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A note on K-theory and triangulated categories

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Abstract. We provide an example of two closed model categories having equivalent homotopy categories but different Waldhausen K-theories. We also show that there cannot exist a functor from small triangulated categories to spaces which recovers Quillen's K-theory for exact categories and which satisfies localization.

Introduction

0.1. In [TT90] Thomason and Trobaugh showed that an exact functor of complicial biWaldhausen categories which induces an equivalence of homotopy categories also induces an equivalence of K-theory spaces. In particular, an exact functor between exact categories inducing an equivalence between the associated bounded derived categories also induces an equivalence of K-theory spaces. Thomason then asked whether two Waldhausen categories having equivalent homotopy categories also have the same K-groups.

It was commonly believed that K-theory can not be defined directly from its homotopy category. In [Nee92], Neeman showed there can not be a functor from triangulated categories to spaces which recovers Waldhausen's K-theory. However, the above question remained open. Jeff Smith suggested looking at an example involving Morava K-theory. But to my knowledge this has never been carried out (in print). We give a fairly simple example which proves that the answer to Thomason's question is no.

Several people, for instance Jens Franke and Amnon Neeman, asked whether there is a functor $\mathbb{K}$ from small triangulated categories to spaces satisfying some very natural axioms so that it deserves the name K-theory. If $\mathcal{E}$ is a small exact category we write $K(\mathcal{E})$ for the Quillen K-theory space $\Omega Q\mathcal{E}$ of $\mathcal{E}$. The two axioms we feel are most natural are the following.

1) *Agreement:* For $\mathcal{E}$ a small exact category let $D_b(\mathcal{E})$ be its bounded derived category. Then there is a homotopy equivalence $K(\mathcal{E}) \simeq \mathbb{K}(D_b(\mathcal{E}))$.
2) *Localization:* Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of triangulated categories, *i.e.* $\mathcal{A}$ is equivalent to the full triangulated subcategory of $\mathcal{B}$ whose objects are sent to 0 in $\mathcal{C}$ and $\mathcal{C}$ is the localization of $\mathcal{B}$ with respect to maps whose cone is isomorphic to an object of $\mathcal{A}$. Then there is a homotopy fibration

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{B}) \to \mathbb{K}(\mathcal{C}).$$

We show in Proposition 2.2 that a functor satisfying agreement and localization cannot exist (see also Remark 2.3).

0.2. Before we come to our example we introduce the following notation. Let $\mathcal{C}$ be a Frobenius category, *i.e.* an exact category having enough projectives and injectives, and whose projectives and injectives coincide. Write $\underline{\mathcal{C}}$ for the stable category of $\mathcal{C}$, *i.e.* the category obtained from $\mathcal{C}$ by identifying two morphisms if their difference factors through a projective-injective object. This is a triangulated category (see Sect. 9 of [Hap87] or [Kel96] for details).

If $\mathcal{C}$ is abelian, we define a morphism in $\mathcal{C}$ to be a weak equivalence if it is a stable isomorphism, *i.e.* its image in the stable category is an isomorphism. Define a morphism in $\mathcal{C}$ to be a cofibration if it is a monomorphism. Define a morphism in $\mathcal{C}$ to be a fibration if it is an epimorphism. It is well known that this makes $\mathcal{C}$ into a closed model category in the sense of Quillen ([Qui67]) whose homotopy category is equivalent to its stable category (see Theorem 2.2.12 [Hov99] for the category of modules over a Frobenius ring). We write $m\mathcal{C}$ for this model category in order to distinguish it from the abelian category $\mathcal{C}$.

0.3. Here are the two model categories. Choose your favorite prime number $p \neq 2$. Let R be the ring $(\mathbb{Z}/p)[\varepsilon]/\varepsilon^2$ or $\mathbb{Z}/p^2$. We write $\mathcal{M}(R)$ for the category of finitely generated R-modules. Then $\mathcal{M}(R)$ is an abelian Frobenius category. In Proposition 1.4 we show that the associated stable categories are equivalent as triangulated categories. Hence the two model categories $m\mathcal{M}(R)$, $R = (\mathbb{Z}/p)[\varepsilon]/\varepsilon^2, \mathbb{Z}/p^2$, have equivalent homotopy categories.

We write $K(m\mathcal{C})$ for the Waldhausen K-theory (see [Wal85]) of the category with cofibrations and weak equivalences $m\mathcal{C}$ (forgetting the fibrations). Proposition 1.7 shows that the Waldhausen K-theories of $m\mathcal{M}(R)$, $R = (\mathbb{Z}/p)[\varepsilon]/\varepsilon^2, \mathbb{Z}/p^2$, differ. This relies on the calculations of [EF82] and [ALPS85]. In fact, the groups $K_4(m\mathcal{M}(R))$ are different for the two rings. Hence the answer to Thomason's question.

Our calculations also show that the existence of a functor from small triangulated categories to spaces satisfying agreement and localization contradicts the results of [EF82] and [ALPS85]. This is done in Proposition 2.2.

Nevertheless, the question whether two exact categories having equivalent bounded derived categories do have equivalent K-theories remains open.

1. The two model categories and their K-theories

1.1. Let k be a field, and let $\mathcal{M}(k)$ be the category of finite dimensional vector spaces over k. We endow $\mathcal{M}(k)$ with a trivial structure of a triangulated category as follows. The suspension functor is the identity functor: $\Sigma = id$. A triangle is distinguished if it is the direct sum of trivial triangles, *i.e.* triangles of the form $A \xrightarrow{1} A \longrightarrow 0 \xrightarrow{[+1]} A$ and rotations there of. It is straightforward to verify that this makes $\mathcal{M}(k)$ into a triangulated category. Moreover, every structure of a triangulated category on $\mathcal{M}(k)$ for which $\Sigma \simeq id$ is equivalent to the above one.

1.2. Let k be a field. It is known that $k[\varepsilon]/\varepsilon^2$ is a Frobenius algebra. Hence the category $\mathcal{M}\big(k[\varepsilon]/\varepsilon^2\big)$ of finitely generated $k[\varepsilon]/\varepsilon^2$-modules is a Frobenius category. The ring homomorphism $k[\varepsilon]/\varepsilon^2 \to k$ sending ε to 0 induces a fully faithful functor $\iota : \mathcal{M}(k) \to \mathcal{M}(k[\varepsilon]/\varepsilon^2)$. Since every finitely generated $k[\varepsilon]/\varepsilon^2$-module is a direct sum of objects of $\mathcal{M}(k)$ and of a free module, we see that ι induces an equivalence of categories $\iota : \mathcal{M}(k) \to \underline{\mathcal{M}(k[\varepsilon]/\varepsilon^2)}$. In order to calculate the suspension functor in $\underline{\mathcal{M}(k[\varepsilon]/\varepsilon^2)}$ we choose for every object M in $\mathcal{M}(k)$ an injective hull $M \hookrightarrow E(M)$ in $\mathcal{M}(k[\varepsilon]/\varepsilon^2)$. The suspension of M is then $E(M)/M$. Multiplication by ε on $E(M)$ induces a natural isomorphism $E(M)/M \xrightarrow{\sim} M$. It follows that the suspension is naturally equivalent to the identity functor. Hence, $\iota : \mathcal{M}(k) \to \underline{\mathcal{M}(k[\varepsilon]/\varepsilon^2)}$ is an equivalence of triangulated categories.

1.3. Let $\mathcal{M}(\mathbb{Z}/p^2)$ be the category of finitely generated $\mathbb{Z}/p^2$-modules. The ring $\mathbb{Z}/p^2$ is self injective and every finitely generated $\mathbb{Z}/p^2$-module is a submodule of a finitely generated free $\mathbb{Z}/p^2$-module. Hence $\mathcal{M}(\mathbb{Z}/p^2)$ is a Frobenius category. If we replace "multiplication by ε" in 1.2 by "multiplication by p" then the arguments of 1.2 carry over *mutatis mutandis* showing that the stable category $\underline{\mathcal{M}(\mathbb{Z}/p^2)}$ is equivalent as a triangulated category to $\mathcal{M}(\mathbb{Z}/p)$.

Summarizing we have the following (certainly well known) proposition.

1.4. Proposition. *The homotopy categories of $m\mathcal{M}(\mathbb{Z}/p^2)$ and $m\mathcal{M}(\mathbb{Z}/p[\varepsilon]/\varepsilon^2)$ are both equivalent as triangulated categories to $\mathcal{M}(\mathbb{Z}/p)$.*

1.5. Let R be the ring $\mathbb{Z}/p^2$ or $\mathbb{Z}/p[\varepsilon]/\varepsilon^2$. The categories $m\mathcal{M}(R)$ are biWaldhausen categories (see 1.2.4 [TT90] for a definition) which implies that the K-theory spaces of $m\mathcal{M}(R)$ and $m\mathcal{M}(R)^{op}$ are equivalent (the Waldhausen $S_\cdot$-constructions are isomorphic). That's why taking K-theory with respect to cofibrations or fibrations leads to the same result. The categories $m\mathcal{M}(R)$ both have natural cocylinder objects since they have natural path objects: $R[M] \overset{m \mapsto m}{\twoheadrightarrow} M$ (recall that a finitely generated R-module M is finite). They satisfy the dual of the cylinder axiom, saturation and extension axiom of [Wal85]. Let $\mathcal{P}(R)$ be the category of finitely generated projective R-modules. The biWaldhausen categories $\mathcal{P}(R)$ and $\mathcal{M}(R)$ have (by definition) isomorphisms as weak equivalences, admissible monomorphisms as cofibrations and admissible epimorphisms as fibrations. Now the dual of Theorem 1.6.4 of [Wal85] asserts that the sequence

$$\mathcal{P}(R) \to \mathcal{M}(R) \to m\mathcal{M}(R)$$

of biWaldhausen categories induces a homotopy fibration of K-theory spaces. Moreover, Quillen's *dévissage* theorem [Qui73] shows that $\iota : \mathcal{M}(\mathbb{Z}/p) \to \mathcal{M}(R)$ induces a K-theory equivalence. Hence there is a homotopy fibration

$$K(R) \to K(\mathbb{Z}/p) \to K(m\mathcal{M}(R)).$$

1.6. Quillen's calculation of $K_n(\mathbb{Z}/p)$ (see [Qui72]) gives $K_4(\mathbb{Z}/p) = 0$ and $K_3(\mathbb{Z}/p) = \mathbb{Z}/(p^2-1)$. Hence there is an exact sequence

$$0 \to K_4(m\mathcal{M}(R)) \to K_3(R) \to \mathbb{Z}/(p^2-1).$$

On one hand, the calculations of [EF82] and [ALPS85] show that $K_3(\mathbb{Z}/p^2) = \mathbb{Z}/p^2 \oplus \mathbb{Z}/(p^2-1)$. Therefore, $K_4(m\mathcal{M}(\mathbb{Z}/p^2))$ contains a subgroup which is isomorphic to $\mathbb{Z}/p^2$ (multiplication by p^2 is an automorphism on $\mathbb{Z}/(p^2-1)$ but zero on $\mathbb{Z}/p^2$). On the other hand, the calculations in the two articles also show that $K_3(\mathbb{Z}/p[\varepsilon]/\varepsilon^2) = \mathbb{Z}/p \oplus \mathbb{Z}/p \oplus \mathbb{Z}/(p^2-1)$. Therefore, $K_4(m\mathcal{M}(\mathbb{Z}/p[\varepsilon]/\varepsilon^2))$ cannot have a subgroup which is isomorphic to $\mathbb{Z}/p^2$.

Summarizing we have the following proposition.

1.7. Proposition. *The Waldhausen K-theories of $m\mathcal{M}(\mathbb{Z}/p[\varepsilon]/\varepsilon^2)$ and $m\mathcal{M}(\mathbb{Z}/p^2)$ are not equivalent.*

1.8. Remark. Although the Theorem 1.9.8 of [TT90] is written down only for "complicial biWaldhausen" categories, its proof carries over to Frobenius categories with (co-) cylinder functor satisfying the (dual) of the cylinder axiom. In fact, the proof becomes easier, since no calculus of fractions is involved.

2. Nonexistence of the functor $\mathbb{K}$

2.1. Suppose there were a functor $\mathbb{K}$ from small triangulated categories to spaces satisfying agreement and localization. Keep the notations of 1.5. According to Theorem 2.1 of [Ric89] (which works for Frobenius categories, not only for those which arise from self-injective *k*-algebras), there is an exact sequence of triangulated categories

$$D_b(\mathcal{P}(R)) \to D_b(\mathcal{M}(R)) \to \underline{\mathcal{M}(R)}$$

(see also [KV87]). By agreement, localization and *dévissage*, it follows that there would be a homotopy fibration

$$K(R) \to K(\mathbb{Z}/p) \to \mathbb{K}(\underline{\mathcal{M}(R)}).$$

Proceeding as in 1.6, we find a contradiction to the calculations of [EF82] and [ALPS85] because the triangulated categories $\underline{\mathcal{M}(R)}$ for $R = \mathbb{Z}/p^2$, $\mathbb{Z}/p[\varepsilon]/\varepsilon^2$, are equivalent by 1.4.

Summarizing we have the following proposition.

2.2. Proposition. *There is no functor from small triangulated categories to spaces satisfying agreement and localization.*

2.3. Remark. If we also want additivity (see 3) below) to hold then a weaker form of agreement, namely 1'), and localization lead to a contradiction as well.

1') *Agreement on K_1:* Let $\mathcal{E}$ be a small exact category. Then $K_1(\mathcal{E}) \overset{\sim}{=} \mathbb{K}_1(D_b(\mathcal{E}))$.
3) *Additivity:* Let $F, G, H : \mathcal{S} \to \mathcal{T}$ be three exact functors between triangulated categories such that there is a natural exact triangle $F \to G \to H \to \Sigma F$ then

$$0 = \mathbb{K}_*(F) - \mathbb{K}_*(G) + \mathbb{K}_*(H) : \mathbb{K}_*(\mathcal{S}) \to \mathbb{K}_*(\mathcal{T}).$$

There is no functor satisfying agreement on K_1, localization and additivity. Suppose on the contrary that a functor $\mathbb{K}$ satisfying 1'), 2) and 3) exists. By additivity, Σ acts on the $\mathbb{K}$-groups as -1. The fact that $\Sigma \simeq id$ for the stable module categories $\underline{\mathcal{M}(R)}$, $R = \mathbb{Z}/p^2$ or $k[\varepsilon]/\varepsilon^2$, then implies that the groups $\mathbb{K}_*(\underline{\mathcal{M}(R)})$ are 2-torsion. Agreement on K_1 shows that $\mathbb{K}_1(\mathcal{P}(R)) = R^\times = \mathbb{Z}/p \oplus \mathbb{Z}/(p-1)$ and $\mathbb{K}_1(\mathcal{M}(R)) = \mathbb{Z}/(p-1)$. So the map $\mathbb{K}_1(\mathcal{P}(R)) \to \mathbb{K}_1(\mathcal{M}(R))$ sends the factor $\mathbb{Z}/p$ to zero. Using localization as before, we see that $\mathbb{Z}/p$ has to be 2-torsion which is clearly wrong for $p > 2$.

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