leading to the tetrahedron, octahedron, icosahedron, cube, dodecahedron, respectively.

The trivial solutions are

$$d=2, \quad c=2, \quad b \text{ arbitrary}$$

(corresponding to the double b-gon), and $b=2$, $d=2c$ (corresponding to an orange peeled in d slices). (For a different argument see Weyl: Symmetry.)

Appendix 4. Hermite's Inequality and the Form E_8

In some sense it is very difficult of exhibit interesting examples of regular symmetric bilinear forms: Over fields all forms can be written down in a trivial manner, namely in diagonal form. On the other hand, it is usually difficult to find (indecomposable) regular forms over rings. The most famous individual form is certainly the form E_8 over $\mathbb{Z}$ which we want to describe now. To do this we will need Hermite's inequality, a basic tool in the theory of integral quadratic forms.

Theorem (Hermite). *Let L be a free $\mathbb{Z}$-module of rank n and b: $L\times L\to\mathbb{R}$ a positive definite $\mathbb{Z}$-bilinear symmetric form of determinant D. Then there exists a vector $0\neq x\in L$ with*

$$b(x,x)=q(x)\leqq\left(\tfrac{4}{3}\right)^{\frac{n-1}{2}}\sqrt[n]{D}.$$

Proof (Compare 6.3.4): We extend b to $V=L\otimes_{\mathbb{Z}}\mathbb{R}$. The case $n=1$ being trivial we proceed by induction on n. Let $0\neq e_1\in L$ be a vector with $a=q(e_1)$ minimal. Consider the hyperplane $H=e_1^{\perp}\subset V$ and let

$$\pi\colon V\to H, \qquad \pi(x)=x-\frac{b(x,e_1)}{b(e_1,e_1)}e_1$$

be the orthogonal projection. Let $e_1, e_2, \ldots, e_n$ be a basis of L. Then

$$L'=\sum_{i=2}^{n}\pi(e_i)\mathbb{Z}$$

is a lattice in H. Let d be the determinant of $b_{L'}$. The change of basis matrix from $e_1,\ldots,e_n$ to $e_1,\pi(e_2),\ldots,\pi(e_n)$ is

$$\begin{pmatrix} 1 & \alpha_2 \ldots \alpha_n \\ & \ddots \quad 0 \\ 0 & \quad\ddots \quad 1 \end{pmatrix}, \qquad \alpha_i=-\frac{b(e_i,e_1)}{b(e_1,e_1)}.$$

Therefore $D=a\,d$. For every $x'\in L'$ there exists an $x\in L$ such that $x=x'+t\,e_1$. Adding a suitable integral multiple of e_1 to x we may assume $|t|\leqq\frac{1}{2}$. Choose $x'\in L'$ such that $a'=q(x')$ is minimal. Then

$$q(x)=q(x'_b+t\,e_1)=q(x')+t^2\,a.$$

Hence

$$a\leqq q(x)\leqq a'+\tfrac{1}{4}a,\qquad a\leqq\tfrac{4}{3}a'.$$

By induction hypothesis we have

$$a'\leqq(\tfrac{4}{3})^{\frac{n-2}{2}}\sqrt[n-1]{d}.$$

Combining this with $a\leqq\frac{4}{3}a'$ and $D=a\,d$ we get the result. □

Corollary. *Let (L,b) be a regular positive definite bilinear space of rank $\leqq 5$. Then*

$$(L,b)\cong\langle 1,\ldots,1\rangle.$$

Proof: By Hermite's inequality, there is a vector $0\neq x$ such that $q(x)=1$. Thus

$$L=x\mathbb{Z}\perp L'=\langle 1\rangle\perp L'$$

with L' regular so that we can proceed with this argument. □

Now we come to the form E_8.

Theorem (Korkine-Zolotareff, Mordell). *There exists over $\mathbb{Z}$ an 8-dimensional positive definite regular symmetric bilinear form b which is even, that is, $b(x,x)\in 2\mathbb{Z}$ for all x. This form is uniquely determined up to isometry.*

Proof: One checks that the form corresponding to the graph E_8 (see appendix 2) is such a form. This establishes the existence part of the theorem. To prove uniqueness we use Hermite's inequality. Since $q(x)$ is even and

$$(\tfrac{4}{3})^{\frac{7}{2}}<4$$

there is a vector $w_1\in L$ with $q(w_1)=2$. Consider the sublattice

$$L':=\{v\in L\,|\,b(v,w_1)\equiv 0\,(\mathrm{mod}\,2)\}=\mathbb{Z}\,w_1\perp L_7.$$

Since $[L:L']=2$ it follows that $\det(L_7,b)=2$. Applying Hermite's estimate to L_7 we get

$$(\tfrac{4}{3})^{\frac{6}{2}}\sqrt[7]{2}<4$$

and find a vector $w_2\in L_7$ with $q(w_2)=2$. As above

$$L'':=\{v\in L_7\,|\,b(v,w_2)\equiv 0\,(\mathrm{mod}\,2)\}=\mathbb{Z}\,w_2\perp L_6,$$

and $[L_7 : L'] \leqq 2$ so that $\det(L_6, b) \leqq 4$. Using Hermite once again we find a vector $w_3 \in L_6$ with $q(w_3) = 2$. So we have found a subspace $\langle 2,2,2 \rangle$ of our lattice. Unfortunately, this process stops at this point.

So we consider $L/2L = \bar{L}$ with the symplectic form $\bar{b}$. The vectors $\overline{w_1}, \overline{w_2}, \overline{w_3}$ lie in a totally isotropic subspace $\bar{U} = \bar{U}^{\perp}$, $\bar{U} = U/2L$ of $\bar{L}$. Clearly $[L:U] = 2^4$ and $b(U,U) \subset 2\mathbb{Z}$ so that $(U, \frac{1}{2}b)$ is a regular bilinear space. We can split off the three lines generated by w_1, w_2, w_3

$$U = w_1 \mathbb{Z} \perp w_2 \mathbb{Z} \perp W_3 \mathbb{Z} \perp U_5$$

and find an orthogonal basis of U_5 by the above corollary. Thus

$$(U, \tfrac{1}{2}b) \cong \langle 1, \ldots, 1 \rangle.$$

Since $U \subset L \subset \frac{1}{2}U$, we see that (L,b) can be constructed in the following way: Start with the unit form

$$(\mathbb{Z}^8, \langle\ ,\ \rangle) \qquad \langle x, y \rangle = \sum x_i y_i.$$

Then there exists a lattice M with $\mathbb{Z}^8 \subset M \subset (\frac{1}{2}\mathbb{Z})^8$ such that

$$(L,b) \cong (M, 2\langle\ ,\ \rangle).$$

It is easy to see that M is uniquely determined (corresponding to the even, selfdual extended [8, 4] - Hamming code). □

Remark. Recall that we have shown in 5.4.2, that the dimension of an even positive definite regular bilinear form is always a multiple of 8.

The following approach to the form E_8 and related ones is due to H.-G. Quebbemann who also pointed out the above proof to me.

Let A be a $n \times n$-matrix over $\mathbb{Z}[X]$ such that $E + A^t A$ has only even coefficients, that is

$$\tfrac{1}{2}(E + A^t A) \in M(n, \mathbb{Z}[X]).$$

We consider the symmetric matrix

$$B = \begin{pmatrix} 2E & A \\ A^t & \frac{1}{2}(A^t A + E) \end{pmatrix} = \begin{pmatrix} 2E & A \\ A^t & D \end{pmatrix}.$$

The following transformation shows that this matrix is positive-definite and unimodular

$$\begin{pmatrix} E & 0 \\ -\frac{1}{2}A^t & E \end{pmatrix} \begin{pmatrix} 2E & A \\ A^t & D \end{pmatrix} \begin{pmatrix} E & -\frac{1}{2}A \\ 0 & E \end{pmatrix} = \begin{pmatrix} 2E & 0 \\ 0 & \frac{1}{2}E \end{pmatrix}.$$

We now choose for A a so called conference matrix, that is $A^t A = (n-1)E$ with n even. If $n = 4k$, we get

$$\begin{pmatrix} 2E & A \\ A^t & 2kE \end{pmatrix},$$

an *even positive definite unimodular form*. Taking for example

$$A=\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

we obtain necessarily the form E_8.

We now come to a variation of this construction: Let C be again a conference matrix, which we assume (in fact, without loss of generality) to be symmetric or skew-symmetric. We now put

$$A=(1-X)\,C+XE.$$

Then

$$E+A^t A=nE+(C^t+C-2(n-1)\,E)+(nE-C^t-C)\,X^2$$

is an even matrix so that we can apply our construction. Thus $\langle B\rangle$ is a regular symmetric bilinear space over the polynomial ring $\mathbb{Z}[X]$.

If we now specialize $X=0$, we get for $n=4k$ the even form

$$B(0)=\begin{pmatrix} 2E & C \\ C^t & \tfrac{1}{2}(C^t C+E) \end{pmatrix}.$$

If we specialize $X=1$, we get the odd form

$$B(a)=\begin{pmatrix} 2E & E \\ E & E \end{pmatrix}$$

which is isometric to the diagonal unit form. *In particular*, $B(0)$ *and* $B(1)$ *are not isometric but homotopic in an obvious sense*. This applies in particular to E_8 and the unit form.