CHAPTER 20

$\pi_3(S^2)$, H. Hopf, W.K. Clifford, F. Klein

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In 1931 there appeared the seminal paper [2] by Heinz Hopf, in which he showed that $\pi_3(S^2)$ (the third homotopy group of the two-sphere S^2) is nontrivial, or more specifically that it contains an element of order ∞. (His language was different. Homotopy groups had not been defined yet; E. Czech introduced them at the 1932 Congress in Zürich. Interestingly enough, both Paul Alexandroff and Hopf persuaded him not to continue with these groups. They had different reasons for considering them as not fruitful; the one because they are Abelian, and the other (if I remember right) because there is no mechanism, like chains say, to compute them. It was not until 1936 that W. Hurewicz rediscovered them and made them respectable by proving substantial theorems with and about them.)

There are two parts to the paper: The first one is the definition of what now is called the Hopf invariant and the proof of its homotopy invariance. The second consists in the presentation of an example of a map from S^3 to S^2 that has Hopf invariant 1 and thus represents an element of infinite order of $\pi_3(S^2)$; it is what is now called the Hopf fibration; the inverse images of the points of S^2 are great circles of S^3. Taking S^3 as the unit-sphere $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$, these circles are the intersections of S^3 with the various complex lines through the origin.

Hopf knew the example from non-Euclidean Geometry and puzzled for years over the question whether it is an "essential" map, i.e. one that is not homotopic to 0, until (so he told me once) one day in 1927 or so, while he was walking along the Spree river in Berlin, the idea "Any two of these circles are linked in S^3" came to him; the rest is history.

This note is concerned only with the second part, the example – where did it come from? Hopf, on p. 655 of the paper (Selecta p. 53), calls it a Clifford parallel congruence, and in a footnote refers to p. 234 of Felix Klein's (posthumous) book [5] on non-Euclidean Geometry. On looking up the reference one finds the very brief statement that these "parallels" had been introduced by Clifford in a talk to the British Association in 1873. (It is not stated which British Association is meant, and there is no reference to any publication.) A little later in the book the Clifford parallels are described with the help of quaternions; and earlier in the book they had been introduced by geometric considerations as families of

*Support by NSF grant DMSD91-02078 is acknowledged.

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Edited by I.M. James

lines in the three-dimensional space of non-Euclidean geometry with spherical metric, that allow a continuous family of motions shifting each line in itself.

In the preface to [5] one learns that Hopf played a substantial role in the preparation of the book, particular in the part that had to do with his own research. It seems quite clear that it was he who put in the reference to Clifford and the description of his parallels. But where did he get it from? First I looked at Clifford's collected works [1], but I could not find any paper that had to do with the parallels (but see below). I started looking through the literature, through earlier books on non-Euclidean Geometry and on Projective Geometry in general, to no avail. I looked through the references to Clifford in the Enzyclopaedie der Mathematischen Wissenschaften (there are very many and I could not check all of them, but they all seemed to refer to other things).

Finally Felix Klein came to the rescue, with two publications. One, [6], is a set of notes of a course on non-Euclidean Geometry that he had given in 1890. It is not exactly a book, although it has a hard cover. It is a dittoed ("autographed") copy of handwritten notes, carefully prepared by one of the listeners (this was Klein's way at that time of making his lectures available to the world at large). The other one is a paper, [7], which amplifies the last few pages of the lecture notes. In both Klein tells of a visit that he made to England in 1873. At that visit he went to a meeting of the British Association for the Advancement of Science in Bradford. He met the young Clifford there, listened to a talk by him, and discussed the content with him afterwards, together with R.S. Ball and others. Unfortunately, he says, the talk was never published; only the title of the talk was published in the Report on the meeting, as "A Surface of Zero Curvature and Finite Extent". Clifford had become interested in elliptic geometry and had found certain interesting congruences in elliptic space (projective 3-space $\mathbb{R}P^3$ with elliptic metric) or in the 3-sphere S^3 (a congruence is a 2-parameter family of disjoint projective lines (or great circles) that covers the space). Each of Clifford's congruences has the property that there exists a one-parameter group of rigid motions of the space that shifts each line of the congruence along itself. In fact there exist two such families of congruences, say "left" and "right". (Each line in space belongs to a left and also to a right congruence.) By taking a line in a left congruence and moving it along the lines of a right congruence by the one-parameter group of motions associated with the latter, Clifford constructed a surface whose induced metric is flat and which thus has Gaussian curvature 0 and which is of finite extent (read compact); it is in fact clearly a torus. This was the first example of what became known as Clifford–Klein space forms. (The name was introduced by Killing in [3], p. 257, [4], p. 314, to denote those space forms, i.e. spaces of constant curvature, that are different from the prime examples sphere and projective space [positive curvature], hyperbolic space [negative curvature], Euclidean space [curvature zero]; the latter are the cases distinguished by free mobility – the isometry group is transitive on the orthonormal frames. As Klein puts it in [7], p. 559, respectively, 367: Just try to turn Clifford's surface around one of its points.)

Klein derives all the formulae needed; he says that he does not know how Clifford proceeded. As pointed out by him, the best way to understand the congruences is probably with quaternions, for the three-sphere formed by the unit-quaternions: A left [respectively, right] congruence consists of the orbits on S^3 under left [respectively, right] multiplication by the elements of a one-parameter group $\cos t + \sin t \cdot u$ with any unit-quaternion u; in other words, the right [respectively, left] cosets of the subgroup. With $u = i$ this is precisely the Hopf fibration, and this must be where Hopf became acquainted with it.

Klein makes a point of saying how glad he was to be able to present these very interesting results of Clifford to the mathematical world, particularly since Clifford died a few years after their meeting prematurely; as noted, Clifford's talk was published by title only; there are only very brief indications of the matter in some of his papers ([1], items XX, XXVI, XLI, XLII, XLIV).

Thus one might wonder: Where would $\pi_3(S^2)$ be today, if Klein had not gone to the meeting of the BAAS in 1873 or if he had not listened to Clifford's talk?

As an appendix we reproduce, with I.M. James's permission, a letter from Hopf to Hans Freudenthal which throws some light on the timing of Hopf's result; the letter was communicated to James by W.T. van Est who has the original.

Princeton, N.J., 30 Murray Place, den 17. August 1928.

Lieber Herr Freudenthal!

Für den Fall, dass Sie sich noch für die Frage nach den Klassen der Abbildungen der 3-dimensionalen Kugel S^3 auf die 2-dimensionale Kugel S^2 interessieren, möchte ich Ihnen mitteilen, dass ich diese Frage jetzt beantworten kann: es existieren unendlich viele Klassen. Und zwar gibt es eine Klasseninvariante folgender Art: x, y seien Punkte der S^2; dann besteht bei hinreichend anständiger Approximation der gegebenen Abbildung die Originalmenge von x aus endlich vielen einfach geschlossenen, orientierten Polygonen $P_1, P_2, \ldots, P_a$ und ebenso die Originalmenge von y aus Polygonen $Q_1, Q_2, \ldots, Q_b$. Bezeichnet v_{ij} die Verschlingungszahl von P_i mit Q_j, so ist $\sum_{i,j} v_{ij} = \gamma$ unabhängig von x, y und von der Approximation und ändert sich nicht bei stetiger Änderung der Abbildung. Zu jedem γ gibt es Abbildungen. Ob es zu einem jeden γ nur eine Klasse gibt, weiss ich nicht. Wird nicht die ganze S^2 von der Bildmenge bedeckt, so ist $\gamma = 0$. Eine Folgerung davon ist dass man die Linienelemente auf einer S^2 nicht stetig in einen Punkt zusammenfegen kann.

Es bleiben noch eine Anzahl von Fragen offen, die mir interessant zu sein scheinen, besonders solche, die sich auf Vektorfelder auf der S^3 beziehen und mit analytischen Fragen zusammenhängen (Existenz geschlossener Integralkurven). Wenn Sie sich dafür interessieren, so schreiben Sie mir doch einmal. Meine Adresse ist bis 20. Mai die oben angegebene, im Juni und Juli: Göttingen, Mathematisches Institut der Universität, Weender Landstrasse.

Mit den besten Grüssen, auch an die übrigen Bekannten im Seminar,

Heinz Hopf.

Translation:
Princeton, N.J., 30 Murray Place, Aug 17 1928

Dear Mr. Freudenthal!

In case you are still interested in the question of the [homotopy] classes of maps of the 3-sphere S^3 onto the 2-sphere S^2 I want to tell you that I now can answer this question: there exist infinitely many classes. Namely there is a class invariant of the following kind: let x, y be points of S^2; then for a sufficiently decent approximation of the given map the counter image of x consists of finitely many simple closed oriented polygons $P_1, P_2, \ldots, P_a$ and likewise the counter image of y consists of polygons $Q_1, Q_2, \ldots, Q_b$. If v_{ij} denotes the linking number of P_i and Q_j, then $\sum_{i,j} v_{ij} = \gamma$ is independent of x, y and of the approximation and does not change under continuous change of the map. For every γ there exist maps. Whether to every γ there is only one map, I do not know. If the whole S^2 is not covered by the image, then γ is $= 0$. A consequence is that one cannot sweep the line elements on S^2 continuously into a point.

A number of questions that seem interesting to me remain open, in particular those that have to do with vector fields on S_3 and are related to analytic questions (existence of closed integral curves). If you are interested in this, then do write me. My address till May 20 is the one given above, in June and July: Göttingen, Mathematical Institute of the University, Weender Landstrasse.

With the best wishes, also to the other acquaintances in the seminar,

Heinz Hopf.

(Note: Freudenthal was Hopf's first student, in Berlin. Hopf told me once that Freudenthal was the "easiest" doctoral student he ever had. One day Freudenthal came to Hopf and said: "Dr. Hopf, I would like to have you as my thesis adviser. And here is my thesis.". It was Freudenthal's work on the ends of topological spaces and groups, in which he proved that a topological group (with suitable conditions, e.g., a connected Lie group) has at most two ends.)

Bibliography

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[3] W. Killing, *Über die Clifford–Kleinschen Raumformen*, Math. Annalen **39** (1891), 257–278.

[4] W. Killing, *Einführung in die Grundlagen der Geometrie*, Part 4, Paderborn (1893).

[5] F. Klein, *Nicht-Euklidische Geometrie*, Springer, Berlin (1928).

[6] F. Klein, *Nicht-Euklidische Geometrie, I*, Vorlesung gehalten während des Wintersemesters 1889–1890, ausgearbeitet von Fr. Schilling, Göttingen (1893).

[7] F. Klein, *Zur Nicht-Euklidischen Geometrie*, Math. Annalen **37** (1890), 544–572; *Gesammelte Mathematische Abhandlungen I*, Springer, Berlin (1921), 353–383.