BORDISM CLASSES OF THE MULTIPLE POINTS MANIFOLDS OF SMOOTH IMMERSION

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Abstract. Let $f : V^n \looparrowright M^m$ be a smooth generic immersion. Then the set of points, that have at least k preimages is an image of a (non-generic) immersion. If the manifolds V^n and M^m are oriented and $m-n$ is even, then the manifold of k-fold points is also oriented. In this paper we compute the oriented bordism class of the manifold of k-fold points in terms of the differential df, provided the tangent bundle of the manifold M^m has a nowhere zero cross-section.

1. Introduction

In this paper we will consider smooth orientable C^∞-manifolds and C^∞-mappings between them. Let V^n and M^m be manifolds without boundary, V^n be compact, $f : V^n \looparrowright M^m$ be a smooth generic immersion and $m-n$ be *even*. From Thom multijet transversality theorem [1] it follows that the set of immersions f such that the map $f^{(k)} : V^{(k)} \to M^{(k)}$ is transverse to the "thin" diagonal $\Delta_k(M) = \{(x, \dots, x) \mid x \in M\} \subset M^{(k)}$ outside the "thick" diagonal $\Delta_2(V) = \{(x_1, \dots, x_k) \mid \exists\ i \neq j : x_i = x_j\} \subset V^{(k)}$ is open and everywhere dense in the set of all immersions $V^n \looparrowright M^m$ (in C^∞ Whitney topology). Therefore, for a generic f the set $V_k = (f^{(k)})^{-1}(\Delta_k(M)) \backslash \Delta_2(V) \subset V^{(k)}$ is an oriented submanifold. Denote by Σ_k the group of permutations on k elements. Permutation of factors in the product $V^{(k)}$ induce the free action of Σ_k on the set $V^{(k)}$. Denote by $\widetilde{V}_k$ and $\widetilde{M}_k$ the quotient manifolds V_k/Σ_{k-1} and V_k/Σ_k, where the subgroup $\Sigma_{k-1} \subset \Sigma_k$ is the stabilizer of the first element. Since $m-n$ is even, the action of Σ_k preserves the orientation of the manifold V_k. Therefore $\widetilde{V}_k$ and $\widetilde{M}_k$ are canonically oriented. Let us define immersions $f_k : \widetilde{V}_k \looparrowright V^n$ and $g_k : \widetilde{M}_k \looparrowright M^m$ by formulae $f_k(x_1, [x_2, \dots, x_k]) = x_1$ and $g_k[x_1, \dots, x_k] = f(x_1)$ (see [2] for details).

For an integer $k > 0$ and an immersion $f : V^n \looparrowright M^m$ we assign the oriented bordism classes $(\widetilde{V}_k, f_k) \in \Omega_{m-k(m-n)}(V^n)$ and $(\widetilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m)$. From [3] it follows that these classes do not change under a regular homotopy of the immersion f. By the fundamental Smale-Hirsch theorem [4], the set of regular homotopy classes of immersions $V^n \looparrowright M^m$ is in 1-1 correspondence with the set of linear monomorphism classes of the tangent bundles $\tau V \to \tau M$. The class of the immersion $f : V^n \looparrowright M^m$ corresponds to the class of the differential $df : \tau V \to \tau M$. Hence, the classes $(\widetilde{V}_k, f_k) \in \Omega_{m-k(m-n)}(V^n)$ and $(\widetilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m)$ have to be computable in terms of the differential $df : \tau V \to \tau M$. In this paper (see corollary 2.3) we compute $(\widetilde{V}_k, f_k)$ and $(\widetilde{M}_k, g_k)$ up to elements of order $(k-1)!$ and $k!$, respectively. This weakening is in some sense natural, for the manifolds $\widetilde{V}_k$ $\widetilde{M}_k$ were constructed as the images of $(k-1)!$- and $k!$-fold coverings.

Recall that all the finite order elements of $\Omega_*(pt)$ have the order 2. Therefore, for $M^m = \mathbb{R}^m$ we compute the classes $(\widetilde{M}_k, g_k)$ up to elements of order 2. It is another approach to [5, Theorem 5], where all the Pontrjagin numbers of the manifolds $\widetilde{M}_k$ were computed in terms of the Pontrjagin classes of the manifold V^n and the integral Euler class of the normal bundle of the immersion f. Lemma 2.1 gives a formula to compute the unoriented bordism class

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$(\widetilde{V}_2, f_2)_2 \in \Re_*(V^n)$, if we do not require $m-n$ to be even, and V^n and M^m to be oriented. For up-to-date reviews of results on the bordism classes of self-intersection manifolds see [5] for oriented case, and [6] for unoriented case.

2. Formulation of results

Denote by $S\tau M$ the spherical fibration, associated to the tangent bundle τM and by $Sdf : S\tau V \to S\tau M$ the fiberwise monomorphism of spherical fibrations, induced by the differential $df : \tau V \to \tau M$. Since the manifolds V^n, M^m, $S\tau V$ and $S\tau M$ are oriented (in usual sense), they are oriented in *oriented bordism theory* [7, 8]. Thus, there is the *Poincare duality* on these manifolds. Denote by $Sdf^! : \Omega_*(S\tau M) \to \Omega_*(S\tau V)$ the Gysin homomorphism, induced by the mapping Sdf.

Let us formulate the key lemma of this paper.

Lemma 2.1. *Let M^m be an oriented manifold without boundary such that there is a nowhere zero cross-section of the tangent bundle τM, or, in other words, a section $s_M : M^m \to S\tau M$. Then for any generic immersion $f : V^n \looparrowright M^m$ of compact oriented manifold without boundary V^n the bordism class $(\widetilde{V}_2, f_2) \in \Omega_{2n-m}(V^n)$ is*

$$(\widetilde{V}_2, f_2) = (-1)^{m-1} i_* Sdf^!(M^m, s_M), \qquad (1)$$

where i is the natural projection $S\tau V \to V^n$.

To formulate our results, it will be convenient to use the oriented cobordism classes $v_k \in \Omega^{(k-1)(m-n)}(V^n)$ and $m_k \in \Omega^{k(m-n)}_{comp.}(M^m)$, the Poincare duals to $(\widetilde{V}_k, f_k)$ and $(\widetilde{M}_k, g_k)$, respectively. Denote by 1_V the identity element of the ring $\Omega^*(V^n)$, and by $f_!$ the Gysin homomorphism, induced by the map f.

Corollary 2.2. *Under the conditions of lemma 2.1 and if $m-n$ is even, the Euler class e of the normal bundle of the immersion f is*

$$e = f^* f_!(1_V) + (-1)^m \gamma i_* Sdf^!(M^m, s_M),$$

where $\gamma : \Omega_*(V^n) \to \Omega^{n-*}(V^n)$ is the Poincare duality.

Corollary 2.3. *Under the conditions of lemma 2.1 and if $m-n$ is even,*

$$(k-1)! \cdot v_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V)$$
$$k! \cdot m_k = f_! \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V),$$

where $\varphi_k(a) = f^* f_!(a) - k \cdot e \cup a$, and e is the Euler class of the normal bundle of the immersion f (which was computed in corollary 2.2).

3. The bordism group of immersions

Let us call two oriented immersions $f_0 : V_0^n \looparrowright M^m$ and $f_1 : V_1^n \looparrowright M^m$ *bordant*, if there exists a compact oriented manifold with boundary W^{n+1} such that $\partial W^{n+1} = V_0^n \sqcup (-V_1^n)$, and an immersion $W^{n+1} \looparrowright M^m \times [0,1]$ such that for a collar $V_0^n \times [0,\varepsilon) \sqcup (-V_1^n) \times (1-\varepsilon, 1]$ of the boundary ∂W^{n+1} the restrictions $F|_{V_0^n \times [0,\varepsilon)} = f_0 \times \mathrm{id}$ and $F|_{(-V_1^n)\times(1-\varepsilon,1]} = f_1 \times \mathrm{id}$. Then the set of equivalence classes of bordant oriented immersions with disjoint union operation is a group $Imm_n^{SO}(M^m)$. The group $Imm_n^{SO}(M^m) = [M^m, QMSO(m-n)]$, where $QX = \varinjlim \Omega^q S^q X$ is the infinite loop space of infinite suspension and MSO is the Thom spectrum [9].

From the results of paper [3] it follows that the map, assigning for any immersion $f : V^n \looparrowright M^m$ the bordism class $(\widetilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m)$, is a well-defined homomorphism $\varepsilon_k : Imm_n^{SO}(M^m) \to \Omega_{m-k(m-n)}(M^m)$. The classes $(\widetilde{M}_k, g_k)$ involve much information about the class of immersion $[f] \in Imm_n^{SO}(M^m)$. The following theorem was proved in [10], using algebraic technics.

Theorem 3.1 ([10, Corollary 1]). *If $3n+1 < 2m$ and $m-n$ is even, then the homomorphism*

$$\varepsilon_1 \oplus \varepsilon_2 : Imm_n^{SO}(\mathbb{R}^m) \to \Omega_n \oplus \Omega_{2n-m}$$

is an isomorphism modulo the class C_2 of finite 2-primary groups.

Our calculations probably clarify the geometric core of theorem 3.1. The matter is that in these very dimensional restrictions ($3n+1 < 2m$) any *skew* map [1] $\tau V \to \tau M$ can be homotoped to a monomorphism of tangent bundles (see details in [11]). Our formula (lemma 2.1) connects the map of spherical fibrations, induced by the differential $df : \tau V \to \tau M$, and the oriented bordism class $2(\widetilde{M}_2, g_2) \in \Omega_{2n-m}(M^m)$. These reasoning was the initial motivation of this paper.

Conjecture 3.2. *Let M^m be an oriented manifold without boundary such that there is a nowhere zero cross-section of the tangent bundle τM, $3n+1 < 2m$, and $m-n$ be even. Then the homomorphism*

$$\varepsilon_1 \oplus \varepsilon_2 : Imm_n^{SO}(M^m) \to \Omega_n(M^m) \oplus \Omega_{2n-m}(M^m)$$

is an isomorphism modulo the class C_2 of finite 2-primary groups.

4. Proofs

Denote the diagonal $\Delta(V) = \{(x,x) \in V\times V \mid x \in V\}$. Since $f : V^n \looparrowright M^m$ is an immersion, there exists a small enough tubular neighborhood U_V of the diagonal $\Delta(V)$ in $V \times V$ such that $f^{(2)}(U_V \backslash \Delta(V)) \cap \Delta(M) = \emptyset$. Note that $\partial(V^{(2)}\backslash U_V) = \partial(U_V)$. Since $f^{(2)}(U_V \backslash \Delta(V)) \cap \Delta(M) = \emptyset$, we get the map $f^{(2)} : (V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)) \to (M^{(2)}, M^{(2)}\backslash\Delta(M))$. Denote by U_M a tubular neighborhood of the diagonal $\Delta(M)$ in $M^{(2)}$. Without loss of generality we may assume that $f^{(2)}(U_V) \subset U_M$. Denote the inclusion

$$j : (U_M, U_M\backslash\Delta(M)) \hookrightarrow (M^{(2)}, M^{(2)}\backslash\Delta(M))$$

By excision axiom [8], the homomorphisms $j_* : \Omega_*(U_M, U_M\backslash\Delta(M)) \to \Omega_*(M^{(2)}, M^{(2)}\backslash\Delta(M))$ and $j^* : \Omega^*(M^{(2)}, M^{(2)}\backslash\Delta(M)) \to \Omega^*(U_M, U_M\backslash\Delta(M))$ are isomorphisms. Note that the pair $(U_M, \Delta(M))$ is canonically isomorphic to the pair $(\tau M, \tau_0 M)$ [12]. Since M^m is oriented, there exists the *Thom class* $t \in \Omega^m(\tau M, \tau M\backslash\tau_0 M)$ of the tangent bundle τM.

Lemma 4.1. *The class $(\widetilde{V}_2, f_2) \in \Omega_{2n-m}(V^n)$ for an immersion $f : V^n \looparrowright M^m$ can be calculated in the following way*

$$(\widetilde{V}_2, f_2) = (\pi_1)_* \left((f^{(2)})^*((j^*)^{-1}t) \cap \left[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)\right] \right), \tag{2}$$

where $\pi_1 : V^{(2)}\backslash U_V \to V$ is the projection on the first factor, and $[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)]$ is the fundamental class.

Proof of lemma 4.1. Let us recall the construction of the class $(\widetilde{V}_2, f_2)$. Since f is an immersion, $\Delta(V)$ is a closed subset in $(f^{(2)})^{-1}(\Delta(M))$. Since f is a generic immersion, $f^{(2)}$ is transversal to $\Delta(M)$ outside $\Delta(V)$. Therefore $(f^{(2)})^{-1}(\Delta(M))\backslash\Delta(V)$ is a compact oriented submanifold without boundary $\widetilde{f}_2 : \widetilde{V}_2 \hookrightarrow V^{(2)}\backslash\Delta(V)$. Then the composition $\pi_1 \circ \widetilde{f}_2$ is $f_2 : \widetilde{V}_2 \to V^n$ (see details in [2]). By definition [8] of *Lefschetz duality* $\gamma : \Omega^*(V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)) \to \Omega_*(V^{(2)}\backslash U_V)$

$$\begin{aligned}(f^{(2)})^*((j^*)^{-1}t) \cap \left[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)\right] &= (-1)^{2n\cdot m}\gamma\left((f^{(2)})^*((j^*)^{-1}t)\right) \\ &= \gamma\left((f^{(2)})^*((j^*)^{-1}t)\right)\end{aligned}$$

[1] a skew map $\tau V \to \tau M$ can be understood as the "fiberwise cone" over a fiber map $h : S\tau V \to S\tau M$ such that $h(-x) = -h(x)$ in each fiber

Since $f^{(2)}$ is transversal to $\Delta(M)$ outside $\Delta(V)$, we have

$$\begin{aligned}(\pi_1)_*\left(\gamma\left((f^{(2)})^*((j^*)^{-1}t)\right)\right) &= (\pi_1)_*\left((\widetilde{f}_2)_*\left[(f^{(2)})^{-1}(\Delta(M))\backslash\Delta(V)\right]\right)\\ &= (\pi_1)_*\left(\widetilde{V}_2,\widetilde{f}_2\right) = (\widetilde{V}_2, f_2)\end{aligned}$$

$\square$

Proof of lemma 2.1. To prove lemma 2.1, it suffices to interpret the right hand side of formula (2) in terms of the differential df. Since $\partial(V^{(2)}\backslash U_V) = \partial(U_V)$, we have

$$\partial_*\left[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)\right] = \left[\partial(V^{(2)}\backslash U_V)\right],$$

where $\partial_* : \Omega_{2n}(V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)) \to \Omega_{2n-1}(\partial(V^{(2)}\backslash U_V))$ is the differential in the exact bordism sequence of pair. Denote by j_1 the inclusion $S\tau M \hookrightarrow \tau M\backslash\tau_0 M$. Obviously, the map j_1 is a homotopy equivalence. Since $j_1 \circ s_M : M^m \to \tau M\backslash\tau_0 M$ is a nowhere zero cross-section of τM, we have

$$\delta^*\left((j_1^*)^{-1}\gamma(M^m, s_M)\right) = t,$$

where $\delta^* : \Omega^{m-1}(\tau M\backslash\tau_0 M) \to \Omega^m(\tau M, \tau M\backslash\tau_0 M)$ is the differential in the exact cobordism sequence of pair, and γ is the Poincare duality on the total manifold $S\tau M$. Denote by j_2 the isomorphism $\partial(V^{(2)}\backslash U_V) \xrightarrow{\sim} S\tau V$. From the explicit formula [12] for the $\mathbb{Z}_2$-equivariant isomorphism, that identify a small neighborhood of zero section of τV with the neighborhood U_V

$$\tau V \ni (x, \vec{v}_x) \mapsto (\exp_x(\vec{v}_x), \exp_x(-\vec{v}_x)) \in V \times V,$$

it follows that the following diagram commutes (double arrows here denote isomorphisms).

$$\begin{tikzcd}
\Omega^{m-1}(S\tau V) \arrow[r, Rightarrow, "j_2^*"] & \Omega^{m-1}(\partial(V^{(2)}\backslash U_V)) \arrow[r, "\delta^*"] & \Omega^m(V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)) \\
 & & \Omega^m(M^{(2)}, M^{(2)}\backslash\Delta(M)) \arrow[u, "(f^{(2)})^*"'] \arrow[d, Rightarrow, "j^*"] \\
\Omega^{m-1}(S\tau M) \arrow[uu, "Sdf^*"'] & \Omega^{m-1}(\tau M\backslash\tau_0 M) \arrow[l, Rightarrow, "j_1^*"'] \arrow[r, "\delta^*"] & \Omega^m(\tau M, \tau M\backslash\tau_0 M)
\end{tikzcd}$$

Figure 1

Therefore, $\delta^*(j_2^* Sdf^*\gamma(M^m, s_M)) = (f^{(2)})^*((j^*)^{-1}t)$. From the naturality of the $\cap$-product [8] we have

$$(j_3)_*\left(j_2^* Sdf^*\gamma(M^m, s_M) \cap \left[\partial(V^{(2)}\backslash U_V)\right]\right) = (f^{(2)})^*((j^*)^{-1}t) \cap \left[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)\right],$$

where $j_3 : \partial(V^{(2)}\backslash U_V) \to V^{(2)}\backslash U_V$ is the inclusion. Then, by lemma 4.1

$$\begin{aligned}(\widetilde{V}_2, f_2) &= (\pi_1)_*\left((f^{(2)})^*((j^*)^{-1}t) \cap \left[V^{(2)}\backslash U_V, \partial(V^{(2)}\backslash U_V)\right]\right)\\ &= (\pi_1 \circ j_3)_*\left(j_2^* Sdf^*\gamma(M^m, s_M) \cap \left[\partial(V^{(2)}\backslash U_V)\right]\right)\\ &= (-1)^{(2n-1)\cdot(m-1)}(\pi_1 \circ j_3)_*\gamma(j_2^* Sdf^*\gamma(M^m, s_M))\\ &= (-1)^{m-1}(\pi_1 \circ j_3)_* \circ (j_2)_*^{-1}\gamma Sdf^*\gamma(M^m, s_M)\\ &= (-1)^{m-1}(\pi_1 \circ j_3)_* \circ (j_2)_*^{-1} Sdf^!(M^m, s_M)\end{aligned}$$

It remains only to note that $i_* = (\pi_1 \circ j_3)_* \circ (j_2)_*^{-1}$. $\square$

To prove the corollaries 2.2 and 2.3 we will need the following results from the paper [2].

Theorem 4.2 ([2])**.** *For any smooth generic immersion $f : V^n \looparrowright M^m$ of the compact oriented manifold without boundary V^n to the oriented manifold without boundary M^m such that $m - n$ is even, we have*

$$v_k = f^*(m_{k-1}) - e \cup v_{k-1}, \tag{3}$$

where e is the Euler class of the normal bundle of the immersion f over V^n.

Corollary 4.3 ([2])**.** *Under the conditions of theorem 4.2, we have*

$$(k-1)! \cdot v_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V),$$

where $\varphi_k(a) = f^* f_!(a) - k \cdot e \cup a$; e is the Euler class of the normal bundle of the immersion f over V^n.

Proof of corollary 2.2. It suffices to substitute (1) into formula (3) with $k = 2$. □

Proof of corollary 2.3. follows immediately from corollaries 4.3 and 2.2. □

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References

[1] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, 1973.
[2] F. Ronga, *On multiple points of smooth immersions*, Comment. Math. Helv. **55:4** (1980), 521–527.
[3] U. Koshcorke and B. Senderson, *Self-intersections and higher Hopf invariants*, Topology **17** (1978), 283–290.
[4] M. Hirsch, *Immersions of manifolds*, Trans. AMS **93** (1959), 242–276.
[5] A. Szücs, *On the multiple points of immersions in euclidean spaces*, Proc. AMS **126:6** (1998), 1873–1882.
[6] M.A. Asadi-Golmankhaneh and P.J. Eccles, *Determining the characteristic numbers of the self-intersection manifolds*, Jour. London Math. Soc. (to appear), available at `http://www.maths.man.ac.uk/~peter/`.
[7] P.E. Conner and E.E. Floyd, *Differentiable periodic maps*, Springer-Verlag, 1964.
[8] R.W. Switzer, *Algebraic topology — homotopy and homology*, Springer-Verlag, 1975.
[9] R. Wells, *Cobordisms of immersions*, Topology **5** (1966), 281–293.
[10] A. Szücs, *Cobordism of maps with simplest singularities*, Topology Symposium, Siegen 1979, Lecture Notes in Math vol. 788, Springer-Verlag, 1980, 223–244.
[11] A. Haefliger and M. Hirsch, *Immersions in the stable range* Annals of Math. **75** (1962), 231–241.
[12] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Ann. of Math. Studies vol.76, Princeton University Press, 1974.

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