type LT_0 and LT_1 easily give the desired result. Thus, we proceed by induction on m and let $m > 1$. By using transformations of type LT_1, LT_2 a) and the hypothesis that $\langle x, L \rangle = s(L)$, it is easy to see that we may assume that $x = x_1 + \sum_{2 \leq i \leq m} \alpha_i x_i + \sum_{1 \leq i \leq p} \gamma_i z_i$, where $\alpha_i, \gamma_i \in \mathfrak{o}$. Using transformations of type LT_2 b), we may first assume that $\alpha_2 = 0$, then $\alpha_2 = 1$. Thus, by induction, we may assume that $x = x_1 + x_2$ and another application of LT_2 b) gives the desired result. Q. E. D.

THEOREM 2.3. *Let x_i, y_i, z_j and x'_i, y'_i, z'_j be the bases elements of two complete decompositions of a quadratic lattice L . Then, there is a sequence of elementary lattice transformations which lead from the first to the second set of basis elements.*

Proof. If $m = 0$, then our assertion reduces to the classical result on changing basis in a free \mathfrak{o} -module of finite type. If $m > 0$, then by Lemma 2.2, we may let $y'_1 = y_1$. It is clear that $\text{ord}(\langle x_1, y_1 \rangle) = \text{ord}(\langle x'_1, y'_1 \rangle) < \infty$. Hence, we may assume that x'_1 has the form given in the proof of Lemma 2.2. We now observe that the transformations carried out in Lemma 2.2 do not alter y_1 . Hence, we may assume that $x'_1 = x_1$. Now by induction, we get the desired result. Q. E. D.

3. Hyperbolic lattices.

Definition 3.1. A quadratic space U is called hyperbolic if it is the orthogonal sum of hyperbolic planes, $\left(\begin{smallmatrix} 1 \\ 0 & 0 \end{smallmatrix} \right)_k$. If $i < \infty$, then a quadratic lattice L is called i -hyperbolic, when it is the orthogonal sum of i -hyperbolic planes, $\left(\begin{smallmatrix} \pi^i \\ 0 & 0 \end{smallmatrix} \right)$. If M is any lattice, H is an i -hyperbolic plane, then $q_i(M \oplus H)$

$= q_i(M)$; in addition, if M is i -modular, then $Q(M \oplus H) = q(M \oplus H) = q(M) = q_i(M)$.

LEMMA 3.2. *A quadratic lattice L is i -hyperbolic if and only if, L is i -modular, kL is a hyperbolic space and $q(L) = \pi^i \mathfrak{o}$.*

Proof. The necessity is clear. Conversely, suppose that the three conditions hold. By Lemma 2.2, we may find $x, y \in L$ such that $M = \mathfrak{o}x + \mathfrak{o}y = \begin{pmatrix} b \\ 0 \ c \end{pmatrix}$, where $\text{ord}(b) = i$, $\text{ord}(c) \geq i$. Replacing y by $\pi^i b^{-1}(cb^{-1}x + y) \in M$, we see that M is i -hyperbolic. By Lemma 1.3, $L = M \oplus M'$. By the cancellation theorem, [1; Satz 6, p. 157], kM' is hyperbolic. Thus, by Lemma 1.3 and Definition 3.1, we see that M' is i -modular and $q(M') = \pi^i \mathfrak{o}$, (provided that $M' \neq 0$.) Thus, by induction on $\dim L$, we get the desired result.

Q. E. D.

THEOREM 3.3. *Let $L = H_j \oplus L_j$, $j = 1, 2$, be two decompositions of a lattice L such that H_j is an i -hyperbolic plane for $j = 1, 2$. Then, $L_1 \cong L_2$.*

Proof. By scaling L , we may assume that $H_j = \mathfrak{o}x_j + \mathfrak{o}y_j = \begin{pmatrix} 1 \\ 0 \ 0 \end{pmatrix}$. Thus, $H_1 + H_2 \subseteq L(0)$ and $\langle H_1, H_2 \rangle \subseteq \mathfrak{o}$.

Case 1. $\langle H_1, H_2 \rangle = \mathfrak{o}$. Without loss of generality, we may assume that $\langle x_1, y_2 \rangle = 1$. Thus, $H' = \mathfrak{o}x_1 + \mathfrak{o}y_2 = \begin{pmatrix} 1 \\ 0 \ 0 \end{pmatrix} \subseteq L(0)$. By Lemma 1.3, $L = H' \oplus L'$. Thus, by symmetry, we may assume that $x_1 = x_2$ and $y_1 \neq y_2$. Let z_j , $1 \leq j \leq m$, be a free \mathfrak{o} -basis for L_1 . Then, $y_2 = ax_1 + by_1 + z$, where $z \in L_1$ and $1 = \langle x_2, y_2 \rangle = \langle x_1, y_1 \rangle = b$. Since $y_2 \in H_2 \subseteq L(0)$, we have $\langle y_2, z_j \rangle \in \mathfrak{o}$ and $w_j = \langle y_2, z_j \rangle x_1 + z_j \in L$ for $1 \leq j \leq m$. It is now trivial to see that $L = (\mathfrak{o}x_2 + \mathfrak{o}y_2) \oplus \sum_{1 \leq j \leq m} \mathfrak{o}w_j$. Thus, $L_2 = \sum_{1 \leq j \leq m} \mathfrak{o}w_j$. By computation, we see that the map $z_j \rightarrow w_j$ leads to an isometry between L_1 and L_2 .

Case 2. $\langle H_1, H_2 \rangle \subseteq \pi \mathfrak{o}$. Let $\{z_1 + z_2\}$ be a pure subset of $H = H_1 + H_2$, where $z_j \in H_j$. Without loss of generality, let $\{z_1\}$ be a pure subset of H_1 . From $\langle H_1, H_2 \rangle \subseteq \pi \mathfrak{o}$, $H \subseteq L(0)$, we see that $\mathfrak{o} = \langle z_1, H_1 \rangle = \langle z_1 + z_2, H_1 \rangle \subseteq \langle z_1 + z_2, H \rangle \subseteq \mathfrak{o}$. Thus, H is 0-modular. By Lemma 1.3, $L = H \oplus M$, $H = H_j \oplus M_j$; thus, we may assume that $L = H = H_1 + H_2 = H_j \oplus L_j$. Hence, $x_2 = w' + z'$ and $y_2 = w'' + z''$, where $w', w'' \in H_1$ and $z', z'' \in L_1$. Since $L = H_1 + H_2$, we see that $L_1 = \mathfrak{o}z' + \mathfrak{o}z''$. From $\langle H_1, H_2 \rangle \subseteq \pi \mathfrak{o}$, we see that $w', w'' \in \pi H_1$. From $Q(x_2) = Q(y_2) = 0$ we get $Q(z') = Q(w')$ and $Q(z'') = Q(w'')$. Thus, $Q(z'), Q(z'') \in q(\pi H_1) = \pi^2 \mathfrak{o}$. We have shown that

$H = L$ is 0-modular, thus, L_1 is 0-modular by Lemma 1.3, hence, $\text{ord}(\langle z', z'' \rangle) = 0$. By Lemma 1.1, $\Delta(kL_1) = 0$, thus, kL_1 is a hyperbolic space, cf. [1; Zusatz 2, p. 153]. It is also clear that $q(L_1) = \mathfrak{o}$. Hence, by Lemma 3.2, L is 0-hyperbolic. By symmetry, L_2 is also 0-hyperbolic. Hence, $L_1 \cong L_2$.
Q. E. D.

4. Isometry of modular lattices.

LEMMA 4.1. Let $B_j = \mathfrak{o}x_j + \mathfrak{o}y_j = \begin{pmatrix} \pi^i & \\ a & c_j \end{pmatrix}$ and $q(B_j) \subseteq a\mathfrak{o}$, $j = 1, 2$. Then $B_1 \cong B_2$ if and only if $\Delta(kB_1) = \Delta(kB_2)$.

Proof. The necessity is obvious. Conversely, by scaling, we may assume that $i = 0$. Since $\mathfrak{o} \subseteq q(B_j) \subseteq a\mathfrak{o}$, we have $\text{ord}(a) \leq 0$ and $\text{ord}(c_1 + c_2) \geq \text{ord}(a)$. Let $d \in k$ such that $a(c_1 + c_2) = \mathfrak{P}(da)$. Then $c_2 = c_1 + d^2a + d$ and $\text{ord}(d) + \text{ord}(1 + da) = \text{ord}(c_1 + c_2) \geq \text{ord}(a)$. If $\text{ord}(d) < 0$, then $\text{ord}(d) + \text{ord}(1 + da) < \text{ord}(1 + da) = \text{ord}(da) < \text{ord}(a)$, a contradiction. Thus, $\text{ord}(d) \geq 0$, $dx_1 + y_1 \in B_1$ and we have $Q(dx_1 + y_1) = d^2a + d + c_1 = c_2$. Hence $B_1 \cong B_2$.
Q. E. D.

COROLLARY 4.2. Let $B \cong \begin{pmatrix} 1 & \\ \pi^u & c \end{pmatrix}$ be a quadratic lattice such that $q(B) \subseteq \pi^u\mathfrak{o}$. Let $\pi^uc = \pi^{-1}b^2 + D + \mathfrak{P}(d)$ as in Lemma 1.1, 5). Then $B \cong \begin{pmatrix} 1 & \\ \pi^u & \pi^{-u}(\pi^{-1}b^2 + D) \end{pmatrix}$.

Proof. By Lemma 4.1 and Lemma 1.1, 5).

LEMMA 4.3. Let M be an i -modular lattice, $i < \infty$. Then, $\pi^{-i} \circ M \cong \begin{pmatrix} 1 & \\ 1 & D \end{pmatrix} \oplus H$, where $D = 0$ or λ and H is 0 or 0-hyperbolic, if and only if, $q(M) = \pi^i\mathfrak{o}$.

Proof. The necessity is obvious. By scaling, we may take $i = 0$. From Corollary 1.4 and that $q(M) = \mathfrak{o}$, we have $M \cong \bigoplus_{1 \leq j \leq m} \begin{pmatrix} 1 & \\ a_j & c_j \end{pmatrix}$, where $a_j, c_j \in \mathfrak{o}$, $1 \leq j \leq m$. If $a_j c_j \in \pi\mathfrak{o}$, then by Lemma 1.1 and Lemma 3.2, $\begin{pmatrix} 1 & \\ a_j & c_j \end{pmatrix} \cong \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$. If $\text{ord}(a_j) = \text{ord}(c_j) = 0$, then an easy application of Lemma 4.1 and Lemma 1.1 shows that we may assume $c_j \neq 0 \in \Omega$. By symmetry, we may assume that $a_j \neq 0 \in \Omega$. Using a transformation of type $LT_1 c$, we may assume that $a_j = 1$. By Corollary 4.2, we may replace c_j by 0 or λ . As we have just seen $c_j = 0$ leads to a 0-hyperbolic plane. By using a transformation of type $LT_2 b$, $\begin{pmatrix} 1 & \\ 1 & \lambda \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. From this we get our assertion immediately.
Q. E. D.

LEMMA 4.4. *Let M be an i -modular lattice, $i < \infty$. If $\dim M \geq 6$, then M contains an i -hyperbolic plane.*

Proof. We may assume that $\dim M = 6$ and $i = 0$. Let $q(M) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ be a standard representation of the norm group. Since $\text{ord}(e) = 0$, by scaling, we may assume that $e = 1$ and $\pi^u = Q(x)$ for some $x \in M$, $u < v \leq 1$, and $u + v \equiv 1 \pmod{2}$. $\{x\}$ is a pure subset of M ; hence, we can find $y \in M$ such that $\text{ord}(\langle x, y \rangle) = 0$. From Lemma 1.3, $M = (\mathfrak{o}x + \mathfrak{o}y) \oplus N$. By the choice of x and using transformations of type $LT_0 d)$ on $\mathfrak{o}x \oplus N$, we may assume that $\mathfrak{o}x \oplus N = (\pi^u) \oplus \begin{pmatrix} 1 & \\ a & b \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ c & d \end{pmatrix}$, where $a, b, c, d \in \pi^v \mathfrak{o}^2$, and $\text{ord}(a) \leq \text{ord}(c) \leq \text{ord}(d)$. By transformations of type $LT_0 d)$ on $(a) \oplus \begin{pmatrix} 1 & \\ c & d \end{pmatrix}$, we see that M contains a 0-hyperbolic plane.

Q. E. D.

LEMMA 4.5. *Let M be an i -module lattice, $i < \infty$, and $\dim M \geq 4$. Let $q(M) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ be standard and $e\pi^u = Q(x)$. Then,*

1). $q(M) = Q(M)$.

2). $e^{-1}\pi^{-i} \circ M \cong \begin{pmatrix} 1 & \\ a_1 & c_1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ a_2 & c_2 \end{pmatrix} \oplus H$, where H is 0-hyperbolic or zero and the following hold:

Case 1. $v \leq i$. $a_1 = \pi^{u-i}$, $c_1 = \pi^{-u+i}D_1 + \pi^{v+i}b^2$, $a_2 = \pi^{v-i}$, and $c_2 = \pi^{-v+i}D_2$, where $D_1, D_2 \in \{0, \lambda\}$ and $\text{ord}(b) \geq 0$.

Case 2. $v = i + 1$ and $u < i$. a_1 and c_1 as in Case 1, $a_2 = c_2 = 0$.

Case 3. $v = i + 1$ and $u = i$. $a_1 = 1$, $c_1 = D_1$, $a_2 = c_2 = 0$.

Proof. 1) follows from 2).

2). By scaling, we may assume that $i = 0$ and $e = 1$. By Lemma 4.4 and Definition 3.1, we may assume that $\dim M = 4$. $\{x\}$ is clearly a pure subset of M , thus, by Lemma 1.3 and Lemma 2.2, we have $M \cong \begin{pmatrix} 1 & \\ \pi^u & f \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ g & h \end{pmatrix}$. Using transformations of type $LT_2 b)$, we may assume that $g, h \in \pi^v \mathfrak{o}^2$.

Case 3 now follows easily from Lemma 4.3.

Case 2. $v = 1$ and $u < 0$. By Lemma 3.2 (also the proof of Lemma 4.3), we may assume that $g = h = 0$. By Corollary 4.2, we may assume that $f = \pi^{-u}(\pi^{-1}b_1^2 + D)$, where $b_1 \in \Omega[\pi^{-1}]$ and $D = 0$ or λ . If $b_1 \neq 0$, then $\text{ord}(f) + u \equiv 1 \pmod{2}$, hence f has the desired form as c_1 .

Case 1. Let $v = 0$. By Lemma 4.3, we may assume that $g = 1$ and $h = 0$ or λ . Repeating Case 2, we get the desired result.

Let $v < 0$ and $\text{ord}(g) = v$. Using transformations of type $LT_1 c)$, we may assume that $g = \pi^v$. As in Case 2, we may replace h by $\pi^{-v}(\pi^{-1}b_1^2 + D)$. By choice, $\pi^{-v-1}b_1^2 \in \pi^u \mathfrak{o}^2$. Thus, using a transformation of type $LT_2 b)$, we may assume that $b_1 = 0$. One more application of Corollary 4.2 on $\begin{pmatrix} 1 \\ \pi^u & f \end{pmatrix}$ gives the desired form.

Finally, let $v < 0$ and $v < \text{ord}(g) \leq \text{ord}(h)$. Let $M' = \alpha x + \beta y = \begin{pmatrix} 1 \\ \pi^u & f \end{pmatrix} \subseteq M$. Since $v < 0$, we have $q(M') = q(M)$. Thus, for some $\alpha, \beta \in \mathfrak{o}$, $\text{ord}(Q(\alpha x + \beta y)) = v$. Clearly, $\{\alpha x + \beta y\}$ is a pure subset of M' . If $\text{ord}(\beta) > 0$, then $\text{ord}(\alpha) = 0$; hence,

$$\text{ord}(Q(\alpha x + \beta y)) = \text{ord}(\alpha^2 \pi^u + \alpha \beta + \beta^2 f) = u < v < 0,$$

a contradiction. Thus, $\text{ord}(\beta) = 0$. Replacing y by $\beta^{-1}(\alpha x + \beta y)$, we may assume that $\text{ord}(f) = v$. Using a transformation of type $LT_2 b)$, we have

$M \cong \begin{pmatrix} 1 \\ \pi^u + g & f \end{pmatrix} \oplus \begin{pmatrix} 1 \\ g & h + f \end{pmatrix}$, where $h + f = \pi^v f_1^2$, $\text{ord}(f_1) = 0$. Using transformations of type LT_1 , $M \cong \begin{pmatrix} 1 \\ \pi^u + g & f \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^v & h' \end{pmatrix}$, where $g, h' \in \pi^v \mathfrak{o}^2$.

Using a transformation of type $LT_2 b)$, we have $M \cong \begin{pmatrix} 1 \\ \pi^u & f' \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^v & h'' \end{pmatrix}$. Using one more transformation of type $LT_2 b)$, we may assume that $h'' \in \pi^v \mathfrak{o}^2$. Repeating the argument of the preceding paragraph, we get the desired result.

Q. E. D.

THEOREM 4.6. *Let M_1 and M_2 be i -modular lattices. Then, $M_1 \cong M_2$ if and only if $kM_1 \cong kM_2$ and $q(M_1) = q(M_2)$.*

Proof. The necessity is obvious. If $i = \infty$, then $q(M_j) = Q(M_j)$, $j = 1, 2$. It is easy to see that either $M_j \cong (e\pi^u)$ or $M_j \cong (e\pi^u) \oplus (e\pi^v)$, where $q(M_j) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ is a standard representation.

Thus, let $i < \infty$, $kM_1 \cong kM_2$, $q(M_1) = q(M_2)$ and M_1, M_2 be i -modular. By Definition 3.1 and the cancellation theorem, cf. [1; Zusatz 2, p. 153], these conditions are "invariant" under addition or deletion of i -hyperbolic planes. Thus, by Theorem 3.3 and Lemma 4.4, we may assume that $\dim M_1 = \dim M_2 = 4$. By scaling, we may assume that $i = 0$ and $q(M_j) = \pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2$, $u < v \leq 1$, $u + v \equiv 1 \pmod{2}$, $j = 1, 2$. If $v = 1$, then by Lemma 4.5, 2) Case 3 and a simple computation of the Arf invariant of kM_1 and kM_2 , we see that $M_1 \cong M_2$. Thus, we may assume that $v \leq 0$. By Corollary 4.2 and Lemma 4.5, 2),

$$M_j \cong \left(\begin{smallmatrix} 1 \\ \pi^u & \pi^{-u}(\pi^{-1}b_j^2 + D'_j) \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 1 \\ \pi^v & \pi^{-v}D_j \end{smallmatrix} \right),$$

where $b_j \in \Omega[\pi^{-1}]$, $D'_j, D_j \in \{0, \lambda\}$, $j = 1, 2$.

If u is even, then v is odd. Thus, $[C(kM_j)] = [\pi^v, \pi^{-u}D_j] = [\pi^{-1}, \pi D_j]$ for $j = 1, 2$. Hence, from $kM_1 \cong kM_2$, we get $D_1 = D_2$. From Lemma 4.1 and that $\Delta(kM_1) = \Delta(kM_2)$, we see that $M_1 \cong M_2$.

If u is odd, then v is even. Thus, $[C(kM_j)] = [\pi^u, \pi^{-u}D'_j] = [\pi^{-1}, \pi D'_j]$ for $j = 1, 2$. As above, $D'_1 = D'_2$. From $kM_1 \cong kM_2$, we get $0 = \Delta(kM_1) + \Delta(kM_2) = \pi^{-1}(b_1 + b_2)^2 + (D_1 + D_2) + \mathfrak{P}k$. By Lemma 1.1, $b_1 = b_2$ and $D_1 = D_2$. Thus, $M_1 \cong M_2$. Q. E. D.

5. Isometry of lattices.

Definition 5.1. Let $L = \bigoplus_{0 \leq i \leq m} L_i \oplus K(L)$ be a canonical decomposition, where L_j is $s(j)$ -modular, $s(0) < \cdots < s(m) < \infty$. It is easy to see that for i such that $s(n) \leq i < s(n+1)$, we have

$$L(i) = \bigoplus_{0 \leq j \leq n} \pi^{i-s(j)} L_j \oplus \bigoplus_{n+1 \leq j \leq m} L_j \oplus K(L).$$

Let $q_i(L) = e_i(\pi^{u(i)}\mathfrak{o}^2 + \pi^{v(i)}\mathfrak{o}^2)$ be standard representations of the i -th norm group for $i \leq \infty$, i.e. they are standard representations of $q(L(i) \oplus H_i)$ or of $q(K(L))$, where H_i is an i -hyperbolic plane, when $i < \infty$, we do not assert that $e_i \pi^{u(i)} \in Q(L(i))$. Clearly, we have $q_i(L) \supseteq q_{i+1}(L) \supseteq \cdots \supseteq q_\infty(L)$ and $\pi^2 q_i(L) \supseteq q_{i+1}(L)$. From these we deduce easily the following possibilities:

- 1). $u(j+1) = u(j)$. Then, $v(j+1) = v(j)$ or $v(j) + 2$ and $= v(j)$ when $v(j) = j+1$, and $q_j(L) = e_{j+1}(\pi^{u(j)}\mathfrak{o}^2 + \pi^{v(j)}\mathfrak{o}^2)$.
- 2). $u(j+1) = u(j) + 1$. Then, $v(j+1) = v(j) + 1 = u(j+1) + 1$ and $q_j(L) = \pi^{u(j)}\mathfrak{o}$, $q_{j+1}(L) = \pi^{u(j)+1}\mathfrak{o} = \pi^{u(j+1)}\mathfrak{o}$.
- 3). $u(j+1) = u(j) + 2$. Then, $q_{j+1}(L) = e_j(\pi^{u(j+1)}\mathfrak{o}^2 + \pi^{v(j)}\mathfrak{o}^2)$.

The given canonical decomposition is said to be saturated in the $s(j)$ -th component, if $q_{s(j)}(L) = Q(L_j) = q(L(s(j)))$. It is said to be saturated, if it is saturated in each $s(i)$ -th component, $0 \leq i \leq m$.

If $L = K(L)$, then let $F(L) = T(L) = \infty$ and $D(L) = (\infty)$. If $L \neq K(L)$, then let $F(L) = s(0)$, $T(L) = s(m) + n$, where n is the largest finite integer among 0 , $u(\infty) - u(s(m))$ and $u(\infty) - u(s(m)) + v(\infty) - v(s(m))$, and $D(L) = (r_{F(L)}, \cdots, r_{T(L)})$, where $r_i = 0$ unless $i = s(j)$, then $r_i = \dim L_j$. $(F(L), T(L))$ and $D(L)$ are called the type and the type dimension of L respectively. L is called a normal lattice when $r_i \geq 8$ for

$F(L) \leq i \leq T(L)$. Thus, either $L = K(L)$ or when $L \neq K(L)$, then $u(\infty) = u(T)$, $v(\infty) = v(T)$ or ∞ , and $s(j) = F + j$, $0 \leq j \leq T - F$. Clearly, $F(L)$, $T(L)$ and $D(L)$ are invariants of L .

LEMMA 5.2. Let $L = \bigoplus_{0 \leq i \leq m} L_i \oplus K(L)$ be a canonical decomposition of the lattice L of type (F, T) . Let $H = 0$ when $F = T = \infty$ and $H = \bigoplus_{F \leq j \leq T} H_j$, where H_j is j -hyperbolic of dimension 8 otherwise. Then,

- 1). $q_i(L \oplus H) = q_i(L)$ for $i = F, F + 1, \dots, T, \infty$.
- 2). $L \oplus H$ is a normal lattice of type (F, T) .

Proof. If $L = K(L)$, then there is nothing to prove. Thus, we assume that $(F, T) \neq (\infty, \infty)$.

1). follows from the definition of $q_i(L)$.

2). If $K(L) = 0$, then the normality follows easily from 1). Thus, let $K(L) \neq 0$ and n be chosen such that $T(L) = s(m) + n = T$.

From Definition 5.1 and 1), $q_T(L \oplus H) = q(\pi^n L(s(m))) + q_\infty(L) + \pi^T \mathfrak{o}$. Since $L_m \neq 0$ is $s(m)$ -modular, let

$$q(L(s(m))) = q_{s(m)}(L) = e_{s(m)}(\pi^{u(s(m))} \mathfrak{o}^2 + \pi^{v(s(m))} \mathfrak{o}^2)$$

be a standard representation. By the choice of n , Definition 5.1 and that $K(L) \neq 0$, we see that $T = s(m) + n \geq u(\infty) + [s(m) - u(s(m))] \geq u(\infty)$ and $2n + u(s(m)) \geq u(\infty) + [u(\infty) - u(s(m))] \geq u(\infty)$. Thus, $q_T(L \oplus H) \mathfrak{o} = q_\infty(L) \mathfrak{o}$, or $u(T) = u(\infty)$. If $v(\infty) = \infty$, then by 1) and definition, $L \oplus H$ is normal. Let $v(\infty) < \infty$. If $u(\infty) \geq v(s(m))$, then $n \geq v(\infty) - u(s(m))$, thus, we get $T \geq v(\infty)$ and $2n + u(s(m)) \geq v(\infty)$. From $\pi^{v(\infty)} \mathfrak{o} \subseteq q_\infty(L)$ we get $q_T(L \oplus H) = q_\infty(L)$, thus $L \oplus H$ is normal by 1). Finally, if $u(\infty) < v(s(m))$, then $u(s(m)) = u(\infty)$ and $v(s(m)) = v(\infty) \pmod{2}$. The argument used in case $v(\infty) = \infty$ shows that we may assume $u(s(m)) = u(\infty)$. Hence, we may assume that $e_{s(m)} = e_\infty$. It is now trivial to see that $q_T(L \oplus H) = q_\infty(L)$, thus, $L \oplus H$ is normal. Q. E. D.

LEMMA 5.3. A normal lattice L admits a saturated canonical decomposition.

Proof. Let $L = \bigoplus_{0 \leq i \leq m} L_i \oplus K(L)$ be a canonical decomposition. If $L = K(L)$, then the decomposition is already saturated. Thus, we may assume that $L \neq K(L)$. By scaling, we may assume that L has type $(0, m)$. If $m = 0$ and $K(L) = 0$, then Lemma 4.5 furnishes the desired result. If

$m = 0$ and $K(L) \neq 0$, then by Lemma 4.4 and Lemma 4.5, we may assume that $L = L'_0 \oplus H \oplus K(L)$, where H is 0-hyperbolic of dimension 4 and $Q(L'_0) = q(L_0)$. Let $q(K(L)) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ be a standard representation. Then by using transformations of type LT_0 on $H \oplus K(L)$, we have $H \oplus K(L) \cong H' \oplus K(L)$, where $H' \cong \begin{pmatrix} 1 \\ e\pi^u & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ e\pi^v & 0 \end{pmatrix}$, since $Q(K(L)) = q(K(L))$. Thus, $L = L' \oplus K(L)$, where $L' \cong L'_0 \oplus H'$. It is clear that $q(L') = q(L) = q_0(L)$ and $L \cong L' \oplus K(L)$ is a saturated canonical decomposition. Hence, we may assume that $m > 0$.

By Lemma 4.4 and Lemma 4.5, let $L_i = L'_i \oplus H_i$, where H_i is i -hyperbolic of dimension 4 and $q(L_i) = Q(L'_i)$, $i = 0, 1$. By Theorem 4.6, $L_0 \cong L'_0 \oplus H'_0$, where $H'_0 \cong \begin{pmatrix} 1 \\ a & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ c & 0 \end{pmatrix}$, $a, c \in q(L_0)$. By using transformations of type LT_2 , we see that $H'_0 \oplus H_1 \cong H'_0 \oplus H'_1$, where

$$H'_1 \cong \begin{pmatrix} 1 \\ \pi^2 a & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^2 c & 0 \end{pmatrix}.$$

Thus, from Definition 5.1, we see that we may assume that $L = L_0 \oplus L'$ with $q_i(L') = q_i(L)$, $i = 1, \dots, m, \infty$. Hence, L' is a normal lattice of type $(1, m)$. By induction, we may then assume that the chosen canonical decomposition is already saturated in the i -th component, $1 \leq i \leq m$. Now, by the same argument, $L_1 \cong L'_1 \oplus H''_1$, where $H''_1 \cong \begin{pmatrix} \pi \\ b & 0 \end{pmatrix} \oplus \begin{pmatrix} \pi \\ d & 0 \end{pmatrix}$ and $H_0 \oplus H''_1 \cong H''_0 \oplus H''_1$, where $H''_0 \cong \begin{pmatrix} 1 \\ b & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ d & 0 \end{pmatrix}$ and $b, d \in q_1(L)$ are arbitrary. Thus, we can saturate the 0-th component without destroying the saturation of the remaining components. Hence our assertion holds.

Q. E. D.

LEMMA 5.4. Let $L = \bigoplus_{0 \leq i \leq m} L_i \oplus K(L) = \bigoplus_{0 \leq i \leq m} M_i \oplus K(L)$ be two canonical decompositions of the normal lattice L of type (F, T) . Let $q_i(L) = e_i(\pi^{u(i)} \mathfrak{o}^2 + \pi^{v(i)} \mathfrak{o}^2)$, $i = F, F+1, \dots, T, \infty$, be standard representations, cf. Definition 5.1. Then,

$$1). \quad \text{Ord} \left(\sum_{0 \leq j \leq i-1} (\Delta(kL_j) + \Delta(kM_j)) \right) \\ \geq u(F+i) + v(F+i) - (2F+2i+1),$$

2). If $v(F+i) = F+i+1$, then

$$\bigoplus_{0 \leq j \leq i-1} kL_j \oplus (e_{F+i} \pi^{u(F+i)})_k \cong \bigoplus_{0 \leq j \leq i-1} kM_j \oplus (e_{F+i} \pi^{u(F+i)})_k,$$

where $1 \leq i \leq T - F = m$. ($\infty - \infty$ is taken to mean 0.)

Proof. Clearly, the assertions are “invariant” under scalings of L . By Theorem 2.3, we may assume that the given decompositions are related by an elementary lattice transformation. By inspection, it is clear that we need only consider transformations of type $LT_2 b$). By scaling L and using transformations of type LT_1 , we may assume that the given decompositions of L differ as follows:

$$\begin{pmatrix} 1 & \\ a & b \end{pmatrix} \oplus \begin{pmatrix} \pi & \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \\ a + cg^2 & b \end{pmatrix} \oplus \begin{pmatrix} \pi & \\ c & g^2\pi^2b + d \end{pmatrix}, \text{ ord}(g) \geq 0;$$

it is to be understood that the given canonical decompositions are, in reality, complete decompositions, and the lattices above are orthogonal summands of the respective complete decompositions.

1). It suffices to show that $\text{Ord}(bcg^2 + \mathfrak{P}k) \geq u(1) + v(1) - 3$. From Definition 5.1, we see that $bcg^2 \in q_0(L)q_1(L) = \pi^{u(0)+u(1)}\mathfrak{o}^2 + \pi^{u(0)+v(1)}\mathfrak{o}^2 + \pi^{v(0)+u(1)}\mathfrak{o}^2 + \pi^{v(0)+v(1)}\mathfrak{o}^2$. If $u(1) = u(0) + 1$, then by 2) of Definition 5.1, $\text{ord}(bcg^2) \geq u(0) + u(1) \geq u(1) + v(1) - 3$. Thus, we may assume that $u(1) - u(0), v(1) - v(0) \in \{0, 2\}$. Hence, $bcg^2 = \pi^{u(0)+u(1)}f^2 + h$, where $\text{ord}(f) \geq 0$ and $\text{ord}(h) \geq u(1) + v(1) - 3$. Let $n = \frac{1}{2}(u(0) + u(1)) \geq u(1) - 1$, $b' = bcg^2 + \mathfrak{P}(\pi^n f)$. Since $v(1) \leq 2$, we have $n \geq u(1) + v(1) - 3$ and $\text{Ord}(bcg^2 + \mathfrak{P}k) \geq \text{ord}(b') = \text{ord}(\pi^n f + h) \geq u(1) + v(1) - 3$.

2). By scaling, we may assume that $e_1 = 1$. By the cancellation theorem [1; Folgerung, p. 160], it suffices to show that,

$$\begin{pmatrix} 1 & \\ a & b \end{pmatrix}_k \oplus (\pi^{u(1)})_k \cong \begin{pmatrix} 1 & \\ a + cg^2 & b \end{pmatrix}_k \oplus (\pi^{u(1)})_k$$

when $v(1) = 2$. From [1; Satz 8, p. 161], it is enough to show that $[\pi^{-u(1)}cg^2, \pi^{u(1)}b] = 1$. From Definition 5.1 and that $v(1) = 2$, we see that $v(0) = 0$ or 1 (note that $v(0) \leq 1$ by definition).

If $v(0) = 1$, then $u(1) = 1$ and $b \in \mathfrak{o}$, $cg^2 \in \pi\mathfrak{o}$. Thus, $[\pi^{-1}cg^2, \pi b] = [C(V)]$, where $V = \begin{pmatrix} 1 & \\ \pi^{-1}cg^2 & \pi b \end{pmatrix}_k$. By Lemma 1.1, $\Delta(V) = 0$, thus V is a hyperbolic plane [1; Zusatz 2, p. 153], hence $[C(V)] = 1$ as observed in Section 1.

If $v(0) = 0$, then $u(1) = u(0)$ or $u(0) + 2$. If $u(1) = u(0)$, then we may take $e_0 = 1$, by using Definition 5.1, 1). If $u(1) = u(0) + 2$, then by Definition 5.1, we have $\pi^2q_0(L) = e_0q_1(L) \subseteq q_1(L)$; now, from $\text{ord}(e_0) = 0$, we have $q_1(L) = e_0^2q_1(L) \subseteq e_0q_1(L)$; hence, $q_0(L) = \pi^2q_1(L)$ and we may take $e_0 = 1$. Thus, in all cases, $b \in \pi^{u(0)}\mathfrak{o}^2 + \pi^{v(0)}\mathfrak{o}^2$ and $cg^2 \in \pi^{u(1)}\mathfrak{o}^2 + \pi^{v(1)}\mathfrak{o}^2$, where $v(1) = 2$ and $v(0) = 0$. By choice, $u(0)$ and $u(1)$ must both be odd.

Thus, $[\pi^{-u(1)}cg^2, \pi^{u(1)}b] = [\pi^{-u(1)+v(1)}c_2^2, \pi^{u(1)+v(0)}b_2^2]$, where $b_2, c_2 \in \mathfrak{o}$. Using the same argument of the preceding paragraph, we get the desired result.

Q. E. D.

THEOREM 5.5. *Let $L = \bigoplus_{0 \leq i \leq m} L_i \oplus K(L)$ and $M = \bigoplus_{0 \leq i \leq n} M_i \oplus K(M)$ be canonical decompositions. Then $L \cong M$ if and only if,*

$$1). \quad kL \cong kM.$$

2). L and M have the same type (F, T) and $D(L) = D(M)$. In particular, $m = n \leq T - F$ when $(F, T) \neq (\infty, \infty)$.

3). $q_i(L) = q_i(M) = e_i(\pi^{u(i)}\mathfrak{o}^2 + \pi^{v(i)}\mathfrak{o}^2)$, $i = F, \dots, T, \infty$, where the representations are standard, cf. Definition 5.1.

$$4). \quad \text{Ord} \left(\sum_{0 \leq j \leq p(i)} (\Delta(kL_j) + \Delta(kM_j)) \right) \\ \geq u(F+i) + v(F+i) - (2F + 2i + 1),$$

where $p(i)$ satisfies $s(p(i)) < F+i \leq s(p(i)+1)$, L_j is $s(j)$ -modular and $1 \leq i \leq T - F$.

5). If $v(F+i) = F+i+1$, then

$$\bigoplus_{0 \leq j \leq p(i)} kL_j \oplus (e_{F+i}\pi^{u(F+i)})_k \cong \bigoplus_{0 \leq j \leq p(i)} kM_j \oplus (e_{F+i}\pi^{u(F+i)})_k,$$

where $p(i)$ is as in 4), and $1 \leq i \leq T - F$.

Proof. Using Theorem 3.3, we see that our assertion is invariant under simultaneous addition of i -hyperbolic planes to L and M , provided that $F \leq i \leq T$. Thus, by Lemma 5.2, we may assume that L and M are both normal lattices.

Necessity. By Lemma 5.4.

Sufficiency. $A_1)$. By the necessity above and Lemma 5.3, we may assume that the given decompositions are saturated.

$A_2)$. By 1) and 3), we may assume that

$$K(L) \cong K(M) \cong (e_{\infty}\pi^{u(\infty)}) \oplus (e_{\infty}\pi^{v(\infty)}),$$

where (0) is understood to be the zero lattice. In particular, we may assume that $(F, T) \neq (\infty, \infty)$.

$A_3)$. By Lemma 4.4, we may assume that $L_i = L'_i \oplus H_i$, where $\dim L'_i = 4$ and H_i is i -hyperbolic. Using Lemma 4.5, we may assume that L'_i is

presented in the form stated in 2) of Lemma 4.5. By A_1), the quantities e , u and v of Lemma 4.5 correspond to e_i , $u(i)$ and $v(i)$ respectively. From Theorem 4.6, we see that $L_i \cong L'_i \oplus H'_i \oplus H''_i$, where

$$H'_i \cong \begin{pmatrix} \pi^i & \\ e_i \pi^{u(i)} & 0 \end{pmatrix} \left(\oplus_{e_i \pi^{v(i)}} \begin{pmatrix} \pi^i & \\ & 0 \end{pmatrix} \right)$$

and H''_i is either zero or i -hyperbolic. Similarly for M_i .

A_4). We may always consider $a \circ L$ and $a \circ M$ in place of L and M , since the given conditions imply a similar set of conditions for the scaled lattices and the latter will also be normal and satisfy A_1), A_2) and A_3). In particular, we may assume that $(F, T) = (0, m)$ and proceed by induction on $T - F = m$.

A_5). $(F, T) = (0, 0)$. Then, the following cases occur:

Case 1. $K(L) = K(M) = 0$. By Theorem 4.6, $L \cong M$.

Case 2. $K(L) \cong K(M) \cong (e_\infty \pi^{u(\infty)})$, $u(\infty) < \infty$.

By normality, $u(\infty) = u(0)$. By scaling, we may assume that $e_\infty = e_0 = 1$. From $u(0) \leq 0$, Lemma 4.5 and using transformations of type $LT_0 d$) on $L'_0 \oplus K(L)$, we have,

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^{v(0)} & \pi^{-v(0)D} \end{pmatrix} \oplus K(L), \text{ if } v(0) \leq 0,$$

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus K(L), \text{ if } u(0) < 0 \text{ and } v(0) = 1,$$

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus K(L), \text{ if } u(0) = 0 \text{ and } v(0) = 1,$$

where $c \in \pi^{v(0)} \mathfrak{o}^2$ and $D = 0$ or λ . Applying Theorem 4.6 to $\begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus H'_0$, we may assume that $c = 0$. Since $u(0) + v(0) \equiv 1 \pmod{2}$ and $u(\infty) = u(0)$, we see that $[\pi^{-1}, \pi D] = [C(\pi^{u(\infty)} \circ kL)/\mathfrak{R}]$, where \mathfrak{R} is the radical of the Clifford algebra $C(\pi^{u(\infty)} \circ kL)$. By condition 1) and interchanging L and M , we see that $L \cong M$.

Case 3. $K(L) \cong K(M) \cong (e_\infty \pi^{u(\infty)}) \oplus (e_\infty \pi^{v(\infty)})$, $v(\infty) < \infty$. By normality, $u(\infty) = u(0)$ and $v(\infty) = v(0)$. Thus, we see easily that $q_0(L) = q_\infty(L) = q_\infty(M) = q_0(M)$. By using transformations of type $LT_0 d$), we may assume that L'_0 and M'_0 are 0-hyperbolic. Thus, $L \cong M$.

A_6). $(F, T) = (0, m)$, $m > 0$. We assert that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ contain isometric 0-modular sublattices of dimension 4, say, $L''_0 \cong M''_0$. We have the following cases to consider:

Case 1. $v(0) = v(1)$. Since $v(0) \leq 1$, condition 5) is vacuous on the 0-modular components. Thus, by Definition 5.1 and scaling by e_0 or e_1 , we may assume that $e_0 = e_1 = 1$. Now $L'_0 \oplus H'_1$ contains the sublattice $N = L'_0 \oplus (\pi^{u(1)}) \oplus (\pi^{v(1)})$. By Lemma 4.5 and using transformations of type $LT_0 d)$ on N , we see easily that N contains $\begin{pmatrix} 1 \\ \pi^{u(0)} & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}$. Interchanging L and M , we get A_6).

Case 2. $v(0) + 1 = v(1)$. By Definition 5.1, $v(0) = u(0) + 1 = u(1)$ and scaling by e_1 allows us to assume that $e_0 = e_1 = 1$. If $v(0) \leq 0$, then the argument of Case 1 may be used to get A_6). Thus, we may assume that $v(0) = 1$, hence $v(1) = 2$. By Lemma 4.5, we see that $L'_0 \cong \begin{pmatrix} 1 \\ 1 & D \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}$ and $M'_0 \cong \begin{pmatrix} 1 \\ 1 & D' \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}$, where $D, D' \in \{0, \lambda\}$. Condition 5), [1; Satz 8, p. 161] and the remarks preceding Lemma 1.1 show that $D = D'$. Thus A_6) holds.

Case 3. $v(0) + 2 = v(1)$. Since $v(1) \leq 2$, we have $v(0) \leq 0$. By Definition 5.1, $u(0) = u(1)$ or $u(1) - 2$. By scaling, we may assume that $e_1 = 1$.

Let $u(0) = u(1)$. If $v(1) < 2$, then $v(0) < 0$. Thus, an argument similar to that of Case 1 shows that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ contain the sublattice $\begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^{v(0)} & 0 \end{pmatrix}$. Hence, we may assume that $v(1) = 2$ and $v(0) = 0$. By Lemma 4.5, we see that $L'_0 \oplus (\pi^{u(1)}) \cong \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 & D \end{pmatrix} \oplus (\pi^{u(1)})$, where $u(1)$ is odd. By condition 5) and the second argument of Case 2, we see that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ both contain the sublattice $\begin{pmatrix} 1 \\ 1 & D \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}$. Thus, A_6) holds in either case.

Let $u(0) = u(1) - 2$. By Definition 5.1, we then have

$$e_0 q_1(L) = \pi^2 q_0(L) \subseteq q_1(L) = e_0^2 q_1(L) \subseteq e_0 q_1(L).$$

Thus, $q_0(L) = \pi^{-2} q_1(L)$ and we may assume that $e_0 = 1$ as well. If $v(1) < 2$, then $u(0) < v(0) < 0$, $-u(0) \geq u(1)$ and $-v(0) \geq v(1)$. Repeating the argument of Case 1, we see that $L'_0 \oplus H'_1$ contains the sublattice

$$\begin{pmatrix} 1 \\ \pi^{u(0)} & \pi^{v(0)} b \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^{v(0)} & 0 \end{pmatrix}, \text{ where } b \in \Omega.$$

Similarly, $M'_0 \oplus H'_1$ contains the sublattice

$$\begin{pmatrix} 1 \\ \pi^{u(0)} & \pi^{v(0)}b' \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \pi^{v(0)} & 0 \end{pmatrix}, \text{ where } b' \in \Omega.$$

From $0 > u(0) + v(0) \equiv 1 \pmod{2}$, condition 4) and Lemma 1.1, we see that $b = b'$. Finally, if $v(1) = 2$, then $u(0) < v(0) = 0$ and $-u(0) \geq u(1)$. Repeating the argument of Case 1, we see that $L'_0 \oplus H'_1$ contains the sublattice $\begin{pmatrix} 1 \\ \pi^{u(0)} & b \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 & D \end{pmatrix}$ and that $M'_0 \oplus H'_1$ contains the sublattice $\begin{pmatrix} 1 \\ \pi^{u(0)} & b' \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 & D' \end{pmatrix}$, where $b, b' \in \Omega$; $D, D' \in \{0, \lambda\}$; and $u(0) < 0$ is odd. By condition 5) and the second argument of Case 2, we get $D = D'$. By condition 4) and the last argument of the preceding paragraph, we see that $b = b'$. Thus $A_6)$ holds.

From $A_6)$, applying Lemma 1.3, we see that L''_0 and M''_0 are orthogonal summands of $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ respectively. Thus, we may "replace" L'_0 and M'_0 by L''_0 and M''_0 respectively. Hence, we have obtained decompositions of L and M satisfying $A_1), A_2)$ as well as $L_0 \cong M_0$. Let L' and M' be the corresponding orthogonal complement of L_0 and M_0 in L and M . It is clear that L' and M' are normal lattices of type $(1, m)$ and satisfy a similar set of conditions as stated in our assertion. Hence, by induction, $L' \cong M'$. Thus, $L \cong M$. Q. E. D.

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