J. Math. Soc. Japan
Vol. 26, No. 4, 1974

The Seifert matrices of Milnor fiberings defined by holomorphic functions

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(Received Aug. 7, 1973)

§ 1. Introduction.

A "spinnable structure" defined by I. Tamura is a generalization of the structure of a Milnor fibering [4] for a holomorphic function at an isolated critical point. M. Kato [2] has shown that there is a one to one correspondence of "simple spinnable structures" on S^{2n+1} $(n \geqq 3)$ with congruence classes of unimodular matrices via Seifert matrices.

The purpose of this paper is to prove "Join theorem" about the Seifert matrices of Milnor fiberings at isolated critical points. As a corollary, we calculate the Seifert matrices of the Milnor fiberings of the Brieskorn polynomials. Essentially, we make use of the facts obtained in [5].

DEFINITION 1. A *spinnable structure* on a closed manifold M is a triple $\mathcal{S}=\{F, h, g\}$: F is a compact manifold, $h: F \rightarrow F$ is a diffeomorphism such that $h|\partial F=\mathrm{id}$, and $g: T(F, h) \rightarrow M$ is a diffeomorphism, where $T(F, h)$ is a closed manifold obtained from $F \times [0, 1]$ by identifying $(x, 1)$ with $(h(x), 0)$ for all $x \in F$ and (x, t) with (x, t') for all $x \in \partial F$ and $t, t' \in [0, 1]$. When F is a handlebody obtained from a ball by attaching handles of index $\leqq \left[\frac{\dim M}{2}\right]$, $\mathcal{S}$ is called a *simple spinnable structure*.

DEFINITION 2. A closed oriented $(2n+1)$-manifold is an *Alexander manifold*, if $H_n M = H_{n+1} M = 0$.

If $\mathcal{S}=\{F, h, g\}$ is a simple spinnable structure on an Alexander manifold M^{2n+1}, then $H_n F$ is torsion free.

DEFINITION 3. Let $\mathcal{S}=\{F, h, g\}$ be a simple spinnable structure on M^{2n+1}. For a basis $\alpha_1, \cdots, \alpha_m$ of $\tilde{H}_n(F)$, a matrix $\Gamma(\mathcal{S})=(L(g_{\#}(\alpha_i \times 0), g_{\#}(\alpha_j \times 1/2)))$ is called a *Seifert matrix* of $\mathcal{S}$, where $L(\xi, \eta)=$ the linking number of ξ and η in $M^{2n+1}=$ intersection number $\langle \lambda, \eta \rangle$. ($\lambda$ is a chain in M such that $\partial \lambda = \xi$.)

THEOREM 1 (M. Kato [2]). *There is a one to one correspondence of isomorphism classes of simple spinnable structures on a* 1*-connected Alexander* $(2n+1)$*-manifold M with congruence classes of unimodular matrices via Seifert matrices, provided that* $n \geqq 3$.

Therefore, the Seifert matrices of the Milnor fiberings give a topological characterization of isolated singularities of hypersurfaces.

§2. Statement of results.

Let $g(x)$ and $h(y)$ be holomorphic functions defined on neighborhoods of the origins of $\boldsymbol{C}^m$ and $\boldsymbol{C}^n$, with $g(0)=h(0)=0$. Suppose that g and h have isolated singularities at the origins.

THEOREM 2. *Let f be a holomorphic function on a neighborhood of the origin of $\boldsymbol{C}^m \times \boldsymbol{C}^n$, such that $f(x, y)=g(x)+h(y)$. Denote $\Gamma(f)$, $\Gamma(g)$ and $\Gamma(h)$ be Seifert matrices of Milnor fiberings defined by f, g and h, then*

$$\Gamma(f) \equiv (-1)^{mn} \Gamma(g) \otimes \Gamma(h) .$$

("$\equiv$" means "belongs to the same congruence class with".)

COROLLARY 3. *Let Γ be the Seifert matrix of the Milnor fibering of a Brieskorn polynomial $f(z)=(z_1)^{a_1}+\cdots+(z_n)^{a_n}$, $(a_i \geqq 2)$. Then,*

$$\Gamma \equiv (-1)^{\frac{n(n+1)}{2}} A_{a_1} \otimes \cdots \otimes A_{a_n}$$

where, A_a $(a \geqq 2)$ is the $(a-1)\times(a-1)$ matrix given by

$$A_a = \begin{pmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{pmatrix} .$$

REMARK. Let Γ be the Seifert matrix of a simple spinnable structure $\mathcal{S}=\{F, h, g\}$ on an Alexander manifold M^{2n-1}. Then the monodromy $h_*: \tilde{H}_{n-1}(F) \to \tilde{H}_{n-1}(F)$ is given by $h_*=(-1)^n \Gamma^t \cdot \Gamma^{-1}$, and the intersection matrix of F is given by $-\Gamma+(-1)^n \Gamma^t$, where Γ^t is the transpose of Γ. (See M. Kato [2] for the details.) Applying these facts to Corollary 3, we can obtain well known results about the monodromy of the Milnor fibering of a Brieskorn polynomial, and the intersection matrix of the fiber [1].

§3. Proof.

PROOF OF COROLLARY 3. By Theorem 2, it is enough to prove only in the case of $n=1$. Let $f(z)=z^a$ $(a \geqq 2)$. In this case the Milnor fiber is $\Omega_a=\{1, \omega, \cdots, \omega^{a-1}\}$ where $\omega=e^{\frac{2\pi i}{a}}$. Hence, $\{1-\omega, \omega-\omega^2, \cdots, \omega^{a-2}-\omega^{a-1}\}$ is a basis of $\tilde{H}_0(\Omega_a)$. It is easy to see that for this basis,

$$\Gamma(f) = -A_a . \qquad \text{q.e.d.}$$

For the proof of Theorem 2, we must prove some lemmas. Let f be a holomorphic function defined on a neighborhood of the origin of $\boldsymbol{C}^n$ with $f(0)=0$, and assume $V=f^{-1}(0)$ has an isolated singularity at the origin.

LEMMA 1. *Assume $n \geqq 2$, then there exist a neighborhood N of $V-\{0\}$ and a small number $\varepsilon>0$, such that, for all $z \in D_{2\varepsilon} \cap N$, the two vectors z and* grad *$f(z)$ are linearly independent over the complex number.*

This is an easy corollary of the Curve Selection Lemma [**4**, Lemma 3.1].

By [**4**, Lemma 4.3] and the above lemma, there is a smooth vector field $\boldsymbol{v}$ on $D_{2\varepsilon}-\{0\}$ so that

$$\langle \boldsymbol{v}(z), \operatorname{grad} f(z) \rangle = f(z)$$

and

$$\operatorname{Re} \langle \boldsymbol{v}(z), z \rangle > 0 ,$$

and that, if $z \in N \cap D_{2\varepsilon}$

$$\langle \boldsymbol{v}(z), z \rangle = |z|^2$$

where N is a neighborhood of $V-\{0\}$. Let $p(t)$ be any integral curve of $\boldsymbol{v}$, then

$$\frac{d}{dt} f(p(t)) = f(p(t))$$

and

$$\frac{d}{dt} |p(t)| > 0 ,$$

and if $p(t) \in N \cap D_{2\varepsilon}$

$$\frac{d}{dt} |p(t)| = |p(t)| .$$

Therefore, $f(p(t)) = e^t f(p(0))$, and $|p(t)|$ is strictly increasing. Let $p(t\,;z)$ be the integral curve of $\boldsymbol{v}$ with $p(0\,;z)=z$, and define $r \circ z = p(\log r\,; z)$, and $0 \circ z = 0$. Then the map $(r, z) \mapsto r \circ z$ is defined on $[0, 1] \times (D_{2\varepsilon}-\{0\})$.

Now assume $n=1$. There is a holomorphic function g defined on a neighborhood of the origin such that $f(z)=(g(z))^k$ (k is a positive integer), $g(0)=0$ and $g'(0) \neq 0$. Then for a small number $\varepsilon>0$, the inverse function g^{-1} of g is defined on $D_{2\varepsilon}$. Define $r \circ z = g^{-1}(r^{\frac{1}{k}} g(z))$ for $(r, z) \in [0, 1] \times D_{2\varepsilon}$. Note that the absolute value of $g^{-1}(rz)$ is a strictly increasing function of r $(0 \leqq r \leqq 1)$ if $|z|$ is sufficiently small. In fact, let h be a holomorphic function such that $g^{-1}(z) = zh(z)$ and let $F(r, z) = |g^{-1}(rz)|^2$. Then $h(0) \neq 0$ and

$$\begin{aligned} \frac{1}{|z|^2} \frac{\partial F}{\partial r}(1, z) &= 2h\bar{h} + zh'\bar{h} + h\overline{zh'} \\ &= |h|^2 + |h+zh'|^2 - |zh'|^2 > 0 \end{aligned}$$

for any small z, hence

$$\frac{1}{|z|^2}\frac{\partial F}{\partial r}(r, z)=\frac{1}{|z|^2}\frac{1}{r}\frac{\partial F}{\partial r}(1, rz)>0$$

for $0<r<1$.

LEMMA 2. *The map* $[0, 1]\times(D_{2\varepsilon}-\{0\})\to D_{2\varepsilon}$; $(r, z)\mapsto r\circ z$ *is continuous and satisfies the following properties.*

i) $1\circ z=z$ *and* $(rs)\circ z=r\circ(s\circ z)$,

ii) $f(r\circ z)=rf(z)$,

iii) $|r\circ z|$ *is a strictly increasing function of* r.

LEMMA 3. *Let* $0<\rho\leqq 2\varepsilon$, *then the map*

$$c*S_\rho \longrightarrow D_\rho\,;\ [r, z]\longmapsto r\circ z$$

is a homeomorphism, where $c*S_\rho=[0, 1]\times S_\rho/0\times S_\rho$ *is the cone over* S_ρ.

The proof is easy.

PROOF OF THEOREM 2. Let f, g and h be as in Theorem 2.

LEMMA 4. *The map*

$$c*S_{2\varepsilon}^{2m+2n-1}\longrightarrow D_{2\varepsilon}^{2m+2n}\,;\ [r, x, y]\longmapsto (r\circ x, r\circ y)$$

is a homeomorphism.

The proof is easy.

By Lemma 4, there is a homeomorphism

$$\varphi: (0, 1]\times S_{2\varepsilon}^{2m+2n}\longrightarrow D_{2\varepsilon}^{2m+2n}-\{0\}$$

defined by $\varphi(r, x, y)=(r\circ x, r\circ y)$. Then

$$(f\circ\varphi)(r, x, y)=rf(x, y)\,.$$

Define a map

$$\sigma:\ (D_\varepsilon^{2m}-\{0\})*(D_\varepsilon^{2n}-\{0\})\longrightarrow D_{2\varepsilon}^{2m+2n}-\{0\}$$

by $\sigma([x, s, y])=(s\circ x, (1-s)\circ y)$ and let

$$p_2:\ (0, 1]\times S_{2\varepsilon}\longrightarrow S_{2\varepsilon}$$

be the natural projection. Then,

$$\psi=p_2\circ\varphi^{-1}\circ\sigma\,|\,S_\varepsilon*S_\varepsilon:\ S_\varepsilon^{2m-1}*S_\varepsilon^{2n-1}\longrightarrow S_{2\varepsilon}^{2m+2n-1}$$

is an orientation preserving homeomorphism, and

$$(r\cdot f)(\psi([x, s, y]))=sg(x)+(1-s)h(y)$$

where $r=r(z)$ $(=p_1\circ\varphi^{-1}\circ\sigma\circ\psi^{-1}(z))$ is a positive real-valued continuous function on $S_{2\varepsilon}^{2m+2n-1}$. Let $g*h$ be a continuous function on $S_\varepsilon^{2m-1}*S_\varepsilon^{2n-1}$ defined by the right hand side of the above equality. We have obtained,

LEMMA 5. *There is a homeomorphism* ψ *from* $S_\varepsilon^{2m-1}*S_\varepsilon^{2n-1}$ *onto* $S_\varepsilon^{2m+2n-1}$ *with* $(r\cdot f)\circ\psi=g*h$.

Therefore, we can identify the spinnable structures defined by f and

$g*h$ via ψ.

Let $X=\{g>0\}\cap S_\varepsilon^{2m-1}$, $Y=\{h>0\}\cap S_\varepsilon^{2n-1}$, $Z=\{f>0\}\cap S_{2\varepsilon}^{2m+2n-1}$ and $Z'=\{g*h>0\}\cap(S_\varepsilon^{2m-1}*S_\varepsilon^{2n-1})$, where $\{g>0\}=\{x\in \boldsymbol{C}^m\,;\,g(x)>0\}$ etc.

LEMMA 6. *The map $j=\psi|X*Y\,;\ X*Y\to Z$ is a homotopy equivalence. Therefore, the natural inclusion $(=\psi^{-1}\circ j)$ $X*Y\subsetneq Z'$ is a homotopy equivalence.*

PROOF. By Lemmas 2, 3 and 4, the following inclusion maps are homotopy equivalences.

$$X\subsetneq \{g>0\}\cap D_\varepsilon^{2m}\supsetneq X_t\,,$$

$$Y\subsetneq \{h>0\}\cap D_\varepsilon^{2n}\supsetneq Y_t\,,$$

$$Z\subsetneq \{f>0\}\cap D_\varepsilon^{2m+2n}\supsetneq Z_t\,,$$

where t is a small positive number, and $X_t=g^{-1}(t)\cap D_\varepsilon^{2m}$, $Y_t=h^{-1}(t)\cap D_\varepsilon^{2n}$ and $Z_t=f^{-1}(t)\cap D_\varepsilon^{2m+2n}$. Hence, the following diagram is homotopy commutative

$$\begin{tikzcd}
X*Y \arrow[r, "j"] \arrow[d, hook, "\simeq"] & Z \arrow[d, hook, "\simeq"] \\
(\{g>0\}\cap D_\varepsilon)*(\{h>0\}\cap D_\varepsilon) \arrow[r, "\sigma"] & \{f>0\}\cap D_{2\varepsilon} \\
X_t*Y_t \arrow[r, "\sigma"] \arrow[u, hook, "\simeq"] & G^{-1}(J)\underset{\simeq}{\subsetneq} Z_t \arrow[u, hook, "\simeq"]
\end{tikzcd}$$

where the map $G: Z_t\to \boldsymbol{C}$ is defined by $G(x, y)=g(x)$ and J is a line segment from 0 to t. By Step 2 in § 2 of [**5**], the inclusion $G^{-1}(J)\subsetneq Z_t$ is a homotopy equivalence. Therefore, it is enough to show that the map

$$\sigma|X_t*Y_t:\ X_t*Y_t\longrightarrow G^{-1}(J)$$

is a homotopy equivalence. Let

$$\phi_1:\ (I-\{0\})\times X_t\longrightarrow g^{-1}(J-\{0\})\cap D_\varepsilon^{2m}$$

and

$$\phi_2:\ (I-\{0\})\times Y_t\longrightarrow h^{-1}(J-\{0\})\cap D_\varepsilon^{2m}$$

$(I=[0, 1],\ J=[0, t])$ be the homeomorphisms used in Step 4 in § 2 of [**5**]. If $(s, x)\in(I-\{0\})\times X_t$, $st=g(\phi_1(s, x))=g(s\circ x)$. Therefore ϕ_1 and the map $(s, x)\mapsto s\circ x$ are fiber homotopic to each other with respect to g, that is, there is a homotopy

$$P_u:\ (I-\{0\})\times X_t\longrightarrow g^{-1}(J-\{0\})\cap D_\varepsilon^{2m}$$

$(0\leqq u\leqq 1)$ such that $P_0=\phi_1$, $P_1(s, x)=s\circ x$ and $(g\circ P_u)(s, x)=st$, for all u, s and x. Similarly, there is a homotopy

$$Q_u:\ (I-\{0\})\times Y_t\longrightarrow h^{-1}(J-\{0\})\cap D_\varepsilon^{2m}$$

$(0\leqq u\leqq 1)$ such that $Q_0=\phi_2$, $Q_1(s, y)=s\circ y$ and $(h\circ Q_u)(s, y)=st$, for all u, s

and y. Consider the identification map $\pi: U\times V\to U/g^{-1}(0)\times V/h^{-1}(0)$ and let $[x, y]=\pi(x, y)$. Define a homotopy

$$H_u:\ X_t*Y_t=X_t\times I\times Y_t/\sim\ \longrightarrow \pi(G^{-1}(J))$$

$(0\leqq u\leqq 1)$ by

$$H_u([x, s, y])=[P_u(s, x),\ Q_u(1-s, y)]\,.$$

(Define $P_u(0, x)=0$ and $Q_u(0, y)=0$.) This is well defined and continuous. By the definition

$$H_0([x, s, y])=[\phi_1(s, x),\ \phi_2(1-s, y)]$$

$$H_1([x, s, y])=[s\circ x,\ (1-s)\circ y]=(\pi\circ\sigma)([x, s, y]).$$

$$\begin{tikzcd} X_t*Y_t \arrow[rr, "H_u"] \arrow[dr, "\sigma"'] & \arrow[d, phantom, "\simeq"] & \pi(G^{-1}(J)) \\ & G^{-1}(J) \arrow[ur, "\pi"'] & \end{tikzcd}$$

H_0 is a homeomorphism constructured in Step 4 in § 2 of [**5**]. Therefore, H_1 is a homotopy equivalence. Hence $\sigma|X_t*Y_t: X_t*Y_t\to G^{-1}(J)$ is a homotopy equivalence, since so is $\pi: G^{-1}(J)\to\pi(G^{-1}(J))$ (Step 3 in § 2 of [**5**]). This proves Lemma 6.

By Lemma 5, we have only to show that

$$\Gamma(g*h)=(-1)^{mn}\Gamma(g)\otimes\Gamma(h)\,.$$

Let $\{e_i\}$ and $\{f_j\}$ be bases of $\tilde{H}_{m-1}(X)$ and $\tilde{H}_{n-1}(Y)$ respectively. Then $\{e_i\otimes f_j\}$ is a basis of $\tilde{H}_{m+n-1}(Z')\cong\tilde{H}_{m+n-1}(X*Y)\cong\tilde{H}_{m-1}(X)\otimes\tilde{H}_{n-1}(Y)$ (Lemma 6).

Let

$$\alpha_\theta:\ X\longrightarrow S_\varepsilon^{2m-1}-g^{-1}(0)$$

and

$$\beta_\theta:\ Y\longrightarrow S_\varepsilon^{2n-1}-h(0)$$

be continuous one-parameter families of embeddings such that α_0 and β_0 are the natural inclusions, and that

$$\arg g(\alpha_\theta(x))=\arg h(\beta_\theta(y))=\theta$$

for all x, y and θ. Then,

$$\alpha_\theta*\beta_\theta:\ X*Y\longrightarrow S_\varepsilon^{2m-1}*S_\varepsilon^{2n-1}-(g*h)^{-1}(0)$$

is a continuous one-parameter family of embeddings such that $\alpha_0*\beta_0$ is the natural inclusion, and that

$$\arg((g*h)\circ(\alpha_\theta*\beta_\theta))([x, s, y])=\theta$$

for all $[x, s, y] \in X*Y$ and θ. Therefore, for the bases $\{e_i\}$, $\{f_j\}$ and $\{e_i \otimes f_j\}$,

$$\Gamma(g) = (L(e_i, (\alpha_\pi)_\# e_j))$$

$$\Gamma(h) = (L(f_k, (\beta_\pi)_\# f_l))$$

and

$$\begin{aligned}\Gamma(g+h) &= (L(e_i \otimes f_k, (\alpha_\pi * \beta_\pi)_\# (e_j \otimes f_l))) \\ &= (L(e_i \otimes f_k, (\alpha_\pi)_\# e_j \otimes (\beta_\pi)_\# f_l)) .\end{aligned}$$

Therefore, we have only to prove

LEMMA 7. *Let*

$$S^m,\ S^n \subsetneq S^{m+n+1},\ S^m \cap S^n = \phi ,$$

$$S^p,\ S^q \subsetneq S^{p+q+1},\ S^p \cap S^q = \phi$$

be embeddings, then,

$$\begin{aligned}&L_{S^{m+n+1} * S^{p+q+1}}(S^m * S^p, S^n * S^q) \\ &\qquad = (-1)^{(n+1)(p+1)} L_{S^{m+n+1}}(S^m, S^n) L_{S^{p+q+1}}(S^p, S^q) .\end{aligned}$$

PROOF. The following diagram is commutative up to sign.

$$\begin{tikzcd}
\widetilde{H}_m(S^m) \otimes \widetilde{H}^p(S^p) \arrow[d, "(\mathrm{inclusion})_*"'] & & \widetilde{H}_{m+p+1}(S^m * S^p) \arrow[ll, "\cong"'] \arrow[d, "(\mathrm{in})_*"] \\
\widetilde{H}_m(S^{m+n+1} - S) \otimes \widetilde{H}_p(S^{p+q+1} - S^q) \arrow[d, "\text{Alexander Dual}"', "\cong"] & & \widetilde{H}_{m+p+1}(S^{m+n+1} * S^{p+q+1} - S^n * S^q) \arrow[d, "\text{A. D.}"', "\cong"] \\
\widetilde{H}^n(S^n) \otimes \widetilde{H}^q(S^q) \arrow[rr, "\cong"] & & \widetilde{H}^{n+q+1}(S^n * S^q)
\end{tikzcd}$$

Therefore the lemma is true up to sign. Hence we may assume $S^m * S^n = S^{m+n+1}$, $S^p * S^q = S^{p+q+1}$. We can see $L_{S^a * S^b}(S^a, S^b) = (-1)^{a+1}$. In fact, let $\sigma = (x_0, \cdots, x_a)$ and $\tau = (y_0, \cdots, y_b)$ be top dimensional simplices of S^a and S^b with the compatible orientations. Then,

$$\begin{aligned}L_{S^a * S^b}(S^a, S^b) &= \text{Intersection number } \langle y_0 * S^a, S^b \rangle \\ &= \text{Intersection number } \langle y_0 * \sigma, \tau \rangle \\ &= \text{Signature of the permutation}\end{aligned}$$

$$\begin{pmatrix} y_0 & x_0 & x_1 \cdots x_a & y_1 \cdots y_b \\ x_0 & x_1 \cdots x_a & y_0 & y_1 \cdots y_b \end{pmatrix}$$

$$=(-1)^{a+1}.$$

$$\therefore L_{(S^m*S^n)*(S^p*S^q)}(S^m*S^p, S^n*S^q)$$

$$=(-1)^{(n+1)(p+1)}L_{(S^m*S^p)*(S^n*S^q)}(S^m*S^p, S^n*S^q)$$

$$=(-1)^{(n+1)(p+1)}(-1)^{m+p+2}$$

$$=(-1)^{(n+1)(p+1)}L_{S^m*S^n}(S^m, S^n)\cdot L_{S^p*S^q}(S^p, S^q). \qquad \text{q.e.d.}$$

This completes the proof of Theorem 2.

REMARK. Using the "good stratification" [3], we can prove Lemmas 2~6 without any assumption about the isolatedness of singularities.

References

[1] E. Brieskorn, Beispiele zur Differentialtopologie von Singularitäten, Invent. Math., 2 (1966),

[2] M. Kato, A classification of simple spinnable structures on a 1-connected Alexander manifold, J. Math. Soc. Japan, 26 (1974), 454-463.

[3] Lê Dung Tráng, Un théorème de Zariski du type de Lefschetz, Centre de Mathématiques de l'École Polytechnique, Paris, 1971.

[4] J. Milnor, Singular points of complex hypersurfaces, Ann. of Math. Studies, 61 (1968).

[5] K. Sakamoto, Milnor fiberings and their characteristic maps, to appear in Proc. Intern. Conf. on Manifolds and Related Topics in Topology, Tokyo, 1973.

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