# MULTIPLE POINTS OF IMMERSIONS

KONSTANTIN SALIKHOV

Abstract. Given smooth manifolds $V^n$ and $M^m$, an integer $k > 1$, and an immersion $f : V \looparrowright M$, we have constructed an obstruction for existence of regular homotopy of $f$ to an immersion $f' : V \looparrowright M$ without $k$-fold points. It takes values in certain bordism group, and for $(k+1)(n+1) \le km$ turns out to be complete. As a byproduct, under certain dimensional restrictions we also constructed a complete obstruction for eliminating by regular homotopy the points of common intersection of several immersions $f_1 : V_1 \looparrowright M, \dots, f_k : V_k \looparrowright M$.

## 1. Introduction

Let $V^n$ and $M^m$ be smooth manifolds without boundary, and $V^n$ be compact. We will call an immersion $V \looparrowright M$ by *k-immersion*, if it has no $k$-to-1 points. Then 2-immersions $V \looparrowright M$ are exactly embeddings $V \hookrightarrow M$. Consider the following question. Given manifolds $V^n$, $M^m$ and integer $k > 1$, do they admit a $k$-immersion $V \looparrowright M$? Note that if $(k+1)n < km$, then any generic immersion $V \looparrowright M$ is a $(k+1)$-immersion. The problem of existence of an immersion $V \looparrowright M$ was solved by Hirsch [Hir]. He proved that regular homotopy classes of immersions $V \looparrowright M$ are in 1-to-1 correspondence with classes of linear monomorphisms of tangent bundles of $\tau V \to \tau M$. So, the natural problem would be to find out if a given regular homotopy class of immersions $V^n \looparrowright M^m$ contains a $k$-immersion, provided $(k+1)n < km$. Without loss of generality we can assume that $M$ is connected.

For the case $k = 2$, under a little bit stronger restrictions $3(n+1) \le 2m$, in 1962 Haefliger gave complete answer whenever it is possible [Hae]. Haefliger's method generalizes the original purely geometric idea of Whitney [Wh] of eliminating double points of immersion. A decade later, in 1974, Hatcher and Quinn [HQ] rewrote the the Haefliger's reasonings in the "right" language of bordism theory. In 1982 Szücs [Sz 82] gave a new proof of Haefliger's theorem in the special case $M^m = \mathbb{R}^m$ using the ideas of theory of singularities. Szücs informed us that he was trying to apply his method for $k > 2$. In the present paper we generalize the ideas from [HQ] and in the range $(k+1)(n+1) \le km$ give the complete answer whenever a given regular homotopy class of immersions $V^n \looparrowright M^m$ contains a $k$-immersion.

For a generic immersion $f : V^n \looparrowright M^m$, its set of $k$-fold points is itself an immersed manifold $\overline{\pitchfork}(f^{(k)}) \looparrowright M$. This immersed manifold, together with some additional structure in normal bundle of this immersion, define an element $b(f^{(k)})$ in certain bordism group $\Omega = \Omega_{kn-(k-1)m}(E_k; \xi(f))$. Both the group $\Omega$ and the element $b(f^{(k)}) \in \Omega$ depend only on the regular homotopy class of $f$ (see Theorem 3.1). If $(k+1)(n+1) \le km$, then the converse is true (Theorem 3.2). Namely, if $N$ is another $(kn-(k-1)m)$-dimensional manifold, which represents the same element $[N] = b(f^{(k)}) \in \Omega$, then there exists an immersion $f_1$, regularly homotopic to $f$, such that $\overline{\pitchfork}(f_1^{(k)}) = N$.

Suppose $f : V^{(k-1)r} \looparrowright M^{kr}$ is a generic immersion. Then the manifold of $k$-fold points of $f$ is zero-dimensional. If $V, M$ are oriented and $r$ is even, to each $k$-fold point $x \in M$ of $f$ we can attach a sign. Since $V, M$ are oriented, the normal bundle of the immersion $f$ is oriented. Since $r$ is even, the orientation of the normal spaces to $k$ "sheets" of $V$, intersecting at $x$, gives an orientation of the tangent space $\tau_x M$. We put sign "+" if this orientation coincides with the orientation of $M$, and "−" otherwise. Define $I(f) \in \mathbb{Z}$ as the number of $k$-fold points, counted with signs. If either $V^{(k-1)r}$ or $M^{kr}$ is not orientable, or $r$ is odd, define $I(f) \in \mathbb{Z}_2$ as the number of $k$-fold points modulo 2.

**Corollary 1.1.** *Let $V^{(k-1)r}, M^{kr}$ be connected smooth manifolds without boundary, $V$ is compact, $M$ is simply connected, and $r > k$. Then a generic immersion $f : V \looparrowright M$ is regularly homotopic to a $k$-immersion iff $I(f) = 0$.*

**Corollary 1.2.** *Let $V^{(k-1)r}$ be a smooth compact orientable manifold without boundary, and $r > k$ are odd integers. Then any immersion $f : V \looparrowright \mathbb{R}^{kr}$ is regularly homotopic to a $k$-immersion.*

A map $f : V \to M$ is called a *"topological immersion"*, if for any point $x \in V$ there exists an open neighborhood $U_x \ni x$ such that for any $y_1 \neq y_2 \in U_x$ we have $f(y_1) \neq f(y_2)$.

**Corollary 1.3.** *Let $V^n, M^m$ be smooth manifolds without boundary, $V^n$ be compact, $f : V \looparrowright M$ be an immersion, and $(k+1)(n+1) \leq km$. Then $f$ is regularly homotopic to a $k$-immersion iff there exists a "regular homotopy" of $f$ through topological immersions to a map without $k$-fold points.*

The main problem of constructing the desired regular homotopy virtually splits into two parts: geometrical and combinatorial. The first part deals with construction of this regular homotopy locally. The second part, which in general turns out to be harder, deals with the fact that the natural $k$-fold covering map $\widetilde{\overline{\pitchfork}(f^{(k)})} \to \overline{\pitchfork}(f^{(k)})$ can be non-trivial. In the section 2 we restrict ourselves to the case, then the second part is trivial. Namely, we try to figure out whenever immersions of $k$ different manifolds $f_1 : V_1 \looparrowright M, \dots, f_k : V_k \looparrowright M$ can be regularly homotoped to immersions without common intersection.

**Corollary 1.4.** *Let $V_i^{n_i}$, $i = 1 \dots k$ and $M^m$ be smooth $C^\infty$ manifolds without boundary, all $V_i$ are compact, $2(p+1) < n_i < m-p-k$ for all $i = 1 \dots k$, and $2(p+k) < m$, where $p = n_1 + \dots + n_k - (k-1)m$. Suppose $f_i : V_i \looparrowright M$ are smooth immersions. Then the immersions $f_1, \dots, f_k$ are regularly homotopic to immersions without common intersection iff $f_1, \dots, f_k$ can be continuously homotoped to continuous maps with empty common intersection.*

**Corollary 1.5.** *Let $V_i^{n_i}$, $i = 1 \dots k$ and $M^m$ be smooth $C^\infty$ manifolds without boundary, all $V_i$ are compact, $2(p+1) < n_i < m-p-k$ for all $i = 1 \dots k$, and $2(p+k) < m$, where $p = n_1 + \dots + n_k - (k-1)m$. Suppose $p \in \{4, 5, 12\}$, all $V_i$ are $p$-connected, and $M$ is $(p+1)$-connected. Then any immersions $f_i : V_i \looparrowright M$ can be regularly homotoped to immersions without common intersection.*

## 2. Simple case: mutual intersections of $k$ manifolds

Let $T$ be $(k-1)$-dimensional simplex, $\sigma_1, \dots, \sigma_k$ be its vertices and $\sigma$ be its barycenter. For any topological spaces $V_1, V_2, \dots, V_k, M$ and continuous maps $f_1 : V_1 \to M, \dots, f_k : V_k \to M$, consider the space

$$E = E(f_1, \dots, f_k) = \{(x_1, \dots, x_k, \theta) \mid x_i \in V_i \text{ and } \theta : T \to M \text{ such that } \theta(\sigma_i) = f_i(x_i)\}$$

Then there are obvious projections $\pi_i : E \to V_i$ and $\pi : E \to \{\theta : T \to M\}$. The map $\Phi : E \times T \to M$, defined by $(e, t) \mapsto \pi(e)(t)$, where $e \in E$ and $t \in T$, makes the diagram below homotopy commutative

$$\begin{tikzcd} & & E \arrow[dll, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[dr, "\pi_k"] & \\ V_1 \arrow[drr, "f_1"'] & V_2 \arrow[dr, "f_2"] & \cdots & V_k \arrow[dl, "f_k"] \\ & & M & \end{tikzcd}$$

The space $E$ is universal in the following sense. If we are given a space $X$, maps $h_i : X \to V_i$, and a homotopy $H : X \times T \to M$ such that $H(\cdot, \sigma_i) = f_i h_i$ (which connects all $f_i h_i$ with some map $H(\cdot, \sigma)$), then there is a unique map $j : X \to E$ such that $H(x, t) = \Phi(j(x), t)$, which is defined by $j(x) = (h_1(x), \dots, h_k(x), H(x, \cdot))$.

2.1. **Bordism group.** From now, suppose all $V_i$ and $M$ are smooth $C^\infty$ manifolds without boundary and $V_i$ are compact. Denote by $\nu_{V_i}$ normal bundles over $V_i$, and by $\tau_M$ the tangent bundle over $M$. Over $E$, consider the bundle $\xi = \pi_1^*(\nu_{V_1} \oplus f_1^* \tau_M) \oplus \dots \oplus \pi_k^*(\nu_{V_k} \oplus f_k^* \tau_M) \oplus \Phi(\cdot, \sigma)^* \nu_M$. Define the bordism group $\Omega_p(E; \xi)$, whose objects are tuples $(N^p, \nu_N, \gamma, \omega_N)$, where $N$ is a compact manifold without boundary, $\nu_N$ is a normal bundle over $N$, $\gamma : N \to E$ is a map, and $\omega_N : \nu_N \xrightarrow{\sim} \gamma^* \xi$ is a stable isomorphism. Note that this group depends only on the homotopy classes of $f_1, \dots, f_k$. By Pontryagin-Thom construction, $\Omega_p(E; \xi) = \lim_{s \to \infty} \pi_{p + \dim \xi + s} T(\xi \oplus \varepsilon^s)$, where $T$ denotes the Thom space, and $\varepsilon$ is the trivial bundle.

Denote by $\Delta_M^k$ the diagonal $\{(x,\dots,x)\in M^{(k)} \mid x\in M\}$. If the map $f_1\times\dots\times f_k : V_1\times\dots\times V_k\to M^{(k)}$ is smooth and transversal to the diagonal $\Delta_M^k$, then we say *"the maps $f_1,\dots,f_k$ are transversal"* and denote the manifold $(f_1\times\dots\times f_k)^{-1}(\Delta_M^k)\subset V_1\times\dots\times V_k$ by $\overline{\pitchfork}(f_1\times\dots\times f_k)$. Since all $V_i$ are compact, $\overline{\pitchfork}(f_1\times\dots\times f_k)$ is also compact. Note that the notion for maps $f_1,\dots,f_k$ to be transversal is generic, and any kit of maps $f_1,\dots,f_k$ can be approximated by a transversal one.

**Theorem 2.1.** *Suppose $V_1,\dots,V_k,M$ are smooth $C^\infty$ manifolds without boundary, $V_1,\dots,V_k$ are compact, and the maps $f_i : V_i\to M$, $i=1\dots k$ are continuous. Then this defines a canonical element $b(f_1\times\dots\times f_k)\in\Omega_*(E(f_1,\dots,f_k);\xi)$. This element depends only on the homotopy classes of $f_1,\dots,f_k$.*

**Theorem 2.2.** *Let $V_i^{n_i}$, $i=1\dots k$ and $M^m$ be smooth $C^\infty$ manifolds without boundary, all $V_i$ are compact, $2(p+1)<n_i<m-p-k$ for all $i=1\dots k$, and $2(p+k)<m$, where $p=n_1+\dots+n_k-(k-1)m$. Suppose $f_i : V_i\looparrowright M$ are smooth immersions. Let $\gamma : N^p\to E$ be a singular manifold, and $\omega_N:\nu_N\xrightarrow{\sim}\gamma^*\xi$ be a stable isomorphism such that $[N]=b(f_1\times\dots\times f_k)\in\Omega_p(E;\xi)$. Then there exists a regular homotopy of the immersions $f_1,\dots,f_k$ to transversal immersions $f_1',\dots,f_k'$ such that $\overline{\pitchfork}(f_1'\times\dots\times f_k')=N$.*

## 3. General case: $k$-fold self-intersections

Let $V$ and $M$ be topological spaces and $f : V\to M$ be a continuous map. The group $\Sigma_k$ of permutations on $k$ elements $\{1,\dots,k\}$ acts on the $(k-1)$-simplex $T$ linearly, permuting its vertices $\sigma_1,\dots,\sigma_k$. Note that the only fixed point for this action is the barycenter $\sigma$ of $T$. Take $k$ copies of the manifold $V$ and enumerate them by $1\dots k$ in arbitrary way $V_1,\dots,V_k$. Consider the space

$$\widehat{E_k}=\widehat{E_k}(f)=\{(x_1,\dots,x_k,\theta)\mid x_i\in V_i, x_i\neq x_j \text{ for } i\neq j, \text{ and } \theta : T\to M \text{ such that } \theta(\sigma_i)=f(x_i)\}$$

Define the action of element $g\in\Sigma_k$ on $\widehat{E_k}$ by the formula $g(x_1,\dots,x_k,\theta)=(x_{g^{-1}(1)},\dots,x_{g^{-1}(k)},\theta\circ g^{-1})$. Note that this action is free. Put $E_k=\widehat{E_k}/\Sigma_k$. Define the projections $\pi_i:\widehat{E_k}\to V$ and $\pi:\widehat{E_k}\to\{\theta : T\to M\}$ by the formulas $\pi_i(x_1,\dots,x_k,\theta)=x_i$ and $\pi(x_1,\dots,x_k,\theta)=\theta$. It is easy to see, the map $\Phi:\widehat{E_k}\times T\to M$, defined by $(e,t)\mapsto\pi(e)(t)$, is $\Sigma_k$-invariant with respect to the diagonal $\Sigma_k$-action on $\widehat{E_k}\times T$. Therefore the map $\phi : E_k\to M$ is well-defined by the formula $[e]\mapsto\Phi(e,\sigma)$. Note that the homotopy types of $\widehat{E_k}$ and $E_k$ depend only on the homotopy class of $f$.

The space $\widehat{E_k}$ is universal in the following sense. For any space $X$ with a free $\Sigma_k$-action denote by $X_{(i)}$ the quotion space $X/\Sigma_{k-1}^i$, where $\Sigma_{k-1}^i\subset\Sigma_k$ is the stabilizer of the element $i$. Denote by $\pi_i' : X\to X_{(i)}$ and by $\pi' : X\to X/\Sigma_k$ the natural projections. Denote by $\Sigma_{k-1}^i(x)$ the orbit of $x\in X$ under $\Sigma_{k-1}^i$-action. Note that $g(\Sigma_{k-1}^i(x))=\Sigma_{k-1}^{g(i)}(g(x))$. So, for any $g\in\Sigma_k$ the map $g : X_{(i)}\to X_{(g(i))}$ is well-defined, and it is identity if $g=1\in\Sigma_k$.

$$\begin{tikzcd}
 & & X \arrow[dll, "\pi_1'"'] \arrow[dl, "\pi_2'"] \arrow[dr, "\pi_k'"'] \arrow[drr, "\pi'"] \arrow[rrr, "\hat{\jmath}"] & & & \widehat{E_k} \arrow[d] \\
X_{(1)} \arrow[d, "h_1"'] & X_{(2)} \arrow[d, "h_2"] & \cdots & X_{(k)} \arrow[d, "h_k"'] & X/\Sigma_k \arrow[r, "j"] \arrow[ddll, bend left, "h"'] & E_k \arrow[ddlll, bend left, "\phi"] \\
V \arrow[drr, "f"'] & V \arrow[dr, "f"] & \cdots & V \arrow[dl, "f"'] & & \\
 & & M & & &
\end{tikzcd}$$

Suppose, we are given the following data:

- a space $X$ with a free $\Sigma_k$-action
- a map $h : X/\Sigma_k\to M$
- maps $h_i : X_{(i)}\to V_i(\cong V)$ such that $h_{g(i)}\circ g=h_i$ for any $g\in\Sigma_k$, and $h_i\pi_i'(x)\neq h_j\pi_j'(x)$ for $x\in X$ and $i\neq j$

- a $\Sigma_k$-invariant homotopy $H : X \times T \to M$ such that $H(\cdot, \sigma) = h\pi'$ and $H(\cdot, \sigma_i) = fh_i\pi'_i$

Then there is a unique $\Sigma_k$-equivariant map $\widehat{\jmath} : X \to \widehat{E_k}$ such that $H(x,t) = \Phi(\widehat{\jmath}(x), t)$, which is defined by $\widehat{\jmath}(x) = (h_1\pi'_1(x), \dots, h_k\pi'_k(x), H(x, \cdot))$. Now we can get rid of the ambiguity of the choice of enumeration $V_1, \dots, V_k$. Clearly, if we will take another enumeration, which differs from the original one by $g \in \Sigma_k$, then the "classifying map" $\widehat{\jmath}$ will become $\widehat{\jmath} \circ g = g \circ \widehat{\jmath}$. Thus, all the information about $X$ is actually equivalent to a $\Sigma_k$-orbit of $\Sigma_k$-equivariant maps $\widehat{\jmath} : X \to \widehat{E_k}$. Since $\Sigma_k$-actions on $X$ and $\widehat{E_k}$ are free, this is equivalent to the map $j := \widehat{\jmath}/_{\Sigma_k} : X/\Sigma_k \to \widehat{E_k}/\Sigma_k = E_k$.

3.1. **Bordism group.** From now, suppose $V^n$ and $M^m$ are smooth $C^\infty$ manifolds without boundary, $V$ is compact. Let $f : V \looparrowright M$ be an immersion. Denote by $\nu_f = \nu_f(V, M)$ the normal bundle over $V$, induced by the immersion $f$, and by $\nu_M$ a normal bundle over $M$. Over $\widehat{E_k}$, consider the bundle $\pi_1^*\nu_f \oplus \dots \oplus \pi_k^*\nu_f$. Define the $\Sigma_k$-action in the total space of this bundle by $g(e, \vec{v}_1, \dots, \vec{v}_k) = (g(e), \vec{v}_{g^{-1}(1)}, \dots, \vec{v}_{g^{-1}(k)})$. Since this action covers the $\Sigma_k$-action on $\widehat{E_k}$, this gives a well-defined vector bundle $(\pi_1^*\nu_f \oplus \dots \oplus \pi_k^*\nu_f)/\Sigma_k$ over $E_k = \widehat{E_k}/\Sigma_k$. Note that the structural group of the last bundle is $O(m-n) \wr \Sigma_k$ (see [AE, KS] for details). As a set, $O(m-n) \wr \Sigma_k = O(m-n)^{(k)} \times \Sigma_k$, and the multiplication is defined by $(A_1, \dots, A_k, \sigma) * (B_1, \dots, B_k, \tau) = (A_1B_{\sigma^{-1}(1)}, \dots, A_kB_{\sigma^{-1}(k)}, \sigma\tau)$. Over $E_k$, consider the bundle $\xi = \xi(f) = (\pi_1^*\nu_f \oplus \dots \oplus \pi_k^*\nu_f)/\Sigma_k \oplus \phi^*\nu_M$. Its structural group is $O(m-n) \wr \Sigma_k \oplus O$. Denote $p = kn - (k-1)m$. Define the bordism group $\Omega_p(E_k; \xi)$, whose objects are tuples $(N^p, \nu_N, \gamma, \omega_N)$, where $N$ is a compact manifold without boundary, $\nu_N$ is a normal bundle over $N$, $\gamma : N \to E_k$ is a map, and $\omega_N : \nu_N \xrightarrow{\sim} \gamma^*\xi$ is a stable isomorphism. Note that the bundle $\nu_f$, and, therefore, the group $\Omega_p(E_k; \xi(f))$, depend only on the regular homotopy class of $f$. By Pontryagin-Thom construction, $\Omega_p(E_k; \xi) = \lim_{s\to\infty} \pi_{p+\dim\xi+s} T(\xi \oplus \varepsilon^s)$, where $T$ denotes the Thom space, and $\varepsilon$ is the trivial bundle.

Denote by $\Delta_V^2$ the diagonal $\{(x_1, \dots, x_k) \in V^{(k)} \mid x_i \neq x_j \text{ for } i \neq j\}$. Note that in the important special case $M^m = \mathbb{R}^m$ the space $E_k$ reduces to $kn$-dimensional manifold $(V^{(k)} - \Delta_V^2)/\Sigma_k$, and $\xi$ reduces to the bundle $\nu_{f^{(k)}}/\Sigma_k$.

We say that a $\Sigma_k$-equivariant map $F : V^{(k)} \to M^{(k)}$ is "*k-disjoint*", if $F^{-1}(\Delta_M^k) - \Delta_V^2$ is disjoint from $\Delta_V^2$. Note that for any topological immersion $f : V \to M$, the map $f^{(k)}$ is $k$-disjoint. We say that a $k$-disjoint map $F$ is "*k-transversal*", if either $F^{-1}(\Delta_M^k) - \Delta_V^2 = \emptyset$, or the map $F|_{V^{(k)} - \Delta_V^2}$ is smooth and transversal to $\Delta_M^k$. By equivariant transversality theorem [GG], for a generic smooth immersion $f : V \looparrowright M$, the map $f^{(k)}$ is $k$-transversal. Denote the manifold $F^{-1}(\Delta_M^k) - \Delta_V^2$ by $\widehat{\overline{\pitchfork}(F)}$. Since $V^{(k)}$ is compact and $\widehat{\overline{\pitchfork}(F)}$ is disjoint from $\Delta_V^2$, the manifold $\widehat{\overline{\pitchfork}(F)}$ is also compact. Since $\widehat{\overline{\pitchfork}(F)}$ is the intersection of $\Sigma_k$-invariant set $V^{(k)} - \Delta_V^2$ with the preimage of $\Sigma_k$-equivariant map $F : V^{(k)} \to M^{(k)}$ of the $\Sigma_k$-invariant set $\Delta_M^k$, it is invariant under $\Sigma_k$-action on $V^{(k)}$. Since the $\Sigma_k$-action on $V^{(k)} - \Delta_V^2$ is free, its restriction on $\widehat{\overline{\pitchfork}(F)}$ is also free. Denote $\overline{\pitchfork}(F) = \widehat{\overline{\pitchfork}(F)}/\Sigma_k$. Note that if $f : V \looparrowright M$ is a generic immersion, then $\overline{\pitchfork}(f^{(k)}) \looparrowright M$ is the locus of its $k$-fold points.

**Theorem 3.1.** *Suppose $V^n, M^m$ are smooth $C^\infty$ manifolds without boundary, $V$ is compact, and $F : V^{(k)} \to M^{(k)}$ is a $k$-transversal map, $\Sigma_k$-equivariantly homotopic to to $f^{(k)}$ for some immersion $f : V \looparrowright M$. Then $\overline{\pitchfork}(F)$ defines a canonical element $b(F) \in \Omega_p(E_k; \xi(f))$. This element depends only on the $k$-disjoint homotopy class of $F$.*

Note that for an immersion $f : V \looparrowright M$, the element $b(f^{(k)}) \in \Omega_p(E_k; \xi(f))$ is well-defined even if $f^{(k)}$ is not $k$-transversal. Indeed, let $f_1$ be an immersion, regularly homotopic to $f$, such that $f_1^{(k)}$ is $k$-transversal. From the second part of Theorem 3.1 it follows that $b(f_1^{(k)})$ doesn't depend on the choice of $f_1$. Put $b(f^{(k)}) = b(f_1^{(k)}) \in \Omega_p(E_k; \xi(f))$.

**Theorem 3.2.** *Let $V^n, M^m$ be smooth $C^\infty$ manifolds without boundary, $V$ be compact, $(k+1)(n+1) \leq km$, and $f : V \looparrowright M$ be an immersion. Let $\gamma : N^p \to E_k$ be a singular manifold, and $\omega_N : \nu_N \xrightarrow{\sim} \gamma^*\xi$ be a stable isomorphism such that $[N] = b(f^{(k)}) \in \Omega_p(E_k; \xi(f))$. Then there exists a regular homotopy of the immersion $f$ to an immersion $f_1$ such that $f_1^{(k)}$ is $k$-transversal and $N = \overline{\pitchfork}(f_1^{(k)})$.*

## 4. Proofs of Corollaries

*Proof of Corollary 1.1.* Note that $\pi_1 \times \cdots \times \pi_k : \widehat{E_k} \to V^{(k)} - \Delta_V^2$ is a Serre fibration with the fiber, homotopy equivalent to $F = \{\theta : T \to M \mid \theta(\sigma_i) = *\}$, where $* \in M$ is a base point. In $T$, consider 1-dimensional subcomplex $T_1$, consisting of vertices $\sigma_1, \dots, \sigma_k$ and $(k-1)$ edges $\sigma_1\sigma_2, \dots, \sigma_1\sigma_k$. Since $T_1$ is a deformation retract of $T$, the space $F$ is homotopy equivalent to $F_1 = \{\theta : T_1 \to M \mid \theta(\sigma_i) = *\} = (\Omega M)^{(k-1)}$. Since $M$ is 1-connected, then the loopspace $\Omega M$ is connected. Since $V$ is connected and $\operatorname{codim}_{V^{(k)}}(\Delta_V^2) = n > 1$, then $V^{(k)} - \Delta_V^2$ is connected. Hence $\widehat{E_k}$ and $E_k = \widehat{E_k}/\Sigma_k$ are connected.

Since $M$ is 1-connected, then $\phi^*\nu_M$ is always orientable. Therefore $\xi = (\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k \oplus \phi^*\nu_M$ is orientable iff $(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k$ is orientable. Suppose $V$ is orientable and $(m-n)$ is even. Then $\nu_f$ is orientable, fix its orientation. Then $\pi_i^*\nu_f$ are oriented. Since $(m-n)$ is even, the $\Sigma_k$-action on $\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f$ preserves orientation. Hence $(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k$ is orientable.

Suppose $V$ is orientable and $(m-n)$ is odd. Then $\nu_f$ is orientable, fix its orientation. Then $\pi_i^*\nu_f$ are oriented. Choose points $b_0, b_1 \in \widehat{E_k}$ such that $g(b_0) = b_1$, where $g \in \Sigma_k$ is an odd permutation. Since $\widehat{E_k}$ is connected, there exists a path $b : [0,1] \to \widehat{E_k}$ such that $b(0) = b_0, b(1) = b_1$. Then the projection of $b_t$ into $E_k$ defines a loop. Since $r$ is odd and $g$ is odd, then the restriction of $(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k$ on this loop is non-orientable.

Suppose $V$ is non-orientable. Then $\nu_f$ is non-orientable. Choose a loop $c : [0,1] \to V$ such that the restriction of $\nu_f$ on $c_t$ in non-orientable. Choose points $c_2, \dots, c_k \in V$ such that $c_i \neq c_j$ if $i \neq j$, and $c_i \neq c(t)$ for all $t \in [0,1]$. This defines a loop $C : [0,1] \to V^{(k)} - \Delta_V^2$ via $C(t) = (c(t), c_2, \dots, c_k)$. Since $\pi_1 \times \cdots \times \pi_k : \widehat{E_k} \to V^{(k)} - \Delta_V^2$ is a Serre fibration with connected fiber, then there exists a loop $B : [0,1] \to \widehat{E_k}$ such that $\pi_1 \times \cdots \times \pi_k \circ B_t \simeq C_t$. Then the projection of the loop $B_t$ into $E_k$ defines a loop, along which $(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k$ is non-orientable.

Choose a base point $* \in E_k$. Since $E_k$ is connected, any map of a 0-dimensional manifold $\gamma : N^0 \to E_k$ can be homotoped to the constant map $N^0 \to *$. Since the orthogonal group $O$ has two connected components, then a two-point singular manifold $\{a_1, a_2\} \to *$ bounds iff the orientations of normal bundles of $a_1, a_2$ at $*$ are opposite. Suppose the bundle $\xi$ is orientable. Fix its orientation. Then the orientation of the normal bundle of $a_1$ at $*$ does not depend on the choice of homotopy $\gamma \rightsquigarrow (N^0 \to *)$. Therefore the bordism class of a singular manifold $\gamma : N^0 \to E_k$ is completely characterized by the number $l_1 - l_2 \in \mathbb{Z}$, where $l_1$ is the number of points in $N^0$, whose orientation coincides with the orientation of $\xi$, and $l_2$ — with opposite orientation. Hence $\Omega_0(E_k; \xi) = \mathbb{Z}$. If $\xi$ is non-orientable, then any singular manifold $a \to *$ is bordant to itself with opposite orientation via a loop, that changes the orientation of $\xi$. Therefore $\Omega_0(E_k; \xi) = \mathbb{Z}_2$. It is easy to see that the invariant $b(f^{(k)}) \in \Omega_0(E_k; \xi)$ coincides with $I(f)$. Since $r > k$, then $(k+1)((k-1)r+1) \leq k^2 r$. Then Corollary 1.1 follows from Theorems 3.1 and 3.2. $\square$

*Proof of Corollary 1.2.* WLOG we may assume that $V$ is connected. By Corollary 1.1, it suffices to show that the number of $k$-fold points of any generic immersion $V \looparrowright \mathbb{R}^{kr}$ is even. This is exactly the result of [Sz 90]. $\square$

*Proof of Corollary 1.3.* Let $f_t : V \to M, t \in [0,1]$ be a topological regular homotopy, $f_0 = f$ and $f_1$ has no $k$-to-1 points. Then $f_t^{(k)}$ is a $k$-disjoint homotopy from $f^{(k)}$ to a map $f_1^{(k)}$ such that $(f_1^{(k)})^{-1}(\Delta_M^k) - \Delta_V^2 = \emptyset$. Then by Theorem 3.1, the class $b(f^{(k)}) = 0 \in \Omega_p(E_k; \xi)$, and Theorem 3.2 gives the desired regular homotopy. $\square$

*Proof of Corollary 1.4.* follows immediately from Theorem 2.1 and Theorem 2.2. $\square$

*Proof of Corollary 1.5.* As we saw above, $E \to V_1 \times \cdots \times V_k$ is a Serre fibration with fiber, homotopy equivalent to $(\Omega M)^{(k-1)}$. Since all $V_i$ are $p$-connected and $M$ is $(p+1)$-connected, then $E$ is $p$-connected. Then any map $\gamma : N^p \to E$, representing a bordism class $\Omega_p(E; \xi)$, is null-homotopic. Therefore $\nu_N$ is trivial, and, by Pontryagin-Thom construction, $(N, \omega_N)$ represents an element of $p$-th stable homotopy group of spheres. Since this group vanishes for $p \in \{4, 5, 12\}$, then $(N, \omega_N) = 0$. Therefore $\Omega_p(E; \xi) = 0$, and by Theorem 2.2, any immersions $f_i : V_i \looparrowright M$ are regularly homotopic to immersions without common intersection. $\square$

## 5. Proofs of Theorems

*Proof of Theorem 2.1.* First, suppose the maps $f_1, \dots, f_k$ are transversal. Then there are obvious projections $h_i : \overline{\pitchfork}(f_1 \times \cdots \times f_k) \to V_i$ such that all $f_i h_i$ are connected by constant homotopy. By universal property of $E(f_1, \dots, f_k)$, there is a canonical map $j : \overline{\pitchfork}(f_1 \times \cdots \times f_k) \to E$. Denote by $\nu(\Delta_M^k, M^{(k)})$ the normal bundle of $\Delta^k_M$ in $M^{(k)}$. By construction of $\overline{\pitchfork}(f_1 \times \cdots \times f_k)$, the normal bundle $\nu(\overline{\pitchfork}(f_1 \times \cdots \times f_k), V_1 \times \cdots \times V_k)$ is $(f_1 \times \cdots \times f_k)^*\nu(\Delta^k_M, M^{(k)}) \cong j^*(f_1\pi_1 \times \cdots \times f_k\pi_k)^*\nu(\Delta^k_M, M^{(k)}) \simeq j^*(f_1\pi_1 \times \cdots \times f_k\pi_k)^*(\tau_{M^{(k)}} \oplus \nu_{\Delta^k_M})$. But $(f_1\pi_1 \times \cdots \times f_k\pi_k) \circ j = D \circ \Phi(\cdot, \sigma) \circ j$, where $D : M \to \Delta^k_M$ is the diffeomorphism $x \mapsto (x, \ldots, x)$. Then $\nu_{\overline{\pitchfork}(f_1 \times \cdots \times f_k)} = \nu_{V_1 \times \cdots \times V_k}|_{\overline{\pitchfork}(f_1 \times \cdots \times f_k)} \oplus \nu(\overline{\pitchfork}(f_1 \times \cdots \times f_k), V_1 \times \cdots \times V_k) \simeq j^*(\pi_1 \times \cdots \times \pi_k)^*(\nu_{V_1 \times \cdots \times V_k}) \oplus j^*(f_1\pi_1 \times \cdots \times f_k\pi_k)^*\tau_{M^{(k)}} \oplus j^*\Phi(\cdot, \sigma)^*\nu_M = j^*\xi$. Thus the map $j : \overline{\pitchfork}(f_1 \times \cdots \times f_k) \to E$ defines an element $b(f_1 \times \cdots \times f_k) \in \Omega_p(E; \xi)$.

Now, suppose we have a kit of homotopies $f_{1,t}, \ldots, f_{k,t}$, $t \in [0,1]$, such that the kits $f_{1,0}, \ldots, f_{k,0}$ and $f_{1,1}, \ldots, f_{k,1}$ are transversal. Then it can be approximated by a new one such that $F = f_{1,\cdot} \times \cdots \times f_{k,\cdot} : V_1 \times \cdots \times V_k \times [0,1] \to M^{(k)}$ is transversal to $\Delta^k_M$ and is left fixed on both ends $t = 0, 1$ (where we already have transversality). Let the manifold $W = F^{-1}(\Delta^k_M)$. Then $\partial W = \overline{\pitchfork}(f_{1,0} \times \cdots \times f_{k,0}) \cup \overline{\pitchfork}(f_{1,1} \times \cdots \times f_{k,1})$.

Denote by $t : V_1 \times \cdots \times V_k \times [0,1] \to [0,1]$ the projection on the last factor. Here we again have obvious projections $\tilde{h}_i : W \to V_i$ such that $f_{i,0}\tilde{h}_i$ are homotopic to $D^{-1} \circ F = f_{i,t(\cdot)}(h_i(\cdot))$ by homotopy $x \mapsto f_{i,t(x)t'}(h_i(x))$, $t' \in [0,1]$. This gives a homotopy $H : W \times T^1 \to M$, where $T^1$ is the 1-dimensional subcomplex of $T$, consisting of vertices $\sigma, \sigma_1, \ldots, \sigma_k$ and straight intervals $\sigma\sigma_1, \ldots, \sigma\sigma_k$. Since there exists a canonical conical retraction of $T$ onto $T^1$, there is a canonical extension of $H$ to a homotopy $H : W \times T \to M$. By universal property of $E$, this gives a canonical map $J : W \to E$.

The normal bundle $\nu(W, V_1 \times \cdots \times V_k \times [0,1])$ is $F^*\nu(\Delta^k_M, M^{(k)}) \cong (F \circ (\pi_1 \times \cdots \times \pi_k \times t) \circ J)^*\nu(\Delta^k_M, M^{(k)}) \simeq J^*(\pi_1 \times \cdots \times \pi_k \times t)^*F^*(\tau_{M^{(k)}} \oplus \nu_{\Delta^k_M})$. But $F \circ (\pi_1 \times \cdots \times \pi_k \times t) \circ J = D \circ \Phi(\cdot, \sigma) \circ J$. Note that $(\pi_1 \times \cdots \times \pi_k \times t \circ J)^*(\nu_{V_1 \times \cdots \times V_k \times [0,1]}) \simeq (\pi_1 \times \cdots \times \pi_k \circ J)^*(\nu_{V_1 \times \cdots \times V_k})$ and $(\pi_1 \times \cdots \times \pi_k \times t \circ J)^*F^*\tau_{M^{(k)}} \simeq (\pi_1 \times \cdots \times \pi_k \circ J)^*(f_{1,0} \times \cdots \times f_{k,0})^*\tau_{M^{(k)}}$. Then $\nu_W = \nu_{V_1 \times \cdots \times V_k \times [0,1]}|_W \oplus \nu(W, V_1 \times \cdots \times V_k \times [0,1]) \simeq J^*(\pi_1 \times \cdots \times \pi_k \times t)^*(\nu_{V_1 \times \cdots \times V_k \times [0,1]}) \oplus J^*(\pi_1 \times \cdots \times \pi_k \times t)^*F^*\tau_{M^{(k)}} \oplus J^*\Phi(\cdot, \sigma)^*\nu_M \simeq J^*\xi$. So, $J : W \to E$ gives a bordism between $b(f_{1,0} \times \cdots \times f_{k,0})$ and $b(f_{1,1} \times \cdots \times f_{k,1})$ in $\Omega_p(E(f_1, \ldots, f_k); \xi)$.

Finally, recall that any kit of maps $f_1, \ldots, f_k$ can be approximated by a transversal kit $f'_1, \ldots, f'_k$, homotopic to $f_1, \ldots, f_k$. Define $b(f_1 \times \cdots \times f_k) := b(f'_1 \times \cdots \times f'_k)$. The second half of the theorem shows, that it does not depend on the choice of the transversal approximation $f'_1, \ldots, f'_k$. $\square$

*Proof of Theorem 2.2.* First, make a small regular perturbation of the immersions $f_1, \ldots, f_k$ to put them into general position. By Theorem 2.1 it will not change the class $b(f_1 \times \cdots \times f_k) \in \Omega_p(E; \xi)$. Now the immersions $f_1, \ldots, f_k$ are transversal, and the manifold $\overline{\pitchfork}(f_1 \times \cdots \times f_k)$ is defined. Since $2p < n_i < m$ for all $i = 1 \ldots k$, by general position we may assume that $\overline{\pitchfork}(f_1 \times \cdots \times f_k)$ is embedded into each $V_i$ and into $M$. Since $n_i < m - p$, we have $2n_i - m + (n_1 + \cdots + \widehat{n_i} + \cdots + n_k - (k-2)m) < m$ for all $i = 1 \ldots k$. This means that by general position we may assume that $(2n_i - m)$-dimensional manifold of self-intersections of $f_i : V_i \looparrowright M$ is disjoint from $(n_1 + \cdots + \widehat{n_i} + \cdots + n_k - (k-2)m)$-dimensional immersed manifold of mutual intersections of $f(V_1), \ldots, \widehat{f_i(V_i)}, \ldots, f_k(V_k)$ in $M^m$.

Let $J : W^{p+1} \to E$ be the bordism between $\overline{\pitchfork}(f_1 \times \cdots \times f_k)$ and $N$ in $\Omega_p(E; \xi)$. Define the map $H : W \times T \to M$ by the formula $H(w, t) = \Phi(J(w), t)$. Note that $H(W, \sigma_i) \subset f_i(V_i)$ and $H|_{\overline{\pitchfork}(f_1 \times \cdots \times f_k)}$ is a constant homotopy.

Since for all $i = 1 \ldots k$ we have $2(p+1) < n_i < m - (p+1)$, then $2 \dim W < \dim(V_i)$, the dimension $\dim W + \dim(\text{self-intersections of } V_i \text{ in } M) < \dim(V_i)$, and $\dim W + \dim(\text{intersections of } V_i \text{ with } V_j) < \dim(V_i)$. Again, applying general position argument, we may $C^0$-perturb the map $H : W \times T \to M$ (leaving it fixed on $\overline{\pitchfork}(f_1 \times \cdots \times f_k) \times T$) such that $H|_{W \times \sigma_i}$ will be smooth embeddings $W \times \sigma_i \hookrightarrow f_i(V_i)$, disjoint from self-intersections of $f_i : V_i \looparrowright M$, and intersections with over $V_j$, distinct from $\overline{\pitchfork}(f_1 \times \cdots \times f_k)$.

Finally, since $2(p+k) < m$ and $n_i < m - p - k$ for all $i = 1 \ldots k$, we have $2 \dim(W \times T) < \dim(M)$ and $\dim(W \times T) + \dim(V_i) < \dim(M)$. This means that by general position we can $C^0$-perturb the map $H$ (leaving it fixed on $\overline{\pitchfork}(f_1 \times \cdots \times f_k) \times T$ and $W \times \sigma_i$) such that $H : (W - \overline{\pitchfork}(f_1 \times \cdots \times f_k)) \times T \hookrightarrow M$ will be a smooth embedding with $H(W \times T - \overline{\pitchfork}(f_1 \times \cdots \times f_k) \times T - W \times \{\sigma_1, \ldots, \sigma_k\}) \cap (\bigcup_{i=1\ldots k} f_i(V_i)) = \emptyset$.

Denote by $W^+$ and $T^+$ open manifolds without boundary, which are obtained from $W$ and $T$ by attaching a small collar neighborhood of the boundary. Since $W \times T$ is a deformation retract of $W^+ \times T^+$, we can extend $H$ to an embedding $H : W^+ \times T^+/(x \times t \sim x \times \sigma \mid x \in \overline{\pitchfork}(f_1 \times \cdots \times f_k), t \in T) \cong W^+ \times T^+ \hookrightarrow M$ such that for this extended $H$ we still have $H|_{W^+ \times \sigma_i}$ — a smooth embedding into $f_i(V_i)$, disjoint from self-intersections of $f_i(V_i)$, and $H(W^+ \times T^+ - \overline{\pitchfork}(f_1 \times \cdots \times f_k) \times T - W^+ \times \{\sigma_1, \ldots, \sigma_k\}) \cap (\bigcup_{i=1\ldots k} f_i(V_i)) = \emptyset$.

Denote by $\pi_W : W^+ \times T^+ \to W^+$ the natural projection. By definition, $\nu_{W\times T} \simeq \pi_W^*(J^*(\pi_1^*\nu_{f_1} \oplus \cdots \oplus \pi_k^*\nu_{f_k}) \oplus J^*\Phi(\cdot,\sigma)^*\nu_M)$. On the other hand, $\nu_{W\times T} = \nu_H \oplus H^*\nu_M$. It is easy to see that $H$ is homotopic to $\Phi(\cdot,\sigma)\circ J\circ\pi_W$. Therefore $\nu_H \simeq \pi_W^* J^*(\pi_1^*\nu_{f_1} \oplus \cdots \oplus \pi_k^*\nu_{f_k})$. Since $\dim(W\times T) = p+k < m-n_i = \dim(\pi_W^* J^* \pi_i^*\nu_{f_i})$, then $\pi_W^* J^*\pi_i^*\nu_{f_i} = \eta_i \oplus \epsilon_i$ for some bundle $\eta_i$ and a trivial line bundle $\epsilon_i$. So, $\nu_H \simeq \eta_1 \oplus \cdots \oplus \eta_k$. Since $\dim(\nu_H) = m-p-k = \dim(\eta_1\oplus\cdots\oplus\eta_k) > p+k = \dim(W\times T)$, then $\nu_H \cong \eta_1\oplus\cdots\oplus\eta_k$. Fix such an isomorphism. Denote by $\delta_i$ the barycenter of the face, opposite to $\sigma_i$ in the simplex $T$. Denote by $\varepsilon_i$ the trivial 1-dimensional bundle over $W^+\times T^+$, parallel to the line $\sigma_i\sigma\delta_i \subset T$. Note that $\varepsilon_1,\ldots,\varepsilon_k$ are linearly dependent ($\vec{\varepsilon}_1 + \cdots + \vec{\varepsilon}_k = 0$), but any $(k-1)$ of them are linearly independent. Denote by $\Xi$ the bundle, spanned by $\varepsilon_1,\ldots,\varepsilon_k$, then $\Xi$ is the trivial bundle, tangent to $T^+$ in $W^+\times T^+$. Recall that $H(W^+\times\sigma_i) \subset f_i(V_i)$. Therefore $\nu_{f_i}(V_i,M)|_{H(W^+\times\sigma_i)} \subset \nu_{H|_{W^+\times\sigma_i}} = \nu_H(W^+\times T^+, M)|_{W^+\times\sigma_i} \oplus \nu(W^+\times\sigma_i, W^+\times T^+)$. In $\nu_H|_{W^+\times\sigma_i} \oplus \nu(W^+\times\sigma_i, W^+\times T^+)$ we already have a summand, isomorphic to the pull-back of $\nu_{f_i}$, namely $\eta_i\oplus\varepsilon_i$. Therefore the complement to $\nu_{f_i}(V,M)$ in $\nu_{H|_{W^+\times\sigma_i}}$ is stably isomorphic to the complement of $\eta_i\oplus\varepsilon_i$. Since $\dim(\nu_{H|_{W^+\times\sigma_i}}) - \dim(\nu_{f_i}(V_i,M)) = n_i - p - 1 > p+1 = \dim(W^+\times\sigma_i)$, then these complements are isomorphic. So, in $\nu_H\oplus\Xi$ we have $k$ bundles $\chi_i = (\eta_1\oplus\cdots\oplus\widehat{\eta_i}\oplus\cdots\oplus\eta_k)\oplus\varepsilon_i^\perp(\Xi)$, and $\chi_i|_{W^+\times\sigma_i}$ is isomorphic to the complement to $\nu_{f_i}(V_i,M)$ in $\nu_{H|_{W^+\times\sigma_i}} = (\nu_H\oplus\Xi)|_{W^+\times\sigma_i}$. Here $\varepsilon_i^\perp(\Xi)$ is the orthogonal complement to $\varepsilon_i$ in $\Xi$, then $\varepsilon_i^\perp(\Xi)|_{W^+\times\sigma_i\sigma\delta_i} = \nu(W^+\times\sigma_i\sigma\delta_i, W^+\times T^+)$. Note that common intersection of $\chi_i|_{W^+\times\sigma}$ is empty. Since the homotopy group of the Stiefel manifold $\pi_d(V_{\dim(\nu_H\oplus\Xi),\dim(\chi_i)}) = 0$ if $d < \dim(\nu_H\oplus\Xi) - \dim(\chi_i)$ [Whd], and $\dim(W\times T) = p+k < m-n_i = \dim(\nu_H\oplus\Xi) - \dim(\chi_i)$, then there is no obstruction for homotopying $\chi_i$ in a small neighborhood of $W^+\times\sigma_i$ in $W^+\times T^+$ to a bundle $\chi_i'$ such that $\chi_i'|_{W^+\times\sigma_i} = \nu_{f_i}(V_i,M)^\perp(\nu_{H|_{W^+\times\sigma_i}})$.

Let $\varphi : W^+ \to \mathbb{R}$ be a smooth function such that $\varphi^{-1}(0) = \bar{\pitchfork}(f_1\times\cdots\times f_k)$, $\varphi^{-1}(1) = N$ and $\varphi^{-1}[0,1] = W$. Since $H$ is an embedding, then on each sheet $H(W^+\times\delta_i\sigma\sigma_i^+)$ we can introduce a coordinate system $W^+\times\mathbb{R}$, such that $H(W^+\times\sigma)$ have coordinates $(w,0)$, and $H(W^+\times\sigma_i)$ have coordinates $(w,\varphi(w))$. Let $U_i$ be a small enough neighborhood of the zero-section of $\chi_i|_{W^+\times\delta_i\sigma\sigma_i^+}$ such that $\exp\circ dH : U_i \to M$ is a diffeomorphism on its image. On the manifold $\exp\circ dH(U_i)$ we have coordinate system $(w,t,\vec{v})$, where $w\in W, t\in\mathbb{R}, \vec{v}\in U_i$. Since $\chi_i|_{W^+\times\sigma_i} = \nu_{f_i}(V_i,M)^\perp(\nu_{H|_{W^+\times\sigma_i}})$, then $\exp\circ dH(U_i|_{W^+\times\sigma_i})$ is a tubular neighborhood of $H(W^+\times\sigma_i)$ in $f_i(V_i)$. Let $\psi : W^+ \to [0,1]$ be a smooth function such that $\psi\geq 0$, $\psi|_W \equiv 1$ and $\psi|_{W^+-U} \equiv 0$, where open set $U$ is such that $W\subset U\subset W^+$ and $\overline{U}\subset W^+$. Let $\kappa : \mathbb{R}\to\mathbb{R}$ be a smooth bell-shaped function such that $\kappa(0) = 1$ and $\kappa(\|\vec{v}\|) = 0$ for $\vec{v}$ outside of $U_i$. Then the regular homotopy from $f_i$ to $f_i'$ is given by $(w,\varphi(w),\vec{v}) \mapsto (w, \varphi(w) - t\psi(w)\kappa(\|\vec{v}\|), \vec{v})$, $t\in[0,1]$ for points in $\exp\circ dH(U_i|_{W^+\times\sigma_i}) \subset f_i(V_i)$, and is constant outside. Since common intersection of $\chi_i|_{W^+\times\sigma}$ is empty, then common intersection of $\exp\circ dH(U_i)$ is empty, and at the last moment we will get exactly $\bar{\pitchfork}(f_1'\times\cdots\times f_k') = N$. $\square$

*Proof of Theorem 3.1.* If $\widehat{\bar{\pitchfork}(F)} = \emptyset$, put $b(F) = 0 \in \Omega_p(E_k;\xi(f))$. Now suppose $\widehat{\bar{\pitchfork}(F)} \neq \emptyset$. Denote by $D : M \to \Delta_M^k$ the diffeomorphism $x\mapsto(x,\ldots,x)$. Obviously, the restriction on $\widehat{\bar{\pitchfork}(F)}$ of the projection $V^{(k)}\to V$ on the $i$-th factor is a composite $h_i\pi_i'$ for a uniquely determined $h_i : \widehat{\bar{\pitchfork}(F)}_{(i)} \to V$, and $D^{-1}F|_{\widehat{\bar{\pitchfork}(F)}}$ is a composite $h\pi'$ for a uniquely determined $h : \bar{\pitchfork}(F)\to M$. Since $\widehat{\bar{\pitchfork}(F)} \subset (V^{(k)} - \Delta_V^2)$, then $h_i\pi_i'(x) \neq h_j\pi_j'(x)$ for $x\in\widehat{\bar{\pitchfork}(F)}$ and $i\neq j$. Let $\bar{F} : V^{(k)}\times[0,1]\to M^{(k)}$ be a $\Sigma_k$-equivariant homotopy between $\bar{F}_0 = f^{(k)}$ and $\bar{F}_1 = F$. Denote by $\pi_i^M : M^{(k)}\to M$ the projection on the $i$-th factor. Then $fh_i\pi_i'$ and $D^{-1}F = \pi_i^M F$ are connected by the homotopy $x\mapsto \pi_i^M\bar{F}_t(h_1\pi_1'(x),\ldots,h_k\pi_k'(x)), t\in[0,1]$. This gives a $\Sigma_k$-invariant homotopy $H : \widehat{\bar{\pitchfork}(F)}\times T^1 \to M$, where $T^1$ is the 1-dimensional subcomplex of $T$, consisting of vertices $\sigma,\sigma_1,\ldots,\sigma_k$ and straight intervals $\sigma\sigma_1,\ldots,\sigma\sigma_k$. Since $T^1$ is an $\Sigma_k$-invariant deformation retract of $T$ under $\Sigma_k$-equivariant deformation, we can extend $H$ to a $\Sigma_k$-invariant homotopy $H : \widehat{\bar{\pitchfork}(F)}\times T\to M$. Note that if $F = f^{(k)}$, we can take $H$ to be the constant homotopy. By universal property of $\widehat{E_k}$, this gives a canonical map $j : \bar{\pitchfork}(F)\to E_k$.

Since the manifold $\widehat{\bar{\pitchfork}(F)} \hookrightarrow V^{(k)}$, then $\nu_{\widehat{\bar{\pitchfork}(F)}} = \nu_{V^{(k)}}|_{\widehat{\bar{\pitchfork}(F)}} \oplus \nu(\widehat{\bar{\pitchfork}(F)}, V^{(k)})$. By construction, $\nu(\widehat{\bar{\pitchfork}(F)}, V^{(k)}) \cong (F|_{\widehat{\bar{\pitchfork}(F)}})^*\nu(\Delta_M^k, M^{(k)})$ and $\nu_{V^{(k)}}|_{\widehat{\bar{\pitchfork}(F)}} \cong \nu_{f^{(k)}}|_{\widehat{\bar{\pitchfork}(F)}} \oplus (f^{(k)}|_{\widehat{\bar{\pitchfork}(F)}})^*\nu_{M^{(k)}}$. Since $f^{(k)}$ is $\Sigma_k$-equivariantly homotopic to $F$ and $F(\widehat{\bar{\pitchfork}(F)}) \subset \Delta_M^k$, then $\nu_{\widehat{\bar{\pitchfork}(F)}} \cong \nu_{f^{(k)}}|_{\widehat{\bar{\pitchfork}(F)}} \oplus (F|_{\widehat{\bar{\pitchfork}(F)}})^*(\nu_{M^{(k)}}|_{\Delta_M^k} \oplus \nu(\Delta_M^k, M^{(k)}))$. Note that $F|_{\widehat{\bar{\pitchfork}(F)}} =$

$D \circ \Phi(\cdot,\sigma) \circ \widehat{\jmath}$ and $f^{(k)}|_{\widehat{\overline{\pitchfork}(F)}} = f^{(k)} \circ (\pi_1 \times \cdots \times \pi_k) \circ \widehat{\jmath}$. Since $\nu_{M^{(k)}}|_{\Delta^k_M} \oplus \nu(\Delta^k_M, M^{(k)}) \simeq \nu_{\Delta^k_M}$ and $\Phi(\cdot,\sigma)$ is $\Sigma_k$-invariant, then $\nu_{\widehat{\overline{\pitchfork}(F)}} \simeq \widehat{\jmath}^*(\pi_1 \times \cdots \times \pi_k)^*\nu_{f^{(k)}} \oplus \widehat{\jmath}^*\Phi(\cdot,\sigma)^*\nu_M$. Therefore $\nu_{\overline{\pitchfork}(F)} = \nu_{\widehat{\overline{\pitchfork}(F)}}/\Sigma_k \simeq j^*(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k \oplus j^*\phi^*\nu_M = j^*\xi$. Thus the map $j : \overline{\pitchfork}(F) \to E_k$ defines an element $b(F) \in \Omega_p(E_k;\xi(f))$.

Now, suppose we have a $k$-disjoint homotopy $\bar{F} : V^{(k)} \times [1,2] \to M^{(k)}$, where $F_1 = \bar{F}_1, F_2 = \bar{F}_2$ are $k$-transversal. Since $\bar{F}$ is $k$-disjoint, then $\bar{F}^{-1}(\Delta^k_M) - \Delta^2_V \times [1,2]$ is disjoint from $\Delta^2_V \times [1,2]$. WLOG we may assume $\widehat{\overline{\pitchfork}(F_1)} \neq \emptyset$. If $\widehat{\overline{\pitchfork}(F_2)} \neq \emptyset$, choose an open $\Sigma_k$-invariant neighborhood $U$ of $\Delta^2_V \times [1,2]$ in $V^{(k)} \times [1,2]$ such that $\overline{U}$ is disjoint from $\bar{F}^{-1}(\Delta^k_M) - \Delta^2_V \times [1,2]$. If $\widehat{\overline{\pitchfork}(F_2)} = \emptyset$, we may assume that $(\bar{F}_t)^{-1}(\Delta^k_M) - \Delta^2_V = \emptyset$ for $t$ close to 2. Then take $U$ to be an open $\Sigma_k$-invariant neighborhood of $\Delta^2_V \times [1,2] \cup V^{(k)} \times \{2\}$ in $V^{(k)} \times [1,2]$ such that $\overline{U}$ is disjoint from $F^{-1}(\Delta^k_M) - \Delta^2_V \times [1,2]$.

Note that the $\Sigma_k$-action on $V^{(k)} \times [1,2] - U$ is free. Therefore we can approximate $\bar{F}|_{V^{(k)}\times[1,2]-U}$ by a smooth $\Sigma_k$-equivariant map $\tilde{F} : V^{(k)} \times [1,2] - U \to M^{(k)}$, which is transversal to $\Delta^k_M$, keeping it fixed on the end $V^{(k)} \times \{1\} - U$, where it is already transversal (and on $V^{(k)} \times \{2\} - U$, if $\widehat{\overline{\pitchfork}(F_2)} \neq \emptyset$). Since $\bar{F}(\partial U) \cap \Delta^k_M = \emptyset$, then $\tilde{F}(\partial U) \cap \Delta^k_M = \emptyset$, and $\widehat{W} = \tilde{F}^{-1}(\Delta^k_M)$ is a proper compact submanifold of $V^{(k)} \times [1,2] - U$ with a free $\Sigma_k$-action, and $\partial \widehat{W} = \widehat{\overline{\pitchfork}(F_1)} \cup \widehat{\overline{\pitchfork}(F_2)}$. Denote $W = \widehat{W}/\Sigma_k$.

Obviously, the restriction on $\widehat{W}$ of the projection $V^{(k)} \times [1,2] \to V$ on the $i$-th factor is a composite $h_i\pi_i'$ for a uniquely determined $h_i : \widehat{W}_{(i)} \to V$, and $D^{-1}\tilde{F}|_{\widehat{W}}$ is a composite $h\pi'$ for a uniquely determined $h : W \to M$. Since $\widehat{W} \subset (V^{(k)} - \Delta^2_V) \times [1,2]$, then $h_i\pi_i'(x) \neq h_j\pi_j'(x)$ for $x \in \widehat{W}$ and $i \neq j$. Let $\tilde{F} : V^{(k)} \times [0,1] \to M^{(k)}$ be a $\Sigma_k$-equivariant homotopy between $\tilde{F}_0 = f^{(k)}$ and $\tilde{F}_1 = \bar{F}_1 = F_1$. Denote by $t : V^{(k)} \times [0,2] \to [0,2]$ the projection on the last factor. Then $fh_i\pi_i'$ and $D^{-1}(\tilde{F}|_{\widehat{W}}) = \pi_i^M(\tilde{F}|_{\widehat{W}})$ are connected by the homotopy $x \mapsto \pi_i^M\tilde{F}(h_1\pi_1'(x),\ldots,h_k\pi_k'(x),t't(x)), t' \in [0,1]$. This gives a $\Sigma_k$-invariant homotopy $H : \widehat{W} \times T^1 \to M$. Since $T^1$ is an $\Sigma_k$-invariant deformation retract of $T$ under $\Sigma_k$-equivariant deformation, we can extend $H$ to a $\Sigma_k$-invariant homotopy $H : \widehat{W} \times T \to M$. By universal property of $\widehat{E_k}$, this gives a canonical map $J : W \to E_k$.

Since the manifold $\widehat{W} \hookrightarrow V^{(k)} \times [1,2]$, then $\nu_{\widehat{W}} = \nu_{V^{(k)}\times[1,2]}|_{\widehat{W}} \oplus \nu(\widehat{W}, V^{(k)} \times [1,2])$. By construction, $\nu(\widehat{W}, V^{(k)} \times [1,2]) \cong (\tilde{F}|_{\widehat{W}})^*\nu(\Delta^k_M, M^{(k)})$ and $\nu_{V^{(k)}\times[1,2]}|_{\widehat{W}} \cong (\pi_1^V \times \cdots \times \pi_k^V|_{\widehat{W}})^*(\nu_{f^{(k)}} \oplus (f^{(k)})^*\nu_{M^{(k)}})$, where $\pi_i^V : V^{(k)} \times [0,2] \to V$ is the projection on the $i$-th factor. Since $f^{(k)} \circ (\pi_1^V \times \cdots \times \pi_k^V)|_{\widehat{W}}$ is $\Sigma_k$-equivariantly homotopic to $\tilde{F}|_{\widehat{W}}$ and $\tilde{F}(\widehat{W}) \subset \Delta^k_M$, then $\nu_{\widehat{W}} \cong (\pi_1^V \times \cdots \times \pi_k^V|_{\widehat{W}})^*\nu_{f^{(k)}} \oplus (\tilde{F}|_{\widehat{W}})^*(\nu_{M^{(k)}}|_{\Delta^k_M} \oplus \nu(\Delta^k_M, M^{(k)}))$. Note that $\tilde{F}|_{\widehat{W}} = D \circ \Phi(\cdot,\sigma) \circ \widehat{J}$ and $f^{(k)} \circ (\pi_1^V \times \cdots \times \pi_k^V)|_{\widehat{W}} = f^{(k)} \circ (\pi_1 \times \cdots \times \pi_k) \circ \widehat{J}$. Since $\nu_{M^{(k)}}|_{\Delta^k_M} \oplus \nu(\Delta^k_M, M^{(k)}) \simeq \nu_{\Delta^k_M}$ and $\Phi(\cdot,\sigma)$ is $\Sigma_k$-invariant, then $\nu_{\widehat{W}} \simeq \widehat{J}^*(\pi_1 \times \cdots \times \pi_k)^*\nu_{f^{(k)}} \oplus \widehat{J}^*\Phi(\cdot,\sigma)^*\nu_M$. Therefore $\nu_W = \nu_{\widehat{W}}/\Sigma_k \simeq J^*(\pi_1^*\nu_f \oplus \cdots \oplus \pi_k^*\nu_f)/\Sigma_k \oplus J^*\phi^*\nu_M = J^*\xi$. Thus the map $J : W \to E_k$ gives a bordism between $b(F_1)$ and $b(F_2)$ in $\Omega_p(E_k;\xi(f))$. $\square$

*Proof of Theorem 3.2.* Making a small regular perturbation of $f$, we may assume that it is in general position. By Theorem 3.1, this will not change the class $b(f^{(k)}) \in \Omega_p(E_k;\xi)$. From $(k+1)(n+1) \leq km$ it follows that $2p < n$, or $2\dim(\overline{\pitchfork}(k,f)) < \dim(V)$. Since $\widehat{\overline{\pitchfork}(f^{(k)})} \subset V^{(k)} - \Delta^2_V$, by general position we may assume that the manifold $\widehat{\overline{\pitchfork}(f^{(k)})}_{(i)}$ is embedded by $h_i$ into $V$ for all $i = 1 \ldots k$, the manifold $\overline{\pitchfork}(f^{(k)})$ is embedded by $h$ into $M$, and $f : \left(\bigcup_i h_i(\widehat{\overline{\pitchfork}(f^{(k)})}_{(i)})\right) \to h(\overline{\pitchfork}(f^{(k)}))$ is a $k$-fold covering (possibly, some of $h_i(\widehat{\overline{\pitchfork}(f^{(k)})}_{(i)})$ coincide).

Let $J : W \to E_k$ be the bordism between $\overline{\pitchfork}(f^{(k)})$ and $N$ in $\Omega_p(E_k;\xi)$. Choose a lifting $\widehat{J} : \widehat{W} \to \widehat{E_k}$. Since the $\Sigma_k$-action on $\widehat{W}$ is free, the diagonal $\Sigma_k$-action on $\widehat{W} \times T$ is also free. Therefore $(\widehat{W} \times T)/\Sigma_k$ is a smooth manifold. Note that $(\widehat{W} \times \sigma)/\Sigma_k = W$. Since $\Phi : \widehat{E_k} \times T \to M$ is $\Sigma_k$-invariant, the map $B : (\widehat{W} \times T)/\Sigma_k \to M$ is well-defined by the formula $[w,t] \mapsto \Phi(\widehat{J}(w),t)$. Note that $B((\widehat{W} \times \sigma_i)/\Sigma_k) \subset f(V)$. By construction of $b(f^{(k)})$, for $x \in \widehat{\overline{\pitchfork}(f^{(k)})}$ and $t, t' \in T$ we have $B([x,t]) = B([x,t'])$.

Since $2(p+1) < n$ and $(p+1) + (2n-m) < n$, we have $2\dim(W) < \dim(V)$, and $\dim(W) + \dim($ self-intersections of $V$ in $M) < \dim(V)$. Applying general position argument, we may $C^0$ perturb the map

$B : (\widehat{W} \times T)/\Sigma_k \to M$ (leaving it fixed on $(\widehat{\overline{\pitchfork}(f^{(k)})} \times T)/\Sigma_k$) so that $B|_{(\widehat{W}\times\sigma_i)/\Sigma_k}$ will be a smooth embedding into $f(V)$, disjoint from self-intersections of $f : V \looparrowright M$, distinct from $\overline{\pitchfork}(f^{(k)})$.

Since $2(p+k) < m$ and $(p+k)+n < m$, we have $2\dim(W \times T) < \dim(M)$, and $\dim(W \times T) + \dim(V) < \dim(M)$. This means that by general position we can $C^0$-perturb the map $B$ (leaving it fixed on $(\widehat{\overline{\pitchfork}(f^{(k)})} \times T)/\Sigma_k$ and $(\widehat{W} \times \sigma_i)/\Sigma_k$) so that $B : \left(\widehat{W} \times T - \widehat{\overline{\pitchfork}(f^{(k)})} \times T\right)/\Sigma_k \hookrightarrow M$ will be a smooth embedding with $B\left(\left\{\widehat{W} \times T - \widehat{\overline{\pitchfork}(f^{(k)})} \times T - \widehat{W} \times \{\sigma_1, \ldots, \sigma_k\}\right\}/\Sigma_k\right) \cap f(V) = \emptyset$.

Denote by $\widehat{W}^+$ and $T^+$ open manifolds without boundary, which are obtained from $\widehat{W}$ and $T$ by attaching a small collar neighborhood of the boundary. Then $\left(\widehat{W}^+ \times T^+/\left\{x \times t \sim x \times \sigma \mid x \in \widehat{\overline{\pitchfork}(f^{(k)})}, t \in T\right\}\right)/\Sigma_k \cong (\widehat{W}^+ \times T^+)/\Sigma_k$ is a smooth manifold. Since $\widehat{W} \times T$ is a $\Sigma_k$-equivariant deformation retract of $\widehat{W}^+ \times T^+$, we can extend $B$ to an embedding $B : \left(\widehat{W}^+ \times T^+/\left\{x \times t \sim x \times \sigma \mid x \in \widehat{\overline{\pitchfork}(f^{(k)})}, t \in T\right\}\right)/\Sigma_k \hookrightarrow M$ such that for this extended $B$ we still have $B|_{(\widehat{W}^+\times\sigma_i)/\Sigma_k}$ — a smooth embedding into $f(V)$, disjoint from self-intersections of $f(V)$, and $B\left(\left\{\widehat{W}^+ \times T^+ - \widehat{\overline{\pitchfork}(f^{(k)})} \times T - \widehat{W}^+ \times \{\sigma_1, \ldots, \sigma_k\}\right\}/\Sigma_k\right) \cap f(V) = \emptyset$.

Denote by $H : \widehat{W}^+ \times T^+ \looparrowright M$ the immersion, defined as a composition of the natural covering $\widehat{W}^+ \times T^+ \to (\widehat{W}^+ \times T^+)/\Sigma_k$ with $B$. Strictly speaking, this is not quite immersion, it has "singularities" at $\overline{\pitchfork}(f^{(k)}) \times T$. Obviously, $H$ is $\Sigma_k$-invariant. The rest of the proof follows the proof of Theorem 2.2, which is expressly written in a $\Sigma_k$-invariant language. Indeed, substitute $W \mapsto \widehat{W}$, $\nu_{f_i} \mapsto \nu_f$, $V_i \mapsto V$, $J \mapsto \widehat{J}$. Locally, in a small neighborhood of $w \in W$, the construction from the proof of Theorem 2.2 translates without changes. Then the fact that $H$ is $\Sigma_k$-invariant guarantee us that the desired regular homotopy will be well-defined globally. $\square$

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Department of Mathematics, University of Maryland, College Park, MD 20742

*E-mail address*: `salikhov@math.umd.edu`