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Transversality Theories at Dimension Four

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In [4] Kirby and Siebenmann prove a relative version of the following theorem:

TOP Transversality Theorem. If $f\colon M^m \to \xi^s$ is a continuous map from a topological m-manifold M to an R^s microbundle $X \hookrightarrow \xi \to X$ then if $m \neq 4 \neq m-s$ there is an arbitrarily small homotopy of f to a function transverse to $X \subset \xi$.

This paper will consider the case $m-s=4$, and prove relative versions of the following three theorems:

Theorem A. *If $f\colon M^m \to \xi^{m-4}$ is a continuous map from a topological m-manifold M to a* TOP $m-4$ *microbundle $X \subset \xi \to X$ and H is a homology* 3*-sphere of Rohlin invariant* 1, *then there is an arbitrarily small homotopy of f to a map f' such that*

i) *$f'^{-1}(X)$ is a homology manifold;*

ii) *the link in $f'^{-1}(X)$ of each point s_i at which $f'^{-1}(X)$ fails to be locally Euclidean is homeomorphic to H;*

iii) *$f'|(M - \bigcup_i s_i)$ is* TOP *transverse to X;*

iv) *any neighborhood of $f'^{-1}(X)$ contains a neighborhood Z such that the inclusion $f'^{-1}(X) \to Z$ is a proper homotopy equivalence, and the diagram*

$$\begin{array}{ccc} Z - f'^{-1}(X) & \xrightarrow{f} & \xi - X \\ \downarrow\cap & & \downarrow\cap \\ Z & \xrightarrow{f} & \xi \end{array}$$

is homotopy equivalent to an $(m-5)$-spherical fiber space map.

Theorem B. *Hypotheses and f' as in Theorem* A. *If the double suspension of H is homeomorphic to S^5 then there is a microbundle $f'^{-1}(X) \subset \nu \to f'^{-1}(X)$ contained in M such that $f'|\nu$ is a microbundle map.*

Define a manifold to be almost smooth if it is smooth except at isolated points.

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Theorem C. *Hypotheses as in* A. *If there is a closed almost parallelizable* 4-*manifold of index* 8, *then there is an arbitrarily small homotopy of f to a map f' such that f' is* TOP *transverse to X and $f'^{-1}(X)$ is almost smooth.*

The proof of the converse will also be sketched.

For the convenience of the reader familiar with [4], statements and proofs here run parallel to those in [4] when this is possible.

The interior of a space X will be denoted $\mathring{X}$, the cone on X by $c(X)$ and the identity map on X by id_X. In particular, $\mathring{c}(X)$ denotes the open cone on X. A homology m-manifold will mean a space homeomorphic to an m-dimensional simplicial complex in which the link of any r-simplex has the integral homology of S^{m-r-1}.

Since any microbundle contains a bundle [5], we will take ξ to be a bundle. The statement $f: v \to \xi$ is a microbundle map then implies that f imbeds each fiber of a bundle contained in v into a corresponding fiber of ξ. If v and ξ contain bundles between which f induces a bundle map we say f contains a bundle map. Clearly f contains a bundle map when the base space for v is compact.

The outline of the paper is as follows. § 1: A preliminary construction which reduces each theorem to a local problem. § 2: A local version of Theorem A. § 3: A local version of Theorem B. § 4: A local version of Theorem C. § 5: The proofs of Theorems A, B and C. § 6: Cancelling pairs of singularities. § 7: Concluding remarks.

I am indebted to L. Siebenmann for suggested improvements.

1. δ-Transversality

In this section f is homotoped so that transversality holds everywhere except in certain small cubes scattered about M.

Definition. Up to isotopy there is one smoothing of $S^3 \times R^n$, $n \geqq 2$, not isotopic to the standard smoothing [3]. We denote this structure by $(S^3 \times R^n)_\Omega$.

Let M^m be an m-dimensional manifold, (Y, X) a pair of metric spaces such that X is closed in Y, and X has an R^{m-4} TOP normal microbundle ξ in Y. Let $\delta: M \to (0, \infty)$ and $f: M \to Y$ be continuous functions such that $f|\partial M$ is TOP transverse to X.

1.1. *Definition.* The function f is δ-transverse on an open set V of M to $X \subset Y$ at v if for a countable collection $(B^4 \times B^1 \times B^{m-5})_i$, $i = 1, 2, 3, \ldots$ of disjoint m-cells there is a map $g: \bigcup_i (B^4 \times B^1 \times B^{m-5})_i \to V$ which is a proper imbedding into int M satisfying the conditions:

i) For each i there is a subset X_i in X and a trivialization τ_i: $\xi|X_i \to R^{m-4}$ such that

$$\tau_i f g: (B^4 \times B^1 \times B^{m-5})_i \to R^{m-4} = R^1 \times R^{m-5}$$

is the product of a map f_i: $B^4 \times B^1 \to R^1$ and the inclusion $B^{m-5} \hookrightarrow R^{m-5}$.

ii) In a neighborhood of M-image (g): a) f is TOP transverse to X and b) there is a smoothing of the manifold $f^{-1}(X)$ which extends to the smoothing $g(\bigcup(\mathring{B}^4-0)\times\mathring{B}^1)_{i\Omega}$.

iii) $f_i(B^4\times 1)>0$ and $f_i(B^4\times -1)<0$.

iv) $\operatorname{diam} f(g(B^4\times B^1\times B^{m-5})_i)<\max\delta(g(B^4\times B^1\times B^{m-5})_i)$. The set int(image $(g))\subset V$ will be called the singular neighborhood of the map f.

Remark. If closure (image(g)) is not contained in V, f may be δ-transverse on V considered as a manifold, yet not δ-transverse on V considered as an open set of M, since it is required in the latter case that h be proper into M.

Let C and D be closed subsets of a TOP m-manifold M and U and V be open neighborhoods of C and D respectively. Let ξ^{m-4} be a normal TOP microbundle to a closed subset X of a metric space Y.

1.2. **δ-Transversality Theorem.** *Suppose* $f: M\to Y$, $\varepsilon: M\to(0,\infty)$ *and* $\delta: M\to(0,\infty)$ *are continuous functions and* f *is* δ*-transverse to* X *on* U *at* v_0. *If* $m=5$ *suppose* $\partial M\subset C$.

Then there is an ε*-homotopy* (i.e. *a homotopy moving no point image more than a distance* ε) $f_t: M\to Y$, $0\leqq t\leqq 1$ *of* $f_0=f$ *fixing a neighborhood of* $C\cup(M-V)$ *so that* f_1 *is* δ*-transverse to* X *near* $C\cup D$ *at a microbundle* v *equal* v_0 *near* C.

Proof of 1.2. The requirement that $v=v_0$ near C will not be stressed in the proof, for v_0 near C can always be added to v at any stage in the argument.

We may assume without loss of generality that $\varepsilon<\delta$.

The proof is trivial if $m\leqq 4$.

Case 1. $\partial M=\emptyset$, $m=5$, $Y=E(\xi)=X\times R^1$.

Proof. The map g of 1.1 is a proper imbedding into M, so by including in C the closure of any component of the singular neighborhood S of f which intersects C and deleting from U any component of closure(S) which is disjoint from C we may assume that closure(S) lies in C. Without loss of generality closure(S) may then be deleted from M and we may assume that $S=\emptyset$, and take $\varepsilon=\delta$.

The trivialization of v_0 given by f and the given smoothing of $f^{-1}(X)\cap U$ provide a smoothing for $E(v_0)$. Let N be a closed disk bundle contained in $v_0|(f^{-1}(X)\cap C)$.

There is an obstruction in $H^4(M,N;Z_2)$ to extending the natural smooth structure near N to all of M [3]. The dual in $H_1^{LC}(M-N;Z_2)$ to this obstruction can be represented by a locally flat proper imbedding of L, a countable disjoint union of circles and lines. Extend the smooth structure on N to a smoothing of all of $M-L$. Choose also a smoothing of L and a smooth bundle structure on $\eta(L)$, the normal bundle to L in M. The two induced smoothings on $\eta(L)-L$ are not isotopic.

Since $L\subset M-N$ and N is a neighborhood of $f^{-1}(X)\cap C$ in M, $f|L$ is trivially DIFF transverse to X near C. Using the homotopy extension theorem $\varepsilon/3$ homotope f rel $C\cup(M-V)$ so that $f|L$ is DIFF transverse to X near $C\cup D$. Choose $\eta(L)$ so small that for any fiber F of $\eta(L)$, $\operatorname{diam} f(F)<\varepsilon/3$.

Let $H: \eta(L) \times I \to X \times R$ be a pinching homotopy from $H_0 = f$ to $H_1 = f \cdot$(projection): $\eta(L) \to X$. Perform the homotopy H near $f^{-1}(X) \cap L \cap D$ and extend to an $\varepsilon/3$ homotopy of f rel $C \cup (M - V)$. After the homotopy f has the property that near $L \cap D$, $f^{-1}(X) \cap \eta(L)$ consists of a properly imbedded countable collection F_i of fibers of $\eta(L)$. By transversaltity of $f|L$ near $C \cup D$, for each F_i there is an $\varepsilon_i > 0$ and a bi-collaring $g_i: F_i \times [-1, 1] \to \eta(L)$ of F_i such that the following diagram commutes.

$$\begin{array}{c} F_i \times [-1,1] \xrightarrow{g_i} \eta(L) \xrightarrow{f} X \times R \xrightarrow{p_2} R \\ \llcorner\!\!\!-\!\!\uparrow \\ \varepsilon_i \cdot p_2 \end{array}$$

Choose ε_i so small that for x in F_i, $\operatorname{diam} f(x \times B^1) < \varepsilon/3$. Then $\operatorname{diam} f(F_i \times B^1) < 2\varepsilon/3$.

If both $\eta(L)$ and each ε_i have been chosen small enough the imbeddings g_j such that $g_j(F_j \times [-1, 1]) \cap (C \cup D) \neq \emptyset$ define an imbedding $g = \bigcup_j g_j: \bigcup_j F_j \times B^1 \to$ (neighborhood of $C \cup D$) which is proper into M.

Let λ: $M \to (0, \infty)$ be a function such that on each $g_j(F \times B^1)$, $0 < \lambda < \varepsilon_j/2$. In the smooth manifold $M - L$, f is, by definition, DIFF transverse to X near C and is trivially DIFF transverse to X near $D \cap (\eta(L) - \operatorname{image}(g))$.

Let M' be a smooth codimension zero submanifold of $M - L$ such that M' is closed in M and $\partial M' \subset \eta(L) - L$. Perform a $\min(\lambda, \varepsilon/3)$ homotopy of $f|M'$ rel $(\eta(L) - \operatorname{image}(g)) \cup C \cup (M - V)$ so that $f|M'$ is DIFF transverse to X near $C \cup D$. Extend this homotopy to a $\min(\lambda, \varepsilon/3)$ homotopy of f on all of M rel $(\eta(L) - \operatorname{image}(g)) \cup C \cup (M - V)$. The resulting map, which we still denote f, clearly satisfies near $C \cup D$ 1.1 ii). Property 1.1 iii) is satisfied since $\lambda < \varepsilon_j/2$, and i) is trivially satisfied. Finally $\operatorname{diam} f(g_j(F_j \times B^1)) < \varepsilon$, because before the last homotopy the diameter was less than $2\varepsilon/3$. This completes the proof of Case 1.

Case 2. $\partial M = \emptyset$, $E(\xi)$ contains $X \times R^{m-4}$ as an open sub-microbundle, $m > 5$, and f: $v_0 \to \xi$ contains a bundle map $v_0' \to \xi'$ near C.

Proof. As above we may assume $S = \emptyset$, $\varepsilon = \delta$. Choose a bundle trivialization $\tau: \xi' \mapsto X \times R^{m-4}$. By making the following substitutions: $f \mapsto \tau \cdot f$, $Y \to X \times R^{m-4}$, $M \mapsto$ open neighborhood M' of $f^{-1}(X)$ in $(\tau f)^{-1}(X \times R^{m-4})$ which near C coincides with $E(v_0')$, $V \mapsto$ open neighborhood V' of $f^{-1}(X)$ in $V \cap M'$ such that closure $(V') \subset M'$ (so that any imbedding into V' which is proper into M' is also proper into M), $C \to C \cap M'$, $D \to D \cap (\tau f)^{-1}(X \times B^{m-4}) \cap M'$ we may assume that $Y = X \times R^{m-4}$, $C \subset E(v_0')$ which has been given a smooth structure as in Case 1, and near C, f: $v_0' \to X \times R^{m-4}$ is a DIFF bundle map.

$\varepsilon/3$ homotope f rel $C \cup (M - V)$ so that f is TOP transverse to $X \times R^1$ near $C \cup D$ [4]. Apply Case 1 to $\bar{f} \equiv f | f^{-1}(X \times R^1)$. That is, $\varepsilon/3$ homotope $\bar{f}$ rel $C \cup (M - V)$ to make $\bar{f}$ $\varepsilon/3$-transverse to $X \times 0 \subset X \times R^1$ near D. Extend the homotopy to an $\varepsilon/3$ homotopy of f on all of M rel $C \cup (M - V)$.

Choose a tubular neighborhood

$$h: \bar{f}^{-1}(X \times R^1) \times R^{m-5} \to M$$

so that near C, h is a smooth tubular neighborhood in the smooth manifold $E(v'_0)$ and near C

$$f\cdot h\colon \bar{f}^{-1}(X\times R^1)\times R^{m-5}\to(X\times R^1)\times R^{m-5}$$

is a smooth bundle map. After an $\varepsilon/3$ pinching homotopy rel $C\cup(M-V)$ alter f so that near $D\cup C$ the map $f\cdot h$ is a TOP microbundle map.

Since $\bar{f}$ is $\varepsilon/3$-transverse to X near $C\cup D$ it follows easily that f satisfies 1.1 and so is ε-transverse to X.

Case 3. $\partial M=\emptyset$.

Proof. Cover X with open sets X_a, a some index set, such that ξ is trivial over each X_a. Choose a star-finite cover of M by coordinate charts R^m_j, $j=1,2,\ldots$ such that each $R^m_j\subset V$, $D\subset\bigcup_j B^m_j$, and each set $p(f(R^m_j)\cap E(\xi))$ lies in some X_a denoted X_j.

For each j let n_j be the number of coordinate charts intersecting X_j and $\lambda_j=(1/n_j)\cdot\min(\varepsilon(2B^m_j))$.

Let $U_0=U$, $f'_0=f$ and suppose for $j\geqq 0$ there is an ε'_j-homotopy f'_t, $0\leqq t\leqq j$ from f_0 to a map $f'_j\colon M\to Y$ δ-transverse to X on an open subset $U_j\subset M$ at v_j, where $U_j\supset C_j\equiv C\cup(B^m_1\cup\cdots\cup B^m_j)$ and $\varepsilon'_j\colon M\to(0,\infty)$ is a continuous function such that $\varepsilon'_j(x)<\sum_{\substack{k\leqq j\\ x\text{ in }\mathring{B}^m_k}}\lambda_k$.

Since $3B^m_{j+1}\cap C_j$ is compact, the microbundle map $f'_j\colon v_j|3B^m_{j+1}\to\xi$ contains an open bundle map $v'_j\to\xi'$ near C_j.

Apply Case 2 with the substitutions $M\mapsto 3\mathring{B}^m_{j+1}\cup v'_j|3\mathring{B}^m_{j+1}$,

$$C\mapsto C_j\cap(3\mathring{B}^m_{j+1}\cup v'_j|3\mathring{B}^m_{j+1}),\quad D\mapsto B^m_{j+1},\quad V\mapsto 2\mathring{B}^m_{j+1},$$

$$Y\mapsto(Y-E(\xi))\cup E(\xi|X_{j+1}),\quad \xi\mapsto\xi|X_{j+1},\quad f\mapsto f'_j,\quad \varepsilon\to\lambda_{j+1},$$

$v'\mapsto v'_j|3\mathring{B}^m_{j+1}$ to obtain a homotopy f'_t fixed outside $2\mathring{B}^m_{j+1}$, $j\leqq t\leqq j+1$ to a map which is δ-transverse to X on an open subset $U_{j+1}\supset C_{j+1}$. This completes the induction to stage $j+1$.

Finally define f_t, $0\leqq t\leqq 1$ to be the unique homotopy such that $f_t=f'_{t/(1-t)}$, $0\leqq t\leqq 1$.

Case 4. The general case.

Proof. By definition f is TOP transverse near $\partial M\cap C$. By exploiting a collar of ∂M and applying codimension 3 TOP transversality to $f|\partial M$ we may assume $\partial M\subset C$. Here property 1.1 ii) is satisfied because $\partial M\cap f^{-1}(X)$ is a three manifold and therefore has a smoothing (unique up to isotopy).

The proof is completed by applying Case 3 to int (M).

2. *H*-Transversality

As above, let M be an m-dimensional manifold, (Y, X) a pair of metric spaces such that X is closed in Y, and X has an R^{m-4} TOP normal microbundle ξ in Y. Let H be a homology sphere of Rohlin invariant 1 (i.e. H bounds a smooth parallelizable manifold of index 8 mod 16). Let $f: M \to Y$ be a continuous function such that $f|\partial M$ is TOP transverse to X.

2.1. *Definition.* The function f is H-transverse to $X \subset Y$ at a microbundle v if

i) $f^{-1}(X)$ is a homology 4-manifold;

ii) the singular (i.e. non-manifold) points $\{s_i\}$ of $f^{-1}(X)$ each have link in $f^{-1}(X)$ homeomorphic to H;

iii) $f|(M - \bigcup_i s_i)$ is TOP transverse to X at v;

iv) any neighborhood of $f^{-1}(X)$ contains a neighborhood Z such that the inclusion $f^{-1}(X) \to Z$ is a proper homotopy equivalence and the diagram

$$\begin{array}{ccc} Z - f^{-1}(X) & \xrightarrow{f} & \xi - X \\ \cap\downarrow & & \cap\downarrow \\ Z & \xrightarrow{f} & \xi \end{array}$$

is homotopy equivalent to an $(m-5)$-spherical fiber map.

Let $S(H)$ denote the suspension of H, and I^2 denote $[0, 1] \times [0, 1]$.

2.2. **Lemma A.** *There is a map* $h: R^4 \times R^1 \to \mathbb{R}^1$ *such that* $h(B^4 \times 1) > 0$, $h(B^4 \times -1) < 0$, *h is* DIFF *transverse to* 0 *near* $((R^4 - \mathring{B}^4) \times R^1)_\Omega$ *and h is H-transverse to* $0 \in R^1$ *with one singular point at* $0 \in R^4 \times R^1$. *Furthermore each neighborhood Z given by* 2.1 iv) *may be chosen so that near* $\partial B^4 \times B^1$, $h: Z \to R^1$ *is a microbundle map.*

Proof. Topological surgery provides a non-smoothable cobordism U from $\partial_0(U) = S^3 \times S^1$ to $\partial_1(U) = H \times S^1$ and a homotopy equivalence $U \to S^3 \times S^1 \times I$. There is no obstruction to assuming the map on $\partial_1(U)$ is the product of a homology equivalence $H \to S^3$ and Id_{S^1}. We may further assume the map is a diffeomorphism on $\partial_0(U)$ [13]. For details see [11].

Attach copies of $S^3 \times I^2$ to $\partial_0 U$ and to $S^3 \times S^1 \times 0$ by means of a homeomorphism $s: S^1 \to \partial I^2$ and thereby obtain spaces U' and $S^3 \times I^2$ together with a homotopy equivalence $g': U' \to S^3 \times I^2$.

Let $q: I^2 \to T^2 = S^1 \times S^1$ be the quotient map which identifies $(t, 0)$ with $(t, 1)$ and $(0, t)$ with $(1, t)$ for all t in I.

Let Q be the manifold obtained from U' by identifying points in $\partial U'$ which have the same image under $(\mathrm{id}_H) \times (q\,s)$ and let $q': U' \to Q$ be the resulting quotient map. There is then a homotopy equivalence g defined by the following commutative diagram:

$$\begin{array}{ccc} U' & \xrightarrow{g'} & S^3 \times I^2 \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{S^3} \times q} \\ Q & \xrightarrow{g} & S^3 \times T2 \end{array}$$

The map g is homotopic to a homeomorphism [14] which we continue to denote g.

Let L be the cover of $q(\partial I^2) \subset T^2$ in R^2. Then in the universal cover $\tilde{Q}$ of Q, $q'(\partial U')$ is covered by $H \times L$. Let $r: R^2 \to \mathring{B}^2$ be the homeomorphism which sends the polar coordinates (t, θ) to $(t/1+t, \theta)$ and let $i: S^3 \times R^2 \to S^5$ be the imbedding which is the composition of $\mathrm{id}_{S^3} \times r$ and the quotient map $S^3 \times B^3 \to S^3 * S^1 = S^5$ which identifies $S^3 \times \partial B^2$ to S^1.

Since any homeomorphism $\bar{g}: \bar{Q} \to S^3 \times R^2$ covering g satisfies

$$\|p_2(x) - p_2\,\bar{g}(x)\| < (\text{constant})$$

for x in $H \times L \subset \bar{Q}$, it follows that the closure of $i\bar{g}(H \times (x\text{-axis}))$ in $S^1 * S^3 = S^5$ is homeomorphic to $S(H)$. We label the suspension points (π) and (0) after their angular coordinates in the suspension circle. Then $S(H) - (0)$ is a properly imbedded copy of $\mathring{c}(H)$ in $S^5 - (0)$.

Similarly use $i\bar{g}(H \times (y\text{-axis}))$ to imbed another copy of $S(H)$ in S^5 "perpendicular" to the first. Label the two copies $S_x(H)$ and $S_y(H)$ respectively.

Let $b: H \times R^1 \times B^1 \to S^3 \times R^2$ be a smooth bicollaring of

$$\bar{g}(H \times (x\text{-axis})) \subset \bar{g}(H \times L)$$

in $S^3 \times R^2$ chosen so that $b(H \times R^1 \times B^1) \subset S^3 \times R^1 \times (-n, n)$, for some integer n sufficiently large, and so that $b(H \times \{0\} \times B^1)$ is a bicollaring of $\bar{g}(H \times \{0\})$ in $\bar{g}(H \times (y\text{-axis}))$. This may be done by covering an appropriate collar of $g(H \times S^1)$ in $S^3 \times T^2$. Define $f: S^3 \times R^1 \times [-n, n] \to R^1$ by

$$fb(h, x, y) = y \quad \text{on} \quad b(H \times R^1 \times B^1)$$
$$f = \pm 1 \quad \text{on} \quad (S^3 \times R^1 \times [-n, n]) - \operatorname{int}(\text{image}(b))$$

where the sign of ± 1 is chosen so as to make f continuous.

Define the map

$$\bar{f}: i(S^3 \times R^1 \times [-n, n]) \cup (\pi) \to R^1 \quad \text{by}$$
$$\bar{f}i(z, x, y) = f(z, x, y)/(|x| + 1) \quad \text{for} \quad x \leqq 0$$
$$\bar{f}i(z, x, y) = f(z, x, y) \quad \text{for} \quad x \geqq 0$$
$$\bar{f}(\pi) = 0,$$

and notice that $\bar{f}$ is continuous.

Using Urysohn's lemma extend $\bar{f}$ to a map of all of $S^5 - (0)$ so that no points outside $\mathring{c}(H)$ are mapped to 0.

Let $p: R^4 \times R^1 \to S^5 - (0)$ be a stereographic projection which carries $0 \times R^1$ to the suspension circle via the map $t \mapsto e^{\pi i w(t)}$, where $w(t) = 1 - t/(|t| + 1)$.

A component of $(S^3 \times [-n, n]) - b(H \times R^1 \times \mathring{B}^1)$ is a homology cobordism from $S^3 \times R$ to $H \times R$. An extension to all of the cobordism of the projection $(S^3 \times R) \cup (H \times R) \to R$, if made DIFF transverse to 0, would provide a smooth index 0 parallelizable cobordism from S^3 to H, contradicting Rohlin's theorem. Hence the cobordism is not smoothable. Therefore the map

$$b: H \times R^1 \times B^1 \hookrightarrow S^3 \times R^1 \times (-n, n)$$

is not isotopic to a smooth imbedding. However, b is a homology equivalence, so there is an extension of the natural smoothing of $H \times R$ to $S^3 \times R \times (-n, n)$, an extension which must be the exotic smoothing [3]. It follows that the map $fp|((R^4 - 0) \times R^1)_\Omega$ is DIFF transverse to 0.

Since $fp(0, 1) > 0$ and $fp(0, -1) < 0$ there is an $\varepsilon > 0$ such that the function

$$h: R^4 \times R^1 \to R^1$$

given by $h(x, y) = fp(\varepsilon x, y)$ satisfies $h(B^4 \times 1) > 0$, $h(B^4 \times -1) < 0$.

Clearly h satisfies i–iii) of 2.1. It remains to find appropriate neighborhoods Z of $\mathring{c}(H)$ satisfying iv).

$S_y(H)$ divides S^5 into two contractible components; let C be the closure of that containing π. Let $Z_\infty = C \cup i\,b(H \times [0, \infty) \times B^1)$. Altering p, if necessary, we may assume that $p^{-1}\,C$ is contained in $\mathring{B}^4 \times \mathring{B}^1$.

Since C and the components D^+ and D^- of $C - \mathring{c}(H)$ are contractible, the inclusions of $\mathring{c}(H)$, $D^+ \cup i\,b(H \times [0, \infty) \times (0, 1])$ and $D^- \cup i\,b(H \times [0, \infty) \times [-1, 0))$ into C are proper homotopy equivalences. Hence the neighborhood $p^{-1}(Z_\infty)$ is of the required type.

A family of neighborhoods $\{Z_{q/r}\}$ can be defined similary, as follows: For any rational $q/r \in [0, \infty)$ it is easy to construct an imbedding j: $R \to L = \{(x, y) | x$ or y is an integer$\} \subset R^2$ such that $|q(p_1\,i(x, t)) + r|p_2\,i(x, t)|| < 1$. Then the composition $H \times R \xrightarrow{id_H \times j} H \times L \xrightarrow{\quad \bar{g} \quad} S^3 \times R^2 \xrightarrow{\quad i \quad} S^5$ extends to an imbedding $S_{q/r}(H)$ of $S(H)$ in S^5 and, as $q/r \to 0$, $S_{q/r}(H) \to S_x(H)$.

It is left to the reader to supply details, construct corresponding neighborhoods $\{Z_{q/r}\}$ of $\mathring{c}(H)$ in $S^5 - (0)$ and verify that $\{p^{-1}(Z_{q/r})\}$ is a fundamental collection of neighborhoods, thus completing the proof of 2.2.

Remark. The neighborhoods $\{Z_{q/r}\}$ constructed in 2.2 have the pleasant property that each has boundary homeomorphic to $\mathring{c}(H) \times S^0$. We would like to prove a similar result for the general situation: for f an H-transverse approximation to a map f': $M \to Y$, each neighborhood Z of $f^{-1}(X)$ defined in 2.1 may be chosen so that ∂Z is an $(m-5)$-sphere bundle. Since there are R^n bundles containing no disk bundles, this assertion is too strong [1].

However, it is possible to show that if ξ contains a sphere bundle ξ, then the neighborhoods Z may be chosen so that each has boundary homeomorphic to $(f | f^{-1}(X))^*(\xi)$. The proof is long and will not be given here.

3. Creating a Microbundle Map

The assumption that the double suspension $\Sigma^2 H$ of a homology 3-sphere of non-trivial Rohlin invariant is homeomorphic to S^5 implies that a large number of non-PL manifolds are homology manifolds [12]. There is a transversality theory in the category of homology manifolds [10]. It is therefore not surprising that, under the assumption $\Sigma^2 H \simeq S^5$, $f^{-1}(X)$ of theorem A may be equipped with a normal microbundle.

The necessary analogue to lemma A is

3.1. **Lemma B.** *Suppose $\Sigma^2 H \simeq S^5$. Then the conclusion of 2.2 holds; moreover each Z may be taken to be a normal microbundle to $h^{-1}(X)$, and h: $Z \to R$ to be a microbundle map.*

Proof. The suspension circle in $\Sigma^2 H$ has a neighborhood homeomorphic to $c(H) \times S^1$. Thus if $\Sigma^2 H \simeq S^5$, $c(H) \times S^1$ is a manifold.

Let U be as in the proof of 2.2. Although the natural smoothing of ∂U does not extend through the interior of U, there is no obstruction to extending the natural smoothing of $\partial_1 U = H \times S^1$ to the interior of U. The induced smoothing near $\partial_0 U = S^3 \times S^1$ will be exotic.

Attach $c(H) \times S^1$ to U along $H \times S^1 = \partial_1 U$. Since any h-cobordism of $S^3 \times S^1$ to itself is a product cobordism [13], it follows easily that the resulting manifold is $B^4 \times S^1$.

Regarding $\mathring{B}^1$ as the universal cover of S^1, the lift of this construction defines a proper imbedding $c(H) \times \mathring{B}^1 \to B^4 \times \mathring{B}^1 \subset R^4 \times R^1$ such that the natural smoothing of $H \times \mathring{B}^1$ extends to $(R^4 \times \mathring{B}^1) - (c(H) \times \mathring{B}^1)$ and is the exotic smoothing near $(R^4 - B^4) \times \mathring{B}^1$. Since the imbedding is proper, the projection map $R^4 \times (R^1 - \mathring{B}^1)$ is a continuous extension of the projection $c(H) \times \mathring{B}^1 \to \mathring{B}^1$; the union of these maps is denoted $\varphi\colon R^4 \times (R^1 - \mathring{B}^1) \cup (c(H) \times \mathring{B}^1) \to R^1$.

The projection $c(H) \times \mathring{B}^1 \to \mathring{B}^1$ is clearly DIFF transverse to 0 on $H \times \mathring{B}^1$ and is itself a normal microbundle map. By DIFF transversality applied to the smooth manifold $(R^4 \times \mathring{B}^1) - (c(H) \times \mathring{B}^1)$, φ may be extended to the required map $h\colon R^4 \times R^1 \to R^1$.

4. Rohlin's Theorem and Transversality

Rohlin's theorem states that if M is a PL or DIFF closed orientable 4-manifold with $w_2(M) = 0$, then $\operatorname{index}(M) \equiv 0 \pmod{16}$. In PL and DIFF the hypotheses on M are equivalent to the statement that M is almost parallelizable, but in TOP the latter is a possibly stronger condition.

In [9] it is shown that for $m \geqq 32$ and $\xi = R^{m-4}$, Kirby-Siebenmann transversality is equivalent to the existence of a closed orientable 4-manifold with $w_2(M) = 0$ and $\operatorname{index}(M) = 8$.

Theorem C will show that a somewhat stronger form of transversality, in which the inverse image is almost smooth, is equivalent to the existence of an almost parallelizable closed 4-manifold of index 8. For our purposes, therefore, Rohlin's theorem will be: *Every* PL *closed almost parallelizable* 4-*manifold has index a multiple of* 16.

If there is a topological counterexample to Rohlin's theorem, then the triangulation conjecture is false for 4-manifolds. More important for our purposes, the Hauptvermutung is false for $S^3 \times \mathbb{R}$:

4.1. **Lemma.** *If Rohlin's theorem is false in* TOP, *then there is a smooth structure* $(S^3 \times R)_\Omega$ *on* $S^3 \times R$ *not concordant to the standard smoothing.*

Proof. Let N be a topological counterexample to Rohlin's theorem. Then N-(point) is open and parallelizable, hence smoothable [7]. The induced smoothing on a neighborhood, homeomorphic to $S^3 \times R$, of the deleted point is exotic. Indeed, were it concordant to the standard smoothing, this neighborhood would also be the end of a smooth parallelizable manifold V of index 0 [8]. The union of N-(point) and V along their common end would then be an index 8 smooth almost parallelizable manifold, contradicting Rohlin's theorem.

4.2. **Lemma C.** *If Rohlin's theorem is false in* TOP, *the projection* $h\colon R^4 \times R^1 \to R^1$ *is* DIFF *transverse to* 0 *on* $[(R^4 - 0) \times R^1]_\Omega$.

Proof. Let $(R^4-0)_\Omega$ be the smooth structure given by Lemma 4.1. $(R^4-0)_\Omega\times R^1$ is not ε-isotopic to the standard smoothing, for the resulting diffeomorphism $(R^4-0)_\Omega\times R^1\to(R^4-0)\times R^1$ would provide a concordance from $(R^4-0)_\Omega$ to R^4-0. Thus $(R^4-0)_\Omega\times R^1\simeq[(R^4-0)\times R^1]_\Omega$ and the lemma follows.

5. Proof of Theorems A, B and C

This section contains full statements and proofs of the three main theorems.

Let C and D be closed subsets of a TOP m-manifold M and U and V be open neighborhoods of C and D respectively. Let H be a homology 3-sphere of Rohlin invariant 1. Let $f: M\to Y$, $\varepsilon: M\to(0,\infty)$ be continuous functions.

5.1. **Theorem A.** *Suppose f is H-transverse to X on U at v_0. If $m=5$, suppose $\partial M\subset C$.*

Then there is an ε-homotopy $f_t: M\to Y, 0\leqq t\leqq 1$, of $f_0=f$ fixing a neighborhood of $C\cup(M-V)$ so that f_1 is H-transverse to X near $C\cup D$ at a microbundle v equal v_0 near C.

5.2. **Theorem B.** *Suppose $\Sigma^2 H\simeq S^5$, f is as in* 5.1, *and furthermore $f^{-1}(X)$ has a normal microbundle neighborhood v_0 near C such that $f|v_0$ is a microbundle map.*

Then the function f_1 of 5.1 *may be chosen so that $f^{-1}(X)$ has a normal microbundle v near $C\cup D$, $v=v_0$ near C, and $f|v$ is a microbundle map.*

5.3. **Theorem C.** *Suppose f is* TOP *transverse to X on U at v_0, and $f^{-1}(X)$ is almost smooth. If $m=5$ suppose $\partial M\subset C$.*

Then, if Rohlin's theorem is false *in* TOP, *there is an ε-homotopy f_t: $M\to Y$, $0\leqq t\leqq 1$ of $f_0=f$ fixing a neighborhood of $C\cup(M-V)$ so that f_1 is* TOP *transverse to X near $C\cup D$ at a microbundle v equal v_0 near C, and $f^{-1}(X)$ is almost smooth.*

Proof of 5.1, 5.2, 5.3. Clearly f is $\varepsilon/2$-transverse to X at v_0 near C. Apply an $\varepsilon/2$ homotopy rel $C\cup(M-V)$ to make f $\varepsilon/2$-transverse to X near $C\cup D$, as in 1.2.

The proofs of all three theorems thereby reduce to the following special case: $V=D=M=R^4\times R^1\times B^{m-5}$, $M-C=B^4\times B^1\times B^{m-5}$, $f=f_i\times(\text{identity})$: $(R^4\times R^1)\times B^{m-5}\to R^1\times B^{m-5}$, where $f_i(B^4\times 1)>0$, $f_i(B^4\times -1)<0$ and $\varepsilon=$diameter $f(B^4\times B^1\times B^{m-5})$.

Without loss of generality we may assume that $f_i(B^4\times B^1)\subset B^1$ and, by condition 1.1 ii), that near $\partial(B^4\times B^1\times B^{m-5})$, $f^{-1}(X)$ is a smooth submanifold of $((\mathring{B}^4-0)\times\mathring{B}^1)_\Omega$.

Since $f=f_i\times(\text{identity})$, the general case follows easily from the case $m=5$. Finally, since any R^1 TOP bundle has a unique DIFF structure, we may assume that near ∂M the map f is DIFF transverse on $((\mathring{B}^4-0)\times\mathring{B}^1)_\Omega$ to 0 in R^1.

The proof of this case proceeds as follows: For Theorems A, B or C choose h from Lemma A, B or C respectively and normalize so that $h(B^4\times B^1)\subset f(B^4\times B^1)$. Construct a map f': $R^4\times R^1\to R^1$ such that $f'=f$ near $(R^4\times R^1)-(\mathring{B}^4\times\mathring{B}^1)$, $f'(x,y)=h(2x,2y)$ near $(\frac{1}{2}B^4\times\frac{1}{2}B^1)$ and, finally, on $(B^4\times B^1)-(\frac{1}{2}B^4\times\frac{1}{2}B^1)$, f' is any continuous extension which is non-zero on $\frac{1}{2}B^4\times(B^1-\frac{1}{2}B^1)$ and has $f'(B^4\times B^1)\subset f(B^4\times B^1)$, for example, an approximation of the linear extension.

The function f' then satisfies the conclusion of the appropriate theorem except perhaps in $(\mathring{B}^4 - \frac{1}{2}B^4) \times B^1$. But on a neighborhood of the boundary of $(B^4 - \frac{1}{2}B^4) \times B^1$, given the smoothing induced from $[(\mathring{B}^4 - \frac{1}{2}B^4) \times \mathring{B}^1]_\Omega$, f' is DIFF transverse to 0. Therefore there is a small homotopy of f' with support in $(\mathring{B}^4 - \frac{1}{2}B^4) \times \mathring{B}^1$ to a map which is satisfactory everywhere [4; Theorem 1.2].

Since $f' = f$ near $(R^4 \times R^1) - (B^4 \times B^1)$, and $f'(B^4 \times B^1) = f(B^4 \times B^1)$, the linear homotopy from f to f' is fixed near $(R^4 \times R^1) - (B^4 \times B^1)$ and moves no point more than diameter $f(B^4 \times B^1)$. This completes the proof.

Remark. Here is a sketched proof of the converse to Theorem C: In [14] Siebenmann presents an orientable 5-manifold M with $w_2(M) = 0$ and a proper homotopy equivalence $M \xrightarrow{f} X \times R$ where X is a closed homology manifold of index 8. Suppose $p_2 f\colon M \to R$ is homotopic to a map such that the inverse image of 0 is an almost smooth manifold N. Let B be an open 4-ball in N such that $N - B$ is a smooth manifold. Since $N - B$ is smooth, orientable and $w_2(N - B) = 0$, $N - B$ is parallelizable. Therefore N is an almost parallelizable manifold of index 8.

6. Cancelling Singularities in Pairs

In this section we show how to cancel the singularities which appear in theorem A until there is at most one singular point in each component of $f^{-1}(X)$. The following lemma is the key ingredient.

Let X be a closed subset, with trivial R^1 normal microbundle, of a topological space Y, let $\gamma\colon [0, 1] \to R^5$ be a locally flat imbedding, and let H' be a homology sphere with trivial Rohlin invariant.

6.1. **Lemma.** *Suppose $g\colon R^5 \to Y$ is* TOP *transverse to X on $R^5 - \gamma[0, 1]$ and the quotient space $g^{-1}(X)/\gamma[0, 1]$ is homeomorphic to $\mathring{c}(H')$, then for any $\varepsilon > 0$ and any neighborhood N of $\gamma[0, 1]$, g is ε-homotopic with support in N to a map which is* TOP *transverse to X everywhere.*

Proof. A construction analogous to that of 2.2 provides a proper imbedding $(\mathring{c}(H'), *) \to (R^5, 0)$. Now, however, since H' has trivial Rohlin invariant, the imbedding can be defined to be smooth away from the vertex $*$. Then $\mathring{c}(H') - (*) \simeq H' \times R$ divides $S^4 \times R$ into two components whose closures in $S^4 \times R$ we denote D_1 and D_2. Each D_i has two ends and it follows from the construction in 2.2 that each D_i is $1 - LC$ at ∞.

Since γ is locally flat, the quotient space $R^5/\gamma[0, 1]$ is homeomorphic to R^5. The normal microbundle to X in Y is trivial, so $g^{-1}(X) - \gamma[0, 1]$ divides $R^4 - \gamma[0, 1] \simeq S^4 \times R$ into two components whose closures we denote E_1 and E_2. Van Kampen's theorem and the Mayer Vietoris sequence applied to the pairs (E_1, E_2), (D_1, D_2), (E_1, D_1) and (E_2, D_2) then show that for $i = 1, 2$, $E_i \cup D_i$ is a homotopy $S^4 \times R$ which is $1 - LC$ at ∞. Hence $E_i \cup D_i \simeq S^4 \times R$ [15].

Since both $g^{-1}(X) - \gamma[0, 1]$ and the normal bundle to X are DIFF, there is a natural smoothing of a bicollar of $g^{-1}(X) - \gamma[0, 1]$ in $R^5 - \gamma[0, 1]$ defined by requiring that in this structure g be DIFF transverse to X. Denote the intersection of this bicollaring with E_i by C_i. Since $C_i \cup D_i$ is a homology disk, the natural smoothing of $C_i \cup D_i$ is unique up to isotopy. Hence the smoothing of $E_i \cup D_i$ induced by a homeomorphism $E_i \cup D_i \simeq S^4 \times R$ restricts, up to isotopy, to the natural smoothing of $C_i \cup D_i$. We conclude that the natural smoothing of C_i extends to a smoothing of E_i.

Returning to the map $g\colon R^5 - \gamma[0,1] = E_1 \cup E_2 \to Y$, we conclude that there is a smoothing (isotopic to the standard smoothing) $[R^5 - \gamma[0,1]]_\Sigma$ on which g is DIFF transverse to X. Let $D^5 \subset N$ be a ball containing $\gamma[0,1]$ such that ∂D^5 is smooth in $[R^5 - \gamma[0,1]]_\Sigma$. There is no obstruction to extending $[R^5 - \mathring{D}^5]_\Sigma$ to all of R^5 [3]. Then 6.1 follows by standard DIFF transversality.

Let M, f, X, Y and ε be as in 5.1.

6.2. **Theorem.** *There is an ε-homotopy f_t of $f_0 = f$ to a map f_1, H-transverse to X, such that each component of $f_1^{-1}(X)$ contains at most one singular point.*

Proof. By 5.1 we may assume f has been made H-transverse to X. Suppose s_0 and s_1 are two singular points in the same component of $f^{-1}(X)$.

Three properties of the construction of the H-transversal map f are relevant:

a) By 1.1 i) each s_i has a co-ordinate neighborhood $g_i\colon (R^m, 0) \to (M, s_i)$ and each $f(s_i)$ a neighborhood X_i for which there is a trivialization $\tau_i\colon \xi|X_i \to X_i \times R^{m-4}$ such that $g_i^{-1} f^{-1}(X) \subset R^5 \subset R^m$ and $\tau f g_i\colon R^5 \times R^{m-5} \to X_i \times R^1 \times R^{m-5}$ is an R^{m-5} microbundle map.

b) By 5.1 there is a map $h\colon R^5 \to R^1$ such that $h^{-1}(0) = \mathring{c}(H)$ with vertex $*$ at 0 and, on R^5, $p_2 \tau f g_i = h\colon R^5 \to R$.

c) The function h may be chosen so that in a tubular neighborhood W of the interior of a ray $\rho \subset R^4 \subset R^5$, h is the projection $R^5 \to R^1$ taking $R^4 \subset R^5$ to 0.

The justification for c) is the following: Let p be a point in H, q a point in S^3. By general position in the manifold Q appearing in the proof of 2.2, a tubular neighborhood of $\{p\} \times S^1 \times (\text{point}) \subset S^3 \times S^1 \times S^1$ is ambient isotopic to a tubular neighborhood of $g(\{q\} \times S^1 \times (\text{point}))$ in $S^3 \times S^1 \times S^1$. It is easy to trace the consequences of such an isotopy through the construction, and conclude that a tubular neighborhood W of $\mathring{c}(p) - *$ in $f^{-1}(X)$ is a tubular neighborhood of $\mathring{c}(q) - *$ in R^4. In 2.2 h was defined to be a normal bundle trivialization of $\mathring{c}(H) - *$, so h may be the projection near $\mathring{c}(q) - * \subset R^4$. Property c) then holds for the ray $\rho = \mathring{c}(q)$. Given ρ, let $q \in R^4$ be the point $(\partial B^4 \times B^1) \cap \rho$.

We now proceed with the proof of 6.2.

For a sufficiently small $\widetilde{B^4 \times B^1}$ contained in $B^4 \times B^1$, $g_0(q)$ and $g_1(q)$ *are* connected in $f^{-1}(X) - (g_0(\widetilde{B^4 \times B^1}) \cup g_1(\widetilde{B^4 \times B^1}))$. Hence we may assume without loss of generality that there is a locally flat arc $\gamma\colon [0,1] \to f^{-1}(X)$ from s_0 to s_1 whose image coincides with $g_i(\rho)$ near $g_i(B^4 \times B^1)$. Parameterize γ so that $\gamma^{-1}(g_0(B^4 \times B^1)) = [0, 1/3] \equiv I_0$, $\gamma^{-1}(g_1(B^4 \times B^1)) = [2/3, 1] \equiv I_1$.

Let $\bar{\sigma}\colon (0,1) \times R^3 \to f^{-1}(X)$ be a framing of a normal bundle to $\gamma(0,1)$ in $f^{-1}(X)$ such that near $g_i(B^4 \times B^1)$, image $\sigma \subset g_i(W)$ (W as defined in c) above) and $\bar{\sigma}^{-1}(g_i(B^4 \times B^1)) = I_i \times R^3$.

Extend $\bar{\sigma}$ to a framing $\sigma\colon (0,1) \times R^3 \times R^1 \times R^{m-5} \to M$ such that $f\sigma\colon ((0,1) \times R^3) \times (R^1 \times R^{m-5}) \to \xi$ is a TOP microbundle map and near $\gamma(I_i)$, $\sigma|\gamma(I_i) \times R^3 \times R^1 \times \{0\} \subset g_i(W \cap (B^4 \times B^1))$. This is possible since the space of lines in R^{m-4} (that is RP^{m-5}) is connected.

The general argument now parallels the argument in the special case $m=5$, so henceforth the factor R^{m-5} will be dropped.

Note that since $\tau f \sigma$ is a TOP microbundle trivialization and $p_2 \tau f g_i$ is the projection near $W \cap (B^4 \times B^1)$, $\sigma(\{1/3\} \times R^3 \times R^1) \subset g_0(\partial B^4 \times B^1)$ and $\sigma(\{2/3\} \times R^3 \times R^1) \subset g_1(\partial B^4 \times B^1)$.

Let R_0^5 and R_1^5 be disjoint copies of R^5 and consider the manifold obtained by attaching $[1/3, 2/3] \times R^3 \times R^1$ to $(R_i^5 - W) \cup (B^4 \times B^1)_i$ by $g_i^{-1} \sigma$, for $i=0, 1$. Clearly the interior P of this manifold is homeomorphic to R^5.

Moreover, g_0, σ and g_1 define an imbedding K: $(B^4 \times B^1)_0 \cup ([1/3, 2/3] \times R^3 \times R^1) \cup (B^4 \times B^1)_1 \to M$. Consider the function g: $P \to \xi$ given by $g|(R_i^5 - W) \cup (B^4 \times B^1)_i = f g_i$ and $g|[1/3, 2/3] \times R^3 \times R^1 = f \sigma$. Then g is clearly TOP transverse to X on $P - K^{-1} \gamma[0, 1]$, $fK = g$, and $g^{-1}(X)$ is homeomorphic to the interior of the boundary connected sum of two copies of $\mathring{c}(H)$.

Since γ is locally flat, $P/K^{-1}\gamma[0, 1] \simeq R^5$. Furthermore, $g^{-1}(X)/K^{-1}\gamma[0, 1] \simeq \mathring{c}(H \# H)$, as is shown explicitly in [2].

If ξ were trivial, 6.1 would imply that g is ε-homotopic with support an arbitrarily small neighborhood of $K^{-1}\gamma[0, 1]$ to a map TOP transverse to X. Since near $\gamma[0, 1]$, $f = gK^{-1}$, the same would then be true of f, eliminating this pair of singularities.

Although ξ may not be trivial, f: $M \to \xi$ factors through a trivial microbundle near $\gamma[0, 1]$, as follows. Let $\coprod$ denote disjoint union and X_0, X_1 be the subsets of X defined in a). Define Y' to be the space obtained from $(X_0 \coprod X_1) \times R$ and $\sigma([1/3, 2/3] \times R^3 \times R^1)$ by identifying the two lines $\sigma(\{1/3, 2/3\} \times \{0\} \times R^1)$ with their image under τf. Define $X' \subset Y'$ to be $(X_0 \coprod X_1) \times \{0\} \cup \sigma([1/3, 2/3] \times R^3 \times \{0\}$. Since $\tau f \sigma$ is a microbundle map, X' has a normal R^1 microbundle in Y' which is easily seen to be trivial.

Define the map φ: $Y' \to \xi$ to be τ^{-1} on $(X_0 \coprod X_1) \times R$ and f on $\sigma([1/3, 2/3] \times R^3 \times R^1)$. Define the map ψ: $g_0(B^4 \times B^1) \cup \sigma([1/3, 2/3] \times R^3 \times R^1) \cup g_1(B^4 \times B^1) \to Y'$ to be τf on $g_i(B^4 \times B^1)$ and the identity on $\sigma([1/3, 2/3] \times R^3 \times R^1)$. Clearly $\varphi \cdot \psi = f$ and φ is a microbundle map near X'. Apply 6.1 to ψ and again conclude that f is ε-homotopic with support an arbitrarily small neighborhood of $\gamma[0, 1]$ to a map which is TOP transverse to X.

To complete the proof of 6.2, perform this entire process simultaneously on a properly imbedded family of disjoint arcs in $f^{-1}(X)$ chosen to connect in pairs all but at most one singularity in each component of $f^{-1}(X)$.

7. Concluding Remarks

The simplest example of a homology 3-sphere of Rohlin invariant 1 is the Poincaré 3-sphere, which is the link of the complex singularity $X_0^5 + X_1^3 + X_2^2 = 0$. It can be shown, in a proof reminiscent of [6], that if N is a compact connected homology 4-manifold which is a combinatorial 4-manifold except at a vertex whose link is the Poincaré sphere, then N is a component of a singular real algebraic variety. Hence for M compact we may take $f^{-1}(X)$ to consist of components of some real algebraic variety. Following Tognoli's proof of the Nash conjecture,

H. King and S. Akbulut have announced that if H is the Poincare 3-sphere, $f^{-1}(X)$ in Theorem 6.2 is a real algebraic variety.

We have examined here TOP transversality of maps to a microbundle. For codimension $\geqq 3$ transversality of imbedded submanifolds and transversality of maps to a TOP block bundle will be discussed in a forthcoming paper of A. Marin.

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