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Quadratic Reciprocity Laws*

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Quadratic reciprocity laws for the rationals and rational function fields are proved. An elementary proof for Hilbert's reciprocity law is given. Hilbert's reciprocity law is extended to certain algebraic function fields.

This paper is concerned with reciprocity laws for quadratic forms over algebraic number fields and algebraic function fields in one variable.

This is a very classical subject. The oldest and best known example of a theorem of this kind is, of course, the Gauss reciprocity law which, in Hilbert's formulation, says the following: If (a, b) is a quaternion algebra over the field of rational numbers $\mathbf{Q}$, then the number of prime spots of $\mathbf{Q}$, where (a, b) does not split, is finite and even. This can be regarded as a theorem about quadratic forms because the quaternion algebras (a, b) correspond 1–1 to the quadratic forms $\langle 1, -a, -b, ab\rangle$, and (a, b) is split if and only if $\langle 1, -a, -b, ab\rangle$ is isotropic.

This formulation of the Gauss reciprocity law suggests immediately generalizations in two different directions:

(1) Replace the quaternion forms $\langle 1, -a, -b, ab\rangle$ by arbitrary quadratic forms.

(2) Replace $\mathbf{Q}$ by an algebraic number field or an algebraic function field in one variable (possibly with arbitrary constant field).

While some results are known concerning (1), it seems that the situation has never been investigated thoroughly, not even for the case of the rational numbers. Concerning (2), the main result is, of course, the Hilbert reciprocity law for arbitrary global fields, which is classically proved either by using Gauss sums and theta functions [6, 21, 4], or from the "axioms" of global class field theory, i.e., the first and second inequality [1].

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This paper contains a few contributions to both problems. In particular, we will give a reciprocity law for arbitrary quadratic forms over a rational function field, and a simple proof of Hilbert's reciprocity law for global fields. This proof is based on a recent theorem of Milnor about the behavior of quadratic forms over $\mathfrak{p}$-adic fields under the transfer-homomorphism [8]. By generalizing Milnor's theorem slightly, we obtain a generalization of Hilbert's reciprocity law to algebraic function fields over a "Milnor field".

Definitions, Notations, Known Results

We assume familiarity with basic results about quadratic forms over fields; in particular, over finite and $\mathfrak{p}$-adic fields; and we refer to [11, 17, 22] for general references about quadratic forms. We denote similarity of quadratic forms by $\sim$ and isometry by $\cong$.

Invariants

We consider fields of characteristic $\neq 2$ only. Let K be such a field. Every quadratic form q over K can be written in "diagonal form"

$$q \cong \langle a_1, ..., a_n \rangle = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle,$$

where $\langle a_i \rangle$ denotes the 1-dimensional form $a_i x^2$. We define

$$\text{discriminant}(q) = d(q) = (-1)^{n(n-1)/2} a_1 \cdots a_n .$$

This is a well-defined element of the group of square classes K^*/K^{*2}. If q is even-dimensional, we call the *Witt-invariant* of q the element $c(q)$ of the Brauer group $B(K)$ of K determined by the Clifford algebra of q. If q is odd-dimensional, we call the Witt-invariant the element $c(q)$ determined by the even Clifford algebra of q. The invariants d, c are well defined for elements of the Witt group $W(K)$, i.e., for similarity classes of quadratic forms.

Transfer

Let L/K be a finite algebraic extension, and $s : L \to K$ a K-linear map $\neq 0$. If q is a nondegenerate quadratic form over L, then $s^*q = sq$ is a nondegenerate quadratic form over K. One obtains in this way a *transfer-homomorphism* for the Witt groups

$$s^* : W(L) \to W(K).$$

In particular, if L/K is separable, we may take s to be the trace $\mathrm{Tr}_{L/K}$. If q is a quadratic form over K, denote by q_L the corresponding form over L. One has the "Frobenius reciprocity":

$$s^*(q_L) \cong q \otimes s^*\langle 1\rangle, \qquad q \in W(K).$$

We refer to [16, 17] for more information about the transfer-homomorphism.

Local fields

Let K be a local field, i.e., K is complete with respect to a discrete valuation. We refer to [18] for the theory of local fields. Let k be the residue class field and assume $\mathrm{char}(k) \neq 2$, let π be a prime element. Every quadratic form q over K can be written in the form

$$q \cong \langle a_1, \ldots, a_m\rangle \oplus \langle b_1\pi, \ldots, b_n\pi\rangle,$$

where a_i, b_j are units. The quadratic form

$$\bar{q}_1 = \langle \bar{a}_1, \ldots, \bar{a}_m\rangle$$

over k is well defined up to similarity, and is called the *first residue class form* of q. After a choice of π, the quadratic form

$$\bar{q}_2 = \langle \bar{b}_1, \ldots, \bar{b}_n\rangle$$

is well defined up to similarity and is called the *second residue class form* of q. This construction gives an isomorphism

$$W(K) \cong W(k) \oplus W(k)$$

(depending on the choice of π). It follows, moreover, that if every $(n + 1)$-dimensional form over k is isotropic, every $(2n + 1)$-dimensional form over K is isotropic. Details may be found in [20].

We call a field $\mathfrak{p}$-adic if it is local and if the residue class field is finite. If K is $\mathfrak{p}$-adic or real, we may identify the Witt invariant $c(q)$ with $+1$ or -1 depending on whether $c(q)$ is split or nonsplit.

1. "Easy" and "Difficult" Problems in the Theory of Quadratic Forms

In this section we recall some results which will be used in later sections.

In the theory of quadratic forms over fields one is mostly confronted with the following two questions:

"*Easy*" *problem*: *Is a given quadratic form q isometric to a direct sum of hyperbolic planes*, i.e., *is* $q = 0$ *in the Witt group?*

"*Difficult*" *problem*: *Is a given quadratic form q isotropic*, i.e., *is there a nonzero x such that* $q(x) = 0$*?*

EXAMPLE 1.1 (T. A. Springer). *Let k be an algebraic extension of odd degree.*

(i) (Weak theorem) *The canonical homomorphism*

$$r^* : W(K) \to W(L)$$

is injective.

(ii) (Strong theorem) *If q is anisotropic over K, then q is anisotropic over L.*

The weak result can be proved very easily [14], the strong result is a little bit harder [19].

EXAMPLE 1.2 (Pfister). *Let K be a field of finite level*, i.e., -1 *is a sum of squares in K.*

(i) (Weak theorem) *The order of the form* $\langle 1 \rangle$ *in* $W(K)$ *is a power of* 2.

(ii) (Strong theorem) *The maximal n such that* $\langle 1,..., 1 \rangle$ (n summands) *is anisotropic is a power of* 2.

The proof of the weak result is quite formal [16]; for the strong result one has to use multiplicative forms. A proof —due to Witt—will be given after 1.5.

EXAMPLE 1.3. *Consider the field of rational numbers.*

(i) (Strong theorem. Minkowski) *A quadratic form q over* **Q** *is isotropic if and only if it is isotropic over all p-adic fields and the reals.*

It is somewhat surprising that the corresponding weak theorem is very elementary, the proof is in fact a preliminary step to a proof of the Gauss reciprocity law. This has only recently been shown by Milnor [9], whose proof is based on work of Bass and Tate [2]. We want to describe this result in more detail:

If p is an odd prime, we have a canonical homomorphism

$$\partial_p : W(\mathbf{Q}) \to W(\mathbf{F}_p)$$

which is the composition of the canonical homomorphism $W(\mathbf{Q}) \to W(\mathbf{Q}_p)$ and the second residue homomorphism (with respect to the uniformizing element p) $W(\mathbf{Q}_p) \to W(\mathbf{F}_p)$. (Of course, $\mathbf{F}_p$ denotes the field with p elements.) We define $\partial_2 : W(\mathbf{Q}) \to \mathbf{Z}/2\mathbf{Z}$ by $\partial_2(q) \equiv v_2(d(q)) \pmod 2$ where v_2 denotes the 2-adic valuation. Finally, we define $\partial_\infty : W(\mathbf{Q}) \to \mathbf{Z}$ to be the signature. (Note that $W(\mathbf{F}_2) \cong \mathbf{Z}/2\mathbf{Z}$, and $W(\mathbf{R}) = \mathbf{Z}$.)

(ii) (Weak theorem) *These homomorphisms define a group isomorphism*

$$W(\mathbf{Q}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \coprod_{p \text{ odd}} W(\mathbf{F}_p).$$

The proof is short and elementary; it may be easily adapted from Milnor's proof of a corresponding (slightly more difficult) result for rational function fields (see 1.4. below).

EXAMPLE 1.4. Consider a rational function field $F = K(x)$, where $\mathrm{char}(K) \neq 2$. We will keep the transcendental element x fixed. The "finite" primes $\mathfrak{p}$ of F/K are in 1–1 correspondence to the irreducible polynomials $p(x)$ with leading coefficient 1. We denote by $F_\mathfrak{p}$ the completion of F at the prime $\mathfrak{p}$, and by $K_\mathfrak{p}$ the residue class field of $F_\mathfrak{p}$. We have

$$\begin{aligned} K_\mathfrak{p} &= K[x]/p(x) \qquad \text{if } \mathfrak{p} = p(x) \text{ is finite} \\ K_\mathfrak{p} &= K \qquad\qquad\;\; \text{if } \mathfrak{p} = \infty. \end{aligned}$$

For each prime $\mathfrak{p}$ we have a homomorphism

$$\partial_\mathfrak{p} : W(F) \to W(K_\mathfrak{p})$$

which is the composition of the canonical homomorphism $W(F) \to W(F_\mathfrak{p})$ and the second residue homomorphism $W(F_\mathfrak{p}) \to W(K_\mathfrak{p})$ with respect to the uniformizing element $p(x)$ if $\mathfrak{p}$ is finite, and $1/x$ if $\mathfrak{p}$ is infinite.

1.4. (WEAK THEOREM. HARDER) *The homomorphisms $\partial_\mathfrak{p}$ and the canonical inclusion $W(k) \to W(F)$ give an exact sequence*

$$0 \to W(K) \to W(F) \xrightarrow{\Sigma\partial_\mathfrak{p}} \coprod_{\mathfrak{p} \text{ finite}} W(K_\mathfrak{p}) \to 0.$$

This sequence splits, e.g., *through the homomorphism $\partial_0 : W(F) \to W(K)$ which is the composition of $W(F) \to W(K((t)))$ and the first residue homomorphism $W(K((t))) \to W(K)$.*

Proof. ([9], [5] 3.5, [7] §13).

Pfister has communicated to me the interesting result that the corresponding strong result is false.

Example 1.5. The distinction between the "easy problem" and the "difficult problem" may explain why Pfister's multiplicative forms are so useful, because *for multiplicative forms both questions coincide*. We give Witt's unpublished version of Pfister's theory (see also [17, Chapter II): An isotropic quadratic form q is called multiplicative if $q \sim 0$. An anisotropic form q is called multiplicative if it represents 1 and if for every nonzero element a represented by q ($g(x) = a$ for some x) the forms q and aq are isometric.

It is easy to see that the forms $\langle 1, a\rangle$ are multiplicative.

LEMMA. *If q is multiplicative, $q \otimes \langle 1, a\rangle$ is multiplicative.*

Proof. If q is multiplicative and isotropic, $q \otimes \langle 1, a\rangle \sim 0$, i.e., $q \otimes \langle 1, a\rangle$ is multiplicative. Assume q is anisotropic, and $q \otimes \langle 1, a\rangle = q \oplus aq$ is isotropic, i.e., $q(x) + aq(y) = 0$ for suitable x, y. Since q is multiplicative,

$$q \oplus aq \cong q(x)q \oplus aq(y)q \cong \langle q(x), q(y)a\rangle \otimes q \cong \langle 1, -1\rangle \otimes q \sim 0.$$

Assume $q \otimes \langle 1, a\rangle$ is anisotropic. Let $d = q(x) + aq(y) \neq 0$ and set $b = q(x)$, $c = q(y)$. If $c = 0$,

$$(q \otimes \langle 1, a\rangle)b \cong q \otimes \langle 1, a\rangle.$$

If $b = 0$,

$$(q \otimes \langle 1, a\rangle)ac \cong q \otimes \langle a, a^2\rangle \cong q \otimes \langle a, 1\rangle.$$

If $b, c \neq 0$,

$$\begin{aligned}(q \otimes \langle 1, a\rangle)(b + ac) &\cong (bq \oplus acq)(b + ac) \cong q \otimes \langle b, ac\rangle(b + ac) \\ &\cong q \otimes \langle 1, abc\rangle \cong q \otimes \langle 1, a\rangle.\end{aligned}$$

COROLLARY. *The maximal n such that $\langle 1,..., 1\rangle$ (n summands) is anisotropic is infinite or a power of 2.*

Proof. Let $2^{k-1} \leqslant n < 2^k$. The form $\langle 1,..., 1\rangle$ (2^k summands) is multiplicative and isotropic, hence ~ 0. By Witt's cancellation theorem, $\langle 1,..., 1\rangle \cong \langle -1,..., -1\rangle$ (2^{k-1} summands); hence $n = 2^{k-1}$.

2. RECIPROCITY LAWS OVER Q

This section contains reciprocity laws for rational quadratic forms along the lines of [2, 10].

Let Ω denote the set of all prime spots of $\mathbf{Q} : \Omega = \{\infty, 2, 3, 5, \ldots\}$. Let A be an abelian group, and $\chi : W(\mathbf{Q}) \to A$ a homomorphism. By 1.3. there are uniquely determined homomorphisms

$$\chi_\infty : \mathbf{Z} \to A, \qquad \chi_2 : \mathbf{Z}/2\mathbf{Z} \to A$$
$$\chi_p : W(\mathbf{F}_p) \to A, \qquad p \text{ odd prime}$$

such that

$$\chi = \sum_{p \in \Omega} \partial_p \chi_p .$$

This equation is the "abstract" reciprocity law. We want to determine the χ_p for homomorphisms χ associated with the valuations of $\mathbf{Q}$. Let us recall that $W(\mathbf{Q}_2)$ is isomorphic to $\mathbf{Z}/8\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ with generators $\langle 1 \rangle, \langle 1, 1, 1, 5 \rangle, \langle -1, 2 \rangle$, respectively. (I do not know an explicit reference for this result, but it may be deduced easily from results in [11] or [17].)

THEOREM 2.1. (Gauss, van der Blij [3], Bass-Tate [2].) *If*

$$\chi : W(\mathbf{Q}) \to \mathbf{Z}/8\mathbf{Z}$$

is the composition of the canonical homomorphism $W(\mathbf{Q}) \to W(\mathbf{Q}_2)$ with the projection of $W(\mathbf{Q}_2)$ on the direct summand $\mathbf{Z}/8\mathbf{Z}$, then

$$\begin{aligned} \chi_\infty(n) &\equiv n \pmod 8, \quad n \in \mathbf{Z}, \\ \chi_2 &= 0, \\ \chi_p\langle 1 \rangle &\equiv p - 1 \pmod 8 \quad p \text{ odd prime}, \\ \chi_p(\varphi_p) &\equiv 4 \pmod 8, \end{aligned}$$

where φ_p denotes the unique 2-dimensional anisotropic form over $\mathbf{F}_p$.

For the proof we shall need the following lemma.

LEMMA 2.2 (Gauss). *If p is a prime $\equiv 1 \pmod 8$, then there exists a prime q such that $2 < q < p$ and p is not a square* mod q.

Proof. [2, 10].

Proof of 2.1. It is obvious that χ_∞ is reduction mod 8. Moreover, $\chi\langle -1, 2 \rangle = 0$ implies $\chi_2 = 0$. If p is an odd prime, then it is easy to check that the following equations hold in $W(\mathbf{Q}_2)$:

$$\begin{aligned} \langle -1, p \rangle &= 0 && \text{if } p \equiv 1 \pmod 8 \\ &= \langle 1, 1, 1, 5 \rangle \oplus \langle 1, 1 \rangle && \text{if } p \equiv 3 \pmod 8 \\ &= \langle 1, 1, 1, 5 \rangle \oplus \langle 1, 1, 1, 1 \rangle && \text{if } p \equiv 5 \pmod 8 \\ &= \langle 1, 1, 1, 1, 1, 1 \rangle && \text{if } p \equiv 7 \pmod 8. \end{aligned}$$

Since $\partial_q\langle -1, p\rangle = 0$ for $q \in \Omega$, $q \neq p$, and $\partial_p\langle -1, p\rangle = \langle 1\rangle$, it follows that

$$\chi_p\langle 1\rangle \equiv p - 1 \pmod 8.$$

If $p \equiv 3, 7 \pmod 8$, $\varphi_p \cong \langle 1, 1\rangle$, and we have proved everything in this case.

Assume $p \equiv 5 \pmod 8$. In this case, 2 is not a square in $\mathbf{F}_p : (-1)^{\frac{1}{8}(p^2-1)} = (-1)$. Since -1 is a square, $\varphi_p = \langle -1, 2\rangle$. Moreover, in $W(\mathbf{Q}_2)$ we know that

$$\langle -p, 2p\rangle = \langle -5, 10\rangle = \langle -1, 2\rangle \oplus \langle 1, 1, 1, 1\rangle.$$

But

$$\partial_2\langle -p, 2p\rangle \equiv 1 \pmod 2,$$
$$\partial_p\langle -p, 2p\rangle = \varphi_p,$$
$$\partial_q\langle -p, 2p\rangle = 0 \quad \text{if} \quad q \neq 2, p.$$

Hence,

$$4 \equiv \chi\langle -p, 2p\rangle = \chi_p(\varphi_p) \pmod 8$$

which completes the proof in this case.

Assume finally $p \equiv 1 \pmod 8$. Choose q according to 2.2. and consider the form $\psi = \langle 1, -p, -q, pq\rangle$. Obviously, this form is 0 in $W(\mathbf{Q}_2)$. Moreover, $\partial_p(\psi) = \langle -1, q\rangle$, $\partial_q(\psi) = \langle -1, p\rangle$, $\partial_{p'}(\psi) = 0$ for $p' \in \Omega$, $p' \neq p, q$. Therefore,

$$\chi_q\langle -1, p\rangle \equiv \chi_p\langle -1, q\rangle \pmod 8.$$

We choose q such that $\langle -1, p\rangle = \varphi_q$. Carrying out induction on p we conclude from the induction hypothesis, $\chi_q\langle -1, p\rangle \equiv 4 \pmod 8$, that $\chi_p\langle -1, q\rangle \equiv 4 \pmod 8$, and *a fortiori* $\langle -1, q\rangle = \varphi_p$.

If we apply Theorem 2.1. to the quaternion forms $\langle 1, -a, -b, ab\rangle$, we obtain the Gauss reciprocity law in the form stated in the introduction.

We now want to formulate a complement to Theorem 2.1. To do this choose an odd prime p, and let

$$\chi : W(\mathbf{Q}) \to W(\mathbf{F}_p)$$

be the composition of the canonical homomorphism $W(\mathbf{Q}) \to W(\mathbf{Q}_p)$ and the *first* residue homomorphism $W(\mathbf{Q}_p) \to W(\mathbf{F}_p)$.

PROPOSITION 2.3. *For this χ, the χ_q, $q \in \Omega$ are as follows:*

$$\chi_\infty : 1 \mapsto \langle 1\rangle$$
$$\chi_2 : 1 \mapsto \varphi_p \quad \textit{if} \quad p \equiv 3, 5 \pmod 8$$
$$1 \mapsto 0 \quad \textit{if} \quad p \equiv 1, 7 \pmod 8$$
$$\chi_p = -id$$

For $q \neq \infty, 2, p$

$$\chi_q\langle 1\rangle = \varphi_p \qquad \textit{if} \quad \left(\frac{q}{p}\right) \neq 1$$

$$= 0 \qquad \textit{if} \quad \left(\frac{q}{p}\right) = 1,$$

$$\chi_q(\varphi_q) = 0 \qquad \textit{always}.$$

The proof of this proposition is analogous to the proof of 2.1. However, it is even simpler since it is not necessary to use 2.2. Also, we leave it to the reader to work out the reciprocity laws for other homomorphisms $\chi : W(\mathbf{Q}) \to A$.

We want to conclude this section with a remark on an invariant of integral symmetric bilinear forms which is often used in topology. Let M be a free $\mathbf{Z}$-module of finite rank, let $f : M \times M \to \mathbf{Z}$ be a nonsingular symmetric bilinear form, and $\bar{f} : M/2M \times M/2M \to \mathbf{Z}/2\mathbf{Z}$ the reduction of f mod 2. Since $\bar{x} \mapsto \bar{f}(\bar{x}, \bar{x})$ is a homomorphism, there is a unique $\bar{w} \in M/2M$ such that $\bar{f}(\bar{x}, \bar{x}) = f(\bar{x}, \bar{w})$ for all $\bar{x}$, i.e.,

$$f(x, x) \equiv f(x, w) \pmod 2.$$

Therefore, $f(w, w)$ is uniquely determined modulo 8:

$$f(w + 2x, w + 2x) = f(w, w) + 4(f(x, w) + f(x, x)).$$

It is easy to see that we obtain a homomorphism

$$\tau : W(\mathbf{Z}) \to \mathbf{Z}/8\mathbf{Z}, \qquad f \mapsto f(w, w)$$

defined on the Witt group $W(\mathbf{Z})$ of nonsingular symmetric bilinear forms over $\mathbf{Z}$. Now it is easy to see that $W(\mathbf{Z})$ imbeds into $W(\mathbf{Q})$. (In fact, this is true for any Dedekind domain and its quotient field [7].) We claim that $\partial_p(W(\mathbf{Z})) = 0$ for $p = 2, 3, \ldots$. This is obvious for $p = 2$ because the determinant of a nonsingular integral form is ± 1. For p odd, this follows from the fact that f can be written in diagonal form over the p-adic integers. Since all diagonal coefficients must be units, the second residue form is 0. In view of 1.3, we have shown that $W(\mathbf{Z}) \simeq \mathbf{Z}$.

COROLLARY 2.4. *The invariant* τ *is congruent to the signature modulo* 8.

Proof. It suffices to check this for the generator $\langle 1\rangle$ of $W(\mathbf{Z})$, where it is obviously true.

3. Milnor Fields

Definition 3.1. A field K of characteristic $\neq 2$ will be called a *Milnor field of degree n* if the following conditions are satisfied:

(i) For every finite algebraic extension L/K there exists up to isometry a unique anisotropic form φ_L over L of dimension 2^n. Every form of dimension $>2^n$ over L is isotropic.

(ii) If $N/L/K$ are finite algebraic extensions and $s : N \to L$ is a nonzero L-linear map, then $s^*(\varphi_N) \sim \varphi_L$.

Before we give examples of Milnor fields, we make a remark on condition (ii): Let K be an arbitrary field, L/K a finite algebraic extension and fix a nonzero K-homomorphism $s_0 : L \to K$, which defines the following nondegenerate symmetric bilinear form

$$(x, y) \mapsto s_0(xy), \qquad x, y \in L.$$

Let $s : L \to K$ be a second K-homomorphism $\neq 0$. Then there exists a unique $a_s \in L^*$ such that

$$s(x) = s_0(a_s x)$$

for all $x \in L$. This means

$$s^*(\varphi) \cong s_0{}^*(a_s \varphi)$$

for all $\varphi \in W(L)$.

If we want to prove condition (ii), it suffices to check this condition for a single s_0 because $a_s \varphi_N \cong \varphi_N$. Condition (ii) will usually be proved stepwise by construction of a tower of field extensions

$$N = L_n \supset L_{n-1} \supset \cdots \supset L_0 = L$$

and L_{i-1}-homomorphisms $s_i : L_i \to L_{i-1}$.

Let us also remark *that* (ii) *is always true in the case of an extension of odd degree*. By 1.1 the form $(\varphi_L)_N$ is anisotropic, i.e., $(\varphi_L)_N \cong \varphi_N$. Thus we may use the Frobenius reciprocity and the fact that

$$a\varphi_L \cong \varphi_L \cong -\varphi_L$$

to obtain

$$s^*(\varphi_N) \cong \varphi_L \otimes s^*\langle 1 \rangle \cong \varphi_L \otimes \langle 1, 1, -1, 1, -1, \ldots, 1, -1 \rangle$$
$$\sim \varphi_L .$$

Proposition 3.2. *The finite fields are Milnor fields of degree* 1.

Proof. (i) is well known and trivial.

(ii) It suffices to consider extensions of degree odd or 2. The first case is settled by our general remark. In the quadratic case one has $\varphi_N = \langle 1, \epsilon \rangle$, where ϵ represents the nonsquare of $N = L(\epsilon)$. Define $s(1) = 1$, $s(\epsilon) = 0$. Thus $s^*\langle \epsilon \rangle$ is isotropic i.e., a hyperbolic plane. Therefore

$$s^*\langle 1, \epsilon \rangle \sim s^*\langle 1 \rangle \cong \langle 1, -\mathrm{norm}_{N/L}(\epsilon)\rangle$$

(see [14]). $\mathrm{Norm}_{N/L}(\epsilon)$ is a nonsquare of L. Separating the cases depending upon whether -1 is a square in L or not, one sees easily that

$$\langle 1, -\mathrm{norm}_{N/L}(\epsilon)\rangle$$

is anisotropic, i.e., is φ_L.

Remark. This proposition and its proof remain valid in the case of a quasi-finite field. (See [18] for the notion of a quasi-finite field.)

THEOREM 3.3. *If k is a Milnor field of degree n, and if K is a local field with residue class field k, then K is a Milnor field of degree $n + 1$.*

Proof. (i) Denote the residue class field of L by λ. Let φ_L be the unique 2^{n+1}-dimensional anisotropic form whose first and second residue class forms are φ_λ. (Note that in this case the second residue class form is independent of the choice of the prime π.) It is well known that (i) is satisfied (see Section 1).

(ii) To simplify our notations we will write L/K instead of N/L. Since a purely inseparable extension is of odd degree, this case is settled by our general remark, and we may assume L/K is separable. For the same reason we may assume L/K is tamely ramified. Let K_{nr} be the maximal unramified extension of K contained in L. The extension L/K_{nr} is normal and cyclic. As in the case of a finite field, we may reduce the problem to the quadratic case. Hence we can write $L = K_{tr} = K_{nr}(\sqrt{\pi})$, where π is a prime. If ψ is the uniquely determined anisotropic form over K_{nr} with 1st residue class form $\varphi_{k_{nr}}$ and 2nd residue class form 0, we can write

$$\varphi_L = \psi_L \oplus \sqrt{\pi}\, \psi_L .$$

Of course, k_{nr} is the residue class field of K_{nr}. Define $s : L \to K_{nr}$ by $s(1) = 1$, $s(\sqrt{\pi}) = 0$. Thus we have

$$s^*(\psi_L \oplus \sqrt{\pi}\, \psi_L) \cong s^*(\psi_L \otimes \langle 1, \sqrt{\pi}\rangle) = \psi \otimes s^*\langle 1, \sqrt{\pi}\rangle$$
$$\sim \psi \otimes \langle 1, \pi\rangle \cong \varphi_{K_{nr}} .$$

The only case remains when L/K is unramified, but then the proof is reduced to the residue class fields in the usual way: Write

$$\varphi_K = \psi_K \oplus \pi\psi_K, \qquad \varphi_L = \psi_L \oplus \pi\psi_L,$$

where

$$\overline{(\psi_K)}_1 \cong \varphi_k, \qquad \overline{(\psi_K)}_2 = 0,$$
$$\overline{(\psi_L)}_1 \cong \varphi_\lambda, \qquad \overline{(\psi_L)}_2 = 0.$$

Let $L = K(x)$ with x a unit and define the K-homomorphism s, e.g., by

$$s(1) = 1, \qquad s(x) = \cdots = s(x^{n-1}) = 0.$$

s induces a k-homomorphism $\bar{s} : \lambda \to k$. Thus it is obvious that we have $s^*(\varphi_L) = s^*(\psi_L) \oplus \pi s^*(\psi_L)$. We now have what we want, namely,

$$\overline{(s^*(\psi_L))_1} \cong \bar{s}^*(\varphi_\lambda) \sim \varphi_k,$$
$$\overline{(s^*(\psi_L))_2} \sim 0,$$

i.e.,

$$s^*(\psi_L) \sim \psi_K, \qquad \text{Q.E.D.}$$

As a corollary we see that every $\mathfrak{p}$-adic field, $\mathfrak{p} \nmid 2$, is a Milnor field of degree 2. As usual this is true in the dyadic case also:

THEOREM 3.4. *Every* $\mathfrak{p}$*-adic field is a Milnor field of degree* 2.

Proof. Milnor [8, Theorem 2.3].

Remark. A more general result and a completely different proof can be found in [13]. Milnor's theorem is essentially contained already in [21].

4. Reciprocity Laws over Rational Function Fields

We consider a rational function field $F = K(x)$ and shall use the same notations as in Example 1.4, except for the following simplification: *If* Q *is a quadratic form over* F, *let* $q_\mathfrak{p}$ *be the second residue class form of* $Q_\mathfrak{p} = Q_{F_\mathfrak{p}}$ *with respect to the uniformizing element* $p(x)$ *resp.* $1/x$ at ∞. For $\mathfrak{p}$ finite and $n = \deg(\mathfrak{p})$, define the K-linear map

$$s_\mathfrak{p} : K[x]/p(x) = K_\mathfrak{p} \to K$$

by

$$s_\mathfrak{p}(1) = \cdots = s_\mathfrak{p}(x^{n-2}) = 0, \qquad s_\mathfrak{p}(x^{n-1}) = 1,$$

define $s_\infty = -id$. With these notations the following theorem holds.

THEOREM 4.1. *If Q is a quadratic form over F, then*

$$\bigoplus_{\mathfrak{p}} s_{\mathfrak{p}}{}^*(q_{\mathfrak{p}}) \sim 0.$$

Proof. It is obvious that almost all second residue class forms are ~ 0. It suffices to prove the theorem for 1-dimensional Q. Write without loss of generality

$$Q = \langle p_1(x) \cdots p_n(x) \rangle,$$

where $p_i(x)$ is an irreducible polynomial of degree d_i with leading coefficient 1. If

$$A = K[x]/p_1(x) \cdots p_n(x),$$
$$A_i = K[x]/p_i(x),$$

the K-homomorphism $A_i \to A$ is defined canonically by

$$x^r \mapsto x^r p_1(x) \cdots \hat{p_i}(x) \cdots p_n(x), \qquad r = 0,\ldots, d_i - 1,$$

(^ means: omit this factor.) and we have consequently a canonical direct sum decomposition

$$A = A_1 \oplus \cdots \oplus A^n .$$

Let $s : A \to K$ be defined as above by

$$s(1) = \cdots = s(x^{d-2}) = 0, \qquad s(x^{d-1}) = 1, \qquad d = \Sigma d_i ,$$

and let H be the quadratic form $s^*\langle 1 \rangle$ on A, i.e.,

$$H(f(x)) = s(f(x)^2).$$

It follows immediately from this definition that the A_i are pairwise orthogonal. We claim

$$H|_{A_i} \cong s_{p_i}^*(q_{p_i}), \qquad i = 1,\ldots, n.$$

To see this, write

$$f(x)^2\, p_1(x)^2 \cdots \hat{p_i}(x)^2 \cdots p_n(x)^2$$
$$= p_1(x) \cdots \hat{p_i}(x) \cdots p_n(x)[c_0 + c_1 x + \cdots + c_{d_i-1} x^{d_i-1} + p_i(x)\, g(x)].$$

By the definition of s,

$$H|_{A_i}(f(x)\, p_1(x) \cdots \hat{p_i}(x) \cdots p_n(x)) = c_{d_i-1} .$$

On the other hand, by the definition of s_{p_i}

$$s^*_{p_i}(q_{p_i})(f(x)) = s_{p_i}(f(x)^2\, p_1(x) \cdots \hat{p_i}(x) \cdots p_n(x)) \\ = c_{d_i - 1}\,.$$

Obviously, $q_{\mathfrak{p}} = 0$ for all finite primes different from $p_1(x),\dots, p_n(x)$. We have proved that

$$H \cong \bigoplus_{i=1}^{n} s^*_{p_i}(q_{p_i}) = \bigoplus_{\mathfrak{p}\ \text{finite}} s_{\mathfrak{p}}{}^*(q_{\mathfrak{p}}).$$

If $d = 2e$, we have $H \sim 0$, because $1, x,\dots, x^{e-1}$ span a totally isotropic subspace. We are finished in this case because $q_\infty = 0$. If $d = 2e + 1$, we still have an e-dimensional totally isotropic subspace. By computing the determinant, it is easy to see that

$$H \cong \langle 1\rangle \oplus e \text{ hyperbolic planes.}$$

Since $s^*(q_\infty) = \langle -1\rangle$, we have completed the proof of Theorem 4.1.

We can reformulate Theorem 4.1 as follows: Let $\chi : W(F) \to W(K)$ be the composition of the homomorphism $W(F) \to W(F_\infty)$ and the second residue homomorphism $W(F_\infty) \to W(K)$. Because of $\chi(W(K)) = 0$, we can write

$$\chi = \sum_{\mathfrak{p}\neq\infty} \chi_{\mathfrak{p}} \partial_{\mathfrak{p}}$$

for suitable homomorphisms $\chi_{\mathfrak{p}} : W(K_{\mathfrak{p}}) \to W(K)$. Theorem 4.1 says $\chi_{\mathfrak{p}} = s_{\mathfrak{p}}{}^*$.

Remark. Theorem 4.1 can also be proved using the residue theorem for rational function fields. For if

$$\omega = f(x)\, dx, \\ f(x) = x^{k+m}(p_1(x) \cdots p_r(x))^{-1},$$

then it is not difficult to see that $\operatorname{res}_{p_i}(\omega)$ is the inner product of x^k and x^m with respect to the quadratic form $s^*_{\mathfrak{p}_i}(q_{\mathfrak{p}_i})$. This is because

$$\operatorname{res}_{p_i}(\omega) = a_{d_i-1}\,,$$

in

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_{d_i-1} x^{d_i-1}}{p_i(x)} + \cdots.$$

(We regard $\operatorname{res}_{p_i}(\omega)$ as an element of K, i.e., it is the trace of the usual residuum!) For $k + m < d - 1$ we see easily that $\operatorname{res} \omega = 0$ for all primes different from $p_1(x),\ldots, p_n(x)$. In particular, $\operatorname{res}(\omega) = 0$ at ∞. Using the residue theorem we see that the vectors $(x^r,\ldots, x^r)$, $0 \leqslant r < e$ span a totally isotropic subspace of $A_1 \oplus \cdots \oplus A_n$.

Remark. Results, analogous to 1.4 and 4.1, were proved by Faddeev for the Brauer group of a rational function field (see [15]).

We now use 4.1 to deduce more examples of reciprocity laws.

Let $\chi : W(F) \to W(K)$ be the composition of the canonical homomorphism $W(F) \to W(F_\infty)$ and the *first* residue homomorphism

$$W(F_\infty) \to W(K).$$

By 1.4 we can write

$$\chi = \chi_0 \partial_0 + \sum_{\mathfrak{p} \neq \infty} \chi_\mathfrak{p} \partial_\mathfrak{p} .$$

COROLLARY 4.2. *In this situation, $\chi_0 : W(K) \to W(K)$ is the identity, and $\chi_\mathfrak{p} = W(K_\mathfrak{p}) \to W(K)$ is equal to $t_\mathfrak{p}^*$, where $t_\mathfrak{p} : K_\mathfrak{p} \to K$ is the K-linear map $t_\mathfrak{p}(b) = s_\mathfrak{p}(xb)$.*

Proof. If Q is a quadratic form over F, then

$$\chi(Q) = \overline{(Q_\infty)_1} = \overline{((xQ)_\infty)_2} = \bigoplus_{\mathfrak{p} \neq \infty} s^* \overline{((xQ)_\mathfrak{p})_2} .$$

Inspecting this equation, in particular, at the prime $\mathfrak{p} = x$, we see that this is exactly what we had to prove.

Let now $\mathfrak{q} = (x - a)$ be a prime of degree 1, and let $\chi : W(F) \to W(K_\mathfrak{q})$ be the composition of the canonical homomorphism $W(F) \to W(F_\mathfrak{q})$ and the *first* residue homomorphism $W(F_\mathfrak{q}) \to W(K_\mathfrak{q})$. By 1.4 we write again

$$\chi = \chi_0 \partial_0 + \sum_{\mathfrak{p} \neq \infty} \chi_\mathfrak{p} \partial_\mathfrak{p} .$$

COROLLARY 4.3. *In this situation, χ_0 is the identity, $\chi_\mathfrak{q}$ is multiplication by a, and for $\mathfrak{p} \neq \infty$, $\mathfrak{q}$ we have $\chi_\mathfrak{p} = t_\mathfrak{p}^* + \tau_\mathfrak{p}^*$, where $t_\mathfrak{p}$ is as before and $\tau_\mathfrak{p}(b) = -s_\mathfrak{p}((x - a)b)$.*

Proof. Let Q be a quadratic form over F. Using 4.1 and 4.2 we obtain

$$\begin{aligned}
\chi(Q) &= \overline{(Q_\mathfrak{q})_1} = \overline{(((x - a)\, Q)_\mathfrak{q})_2} \\
&= \overline{(((x - a)\, Q)_\infty)_2} - \bigoplus_{\mathfrak{p} \neq \mathfrak{q}, \infty} s_\mathfrak{p}^* \overline{(((x - a)\, Q)_\mathfrak{p})_2} \\
&= (Q_\infty)_1 - \cdots \\
&= \bigoplus_{\mathfrak{p} \neq \infty} s_\mathfrak{p}^* \overline{((xQ)_\mathfrak{p})_2} - \cdots .
\end{aligned}$$

This is what we had to prove.

Remark. We leave it to the reader to investigate the case of a prime of degree >1. This is related to the behaviour of the whole situation under a constant field extension.

Remark. Theorem 4.1, particularly in its reformulation answers a question of Knebusch [7, Section 13]. That there should exist a reciprocity law of this type (i.e., involving the transfer) has been pointed out to me by Harder. A somewhat similar result for algebraic number fields has been proved by Weil [21, p. 179, Proposition 5]. The arbitrary choice of the transcendental x here corresponds there to a choice of a character of the adele group.

5. Reciprocity Laws over Algebraic Number Fields

In this section we want to give an elementary proof of the Hilbert reciprocity law:

THEOREM 5.1. *Let K be an algebraic number field and (a, b) a quaternion algebra over K. The number of primes $\mathfrak{p}$, where (a, b) does not split, is finite and even.*

For the proof we use the following lemmas:

LEMMA 5.2. *Let K be an algebraic number field. The following statements are equivalent:*

(i) *If $a, b \in K^*$, then the number of primes $\mathfrak{p}$ such that the quaternion algebra (a, b) does not split over $K_\mathfrak{p}$ is finite and even.*

(ii) *For every quadratic form q over K of even dimension and discriminant 1 the number of primes $\mathfrak{p}$ such that $c(q_\mathfrak{p}) \neq 1$ is finite and even.*

Proof. "(i) $\Rightarrow$ (ii)". The invariant $c(q)$ is a product of quaternion algebras. If each of these does not split exactly at an even number of primes, the same is true for their product.

"(ii) $\Rightarrow$ (i)". We only have to observe that

$$c\langle 1, -a, -b, ab\rangle = (a, b).$$

LEMMA 5.3. *Let F be a field and $\mathfrak{p}$ a discrete or archimedean prime. Let E/F be a separable algebraic extension of finite degree. If Q is a quadratic form over E, then*

$$(\mathrm{Tr}^*_{E/F}Q)_\mathfrak{p} = \bigoplus_{\mathfrak{P}|\mathfrak{p}} (\mathrm{Tr}^*_{E_\mathfrak{P}/F_\mathfrak{p}}Q_\mathfrak{P}).$$

This lemma can be easily deduced from the well-known fact

$$\mathrm{Tr}_{E/F}(x) = \sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{Tr}_{E_{\mathfrak{P}}/F_{\mathfrak{p}}}(x).$$

Proof of 5.1. Let q be a quadratic form over the algebraic number field K, assume q has even dimension and discriminant 1. We have to show

$$\prod_{\mathfrak{p}} c(q_{\mathfrak{p}}) = 1.$$

Let $\psi = \mathrm{Tr}^*_{K/\mathbf{Q}}(q)$. Let p be a prime of $\mathbf{Q}$ or ∞. By Lemma 5.3 we have

$$\psi_p \simeq \bigoplus_{\mathfrak{p}|p} \mathrm{Tr}^*_{K_{\mathfrak{p}}/\mathbf{Q}_p}(q_{\mathfrak{p}}).$$

All summands on the right hand side have discriminant 1. By 2.4, $\mathbf{Q}_p(p \neq \infty)$ is a Milnor field. Moreover, $q_{\mathfrak{p}} \sim 0$ or $q_{\mathfrak{p}} \sim \varphi_{K_{\mathfrak{p}}}$. Therefore,

$$c(\psi_p) = \prod_{\mathfrak{p}|p} c(\mathrm{Tr}^*_{K_{\mathfrak{p}}/\mathbf{Q}_p}(q_{\mathfrak{p}})) = \prod_{\mathfrak{p}|p} c(q_{\mathfrak{p}}).$$

Note that this is also correct at ∞. Therefore,

$$\prod_{\mathfrak{p}} c(q_{\mathfrak{p}}) = \prod_{p} \prod_{\mathfrak{p}|p} c(q_{\mathfrak{p}}) = \prod_{p} c(\psi_p).$$

Since we assume the theorem for $\mathbf{Q}$, it remains to check that $d(\psi) = 1$, which is easily proved (or see [8, 2.2]).

6. Reciprocity Laws over Algebraic Function Fields

Let F be an algebraic function field in one variable with constant field K, where K is a Milnor field of degree n. By 3.3, the completions $F_{\mathfrak{p}}$ of F are Milnor fields of degree $n + 1$. A quadratic form Q over F is called *locally universal* if for all primes $\mathfrak{p}$ of F/K either $Q_{\mathfrak{p}} \sim 0$ $(Q_{\mathfrak{p}} := Q_{F_{\mathfrak{p}}})$ or $Q_{\mathfrak{p}} \sim \varphi_{F_{\mathfrak{p}}}$. By Pfister's theorem (Example 1.5) the multiplicative forms

$$\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_{n+1} \rangle$$

are locally universal.

Theorem 6.1. *If F is an algebraic function field in one variable with* *constant field K, where K is a Milnor field of degree n, and if Q is a locally universal quadratic form over F, then the set*

$$S(Q) = \{\mathfrak{p} \mid Q_\mathfrak{p} \not\sim 0\}$$

contains a finite and even number of elements.

In particular, if Q is multiplicative of dimension 2^{n+1}, *then the number of primes* $\mathfrak{p}$ *such that* $Q_\mathfrak{p}$ *is anisotropic is finite and even.*

Proof. With the notations fixed at the beginning of this section, $Q_\mathfrak{p} \sim 0$ if and only if $q_\mathfrak{p} \sim 0$. Therefore, $S(Q)$ is a finite set.

If $s(Q) = \text{card}(S(Q))$, we will first prove that this number is even when $F = K(x)$ is a rational function. By 4.1 we know that

$$\bigoplus_{\mathfrak{p}} s_\mathfrak{p}{}^*(q_\mathfrak{p}) \sim 0.$$

Now, using the Milnor property and the definition of locally universal, we see that all the nonzero summands are similar to φ_K. Therefore, their number must be even.

To settle the case of an arbitrary function field we apply Lemma 5.3. Let E be a finite separable extension of $F = K(x)$, and let Q be a locally universal form over E. From the lemma it follows that $\text{Tr}^*_{E/F}(Q)$ is locally universal because $\text{Tr}^*_{E_\mathfrak{P}/F_\mathfrak{p}}(Q_\mathfrak{P})$ is either ~ 0 or $\sim \varphi_{F_\mathfrak{p}}$, and a sum of such forms is either ~ 0 or $\sim \varphi_{F_\mathfrak{p}}$. It follows from the Milnor property that

$$\begin{aligned} \#\{\mathfrak{P} \mid \mathfrak{p} \text{ such that } Q_\mathfrak{P} \not\sim 0\} &\equiv 0 \pmod 2 \qquad \text{if} \quad (\text{Tr}^*_{E/F}Q)_\mathfrak{p} \sim 0 \\ &\equiv 1 \pmod 2 \qquad \text{if} \quad (\text{Tr}^*_{E/F}Q)_\mathfrak{p} \sim \varphi_{F_\mathfrak{p}} \end{aligned}$$

and these congruences prove what we want.

Finally, we assume that we know the theorem for an algebraic function field F, and we want to extend it to a purely inseparable extension E/F. We may restrict ourselves to the case when $[E:F] = p$. Let $\mathfrak{p}$ be a prime and $\mathfrak{P}$ the unique extension to E. Then two cases may occur:

(i) either we have $[E_\mathfrak{P} : E_\mathfrak{p}] = p$. In this case s induces a map $s_\mathfrak{p} : E_\mathfrak{P} \to F_\mathfrak{p}$ and we have $s^*(Q)_\mathfrak{p} = s_\mathfrak{p}{}^* Q_\mathfrak{P}$, i.e., by the Milnor property $Q_\mathfrak{P} \not\sim 0$ if and only if $s^*(Q)_\mathfrak{p} \not\sim 0$.

(ii) or we have $E_\mathfrak{P} = F_\mathfrak{p}$. In this case $F_\mathfrak{p} \otimes_F E$ is a complete local ring with residue class field $F_\mathfrak{p}$ and s induces an $F_\mathfrak{p}$-linear map $s_\mathfrak{p} : A \to F_\mathfrak{p}$. From the theory of quadratic forms over complete local rings (or using an easy direct argument) it follows that we can write the form Q_A as a diagonal form with all diagonal coefficients contained in $F_\mathfrak{p} \subset A$. Thus

$$s^*(Q)_\mathfrak{p} = s_\mathfrak{p}{}^*(Q_A) = Q_\mathfrak{P} \otimes s_\mathfrak{p}{}^*\langle 1 \rangle \sim Q_\mathfrak{P},$$

i.e., as above, we know that $Q_{\mathfrak{P}} \not\sim 0$ if and only if $s^*(Q)_{\mathfrak{p}} \not\sim 0$. This completes the proof.

As a corollary we obtain the Hilbert reciprocity law for algebraic function fields with a quasi-finite constant field.

Let us finally remark without going into details:

COROLLARY 6.2. (i) *Let K be a Milnor field of degree 2 and F an algebraic function field in one variable over K. Let G be a simple algebraic group of type G_2 defined over F. The number of primes of $F|K$ where G is anisotropic is finite and even.*

(ii) *Let K be a Milnor field of degree 4 and F an algebraic function field in one variable over K. Let G be a simple algebraic group of type F_4 derived from a reduced exceptional simple Jordan algebra which is defined over F. The number of primes of $F|K$ where G is anisotropic is finite and even.*

(The proof follows from the fact that in both cases G is defined via a multiplicative form of the appropriate dimension.)

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W.-D. GEYER, G. HARDER, M. KNEBUSCH, W. SCHARLAU, Ein Residuensatz für symmetrische Bilinearformen, *Inventiones Math.* **11** (1970), 319–328.

M. KNEBUSCH AND W. SCHARLAU, Quadratische Formen und quadratische Reziprozitätsgesetze über algebraischen Zahlkörpern, *Math. Z.* **121** (1971), 346–368.

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