K-Theory **32:** 253–267, 2004.

Hermitian K-theory on a Theorem of Giffen

MARCO SCHLICHTING
Universität GHS Essen, FB 6 Mathematik, 45117 Essen, Germany.
e-mail: marco.schlichting@uni-essen.de

(Received: September 2003)

Abstract. Let $\mathcal{E}$ be an exact category with duality. In [1] a category $\mathcal{W}(\mathcal{E})$ was introduced and the authors asserted that the loop space of the topological realization of $\mathcal{W}(\mathcal{E})$ is homotopy equivalent to Karoubi's U-theory space of $\mathcal{E}$ when $\mathcal{E} = \mathcal{P}(R)$, the category of finitely generated projective modules over a ring R with an involution if 2 is invertible in R. Unfortunately, their proof contains a mistake. We present a different proof which avoids their argument.

Mathematics Subject Classifications (1991): 19DO6, 19G38, 11E70.

Key words: Hermitian K-theory, Q-construction

1. Introduction

Let R be a ring with unit, and let $R \to R : a \mapsto \overline{a}$ be an involution, i.e., $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\overline{a}$, $\overline{1} = 1$ and $\overline{\overline{a}} = a$ (see [9, I 1.1.]). Let $\varepsilon \in R$ be a central element with $\varepsilon\overline{\varepsilon} = 1$. In analogy to algebraic K-theory, Karoubi [6] defined the hermitian K-theory groups of R as the homotopy groups of a space ${}_{\varepsilon}K^{\mathrm{h}}(R) \simeq {}_{\varepsilon}K_0^{\mathrm{h}}(R) \times B_{\varepsilon}O(R)^{+}$. Here ${}_{\varepsilon}K_0^{\mathrm{h}}(R)$ is the Grothendieck group of the abelian monoid of isometry classes of finitely generated projective right R-modules equipped with a non-degenerate ε-symmetric form. There is a hyperbolic functor from algebraic K-theory to Hermitian K-theory $h : K(R) \to {}_{\varepsilon}K^{\mathrm{h}}(R)$ whose homotopy fiber is denoted by ${}_{\varepsilon}U(R)$. There is also a forgetful functor $f : {}_{\varepsilon}K^{\mathrm{h}}(R) \to K(R)$ whose homotopy fiber is denoted by ${}_{\varepsilon}V(R)$. Karoubi's fundamental theorem in Hermitian K-theory states that there is a natural homotopy equivalence $\Omega_{\varepsilon}U(R) \simeq {}_{-\varepsilon}V(R)$ whenever 2 is invertible in R (see [7]).

Giffen attempted to reinterpret the fundamental theorem in a categorical framework. To this end, he introduced a category ${}_{\varepsilon}\mathcal{W}(R)$ which has been proposed independently by Karoubi (unpublished). We refer the reader to Section 2 for a description. In [1, Theorem 3.1, Corollary 3.7], Charney and Lee claimed to have proved the existence of a homotopy fibration

$$K(R) \xrightarrow{h} {}_{\varepsilon}K^{\mathrm{h}}(R) \longrightarrow {}_{\varepsilon}\mathcal{W}(R) \tag{1}$$

if 2 is invertible in R. This would identify, up to homotopy, the loop space of ${}_{\varepsilon}\mathcal{W}(R)$ with ${}_{\varepsilon}U(R)$. Unfortunately, their proof contains an error: the functor σ^* of [1, p. 177] does not act as an inner automorphism as claimed. We do not know whether the mistake can be fixed within their framework. In this paper, we present a different proof which avoids their argument.

The main reason for being interested in the construction of $\mathcal{W}$ is that it can be applied to any exact category with duality, whereas the definition of Karoubi's U-theory presupposes a definition of Hermitian K-theory which is not yet available for exact categories, in general. We feel that the category $\mathcal{W}$ should play a similar role in Hermitian K-theory as Quillen's Q-construction does in algebraic K-theory.

The author's interest in writing this article stems from the attempt to generalize Hermitian K-theory of rings to exact categories or even to chain complexes as in the spirit of Thomason and Trobaugh [17]. A localization theorem for Hermitian K-theory whose proof depends on the result of this article will be published in [5]. This can be considered as a first step to the targeted generalization.

The statement of [1, Theorem 3.1] differs from (1) insofar as the authors of [1] do not assume 2 to be invertible in R. The methods of our proof do not yield this more general version.[1]

The plan of the article is as follows. In Section 2, we recollect some facts about symmetric monoidal categories and the Bar construction all of which are well known. Section 3, introduces a Hermitian analog of Waldhausen's $S_{\cdot}$-construction whose definition emerged from discussions I had with Jens Hornbostel. We also recall the definition of Giffen's category $\mathcal{W}$. In Section 4, we prove the main result which is contained in Theorem 4.2 and Proposition 4.8. The appendix recalls definitions and basic facts about exact categories with duality and shows that, up to equivalence of such categories, we can always "identify an object with its double dual". This simplifies notations and proofs.

We use the word "space" interchangeably for "topological space of the homotopy type of a CW-complex", "simplicial set", or "category", the homotopy categories whereof they are objects are known to be equivalent.

2. Symmetric Monoidal Categories

2.1. Let M be a monoid in the category of spaces which acts on a space X. There is a simplicial space $\mathrm{Bar}_{\cdot}(M, X)$, called Bar construction, whose space of n-simplices is $\mathrm{Bar}_n(M, X) = M^n \times X$ (the simplicial space is $B(*, M, X)$ in

[1]After submitting this article, we have found a proof of Lemma 4.4 avoiding $\frac{1}{2} \in \mathcal{A}$. The results of this article are therefore true even if $\frac{1}{2} \notin \mathcal{A}$. This will be published elsewhere.

[11, Section 7] and $EM \times_M X$ in [3, IV 5.3]). The face maps are given by the multiplication in M and by the action of M on X. We write $\mathrm{Bar}.(M)$ for $\mathrm{Bar}.(M, *)$.

Let M be a monoid in the category of simplicial sets acting on a simplicial set X. We say that it acts invertibly up to homotopy if for every vertex $v \in M_0$ the map $X \to X$ induced by restricting the action of M to the vertex v is a homotopy equivalence. A homology version of the following lemma can also be found in [3, Theorem IV, 5.15].

2.2. LEMMA. *Let M be a monoid in the category of simplicial sets acting on a simplicial set X invertibly up to homotopy, then the sequence*

$$X \to \mathrm{Bar}.(M, X) \to \mathrm{Bar}.(M)$$

is a homotopy fibration. Here the first map is "inclusion of zero simplices", and the second map is induced by the M-equivariant map $X \to *$.

2.3. *Proof*. This is [13, Theorem 2.1] with $\mathbb{C} = M$, $N\mathbb{C} = \mathrm{Bar}.(M)$, $N(X_{\mathbb{C}}) = \mathrm{Bar}.(M, X)$ and $X(\mathbb{C}) = \gamma^*(X(\mathbb{C})) = X$.

2.4. *Remark*. By taking nerves and diagonals when appropriate, Lemma 2.2 remains true when we replace simplicial sets by categories or bisimplicial sets.

2.5. Recall that a symmetric monoidal category is a category $\mathcal{C}$ equipped with a functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object 0 and for all objects A, B, C with natural isomorphisms $\alpha : A \oplus (B \oplus C) \xrightarrow{\sim} (A \oplus B) \oplus C$, $\sigma : A \oplus B \xrightarrow{\sim} B \oplus A$ and $\eta : 0 \oplus A \xrightarrow{\sim} A$ making certain diagrams commutative (see [12, VII, 7, XI]). A symmetric monoidal category is called strict if α and η are identity morphisms. A morphism in the category of small symmetric monoidal categories is a functor $f : \mathcal{C} \to \mathcal{D}$ together with natural isomorphisms $\tau : f(A) \oplus f(B) \xrightarrow{\sim} f(A \oplus B)$, $\varepsilon : 0 \to f(0)$ making certain diagrams commute. A morphism in the category of small strict symmetric monoidal categories is a morphism between symmetric monoidal categories such that τ and ε are the identity maps. For more details see for instance [12, VII 7, XI 16, Section 1]. Every symmetric monoidal category is equivalent to a strict symmetric monoidal category (see [10, 4.2, 12, XI 3]).

Considering a small strict symmetric monoidal category as a monoid in the category of small categories, the Bar construction of 2.1 is then a functor from strict symmetric monoidal categories to simplicial symmetric monoidal categories.

If $\mathcal{C}$ is a category, then we write $i\mathcal{C}$ for the subcategory which has the same objects as $\mathcal{C}$ and whose morphisms are the isomorphisms in $\mathcal{C}$. In the case of $\mathcal{C}$ symmetric monoidal, a category $(i\mathcal{C})^{-1}\mathcal{C}$ has been constructed in [4, p. 219] in a functorial way. There is an inclusion of categories $\mathcal{C} \to (i\mathcal{C})^{-1}\mathcal{C}$ which is a group completion provided that all translations $\oplus A : i\mathcal{C} \to i\mathcal{C}$ are faithful [4, 2, p. 220]. In this case we write $\mathcal{C}^+$ for $(i\mathcal{C})^{-1}\mathcal{C}$. Remark that the functor $\mathcal{C} \mapsto \mathcal{C}^+$ sends strict symmetric monoidal categories to strict ones.

We need the following well-known lemma.

2.6. LEMMA. *Let $\mathcal{C}_*$ be a simplicial strict symmetric monoidal category such that translations in $i\mathcal{C}_n$ are faithful for $n \in \mathbb{N}$ then the map*

$$|\mathcal{C}_*| \to |\mathcal{C}_*^+|$$

is a group completion. In particular, if the monoid $\pi_0(|\mathcal{C}_*|)$ is a group, then the above map is a homotopy equivalence.

2.7. *Proof.* The two topological monoids $|\mathcal{C}_*|$ and $|\mathcal{C}_*^+|$ are both H-spaces which meet the hypothesis of the "group completion theorem", i.e., for which the natural transformation of functors $id \to \Omega|p \mapsto \mathrm{Bar}_p(id)|$ is a group completion [11, Section 15]. It therefore suffices to show that $|\mathrm{Bar}_*(|\mathcal{C}_*|)| \to |\mathrm{Bar}_*(|\mathcal{C}_*^+|)|$ is a homotopy equivalence. Realizing in a different order, we see that the this map is homeomorphic to $|q \mapsto |\mathrm{Bar}_*(\mathcal{C}_q)|| \to |q \mapsto |\mathrm{Bar}_*(\mathcal{C}_q^+)||$. But the last map is degree wise a homotopy equivalence because $\mathcal{C}_q \to \mathcal{C}_q^+$ being a group completion of symmetric monoidal categories implies that $|\mathrm{Bar}_*(\mathcal{C}_q)| \to |\mathrm{Bar}_*(\mathcal{C}_q^+)|$ is a homotopy equivalence.

3. The $\mathcal{R}_\cdot$-Construction and Giffen's Category $\mathcal{W}$

3.1. DEFINITION. *A category with duality* is a triple $(\mathcal{C}, *, \eta)$ with

(1) $\mathcal{C}$ a category,
(2) $* : \mathcal{C} \to \mathcal{C}^{op}$ a functor,
(3) $\eta : id_{\mathcal{C}} => **$ a natural equivalence such that
(4) for all objects A of $\mathcal{C}$ we have $1_{A^*} = \eta_A^* \circ \eta_{A^*}$.

The duality of $(\mathcal{C}, *, \eta)$ is called *strict* if $** = id_{\mathcal{A}}$ and if $\eta_A = 1_A$ for all objects A of $\mathcal{C}$. A map $f : (\mathcal{A}, *) \to (\mathcal{B}, \sharp)$ between categories with strict duality is a functor $f : \mathcal{A} \to \mathcal{B}$ such that $f^{op} \circ * = \sharp \circ f$. A category with duality $(\mathcal{E}, *, \eta)$ is called *exact* if $\mathcal{E}$ is an exact category (see [14, 8, Appendix A]) and if $* : \mathcal{E} \to \mathcal{E}^{op}$ is an exact functor.

Since every (exact) category with duality is equivalent to a (exact) category with strict duality (see A.8) we will assume our (exact) categories with duality to have a strict duality.

3.2. THE $\mathcal{R}_\cdot$-CONSTRUCTION

Let Δ be the category of finite ordered sets and monotonic maps as morphisms. As usual, denote by $[n]$ the ordered set $\{0 < \cdots < n-1 < n\}$. We consider ordered sets as categories. In this way, the terminology "ordered set with duality" is defined in 3.1. Remark that every duality on an ordered set is necessarily strict. Remark furthermore that every ordered set can be made into an ordered set with duality in a unique way. Denote $x \mapsto x'$ the unique duality functor $A \to A^{op}$ on a finite ordered set A. Let $d\Delta$ be the category of finite ordered sets with duality. Maps in this category are the maps of categories with strict duality. Remark that the inclusion functor $d\Delta \subset \Delta$ which forgets the duality is the identity on objects but is not full. For two finite ordered sets A, B let the concatenation AB of A and B be the finite ordered set whose set of elements is the disjoint union of those of A and B ordered in such a way that the elements in A are smaller than the elements of B and such that the natural inclusions of A and B in AB are monotonic. Concatenation defines a functor $\Delta \times \Delta \to \Delta$. We remark that the functor $d : \Delta \to \Delta : A \mapsto A^{op}A,\ f \mapsto f^{op}f$ factors through $d\Delta$.

If $\mathcal{E}$ is an exact category, then the category $S_n\mathcal{E}$ defined by Waldhausen (see [18]) is an exact category, and $n \mapsto S_n\mathcal{E}$ is a simplicial exact category. If $(E, *)$ is an exact category with strict duality, then so is $S_n\mathcal{E}$ by declaring $(A_{i,j})^* = (A^*_{j',i'})$ on objects and $(f_{i,j})^* = (f^*_{j',i'})$ on morphisms. The assignment $n \mapsto S_n\mathcal{E}$ does not yield a simplicial exact category with duality, in general, but the assignment $n \mapsto S_{d(n)}\mathcal{E}$ does. Therefore, we define the simplicial exact category with duality $\mathcal{R}.\mathcal{E}$ to be $S.\mathcal{E} \circ d^{op}$. Remark that its underlying simplicial exact category is $S^e_.\mathcal{E}$, Waldhausen's edgewise subdivision of $S.\mathcal{E}$ [18, 1.9] whose topological realization is homeomorphic to the realization of $S.\mathcal{E}$. To simplify notation, we label the elements of $\underline{n} := [n]^{op}[n] \cong [2n+1]$ as $n' < (n-1)' < \cdots < 0' < 0 < \cdots < n-1 < n$.

3.3. If $(\mathcal{C}, *)$ is a category with strict duality, then we write $\mathcal{C}_h$ for the category whose objects are isomorphisms $\phi : X \to X^*$ in $\mathcal{C}$ such that $\phi^* = \phi$. A morphism from $\phi : X \to X^*$ to $\phi' : X' \to X'^*$ is a map $a : X \to X'$ such that $a^* \circ \phi' \circ a = \phi$. Recall from section 1 that $i\mathcal{C}_h$ is the subcategory of $\mathcal{C}_h$ which has the same objects as $\mathcal{C}_h$ and whose morphisms are the isomorphisms in $\mathcal{C}_h$. There is a forgetful functor $i\mathcal{C}_h \to i\mathcal{C}$ sending the object $\phi : X \to X^*$ to X and the morphism a to a. Applying this observation degree wise to the simplicial exact category with duality $\mathcal{R}.\mathcal{E}$ yields a simplicial category $i(\mathcal{R}.\mathcal{E})_h$ and a forgetful functor of simplicial categories $i(\mathcal{R}.\mathcal{E})_h \to iS^e.\mathcal{E}$.

3.4. *Remark*. For $\mathfrak{M}$ an exact category with duality the simplicial set $\mathrm{Ob}(\mathcal{R}.\mathfrak{M})_h$ is isomorphic to ${}_1s^e.\mathfrak{M}$ of [15].

3.5. THE CATEGORY $\mathcal{W}$

Let $(\mathcal{E}, *)$ be an exact category with duality. We construct a new category $\mathcal{W}(\mathcal{E})$. Its objects are "symmetric bilinear forms" in $\mathcal{E}$, more precisely, an object of $\mathcal{W}(\mathcal{E})$ is an isomorphism $\phi : X \to X^*$ such that $\phi^* = \phi$. A morphism from

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & Y \\ {\scriptstyle \phi\circ p}\downarrow & & \downarrow{\scriptstyle i^*\circ\psi} \\ X^* & \underset{p^*}{\rightarrowtail} & A^* \end{array}$$

(X, ϕ) to (Y, ψ) is an equivalence class of data (p, A, i) with $p : A \twoheadrightarrow X$ an admissible epimorphism and $i : A \rightarrowtail Y$ an admissible monomorphism such that the diagram is bicartesian. The data (p, A, i) is *equivalent* to (p', A', i') if there exists an isomorphism $a : A \to A'$ such that $p'a = p$ and $i'a = i$. If $(p, A, i) : (X, \phi) \to (Y, \psi)$ and $(q, B, j) : (Y, \psi) \to (Z, \rho)$ are two composable maps then its composition (r, C, k) is defined as in Quillen's Q-construction, i.e., C is the pullback of q along i, r and k are the compositions of p and j with the epimorphism and the monomorphism obtained by the pullback square, respectively.

3.6. Given a ring with involution $(R, \bar{\ })$ and $\varepsilon \in R$ a central element such that $\bar{\varepsilon}\varepsilon = 1$, let $(\mathcal{P}(R), \eta, *)$ be the following split exact category with duality. The category $\mathcal{P}(R)$ is the category of finitely generated projective right R-modules, $* = \mathrm{Hom}_R(\ \ , R)$ where $\mathrm{Hom}(M, R)$ is made into a right R-module through the involution of R: $(f \cdot r)(m) = \overline{r}\,\overline{f(m)}$. The identification $\eta_M : M \to M^{**}$ sends $m \in M$ to $f \mapsto \bar{\varepsilon}\overline{f(m)}$ (see [9, II 2.2.1] when $\varepsilon = 1$). By A.8, there is an additive category with strict duality $(\mathcal{P}, *)$ which is equivalent to $(\mathcal{P}(R), \eta, *)$. Then the connected component of the 0-object of $\mathcal{W}(\mathcal{P})$ is equivalent to ${}_{\varepsilon}\hat{\mathcal{W}}(0)$ of [1] which, by the argument of [1, 3.6], is homotopy equivalent to $\mathcal{W}(R)$ of [1].

3.7. LEMMA. *Let $\mathcal{E}$ be an exact category with duality. There is a natural homotopy equivalence $\mathcal{W}(\mathcal{E}) \simeq i(\mathcal{R}.\mathcal{E})_h$.*

3.8. Proof. Waldhausen's argument [18, 1.9] carries over.

4. The Homotopy Fibration

4.1. Suppose $(\mathcal{A}, *)$ is a split exact category with strict duality which, as a symmetric monoidal category, is strict under $\oplus$ and a chosen zero object. Suppose further that direct sum is compatible with the duality: $(A \oplus B)^* = B^* \oplus A^*$. Every split exact category with duality is equivalent to such a category (see A.8). Then $i(\mathcal{R}.\mathcal{A})_h$ and $iS^e_.\mathcal{A}$ are simplicial strict symmetric monoidal categories and the forgetful map $i(\mathcal{R}.\mathcal{A})_h \to iS^e.\mathcal{A}$ becomes a map of monoids. Let f be this forgetful map composed with the map $iS^e.\mathcal{A} \to iS.\mathcal{A}$ which is induced by the natural inclusions $[n] \to [n]^{op}[n] = \underline{n}$, $n \in \mathbb{N}$. Consider $i\mathcal{A}_h$ as a constant simplicial category, and let $\imath : i\mathcal{A}_h \to i(\mathcal{R}.\mathcal{A})_h$ be the inclusion of zero simplices.

4.2. THEOREM. *If* $\frac{1}{2} \in \mathcal{A}$, *then the following sequence is a homotopy fibration*

$$(i\mathcal{A}_h)^+ \xrightarrow{\quad \iota \quad} (i(\mathcal{R}.\mathcal{A})_h)^+ \xrightarrow{\quad f \quad} (iS.\mathcal{A})^+.$$

4.3. Proof. We want to apply Lemma 2.2. To this end consider $\mathcal{A} \times \mathcal{A}^{op}$ as an additive category with duality which is given by interchanging the two factors, ignoring any duality $\mathcal{A}$ might have had. Suppose $\mathcal{A}$ is strict as in 4.1. Then we let $\mathcal{A} \times \mathcal{A}^{op}$ act on $\mathcal{A}$ by

$$((A, B), Z) \mapsto A \oplus Z \oplus B^*.$$

This action is compatible with the duality. It induces a (unital and associative) action of $i(\mathcal{A} \times \mathcal{A}^{op})_h$ on $i\mathcal{A}_h$. By functoriality (see 2.5) we have an induced action of $M = i(\mathcal{A} \times \mathcal{A}^{op})_h^+$ on $X = (i\mathcal{A}_h)^+$. Since X is group complete, M acts invertibly up to homotopy. Therefore, lemma 2.2 applies to our situation. The natural isomorphism $i(\mathcal{S} \times \mathcal{T})^{-1}(\mathcal{S} \times \mathcal{T}) \cong i\mathcal{S}^{-1}\mathcal{S} \times i\mathcal{T}^{-1}\mathcal{T}$ yields isomorphisms of simplicial categories Bar. $(M, X) \cong i$Bar.$(\mathcal{A}\times\ \mathcal{A}^{op}, \mathcal{A})_h^+$, and Bar. $(M) \cong i$Bar.$(\mathcal{A} \times \mathcal{A}^{op})_h^+$. There is a map of simplicial categories with duality

$$\beta. : \mathrm{Bar.}(\mathcal{A} \times \mathcal{A}^{op}, \mathcal{A}) \to \mathcal{R}.\mathcal{A},$$

which on n-simplices is given by

$$\prod_{i=1}^{n} n(A_{i'}, A_i), Z \mapsto (B_{i,j})_{n' \le i < j \le n}\ ,$$

where $B_{i,j} = B_{i,i+1} \oplus \cdots \oplus B_{j-1,j}$ (sum in this order) with $B_{i,i+1} = A_i$ for $i < 0'$, with $B_{0',0} = Z$ and $B_{i,i+1} = A_{i+1}^*$ for $i \ge 0$. The maps $B_{i,j} \to B_{k,l}$ are the natural partial inclusions/projections. Hence, we have an induced map of simplicial categories

$$\mathrm{Bar.}(M, X) \cong i\mathrm{Bar.}(\mathcal{A} \times \mathcal{A}^{op}, \mathcal{A})_h^+ \to i(\mathcal{R}.\mathcal{A})_h^+.$$

By Lemma 4.4, this map is degree-wise a homotopy equivalence, hence it is a homotopy equivalence. We have a homotopy equivalence Bar.$(M) \to$ Bar.$(i\mathcal{A}^+) \to (iS.\mathcal{A})^+$ as it is the composition of two homotopy equivalences. The first map is because $M = i(\mathcal{A} \times \mathcal{A}^{op})_h^+ \to i\mathcal{A}^+$ is an equivalence of categories which is induced by the forgetful functor $(\mathcal{A} \times \mathcal{A}^{op}) \to \mathcal{A}$ sending (A, B) to B^*. The second map is by additivity in K-theory applied degree-wise. Using the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \mathrm{Bar.}(M, X) \arrow[r] \arrow[d] & \mathrm{Bar.}(M) \arrow[d] \\
(i\mathcal{A}_h)^+ \arrow[r, "\iota"] & (i(\mathcal{R}.\mathcal{A})_h)^+ \arrow[r, "f"] & (iS.\mathcal{A})^+.
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences and the first row is a homotopy fibration, the statement of the theorem follows.

4.4. LEMMA. *The map* $\beta_n : \mathrm{Bar}_n(\mathcal{A} \times \mathcal{A}^{op}, \mathcal{A}) \to \mathcal{R}_n\mathcal{A}$ *of additive categories with duality induces a homotopy equivalence of Hermitian K-theory spaces if* 2 *is invertible in* $\mathcal{A}$.

4.5. Proof. Suppose for the moment that $\mathcal{A}$ is idempotent complete. We will use Karoubi's fundamental theorem of Hermitian K-theory [7] to prove the lemma. Let ${}_\varepsilon U(\mathcal{A})$ be the homotopy fiber of the hyperbolic functor $K(\mathcal{A}) \to {}_\varepsilon K^{\mathrm{h}}(\mathcal{A})$ and ${}_\varepsilon V(\mathcal{A})$ be the homotopy fiber of the forgetful functor ${}_\varepsilon K^{\mathrm{h}}(\mathcal{A}) \to K(\mathcal{A})$. Then Karoubi's theorem says that there is a natural homotopy equivalence $\Omega_\varepsilon U(\mathcal{A}) \simeq {}_{-\varepsilon}V(\mathcal{A})$ provided that $1/2 \in \mathcal{A}$, at least if $(A, *)$ is of the form $(\mathcal{P}(R), \eta, *)$ as in 3.6 arising from a ring with involution. The general case of idempotent complete additive categories with duality follows from the fact that (Hermitian) K-theory commutes with filtered colimits and A.12 reducing the general case to the case of rings with involution. Let $\alpha : \mathcal{B} \to \mathcal{C}$ be a map between idempotent complete additive categories with duality which induces a homotopy equivalence of K-theory spaces (forgetting the dualities). Using the four exact sequences associated with the above homotopy fibrations (for $\varepsilon = \pm 1$) and the natural homotopy equivalence of Karoubi's fundamental theorem one finds: if ${}_\varepsilon K^{\mathrm{h}}_j(\alpha)$ is an isomorphism and if ${}_\varepsilon K^{\mathrm{h}}_{j+1}(\alpha)$ a surjection for $\varepsilon = \pm 1$, then ${}_\varepsilon K^{\mathrm{h}}_{j+2}$ is an isomorphism for $\varepsilon = \pm 1$.

By the additivity theorem in K-theory, the β_n's are K-theory equivalences (forgetting the dualities). Since β_n has a retraction ρ_n which sends $(B_{i,j})$ to $\Pi_{i=1}^n (B_{i',(i-1)'}, B^*_{i-1,i}), B_{0',0}$ it suffices to show that ${}_\varepsilon K^{\mathrm{h}}_j(\rho_n)$ is an isomorphism for $j = 0, 1$, $\varepsilon = \pm 1$ to get the induction to start, implying ${}_\varepsilon K^{\mathrm{h}}_j(\rho_n)$ an isomorphism for all $j \in \mathbb{N}$. Now ${}_\varepsilon K^{\mathrm{h}}_j(\rho_n)$ is certainly surjective (with section ${}_\varepsilon K^{\mathrm{h}}_j(\beta_n)$). Suppose for the moment that ${}_\varepsilon K^{\mathrm{h}}_0(\rho_n)$ is also injective for $\varepsilon = \pm 1$ and for all idempotent complete additive categories with duality. Then by the above argument ${}_\varepsilon K^{\mathrm{h}}_2(\rho_n)$ is an isomorphism for all idempotent complete additive categories with duality. Applying the same argument to the suspension of ρ_n (see [7], [5]) we find that ${}_\varepsilon K^{\mathrm{h}}_2(S\rho_n)$ is an isomorphism (remark that $\mathcal{R}_n S\mathcal{A} = S\mathcal{R}_n\mathcal{A}$ and $S(\mathcal{A} \times \mathcal{A}^{op}) = S\mathcal{A} \times (S\mathcal{A})^{op}$, etc.). But ${}_\varepsilon K^{\mathrm{h}}_2(S\rho_n) = {}_\varepsilon K^{\mathrm{h}}_1(\rho_n)$ and the induction gets started.

We are left with proving that ${}_\varepsilon K^{\mathrm{h}}_0(\rho_n)$ is injective, or that ${}_\varepsilon K^{\mathrm{h}}_0(\beta_n)$ is surjective. The natural exact sequence $K_0 \to {}_\varepsilon K^{\mathrm{h}}_0 \to {}_\varepsilon W \to 0$ shows that it suffices to prove surjectivity of ${}_\varepsilon W(\beta_n)$. Now the inclusion of zero simplices induces an isomorphism ${}_\varepsilon W(\mathcal{A}) \to {}_\varepsilon W(\mathrm{Bar}_n(\mathcal{A} \times \mathcal{A}^{op}, \mathcal{A}))$ because ${}_\varepsilon W(\mathcal{A} \times \mathcal{A}^{op}) = 0$. It therefore suffices to show that the inclusion of zero simplices induces a surjection ${}_\varepsilon W(\mathcal{A}) \to {}_\varepsilon W(\mathcal{R}_n\mathcal{A})$. Given an isomorphism $\phi : (B_{i,j}) \to (B_{i,j})^* = (B^*_{j',i'})$ with $\phi^* = \varepsilon\phi$, the submodule $(C_{i,j}) \rightarrowtail (B_{i,j})$ is totally isotropic where $C_{i,j} = B_{i,min\{j,0'\}}$ if $i \leqslant min\{j, 0'\}$, otherwise $C_{i,j} = 0$. The inclusion of $C := (C_{i,j})$ into $(B_{i,j})$ is given by the structure maps of the latter object. We calculate $(C^{\perp}/C, \bar{\phi}) = \iota(B_{0',0}, \phi_{0',0})$ with $\iota : \mathcal{A} \to \mathcal{R}_n\mathcal{A}$ the map "inclusion of zero simplices". Hence the surjectivity of $W(\beta_n)$. And we are done in the idempotent complete case.

If $\mathcal{A}$ is not idempotent complete, then we still have an isomorphism ${}_{\varepsilon}K_j^{\mathrm{h}}(\beta_n)$ for $j \geqslant 1$ by cofinality and passing to the idempotent completion of $\mathcal{A}$. But the above calculation includes the isomorphism ${}_{\varepsilon}K_0^{\mathrm{h}}(\beta_n)$ even if $\mathcal{A}$ is not idempotent complete.

4.6. COROLLARY. *Let $\mathcal{A}$ be a split exact category with duality in which 2 it is invertible. Then there is a homotopy fibration*

$$i\mathcal{A}_h^+ \to i(\mathcal{R}.\mathcal{A})_h \to iS.\mathcal{A}.$$

4.7. Proof. This follows from Theorem 4.2 and the commutativity of the diagram

$$\begin{tikzcd} i(\mathcal{R}.\mathcal{A})_h \arrow[r] \arrow[d] & iS.\mathcal{A} \arrow[d] \\ i(\mathcal{R}.\mathcal{A})_h^+ \arrow[r] & (iS.\mathcal{A})^+ \end{tikzcd}$$

in which the vertical arrows are homotopy equivalences by Lemma 2.6.

4.8. PROPOSITION. *Let $\mathcal{A}$ be a split exact category with duality with $\frac{1}{2} \in \mathcal{A}$. Then there is a homotopy fibration*

$$K(\mathcal{A}) \xrightarrow{h} K^{\mathrm{h}}(\mathcal{A}) \xrightarrow{\iota} \mathcal{W}(\mathcal{A})$$

in which h is induced by the hyperbolic functor and ι is the map "inclusion of zero simplices" $i\mathcal{A}_h^+ \to i\mathcal{R}.\mathcal{A}_h^+$ composed with the homotopy equivalences of Lemmas 2.6 and 3.7.

4.9. Proof. Keep the notation of 4.3. The hyperbolic functor is the map $M \to X$ given by the action of M on $0 \in X$. By functoriality we have a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "h"'] & \mathrm{Bar}.(M, M) \arrow[r] \arrow[d, "(h,id)"] & \mathrm{Bar}.(*, M) \arrow[d, "id"] \\ X \arrow[r] & \mathrm{Bar}.(M, X) \arrow[r] & \mathrm{Bar}.(*, M) \end{tikzcd}$$

in which the rows are homotopy fibrations by Lemma 2.2. Hence the left-hand square is homotopy cartesian. By a standard argument, $EM = \mathrm{Bar}.(M, M)$ is contractible. It follows that

$$M \xrightarrow{h} X \to \mathrm{Bar}.(M, X)$$

is a homotopy fibration. Using the identification $\beta : \mathrm{Bar}.(M, X) \to i\mathcal{R}.\mathcal{A}_h^+$ of 3.3 we are done.

Appendix A

We remind the reader that the terms "(exact) category with duality" and "(exact) category with strict duality" have been defined in 3.1. As every symmetric monoidal category is equivalent to a strict symmetric monoidal category so is every (exact) category with duality equivalent to a (exact) category with strict duality. Moreover, every idempotent complete additive category with duality is equivalent to a filtered colimit of additive categories with duality all of which are of the form $(\mathcal{P}(R), \eta, *)$, the category with duality of 3.6 arising from a ring with involution $(R, ^-)$. The purpose of the appendix is to make these statements precise. Lemma A.8 can also be found in [2].

A.1. DEFINITION. A *map of categories with duality* between $(\mathcal{A}, *, \alpha)$ and $(\mathcal{B}, \sharp, \beta)$ is a couple (f, τ) with

(1) $f : \mathcal{A} \to \mathcal{B}$ a functor and
(2) $\tau : f^{op} \circ * => \sharp \circ f$ a natural equivalence such that
(3) for all objects A of $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd} f(A) \arrow[r, "f(\alpha_A)"] \arrow[d, "\beta_{f(A)}"'] & f(A^{**}) \arrow[d, "\tau_{A^*}"] \\ f(A)^{\sharp\sharp} \arrow[r, "\tau_A^{\sharp}"'] & f(A^*)^{\sharp}. \end{tikzcd}$$

Composition is defined by $(g, \sigma) \circ (f, \tau) = (g \circ f, \rho)$ with $\rho_A = \sigma_{f(A)} \circ g(\tau_A)$.

A *map of additive* (*exact*) *categories with duality* (f, τ) *is a map of categories with duality with f an additive* (*exact*) functor.

A map of categories with strict duality is called *strict map* if $f^{op} \circ * = \sharp \circ f$ and if $\tau = id$.

A.2. *Remark*. Composition of maps between (additive, exact) categories with duality is associative. Composition of strict maps is strict.

A.3. DEFINITION. Given two morphisms $(f_i, \tau_i) : (\mathcal{A}, *, \alpha) \to (\mathcal{B}, \sharp, \beta)$, $i = 0, 1$, between categories with duality, a natural transformation $t : (f_0, \tau_0) => (f_1, \tau_1)$ is a natural transformation of functors $t : f_0 => f_1$ such that for all objects A of $\mathcal{A}$ the following diagram commutes:

$$\begin{tikzcd} f_0(A^*) \arrow[r, "\tau_{0,A}"] \arrow[d, "t_{A^*}"'] & f_0(A)^\sharp \\ f_1(A^*) \arrow[r, "\tau_{1,A}"] & f_1(A)^\sharp. \arrow[u, "t_A^\sharp"'] \end{tikzcd}$$

We call t a natural equivalence if for all objects A of $\mathcal{A}$ the map t_A is an isomorphism.

A.4. The above notions define a category *Dualcat* of small categories and maps of categories with duality. There is also a category *sDualcat* of small categories with strict duality and strict maps of categories with duality. There is an inclusion of categories $\imath : sDualcat \to Dualcat$.

A.5. DEFINITION. A map $(f, \tau) : (\mathcal{A}, *, \alpha) \to (\mathcal{B}, \sharp, \beta)$ is called *an equivalence of categories with duality* if there is a map of categories with duality $(g, \sigma) : (\mathcal{B}, \sharp, \beta) \to (\mathcal{A}, *, \alpha)$ such that $(f, \tau) \circ (g, \sigma)$ and $(g, \sigma) \circ (f, \tau)$ are equivalent to $id_{(\mathcal{B}, \sharp, \beta)}$ and $id_{(\mathcal{A}, *, \alpha)}$, respectively.

A.6. LEMMA. *Let $(f, \tau) : (\mathcal{A}, *, \alpha) \to (\mathcal{B}, \sharp, \beta)$ be a map of catagories with duality. If $f : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories then (f, τ) is an equivalence of catagories with duality.*

A.7. *Proof.* This is left to the reader (hint: use an inverse g of f with good properties).

A.8. LEMMA. *There is a strictifying functor $^s : Dualcat \to sDualcat :$ $(\mathcal{A}, *, \alpha) \mapsto (\mathcal{A}^s, *^s, \alpha^s)$ and a natural transformation $(E, \eta) : id => (\imath \circ ^s)$, $(E, \eta)_{(\mathcal{A}, *, \alpha)} = (E^{\mathcal{A}}, \eta^{\mathcal{A}}) : (\mathcal{A}, *, \alpha) \to (\mathcal{A}^s, *^s, \alpha^s)$ which is an equivalence of categories with duality for any category with duality $(\mathcal{A}, *, \alpha)$. In other words, any category with duality is equivalent to a categories and maps between categories with duality can by strictified in a functorial way.*

A.9. *Proof.* The objects of $\mathcal{A}^s$ are two copies of the objects of $\mathcal{A}$: $Ob\mathcal{A}^s = Ob\mathcal{A} \times \{0, 1\}$. We will write A_0 for $(A, 0)$ and A_1 for the object $(A, 1)$. Morphisms in $\mathcal{A}^s$ are defined as follows:

(a) maps from A_0 to B_0 correspond bijectively to maps from A to B in $\mathcal{A}$,
(b) maps from A_1 to B_0 correspond bijectively to maps from A^* to B in $\mathcal{A}$,
(c) maps from A_0 to B_1 correspond bijectively to maps from A to B^* in $\mathcal{A}$,
(d) maps from A_1 to B_1 correspond bijectively to maps from A^* to B^* in $\mathcal{A}$.

Composition is induced by the composition in $\mathcal{A}$. The duality functor $*^s : \mathcal{A}^s \to \mathcal{A}^{sop}$ is defined as follows. On objects it is $A_i^{*^s} = A_{1-i}$.

(a) Let $\varphi : A_0 \to B_0$, i.e., $\varphi : A \to B$ then $\varphi^* : B^* \to A^*$ and we define $\varphi^{*^s} := \varphi^* : B_1 \to A_1$.
(b) Let $\varphi : A_1 \to B_0$, i.e., $\varphi : A^* \to B$ then $\varphi^* : B^* \to A^{**}$ and we define $\varphi^{*^s} := \alpha_A^{-1} \circ \varphi^* : B_1 \to A_0$.
(c) Let $\varphi : A_0 \to B_1$, i.e., $\varphi : A \to B^*$ then $\varphi^* : B^{**} \to A^*$ and we define $\varphi^{*^s} := \varphi^* \circ \alpha_B : B_0 \to A_1$.
(d) Let $\varphi : A_1 \to B_1$, i.e., $\varphi : A^* \to B^*$ then $\varphi^* : B^{**} \to A^{**}$ and we define $\alpha_A^{-1} \circ \varphi^{*^s} \circ \alpha_B := \varphi^* : B_0 \to A_0$.

Observe that the α^{-1}'s always compose on the left and the α's on the right in *a*), ..*d*). This shows that $*^s : \mathcal{A}^s \to \mathcal{A}^{sop}$ is a functor. We calculate $*^s \circ *^s = id$. This is obviously true on objects. On morphisms we have:

(a) Let $\varphi : A_0 \to B_0$ then $\varphi^{*^s *^s} = (\varphi^*)^{*^s} = \alpha_B^{-1} \varphi^{**} \alpha_A = \varphi$ by 2.1 (3).
(b) Let $\varphi : A_1 \to B_0$ then $\varphi^{*^s *^s} = (\alpha_A^{-1} \varphi^*)^{*^s} = \alpha_B^{-1} \varphi^{**} {\alpha_A^*}^{-1} = \varphi \alpha_{A^*}^{-1} {\alpha_A^*}^{-1} = \varphi$ by 2.1 (3) and (4).
(c) Let $\varphi : A_0 \to B_1$ then $\varphi^{*^s *^s} = (\varphi^* \alpha_B)^{*^s} = \alpha_B^* \varphi^{**} \alpha_A = \alpha_B^* \alpha_{B^*} \varphi = \varphi$ by 2.1 (3) and (4).
(d) Let $\varphi : A_1 \to B_1$ then $\varphi^{*^s *^s} = (\alpha_A^{-1} \varphi^{*^s} \alpha_B)^{*^s} = \alpha_B^* \varphi^{**} {\alpha_A^*}^{-1} = \alpha_B^* \varphi^{**} {\alpha_A}^* = \alpha_B^* {\alpha_B}^* \varphi = \varphi$ by 2.1 (3) and (4).

It follows that $(\mathcal{A}^s, *^s, \alpha^s)$ is a category with duality such that $*^s \circ *^s = id$ and $\alpha^s = 1$. Hence $(\mathcal{A}^s, *^s, \alpha^s)$ is a category with duality.

We define a map of categories with duality $(E^{\mathcal{A}}, \eta^{\mathcal{A}}) : (\mathcal{A}, *, \alpha) \to (\mathcal{A}^s, *^s, \alpha^s)$ in the following way. The underlying functor $E^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^s$ sends A to A_0. It is the identity on morphisms. By definition of the composition in $\mathcal{A}^s$, $E^{\mathcal{A}}$ is an additive functor. We let $\eta_A^{\mathcal{A}} = 1_{A^*} : A_0^* \to A_1$. One verifies that $\eta^{\mathcal{A}} : E \circ * => *^s \circ E$ is a natural equivalence satisfying A.1 (3). Since $E^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^s$ is an equivalence of categories, $(E^{\mathcal{A}}, \eta^{\mathcal{A}}) : (\mathcal{A}, *, \alpha) \to (\mathcal{A}^s, *^s, \alpha^s)$ is an equivalence of categories with duality by the above Lemma A.6.

Given a map of categories with duality $(g, \sigma) : (\mathcal{A}, *, \alpha) \to (\mathcal{B}, \sharp, \beta)$, we construct a map $(g, \sigma)^s = (g^s, \sigma^s) : (\mathcal{A}^s, *^s, \alpha^s) \to (\mathcal{B}^s, \sharp^s, \beta^s)$ making the following diagram commute:

$$\begin{tikzcd}
(\mathcal{A}, *, \alpha) \arrow[r, "{(g,\sigma)}"] \arrow[d, "{(E^{\mathcal{A}}, \eta^{\mathcal{A}})}"'] & (\mathcal{B}, \sharp, \beta) \arrow[d, "{(E^{\mathcal{B}}, \eta^{\mathcal{B}})}"] \\
(\mathcal{A}^s, *^s, \alpha^s) \arrow[r, "{(g^s,\sigma^s)}"'] & (\mathcal{B}^s, \sharp^s, \beta^s).
\end{tikzcd}$$

On objects g^s is $A_i \mapsto g(A)_i$. On morphisms it is defined in the following way:

(a) $\varphi : A_0 \to B_0$, i.e., $\varphi : A \to B$ goes to $g(\varphi)$.
(b) $\varphi : A_1 \to B_0$, i.e., $\varphi : A^* \to B$ goes to $g(\varphi) \circ \sigma_A^{-1} : g(A)_1 \to g(B)_0$.
(c) $\varphi : A_0 \to B_1$, i.e., $\varphi : A \to B^*$ goes to $\sigma_B \circ g(\varphi) : g(A)_0 \to g(B)_1$.
(d) $\varphi : A_1 \to B_1$, i.e., $\varphi : A^* \to B^*$ goes to $\sigma_B \circ g(\varphi) \circ \sigma_A^{-1}$.

As above g^s is a functor. We let $\sigma^s = 1$. It is obviously a natural equivalence satisfying A.1 (3). We leave it to the reader to verify the commutativity of (I) and of the fact that strictifying is a functor, *i.e.*, $[(g, \sigma) \circ (f, \tau)]^s = (g, \sigma)^s \circ (f, \tau)^s$.

A.10. *Remark*. For $(\mathcal{E}, \eta, *)$ an (exact) category with (non necessarily strict) duality, one can extend the definitions of this article to obtain the obvious definitions for $\mathcal{E}_h$, $\mathcal{R}.\mathcal{E}$ and $\mathcal{W}(\mathcal{E})$. The category $\mathcal{E}_h$ for instance has objects isomorphisms $\phi : X \to X^*$ such that $\phi = \phi^* \circ \eta_X$ and morphisms $a : \phi \to \phi'$ such that $a : X \to X'$ with $a^* \circ \phi' \circ a = \phi$ (compare 2.3), similarly for the $\mathcal{W}$ and $\mathcal{R}$.-constructions. The natural transformation (E, η) induces homotopy equivalences between the $\mathcal{R}$. and $\mathcal{W}$-constructions of an exact category with duality and of its strictification.

A.11. REMARK. For $(\mathcal{A}, *)$ an (pre-) additive category with strict duality, we can extend the duality of $\mathcal{A}$ to the associated strict additive category $L\mathcal{A}$. Recall that an object of the category $L\mathcal{A}$ is a finite string $n(A_1, \dots, A_n)$ of objects of $\mathcal{A}$. A morphism from $n(A_1, \dots, A_n)$ to $m(B_1, \dots, B_m)$ is a map $(\dots((A_1 \oplus A_2) \oplus A_3) \cdots \oplus A_n) \to (\dots((B_1 \oplus B_2) \oplus B_3) \cdots \oplus B_m)$. We extend the duality on $\mathcal{A}$ by $n(A_1, \dots, A_n)^* = n(A_n^*, \dots, A_1^*)$. The natural inclusion $\mathcal{A} \to L\mathcal{A} : A \mapsto 1(A)$ is a map of categories with strict duality. It is an equivalence if and only if $\mathcal{A}$ is additive.

A.12. LEMMA. *Every idempotent complete additive category with duality is equivalent to a filtered colimit of additive categories with duality all of which are equivalent to* $(\mathcal{P}(R), \eta, *)$ *for some ring with involution* $(R, ^-)$ *and* $* = \mathrm{Hom}_R(\ \ , R)$ *as in 2.6.*

A.13. *Proof.* According to Lemma A.8 and Remark A.11 we can suppose our additive category with duality $(\mathcal{A}, \alpha, *)$ to be strict and to have a strict duality satisfying $(A \oplus B)^* = B^* \oplus A^*$. Call a full subcategory with duality finitely generated if there are finitely many objects in $\mathcal{A}$, called generators, such that it is the full subcategory of objects of $\mathcal{A}$ which are direct factors of finite sums of the generators. The set of all such subcategories becomes a filtered partially ordered set $\mathcal{I}$ under inclusion. Obviously $\mathcal{A} = \mathrm{colim}_{i \in \mathcal{I}} i$. Taking the sum of the generators we see that every category in $\mathcal{I}$ is actually generated by a single object. It therefore suffices to prove the Lemma for categories $\mathcal{A}$ as in A.11 generated by a single object, say A. Let $R = End_{\mathcal{A}}(A \oplus A^*)$. The ring R is equipped with an involution

$$^{-}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^* & b^* \\ c^* & a^* \end{pmatrix}.$$

The inclusion of the one object category with strict duality $(R,^{-})$ into $(\mathcal{A}, *)$ given by the identity on morphisms respects the duality and extends to an equivalence between the idempotent completion of $L(R,^{-})$ and the one of $L(\mathcal{A}, *)$. The latter category with duality is equivalent to $(\mathcal{A}, *)$. The former category is equivalent to $(\mathcal{P}(R), \mathrm{Hom}_R(\ ,R))$. The equivalence is induced by the functor $(id, \tau) : (R,^{-}) \to (\mathcal{P}(R), \mathrm{Hom}_R(\ ,R))$ where the natural equivalence $\tau_R : R \to \mathrm{Hom}_R(R,R)$ sends a to the map $r \mapsto \bar{a}r$. This functor extends to an equivalence between the idempotent completions of the associated strict additive categories with duality (see (A.11) and (A.6)).

Acknowledgments

I would like to thank Daniel Grayson and Jens Hornbostel for the fruitful discussions I had with them and Rick Jardine for pointing out the reference [13].

References

1. Charney, R. and Lee, R.: On a theorem of Giffen, *Mich. Math. J.* **33**: (1986), 169–186.
2. Giffen, C.: Unitary algebraic Witt- and *K*-theories, preprint, (1995).
3. Goerss, P.G. and Jardine, J.F.: *Simplicial Homotopy Theory*, Birkhäuser Verlag, Basel, 1999.
4. Grayson, D.: Higher algebraic *K*-theory. II (after Daniel Quillen), pp. 217–240. Lecture Notes in Math., Vol. 551, 1976.
5. Hornbostel, J. and Schlichting, M.: Localization in hermitian *K*-theory of rings, *J. London Math. Soc.,* To appear.
6. Karoubi, M.: Périodicité de la *K*-théorie hermitienne, In: *Algebraic K-theory, III: Hermitian K-theory and Geometric Applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973, pp. 301–411.
7. Karoubi, M.: Le théorème fondamental de la *K*-théorie hermitienne, *Ann. of Math. (2)*, **112**(2) (1980), 259–282.
8. Keller, B.: Chain complexes and stable categories, *Manuscripta Math.* **67**(4) (1990), 379–417.
9. Knus, M.-A.: *Quadratic and Hermitian Forms Over Rings*, Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni.
10. May, J.P.: E_∞ spaces, group completions, and permutative categories, London Math. Soc. Lecture Note Ser., No. 11, 1974, pp. 61–93.
11. P. May, J., Classifying spaces and fibrations, *Mem. Amer. Math. Soc.* **1**(1, 155) (1975), xiii + 98.
12. Lane, S.M.: *Categories for the Working Mathematician*, Springer-Verlag, New York, second edition, 1998.
13. Moerdijk, I.: Bisimplicial sets and the group-completion theorem, In: *Algebraic K-theory: Connections with Geometry and Topology (Lake Louise, AB, 1987)*, Kluwer Acad. Publ., Dordrecht, 1989, pp. 225–240.

14. Quillen, D.: Higher algebraic K-theory. I. In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973, pp. 85–147.
15. Shapiro, J.M. and Yao, D.: Hermitian U-theory of exact categories with duality functors. *J. Pure Appl. Algebra* 109(3) (1996), 323–330.
16. Thomason, R.W.: First quadrant spectral sequences in algebraic K-theory via homotopy colimits. *Comm. Algebra* 10(15) (1982), 1589–1668.
17. Thomason, R.W. and Trobaugh, T.: Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.
18. Waldhausen, F.: Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, Springer, Berlin, 1985, pp. 318–419.