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CONSTRUCTING STRANGE MANIFOLDS WITH THE DODECAHEDRAL SPACE

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Since its discovery by Poincaré at the beginning of the century, the dodecahedral manifold K has fueled several of the most fundamental theorems on manifolds. Originally K was presented as an example of a homology 3-sphere which is not a sphere, providing a counterexample to the "homology" Poincaré conjecture [ST]. Milnor's original counterexample Σ to the smooth Poincaré conjecture is closely connected to K in the theory of complex singularities [Mi$_1$], [Mi$_3$]. Since Σ is a PL sphere, it provided the first example of a PL manifold with more than one smooth structure, thus distinguishing between the categories $DIFF$ and PL.

Most recently the manifold K has been used by Kirby and Siebenmann to distinguish between PL manifolds and merely topological manifolds. In particular, their fundamental example of a non-PL manifold is a 5-manifold M homotopy equivalent to $X \times S^1$, where X is a homology 4-manifold whose only non-Euclidean point has link K [Sie].

In this paper three manifolds are constructed which are closely related to the results of Kirby–Siebenmann. Their existence has been demonstrated elsewhere [Sh$_1$], [CS], [HoM]. Here the constructions flow from properties of K. §1 presents two of the many descriptions of K. In §2 fake homotopy structures are constructed for $S^3 \times S^1 \# S^2 \times S^2$ and $S^3 \times S^1 \times S^1$. In §3 a nontriangulable 5-manifold homotopy equivalent to $CP(2) \times S^1$ is constructed, using deep results of Kirby–Siebenmann.

This paper is derived mainly from the author's thesis while at U. C. Berkeley; I would like to express my warmest thanks to Prof. R. Kirby for his guidance during that time.

§1. The dodecahedral manifold K.

The dodecahedral space is remarkable not only for its colorful past, but also for the multitude of ways in which it can be defined. Originally it was constructed by identifying opposite sides of the dodecahedron [ST]. It is also the p-fold branched cyclic covering of a torus knot of type (q, r), where (p, q, r) is any permutation of $(2, 3, 5)$. It can be constructed by plumbing 2-disk bundles of euler characteristic 2 over S^2 along a tree with branches of length 2, 3 and 5.

The descriptions which will be of most interest here are the following. First, K is the intersection of the unit 5-sphere in $\mathbf{C}^3$ with the complex variety $z_0^2 +$

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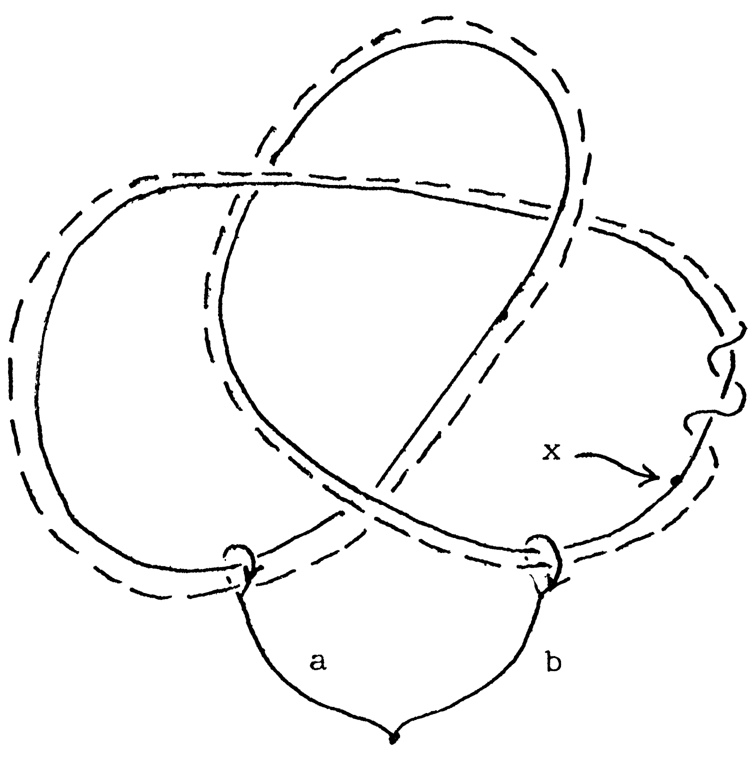
$z_1^3 + z_2^5 = 0$. As a consequence, K admits an effective circle action, induced from the natural action of $\mathbf{C}^* = \mathbf{C} - \{0\}$ on $\mathbf{C}^3$ given by $t(z_0, z_1, z_2) = (t^{15}z_0, t^{10}z_1, t^6z_2)$ for $t \in \mathbf{C}^*$, $(z_0, z_1, z_2) \in \mathbf{C}^3$. Furthermore it can be shown that K is the unique 3-manifold with an effective circle action and fundamental group the binary icosahedral group, the group with presentation $\pi_1(K) = \{x, y \mid x^3 = (xy)^2 = y^5\}$ [Or]. This fact leads to the second description of K.

Let $k : S^1 \to S^3$ be an imbedding of the trefoil knot. Remove an open tubular neighborhood of $k(S^1)$ in S^3. Attach to the boundary torus of the resultant manifold a copy of $S^1 \times D^2$ by a homeomorphism of boundaries chosen so that (pt. $\times\ \partial D^2) \subset S^1 \times D^2$ is attached to the dotted curve shown below:



Scale 3:1

We will show that the constructed manifold N has a circle action and that $\pi_1(N) \simeq \pi_1(K)$. Hence $N = K$.

By Van Kampen's theorem N has fundamental group isomorphic to $\pi_1(S^3 - k(S^1))/r$, where r is the element of $\pi_1(S^3 - k(S^1))$ represented by the dotted circle in the figure.

The group $\pi_1(S^3 - k(S^1))$ is well-known to have the presentation $\{a, b \mid aba = bab\}$, with a and b represented by the illustrated loops [F]. Starting at x, the word r is read off $r = ab^2ab^{-3}$. On substituting $a \to b^{-1}c$ the group $\pi_1(S^3 - k(S^1))/r$ becomes $\{b, c \mid c^2 = bcb, cbc = b^4\}$ or $\{b, c \mid b^5 = (bc)^2 = c^5\}$ $\simeq \pi_1(K)$.

A circle action on N can be defined as follows. The circle action on S^3 induced by the action of C^* on $\mathbf{C}^2$ given by $t(z_0, z_1) = (t^3z_0, t^2z_1)$ leaves the trefoil knot $z_0^2 = z_1^3$ invariant. Choose the tubular neighborhood of the trefoil knot used in constructing N to be invariant under the circle action and choose the attaching homeomorphism of $\partial(S^1 \times D^2)$ to the boundary of the tubular neighborhood so that the induced circle action on $\partial(S^1 \times D^2)$ is a standard linear action. By this is meant an action which is complex coordinates maps $(t, (u, v))$ to $(u + \alpha t, v + \beta t)$ for t in S^1, (u, v) in $S^1 \times \partial D^2$ and α, β integers. Such an action extends linearly over $S^1 \times D^2$, yielding a circle action on all of N. Hence $N \simeq K$.

Remark. A careful look at the described S^1 action shows that N has the same Seifert invariants as K and therefore is homeomorphic to K. The fundamental group calculation is thus unnecessary, but more easily carried out than is the calculation of Seifert invariants.

§2. Fake homotopy structures on $S^3 \times S^1 \# S^2 \times S^2$ and $S^3 \times S^1 \times S^1$.

A homotopy structure on a PL manifold M is a simple homotopy equivalence $f : M^1 \to M$. Two structures $f_0 : M_0 \to M$, $f_1 : M_1 \to M$ are equivalent if there is a PL s-cobordism W from M_0 to M_1 and a homotopy equivalence $F : W \to M \times I$ restricting to f_0 and f_1 on M_0 and M_1 respectively.

A fake homotopy structure will here mean a homotopy structure $f : M_1 \to M$ such that there is a PL cobordism W from M_1 to M and a map $F : W \to M \times I$ covered by a map of the respective normal microbundles and restricting to the maps f and the identity on M_1 and M respectively, yet W cannot be chosen so that F is a homotopy equivalence.

Here we construct fake homotopy structures on $S^3 \times S^1 \# S^2 \times S^2$ and $S^3 \times S^1 \times S^1$. Interest in the second manifold arises from the work of Kirby–Siebenmann, who show that any homotopy equivalence $M \to S^3 \times S^1 \times S^1$ is homotopic to a homeomorphism. The fake homotopy structure on $S^3 \times S^1 \times S^1$ provides a counterexample to the Hauptvermutung, because the homeomorphism $M \to S^3 \times S^1 \times S^1$ is not even homotopic to a PL homeomorphism.

Similarly one might hope that a close study of the fake homotopy $S^3 \times S^1 \# S^2 \times S^2$ might lead to the same conclusions for 4-manifolds. For this a good understanding of its geometry seems to be necessary.

The existence and uniqueness of the fake homotopy structure on $S^3 \times S^1 \# S^2 \times S^2$ was demonstrated by Cappell-Shaneson-Lee [CS], that on $S^3 \times S^1 \times S^1$ by Shaneson [Sh_1].

2.1. LEMMA. *There is a PL cobordism W, with trivial normal bundle, from $\partial_0 W \simeq S^3 \times S^1$ to $\partial_1 W \simeq K \times S^1$ such that for $i = 0, 1$, $H_*(W, \partial_i W; Z) = 0$. Moreover the inclusion $K \times$ (pt.) $\to \partial_1 W \to W$ induces the trivial map $\pi_1(K) \to \pi_1(W)$.*

Proof. K is imbedded in S^5 as the link of the algebraic variety defined above. Since K is a homology 3-sphere, it follows from the Mayer–Vietoris sequence that $S^5 - K$ is a homology circle. Remove from S^5 a tubular neighborhood of K and of a circle in the complement of K representing the generator of $H_1(S^5 - K)$. An easy calculation shows that the resulting cobordism W between the boundaries of the tubular neighborhoods satisfies $H_*(W, \partial_i W) = 0$ for $i = 0, 1$. Furthermore Van Kampen's theorem implies that the inclusion induces the trivial map $\pi_1(K \times (\text{pt.})) \to \pi_1(W)$. This proves 2.1.

Let $\alpha \in \pi_1(K)$ be a generator of $\pi_1(K)$, and let M be the manifold obtained from $K \times S^1$ by doing surgery on α in the normal homology equivalence $K \times S^1 \to S^3 \times S^1$.

2.2. LEMMA. *There is a PL cobordism W' with trivial normal bundle from $\partial_1 W' \simeq M$ to $\partial_0 W' = S^3 \times S^1 \# S^2 \times S^2$ such that for $i = 0, 1$, $H_*(W', \partial_i W'; Z) = 0$.*

Proof. Let $T \subset K \times$ (pt.) be a PL flatly imbedded circle representing α. By 2.1 and general position there is a PL flat imbedding $k : S^1 \times I \to W$ such that $k(S^1 \times \{0\})$ is a trivial circle in $S^3 \times S^1$ and $k(S^1 \times \{1\}) = T$. Do surgery along $k(S^1 \times I)$. That is, remove a tubular neighborhood ν of $k(S^1 \times I)$ and to its boundary attach a copy of $D^2 \times S^2 \times I$ by an S^2-bundle equivalence $h : \partial D^2 \times S^2 \times I \to \nu$ chosen so that the resulting manifold W' still has trivial normal bundle.

The manifold W' is a homology product since W is. Furthermore since $k(S^1 \times \{0\})$ is null-homotopic in $\partial_0 W$ it bounds an imbedded 2-disk in $\partial_0 W$. Therefore $\partial_0 W'$ is the connected sum of $\partial_0 W \simeq S^3 \times S^1$ and $S^2 \times S^2$ along a 4-disk neighborhood of the 2-disk which $k(S^1 \times \{0\})$ bounds [W_1].

The map h defined in the proof of 2.2 extends naturally to an imbedding $h : D^2 \times S^2 \times I \to W'$.

2.3. LEMMA. *The identity map $\partial_0 W' \to \partial_0 W'$ extends to a normal map $F : W' \to \partial_0 W' \times I$ such that $Fh : D^2 \times S^2 \times I \to h(D^2 \times S^2 \times 0)$ is the natural projection and $f = F \mid \partial_1 W'$ is a homotopy equivalence.*

Proof. Since $H^*(W', \partial_0 W'; Z) = 0$, $H^*(W' - h(D^2 \times S^2 \times I), \partial_0 W' - h(D^2 \times S^2 \times 0); \pi_i(\partial_0 W' - h(D^2 \times S^2 \times 0)) = 0$ and there is no obstruction to defining F as above. It remains to show that f is a homotopy equivalence.

The fundamental group of $\pi_1(\partial_1 W')$ is $\pi_1(K) \times S^1/\alpha = \pi_1(S^1) = Z$. The map $f : \partial_1 W' \to \partial_0 W'$ is of degree 1, so Poincaré duality defines a splitting of the induced map $H_1(\partial_1 W'; Z) \xrightarrow{f_*} H_1(\partial_0 W'; Z)$ [W_2, Lemma 2.2]. Therefore since the fundamental groups are both infinite cyclic $f_\# : \pi_1(\partial_1 W') \to \pi_1(\partial_0 W')$ is an isomorphism. Once again [W_2, Lemma 2.2] implies that $f_* : H_*(\partial_1 W'; Z[Z]) \to H_*(\partial_0 W'; Z[Z])$ is also a split surjection. By a Mayer–Vietoris argument, $H_*(\partial_1 W'; Z[Z])$ and $H_*(\partial_0 W'; Z[Z])$ are isomorphic as $Z[Z]$ modules. Hence f_* defines an isomorphism with local coefficients, and so f is a homotopy equivalence.

2.4. PROPOSITION. *The homotopy equivalence f is not PL s-cobordant to the identity.*

Proof. Suppose there is a PL manifold V and a homotopy equivalence $H : V \to \partial_0 W \times I$ which coincides with f on $\partial_1 V \simeq M \simeq \partial_1 W$ and is the identity on $\partial_0 V$.

Since $f : h(D^2 \times S^2 \times 1) \to h(D^2 \times S^2 \times 0) \times 1 \subset \partial_0 W \times 1$ is the natural homeomorphism, H can be extended to a homotopy equivalence between the manifolds obtained by attaching 3-handles to $\partial_1 V$ and $\partial_0 W \times \{1\}$ along $h(D^2 \times S^2 \times 1)$ and $h(D^2 \times S^2 \times 0) \times 1$ respectively. Since $H \mid \partial_0 V$ is the identity, H can further be extended to 3-handles attached to $\partial_0 V$ and $\partial_0 W \times \{0\}$ along copies of S^2 orthogonal to $h(D^2 \times S^2 \times 0)$ in $\partial_0 W \simeq S^3 \times S^1 \# S^2 \times S^2$. The result is a homotopy equivalence $H' : V' \to S^3 \times S^1 \times I$. Here $\partial_1 V'$ has been obtained by undoing the surgery of $\partial_1 W$ in Step 2 and so is PL homeomorphic to $\partial_1 W \simeq K \times S^1$. On $\partial_0 V'$, H' is the identity.

Construct a possibly new homotopy equivalence $V' \to S^3 \times S^1 \times I$, still denoted H', as follows. Let $H' \mid \partial_1 V' : K \times S^1 \to S^3 \times S^1$ be a homology equivalence which fibres over S^1. There is no obstruction to extending to a homotopy equivalence $H' : V' \to S^3 \times S^1 \times I$. $H' \mid \partial_0 V' : S^3 \times S^1 \to S^3 \times S^1$ is a homotopy equivalence, hence we may assume it is a diffeomorphism which fibres over S^1 [Sh$_2$]. Homotope H' rel $\partial V'$ so that H' is PL-transverse to $S^3 \times$ (pt.) $\times I$. A theorem of Novikov implies that $H'^{-1}(S^3 \times \text{(pt.)} \times I)$, a PL cobordism from K to S^3, has the index of $S^3 \times$ (pt.) $\times I$ [N]. But K bounds a parallelizable PL-manifold of index 8 [Mi$_3$]. Attach this manifold to one end of $H'^{-1}(S^3 \times \text{(pt.)} \times I)$ along $H'^{-1}(S^3 \times \text{(pt.)} \times 1) \simeq K$ and attach D^4 to the other end along $H'^{-1}(S^3 \times \text{(pt.)} \times 0) \simeq S^3$. The result is an almost parallelizable PL 4-manifold of index 8. A theorem of Rohlin states that the index of any almost parallelizable PL 4-manifold is a multiple of 16 [Roh.] This contradiction proves the proposition.

A fake homotopy structure on $S^3 \times S^1 \times S^1$ can be constructed analogously. Instead of the cobordism W of 2.1 consider the cobordism $W \times S^1$ from $K \times S^1 \times S^1$ to $S^3 \times S^1 \times S^1$. Do surgery as in 2.2 to obtain a PL normal cobordism from a manifold M to $S^3 \times S^1 \times S^1 \# S^2 \times S^3$ such that the normal cobordism is a homology product (with coefficients in Z). Construct a homotopy equivaluence $M \to S^3 \times S^1 \times S^1 \# S^2 \times S^2$ as in 2.3. By general position there is an imbedded S^2 in M whose image under the homotopy equivalence represents a generator of $\pi_2(S^3 \times S^1 \times S^1 \# S^2 \times S^3) \simeq Z$. Perform a surgery on $S^2 \subset M$ and thereby obtain a homotopy equivalence $N \to S^3 \times S^1 \times S^1$.

Suppose $N \to S^3 \times S^1 \times S^1$ is PL s-cobordant to the identity map. Then by the s-cobordism theorem there is a PL homeomorphism $f : N \to S^3 \times S^1 \times S^1$. Notice, however, that in the construction of N from $K \times S^1 \times S^1$, any $K \times S^1 \times$ (pt.) chosen well away from the circle $T \subset K \times S^1 \times S^1$ on which surgery is first performed (see 2.2) remains intact during the construction of N. Thus any PL homeomorphism $N \to S^3 \times S^1 \times S^1$ would provide a flat PL imbedding of $K \times S^1$ into $S^3 \times S^1 \times S^1$. Lift this to an imbedding $K \times S^1 \to S^3 \times S^1 \times R$ in the covering $S^3 \times S^1 \times R$ of $S^3 \times S^1 \times S^1$, and observe that for t large enough $K \times S^1 \cap S^3 \times S^1 \times \{t\} = \phi$. An elementary argument shows that the region V' of $S^3 \times S^1 \times R$ between $K \times S^1$ and $S^3 \times S^1 \times \{t\}$ has the homotopy type of $S^3 \times S^1$. Just as in the proof of 2.4 this property of V' contradicts Rohlin's theorem.

Remark. One might ask why a similar final surgery on the homotopy equivalence $f : M \to S^3 \times S^1 \# S^2 \times S^2$ of 2.3 cannot be used to construct a fake homotopy structure on $S^3 \times S^1$. Classically the answer has been:

a) It may be impossible to imbed in M an S^2 with trivial normal bundle representing a generator of $H_2(S^3 \times S^1 \# S^2 \times S^2)$;

b) Should such an S^2 be found, surgery on S^2 may alter the fundamental group, since S^2 is a codimension 2 submanifold.

Note, however, that h : (pt.) $\times S^2 \times 1 \to M$ defines an imbedding of S^2 in M as in a) above. Thus the difficulty lies in obtaining an imbedded S^2 with the property $\pi_1(M - S^2) \simeq \pi_1(M)$, in order that the surgery on S^2 not alter the fundamental group.

§3. A non-triangulable homotopy $CP(2) \times S^1$.

Hollingsworth and Morgan calculate in [HoM] that there is a closed 5-manifold M which is homotopy equivalent to $CP(2) \times S^1$ but is not a PL manifold. In order to construct M via the dodecahedral space we cite the following theorems which are corollaries of the classification results of Kirby–Siebenmann [KS]:

3.1. Theorem. *Any homotopy equivalence $h : N \to S^3 \times T^2$ is homotopic to a homeomorphism.*

3.2. Theorem. *Let M be a PL manifold without boundary of dimension $m \geq 5$. If $H^3(M; Z_2) = 0$ then the PL triangulation of M is unique up to isotopy.*

In light of our construction of the fake homotopy equivalence $N \to S^3 \times S^1 \times S^1$, 3.1 has the following important corollary:

3.3. Proposition. *There is a manifold W and a homeomorphism $K \times S^1 \to \partial W$ which extends to a homotopy equivalence* cone$(K) \times S^1 \to W$.

Proof. It was shown in §2 that a *PL* homeomorphism $N \to S^3 \times S^1 \times S^1$ may be used to define a *PL* cobordism V' from $K \times S^1$ to $S^3 \times S^1$ such that the inclusion $S^3 \times S^1 \to V'$ is a homotopy equivalence. Though in the *PL* category this created a contradiction, 3.1 asserts that $N \to S^3 \times S^1 \times S^1$ *is* homotopic to a (non-*PL*) homeomorphism. Thus there is a *topological* cobordism V' as defined above. Construct W from V' by attaching $D^4 \times S^1$ to $S^3 \times S^1 = \partial_0 V'$. There is no obstruction to extending the homeomorphism $K \times S^1 \to \partial_1 V' = \partial W$ to a homotopy equivalence cone$(K) \times S^1 \to W$, proving 3.3.

In §1 K was obtained from S^3 by performing surgery on the trefoil knot. Equivalently, K is the boundary of the 4-manifold U obtained by attaching a 2-handle to D^4 along a trefoil knot in ∂D^4. Attach W defined in 3.3 to $U \times S^1$ along $\partial W \simeq \partial U \times S^1$ by a *PL* homeomorphism and call the resulting manifold P.

3.4. THEOREM. *P is a non-PL manifold of the homotopy type of $CP(2) \times S^1$.*

Proof. Let X be the complex obtained from U by attaching cone(K) to ∂U. X is a simply-connected homology manifold with $H_2(X; Z) \simeq Z$. By Poincaré duality, the self-intersection number of a generator of $H_2(X; Z)$ is ± 1, so, properly oriented, X has the quadratic form of $CP(2)$. Milnor, using a theorem of Whitehead, shows that the homotopy type of a simply-connected 4-dimensional Poincaré complex is determined by its quadratic form [Mi_3], [Wh]. Thus X has the homotopy type of $CP(2)$. It follows from 3.3 that P has the homotopy type of $X \times S^1$, hence the homotopy type of $CP(2) \times S^1$.

Suppose P is *PL*. Then so is the universal cover $\tilde{P}$ of P. $\tilde{P}$ is the union along the boundary of the universal covers $\tilde{W}$ of W and $\tilde{U}$ of U. Since $H^3(\tilde{U}) = 0$, the natural *PL* triangulation of interior $(\tilde{U})$ is unique up to isotopy. Hence, after an isotopy, $\partial\tilde{U} = \partial\tilde{W}$ is *PL* flatly imbedded in $\tilde{P}$. The natural triangulation of $\partial\tilde{W}$ therefore extends over $\tilde{W}$. But as in the proof of 2.4, *PL* transversality of a proper map $\tilde{W} \to \mathbf{R}$ would then produce a counterexample to Rohlin's theorem. The contradiction proves 3.4.

§4. Concluding remarks.

In [Sch] a structure similar to that on P in §3 is obtained for any nontriangulable closed oriented 5-manifold. In fact, it is always possible to remove a copy of interior (W) from a 5-manifold and obtain a *PL* manifold with boundary.

The connection between the fake homotopy $S^3 \times S^1 \# S^2 \times S^2$ in §1 and that of Cappell–Shaneson–Lee is mysterious. The surgery theory of [CS] implies that the two examples are *PL* *h*-cobordant, but I do not know whether they are homotopic. Perhaps an easier problem is the following: In 2.2 the only property of $\alpha \in \pi_1(K)$ which was used was that $\pi_1(K)/\langle\alpha\rangle = 0$. The group $\pi_1(K)$ contains 120 elements but only one non-trivial normal subgroup, the center of order 2. There are thus 118 possible candidates for α. Many are equivalent under homeomorphisms of K, but do all 118 define the same manifold?

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