Chapter 3

Getting Acquainted with Intersection Forms

We define the intersection form of a 4–manifold, which governs intersections of surfaces inside the manifold. We start by representing every homology 2–class by an embedded surface, then, in section 3.2 (page 115), we explore the properties of the intersection form. Among them is unimodularity, which is essentially equivalent to Poincaré duality. An important invariant of an intersection form is its signature, and we discuss how its vanishing is equivalent to the 4–manifold being a boundary of a 5–manifold. After listing a few simple examples of 4–manifolds and their intersection form, in section 3.3 (page 127) we present in some detail the important example of the $K3$ manifold.

Given any closed oriented 4-manifold M, its **intersection form** is the symmetric 2–form defined as follows:

$$\begin{aligned} Q_M\colon H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ Q_M(\alpha,\beta) &= (\alpha \cup \beta)[M] \,. \end{aligned}$$

This form is bilinear[1] and is represented by a matrix of determinant ± 1. While over $\mathbb{R}$ this is a recipe for boredom, since this intersection form is defined over the *integers* (and thus changes of coordinates must be made only through integer-valued matrices), our Q_M is a quite far-from-trivial object.

1. Notice that Q_M vanishes on any torsion element, and thus can be thought of as defined on the free part of $H^2(M;\mathbb{Z})$; since our manifolds are assumed simply-connected, torsion is not an issue.

For convenience, we will often denote $Q_M(\alpha, \beta)$ by $\alpha \cdot \beta$. Further, we will identify without comment a cohomology class $\alpha \in H^2(M; \mathbb{Z})$ with its Poincaré-dual homology class $\alpha \in H_2(M; \mathbb{Z})$.

For defining Q_M more geometrically,[2] we will represent classes α and β from $H_2(M; \mathbb{Z})$ by embedded surfaces S_α and S_β, and then equivalently define $Q_M(\alpha, \beta)$ as the intersection number of S_α and S_β:

$$Q_M(\alpha, \beta) = S_\alpha \cdot S_\beta \ .$$

First, though, we need to argue that any class $\alpha \in H_2(M; \mathbb{Z})$ can indeed be represented by a smoothly embedded surface S_α:

3.1. Preparation: representing homology by surfaces

It is known from general results[3] that every homology class of a 4–manifold can be represented by embedded submanifolds. Nonetheless, we present a direct argument for the case of 2–classes, owing to the useful techniques that it exhibits.

Simply-connected case. Assume first that M is simply-connected. Then by Hurewicz's theorem $\pi_2(M) \approx H_2(M; \mathbb{Z})$, and hence all homology classes of M can be represented as images of maps $f\colon S^2 \to M$. The latter can always be perturbed to yield immersed spheres, whose only failures from being embedded are transverse double-points. These double-points can be eliminated at the price of increasing the genus.

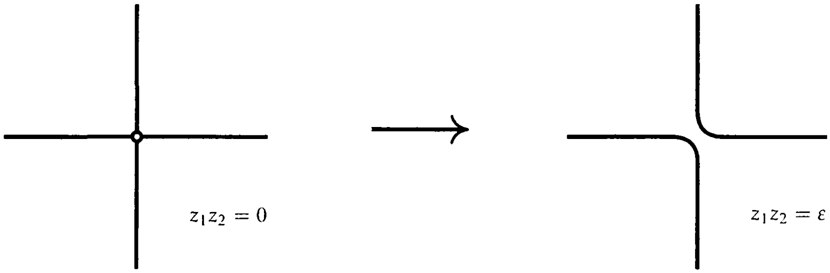
For example, by using complex coordinates, a double-point is isomorphic to the simple nodal singularity of equation $z_1 z_2 = 0$ in $\mathbb{C}^2$: the complex planes $z_1 = 0$ and $z_2 = 0$ meeting at the origin. It can be eliminated by perturbing to $z_1 z_2 = \varepsilon$, as suggested in figure 3.1 on the facing page. (A simple change of coordinates transforms the situation into perturbing $w_1^2 + w_2^2 = 0$ to $w_1^2 + w_2^2 = \varepsilon$.)

More geometrically, imagine two planes meeting orthogonally at the origin of $\mathbb{R}^4$. Their traces in the 3–sphere S^3 are two circles, linking once.[4] We can eliminate the singularity if we discard the portions contained in the open 4–ball bounded by S^3, and instead connect the two circles in S^3 by an annular

2. *"Think with intersections, prove with cup-products."*

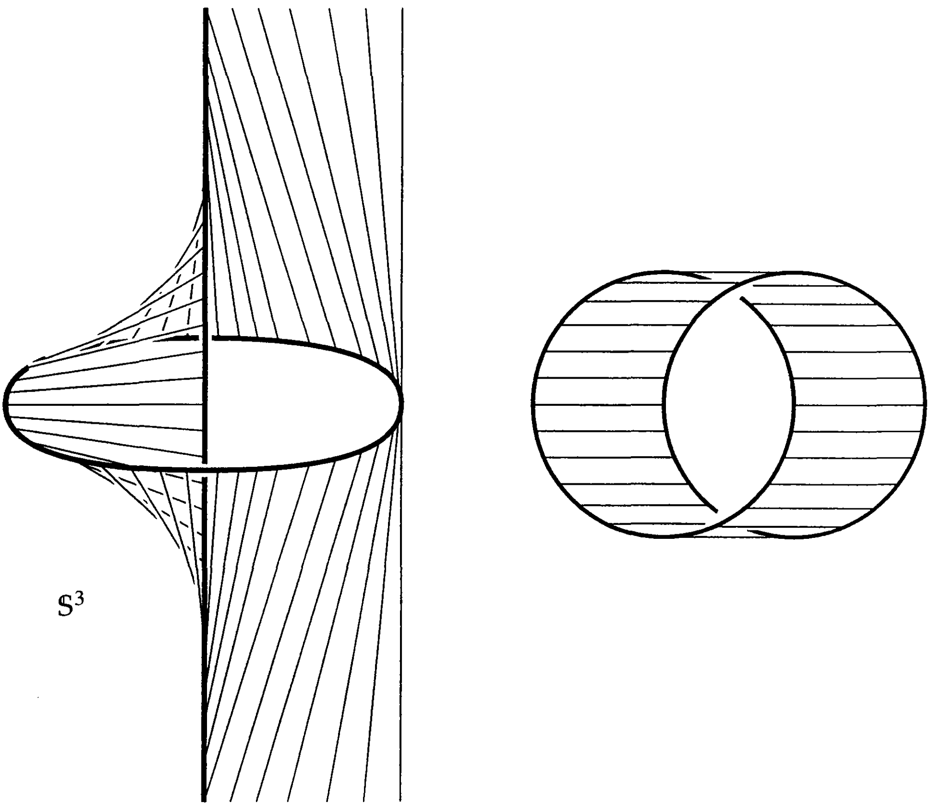
3. For example, for any smooth oriented X^m and any $\alpha \in H^*(X; \mathbb{Z})$, there is some integer k so that $k\alpha$ can be represented by an embedded submanifold; if α has dimension at most 8 or codimension at most 2, then it can be represented directly by a submanifold; if X^m is embedded in $\mathbb{R}^{m+2}$, then X is the boundary of an oriented smooth $(m+1)$–submanifold in $\mathbb{R}^{m+2}$. These results were announced in **R. Thom's** ***Sous-variétés et classes d'homologie des variétés différentiables*** **[Tho53a]** and proved in his celebrated ***Quelques propriétés globales des variétés différentiables*** **[Tho54]**.

4. Think: fibers of the Hopf map $S^3 \to \mathbb{CP}^1$; the Hopf map will be recalled in footnote 34 on page 129.



3.1. Eliminating a double-point, I: complex coordinates

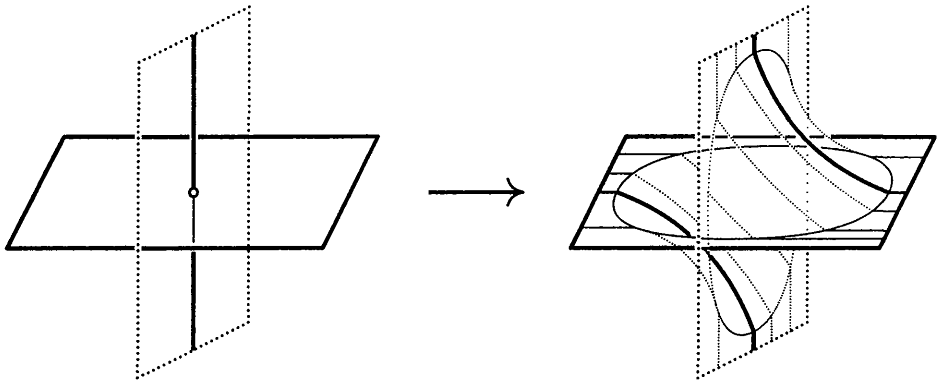
sheet, as suggested in figure[5] 3.2. Thus, we replaced two disks meeting at the double-point by an annulus. A 4–dimensional image is attempted in figure[6] 3.3 on the following page.



3.2. Eliminating a double-point, II: annulus

5. On the left of figure 3.2, one circle is drawn as a vertical line through ∞, after setting $S^3 = \mathbb{R}^3 \cup \infty$.

6. As usual, in figure 3.3, dotted lines represent creatures escaping in the fourth dimension.



3.3. Eliminating a double-point, III

Either way, we can eliminate all double-points of the immersed sphere, and the result is then an embedded surface representing that homology class. Thus, all homology classes can be represented by embedded surfaces, but rarely by spheres.

The failure to represent homology classes by smoothly embedded spheres is of course related to the failure of smoothly embedding disks. The natural question to ask is then: what is the minimum genus needed to represent a given homology class? We will come back to this question later.[7]

In general. The method above only works for simply-connected M^4's. An argument for general 4–manifolds has two equivalent versions:

(1) Since $\mathbb{CP}^\infty$ is an Eilenberg–Maclane $K(\mathbb{Z},2)$–space,[8] it follows that the elements of $H^2(M;\mathbb{Z})$ correspond to homotopy classes of maps $M \to \mathbb{CP}^\infty$. Since M is 4–dimensional, such maps can be slid off the high-dimensional cells of $\mathbb{CP}^\infty$ and thus reduced to maps $M \to \mathbb{CP}^2$. For any class $\alpha \in H^2(M;\mathbb{Z})$, pick a corresponding $f_\alpha \colon M \to \mathbb{CP}^2$ and arrange it to be differentiable and transverse to $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Then $f_\alpha^{-1}[\mathbb{CP}^1]$ is a surface Poincaré-dual to α.

(2) Equivalently, since $\mathbb{CP}^\infty$ coincides with the classifying space[9] $\mathscr{B}U(1)$ of the group $U(1)$, classes in $H^2(M;\mathbb{Z})$ correspond to complex line bundles on M, with α being paired to L_α whenever $c_1(L_\alpha) = \alpha$. If we pick a

7. See ahead, chapter 11 (starting on page 481).

8. An **Eilenberg–Maclane $K(G,m)$–space** is a space whose *only* non-zero homotopy group is $\pi_m = G$; such a space is unique up to homotopy-equivalence. It can be used to represent cohomology as $H^m(X;G) = [X; K(G,m)]$, where $[A;B]$ denotes the set of homotopy classes of maps $A \to B$.

9. A **classifying space** $\mathscr{B}G$ for a topological group G is a space so that $[X;\mathscr{B}G]$ coincides with the set of isomorphisms classes of G–bundles over X. A bit more on classifying spaces is explained in the end-notes of the next chapter (page 204).

generic section σ of L_α, then its zero set $\sigma^{-1}[0]$ will be an embedded surface Poincaré-dual to α.

3.2. Intersection forms

Given a closed oriented 4-manifold M, we defined its **intersection form** as

$$Q_M\colon H_2(M;\mathbb{Z})\times H_2(M;\mathbb{Z})\longrightarrow\mathbb{Z}\qquad Q_M(\alpha,\beta)=S_\alpha\cdot S_\beta\,,$$

where S_α and S_β are any two surfaces representing the classes α and β.

Notice that, if M is simply-connected, then $H_2(M;\mathbb{Z})$ is a free $\mathbb{Z}$–module and there are isomorphisms $H_2(M;\mathbb{Z})\approx\oplus m\,\mathbb{Z}$, where $m=b_2(M)$. If M is not simply-connected, then $H_2(M;\mathbb{Z})$ inherits the torsion of $H_1(M;\mathbb{Z})$, but by linearity the intersection form will always vanish on these torsion classes; thus, when studying intersection form, we can safely pretend that $H_2(M;\mathbb{Z})$ is always free.

Lemma. *The form $Q_M(\alpha,\beta)=S_\alpha\cdot S_\beta$ on $H_2(M;\mathbb{Z})$ coincides modulo Poincaré duality with the pairing $Q_M(\alpha^*,\beta^*)=(\alpha^*\cup\beta^*)[M]$ on $H^2(M;\mathbb{Z})$.*

Proof. Given any class $\alpha\in H_2(M;\mathbb{Z})$, denote by α^* its Poincaré-dual from $H^2(M;\mathbb{Z})$; we have $\alpha^*\cap[M]=\alpha$. We wish to show that the pairing

$$Q_M(\alpha^*,\beta^*)=(\alpha^*\cup\beta^*)[M]$$

on $H^2(M;\mathbb{Z})$ defines the same bilinear form as the one defined above.

We use the general formula[10] $(\alpha^*\cup\beta^*)[M]=\alpha^*\big[\beta^*\cap[M]\big]$, from which it follows that $Q_M(\alpha^*,\beta^*)=\alpha^*[\beta]$, or

$$Q_M(\alpha^*,\beta^*)=\alpha^*[S_\beta]\,.$$

Therefore, we need to show that

$$\alpha^*[S_\beta]=S_\alpha\cdot S_\beta\,.$$

Since Q_M vanishes on torsion classes, it is enough to check the last formula by including the free part of $H^2(M;\mathbb{Z})$ into $H^2(M;\mathbb{R})$ and by interpreting the latter as the de Rham cohomology of exterior 2–forms.

Moving into de Rham cohomology translates cup products into wedge products and cohomology/homology pairings into integrations. We have, for example,

$$Q_M(\alpha^*,\beta^*)=\int_M\alpha^*\wedge\beta^*\qquad\text{and}\qquad\alpha^*[S_\beta]=\int_{S_\beta}\alpha^*$$

for all 2–forms $\alpha^*,\beta^*\in\Gamma\big(\Lambda^2(T^*_M)\big)$.

10. More often written in terms of the Kronecker pairing as $\langle\alpha^*\cup\beta^*,\,[M]\rangle=\langle\alpha^*,\,\beta^*\cap[M]\rangle$.

In this setting, given a surface S_α, one can find a 2–form α^* dual to S_α so that it is non-zero only close to S_α. Further, one can choose some local oriented coordinates $\{x_1, x_2, y_1, y_2\}$ so that S_α coincides locally with the plane $\{y_1 = 0;\ y_2 = 0\}$, oriented by $dx_1 \wedge dx_2$. One can then choose α^* to be locally written $\alpha^* = f(x_1, x_2)\, dy_1 \wedge dy_2$, for some suitable bump-function f on $\mathbb{R}^2$, supported only around $(0,0)$ and with integral $\int_{\mathbb{R}^2} f = 1$.

If S_β is some surface transverse to S_α and we arrange that, around the intersection points of S_α and S_β, we have S_β described by $\{x_1 = 0;\ x_2 = 0\}$, then clearly

$$\int_{S_\beta} \alpha^* = S_\alpha \cdot S_\beta \,,$$

with each intersection point of S_α and S_β contributing ± 1 depending on whether $dy_1 \wedge dy_2$ orients S_β positively or not.[11] □

Unimodularity and dual classes

The intersection form Q_M is $\mathbb{Z}$*–bilinear* and *symmetric*. As a consequence of Poincaré duality, the form Q_M is also **unimodular**, meaning that the matrix representing Q_M is invertible over $\mathbb{Z}$. This is the same as saying that

$$\det Q_M = \pm 1 \,.$$

Unimodularity is further equivalent to the property that, for every $\mathbb{Z}$–linear function $f\colon H_2(M;\mathbb{Z}) \to \mathbb{Z}$, there exists a unique $\alpha \in H_2(M;\mathbb{Z})$ so that $f(x) = \alpha \cdot x$.

Lemma. *The intersection form Q_M of a 4–manifold is unimodular.*

Proof. The intersection form is unimodular if and only if the map

$$\begin{aligned} \widehat{Q}_M\colon H_2(M;\mathbb{Z}) &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}),\ \mathbb{Z}) \\ \alpha &\longmapsto x \mapsto \alpha \cdot x \end{aligned}$$

is an isomorphism. We will argue that this last map coincides with the Poincaré duality morphism. Indeed, Poincaré duality is the isomorphism

$$\begin{aligned} H_2(M;\mathbb{Z}) &\xrightarrow[\approx]{} H^2(M;\mathbb{Z}) \\ \alpha &\longmapsto \alpha^* \,, \end{aligned}$$

with α^* characterized by $\alpha^* \cap [M] = \alpha$. Assume for simplicity that $H_2(M;\mathbb{Z})$ is free.[12] Then the universal coefficient theorem[13] shows that

11. See **R. Bott** and **L. Tu**'s ***Differential forms in algebraic topology*** **[BT82]** for more such play with exterior forms.

12. If not free, a similar argument is made on the free part $H^2(M;\mathbb{Z}) / \operatorname{Ext}(H_1(M;\mathbb{Z});\mathbb{Z})$ of $H^2(M;\mathbb{Z})$, which is all that matters since Q_M vanishes on torsion.

13. The universal coefficient theorem was recalled on page 15.

we have an isomorphism

$$\begin{array}{ccc} H^2(M;\mathbb{Z}) & \xrightarrow[\approx]{} & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\ \mathbb{Z}\big) \\ \alpha^* & \longmapsto & x \mapsto \alpha^*[x]\,. \end{array}$$

Combining Poincaré duality with the latter yields the isomorphism

$$\begin{array}{ccc} H_2(M;\mathbb{Z}) & \xrightarrow[\approx]{} & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\ \mathbb{Z}\big) \\ \alpha & \longmapsto & x \mapsto \alpha^*[x]\,. \end{array}$$

However, as argued in the preceding subsection, we have $Q_M(\alpha, x) = \alpha^*[x]$, and therefore the above isomorphism coincides with the map $\widehat{Q}_M$. That proves that the intersection form Q_M is unimodular. □

Further, the unimodularity of Q_M is equivalent to the fact that, for every basis $\{\alpha_1, \ldots, \alpha_m\}$ of $H_2(M;\mathbb{Z})$, there is a unique **dual basis** $\{\beta_1, \ldots, \beta_m\}$ of $H_2(M;\mathbb{Z})$ so that $\alpha_k \cdot \beta_k = +1$ and $\alpha_i \cdot \beta_j = 0$ if $i \neq j$.

To see this, start with the basis $\{\alpha_1, \ldots, \alpha_m\}$ in $H_2(M;\mathbb{Z})$, pick the familiar dual basis[14] *$\{\alpha_1^*, \ldots, \alpha_m^*\}$ in the dual $\mathbb{Z}$–module* $\mathrm{Hom}(H_2(M;\mathbb{Z}),\ \mathbb{Z})$*, then transport it back to $H_2(M;\mathbb{Z})$ by using Poincaré duality (or $\widehat{Q}_M$) and hence obtain the desired basis $\{\beta_1, \ldots, \beta_m\}$.*

In particular, for every *indivisible* class α (*i.e.*, not a multiple), there exists at least one **dual class** β such that $\alpha \cdot \beta = +1$: complete α to a basis and proceed as above. (Of course, such β's are *not* unique: once you find one, you can obtain others by adding any γ with $\alpha \cdot \gamma = 0$.)

Intersection forms and connected sums

The simplest way of combining two 4–manifolds yields the the simplest way of combining two intersection forms. First, a bit of review:

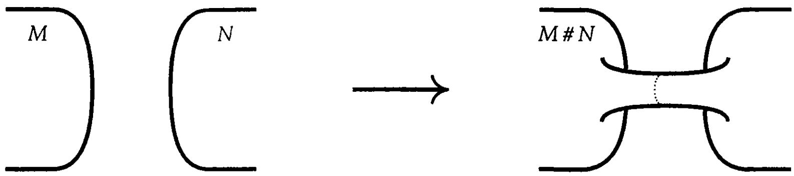
Remembering connected sums. The **connected sum** of two manifolds M and N, denoted by

$$M \# N\,,$$

is the simplest method for combining M and N into one connected manifold, by joining them with a tube as sketched in figure 3.4 on the next page. Notice that the 4–sphere is an identity element for connected sums: $M \# S^4 \cong M$.

Connected sums are described more rigorously by choosing in each of M and N a small open 4–ball and removing it to get two manifolds M° and N°, each with a 3–sphere as boundary, then identifying these 3–spheres to obtain the closed manifold $M \# N$.

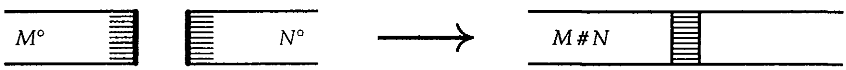
14. Recall that, given a basis $\{e_1, \ldots, e_m\}$ in a module Z, the dual basis $\{e_1^*, \ldots, e_m^*\}$ in Z^* is specified by setting $e_k^*(e_k) = 1$ and $e_i^*(e_j) = 0$ for $i \neq j$.



3.4. The connected sum of two manifolds, I

More about connected sums. *The identification of the two 3–spheres must be made through an orientation-reversing diffeomorphism $\partial M^\circ \cong \overline{\partial N^\circ}$, as was mentioned on page 13. Indeed, if M and N are oriented, then the new boundary 3–spheres will inherit orientations. In order that the orientations of M and N be nicely compatible with an orientation of M # N, we must identify the 3–spheres with an orientation flip.*

Furthermore, to ensure that M # N is a smooth manifold, this gluing must be done as follows: Choose open 4-balls in M and N, then remove them. Embed copies of $S^3 \times [0,1]$ as collars to the new boundary 3–spheres. Take care to embed these collars so that, on the side of M, the sphere $S^3 \times 1$ be sent onto ∂M°, with $S^3 \times [0,1)$ going into the interior of M°. On the N side, $S^3 \times 0$ should be sent onto ∂N° and $S^3 \times (0,1]$ into the interior of N°. Now identify the two collars $S^3 \times [0,1]$ in the obvious manner and thus obtain M # N, as in figure 3.5. This automatically forces the boundary-spheres to be identified "inside-out", reversing orientations, and further makes it clear that M # N is smooth.[15] See figure 3.6 on the next page. The equivalence of this procedure with "joining by a tube" is explained in figure 3.7 on the facing page.



3.5. Gluing by identifying collars

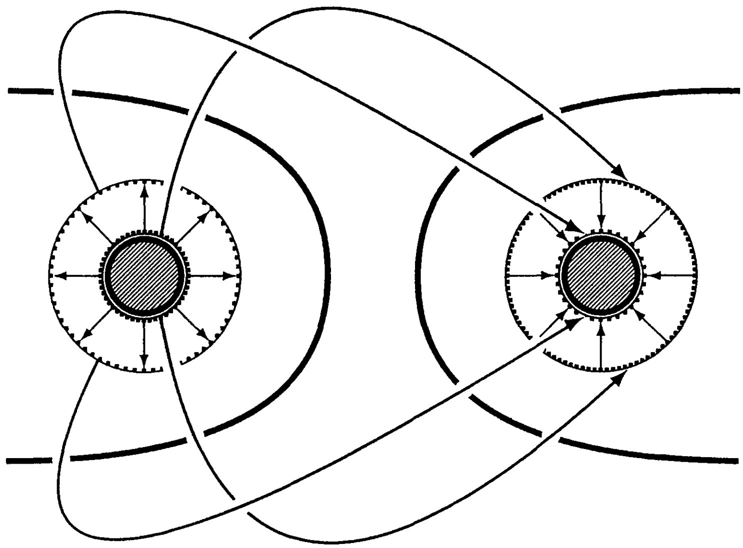
Sums and forms. This connected sum operation is nicely compatible with intersection forms:

Lemma. *If M and N have intersection forms Q_M and Q_N, then their connected sum M # N will have intersection form*

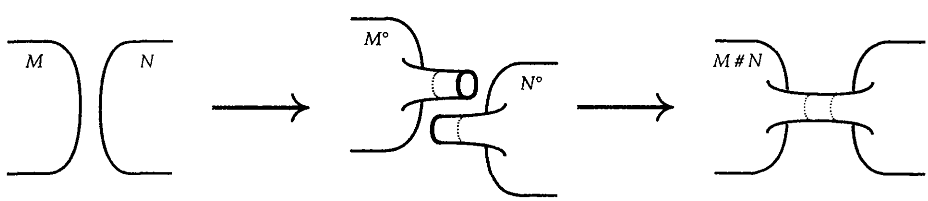
$$Q_{M\#N} = Q_M \oplus Q_N \,.$$

Proof. Since M° and N° can be viewed as M and N without a 4–handle (or a 4–cell), and since 2–homology is influenced only by 1–, 2– and 3–handles, it follows that the 2–homology of $M\#N$ will merely be the friendly gathering of the 2–homologies of M and N, intersections and all. □

15. In fact, each time you read "*A and B both have the same boundary, so we glue A and B along it*", you should understand that the "gluing" is done *via* an orientation-reversing diffeomorphism $\partial A \cong \overline{\partial B}$, and that a collaring procedure as above is used. This was already explained on page 13. For more on the foundation of these gluings, read from **M. Hirsch's *Differential topology*** [**Hir94**, sec 8.2].



3.6. The connected sum of two manifolds, II



3.7. The connected sum of two manifolds, III

Topological heaven. For topological 4–manifolds a converse is true:

Theorem (*M. Freedman*). *If M is simply-connected and Q_M splits as a direct sum $Q_M = Q' \oplus Q''$, then there exist* topological *4–manifolds N' and N'' with intersection forms Q' and Q'' such that $M = N' \# N''$.* □

This is a direct consequence of Freedman's classification that we will present later.[16] *Such a result certainly fails in the smooth case, and its failure spawns exotic*[17] *$\mathbb{R}^4$'s.*

Invariants of intersection forms

To start to distinguish between the various possible intersection forms, we define the following simple algebraic invariants:

16. See ahead section 5.2 (page 239). For a more refined topological sum-splitting result, we refer to **M. Freedman** and **F. Quinn**'s ***Topology of 4–manifolds*** **[FQ90**, ch 10].

17. See ahead section 5.4 (page 250).

— The **rank** of Q_M:

It is the size of Q_M's domain, defined simply as

$$\operatorname{rank} Q_M = \operatorname{rank}_{\mathbb{Z}} H^2(M;\mathbb{Z}) ,$$

or $\operatorname{rank} Q_M = \dim_{\mathbb{R}} H^2(M;\mathbb{R})$. In other words, the rank is the second Betti number $b_2(M)$ of M.

— The **signature** of Q_M:

It is obtained as follows: first diagonalize Q_M as a matrix over $\mathbb{R}$ (or $\mathbb{Q}$), separate the resulting positive and negative eigenvalues, then subtract their counts; that is

$$\operatorname{sign} Q_M = \dim H^2_+(M;\mathbb{R}) - \dim H^2_-(M;\mathbb{R}) ,$$

where $H^2_\pm$ are any maximal positive/negative-definite subspaces for Q_M. We can set partial Betti numbers $b_2^\pm = \dim H^2_\pm$, and thus we can read $\operatorname{sign} Q_M = b_2^+(M) - b_2^-(M)$.

— The **definiteness** of Q_M (*definite* or *indefinite*):

If for all non-zero classes α we always have $Q_M(\alpha,\alpha) > 0$, then Q_M is called **positive definite.**

If, on the contrary, we have $Q_M(\alpha,\alpha) < 0$ for all non-zero α's, then Q_M is called **negative definite.**

Otherwise, if for some α_+ we have $Q_M(\alpha_+, \alpha_+) > 0$ and for some α_- we have $Q_M(\alpha_-, \alpha_-) < 0$, then Q_M is called **indefinite.**

— The **parity** of Q_M (*even* or *odd*):

If, for all classes α, we have that $Q_M(\alpha,\alpha)$ is even, then Q_M is called **even.** Otherwise, it is called **odd.** Notice that it is enough to have *one* class with odd self-intersection for Q_M to be called odd.

Signatures and bounding 4–manifolds

A first remark is that signatures are *additive:* $\operatorname{sign}(Q' \oplus Q'') = \operatorname{sign} Q' + \operatorname{sign} Q''$. In particular,[18]

$$\operatorname{sign}(M \# N) = \operatorname{sign} M + \operatorname{sign} N .$$

Another remark is that changing the orientation of M will change the sign of the signature:

$$\operatorname{sign} \overline{M} = -\operatorname{sign} M ,$$

since it obviously changes the sign of its intersection form: $Q_{\overline{M}} = -Q_M$.

18. The additivity of signatures still holds for gluings $M \cup_\partial N$ more general than connected sums. This result *(Novikov additivity)* and an outline of its proof can be found in the the end-notes of the next chapter (page 224).

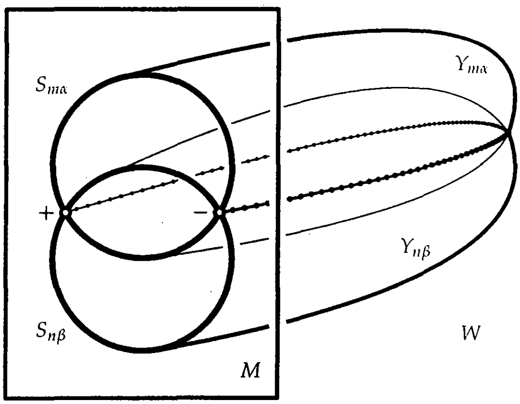
The signature vanishes for boundaries. More remarkably, the vanishing of the signature of a 4–manifold M has a direct topological interpretation:

Lemma. *If M^4 is the boundary of some oriented 5–manifold W^5, then*

$$\operatorname{sign} Q_M = 0 .$$

Proof. Since the signature appears after diagonalizing over some field, we will work here with homology with rational coefficients. Thus, denote by $\iota\colon H_2(M;\mathbb{Q}) \to H_2(W;\mathbb{Q})$ the morphism induced from the inclusion of M^4 as the boundary of W^5.

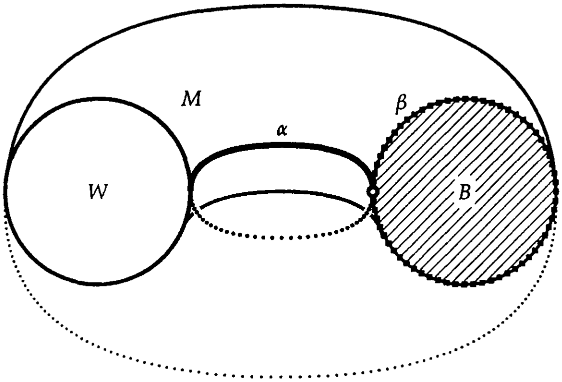
If bounding. First, we claim that if both $\alpha, \beta \in H_2(M;\mathbb{Q})$ have $\iota\alpha = 0$ and $\iota\beta = 0$ then their intersection must be $\alpha \cdot \beta = 0$. Indeed, since α and β are rational, some of their multiples $m\alpha$ and $n\beta$ will be integral. Then $m\alpha$ and $n\beta$ can be represented by two embedded surfaces $S_{m\alpha}$ and $S_{n\beta}$ in M. Since $\iota\alpha = 0$ and $\iota\beta = 0$, this implies that $S_{m\alpha}$ and $S_{n\beta}$ will bound two oriented 3–manifolds $Y_{m\alpha}$ and $Y_{n\beta}$ inside W. The intersection number $\alpha \cdot \beta$ is determined by counting the intersections of the surfaces $S_{m\alpha}$ and $S_{n\beta}$, then dividing by mn. However, the intersection of $Y^3_{m\alpha}$ and $Y^3_{n\beta}$ inside W^5 consists of arcs, which connect pairs of intersection points of $S_{m\alpha}$ and $S_{n\beta}$ with opposite signs, as pictured in figure 3.8. It follows that $S_{m\alpha} \cdot S_{n\alpha} = 0$, and therefore $\alpha \cdot \beta = 0$, as claimed.



3.8. Bounding surfaces have zero intersection

If not bounding. Second, we claim that for every $\alpha \in H_2(M;\mathbb{Q})$ with $\iota\alpha \neq 0$ there must be some $\beta \in H_2(M;\mathbb{Q})$ so that $\alpha \cdot \beta = +1$ but $\iota\beta = 0$.

To see that, we notice that, since $\iota\alpha \neq 0$ in $H_2(W;\mathbb{Q})$, there exists a 3–class $B \in H_3(W,\partial W;\mathbb{Q})$ that is *dual*[19] to our $\iota\alpha \in H_2(W;\mathbb{Q})$, *i.e.*, has $\alpha \cdot B = +1$ in W^5. Its boundary $\partial B = \beta$ is a class in $H_2(M;\mathbb{Q})$, and we have that $\alpha \cdot \beta = \iota\alpha \cdot B = +1$ and also that $\iota\beta = 0$. See figure 3.9.



3.9. A non-bounding class has a bounding dual

Unravel the form. Finally, we are ready to attack the actual intersection form of M. Any class α that bounds in W, *i.e.*, has $\iota\alpha = 0$, must have zero self-intersection $\alpha \cdot \alpha = 0$. We are thus more interested in classes α that do not bound.

Assume we choose some $\alpha \in H_2(M;\mathbb{Q})$ so that $\iota\alpha \neq 0$. Then there is some $\beta \in H_2(M;\mathbb{Q})$ so that $\alpha \cdot \beta = +1$, while $\iota\beta = 0$, and thus $\beta \cdot \beta = 0$. Therefore the part of Q_M corresponding to $\{\alpha, \beta\}$ has matrix

$$Q_{\alpha\beta} = \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix},$$

which has determinant -1 and diagonalizes over $\mathbb{Q}$ as $[+1] \oplus [-1]$.

Since Q_M is unimodular, this means that Q_M must actually split as a direct sum $Q_M = Q_{\alpha\beta} \oplus Q^{\perp}$ for some unimodular form $Q^{\perp}$ defined on a complement of $\mathbb{Q}\{\alpha,\beta\}$ in $H_2(M;\mathbb{Q})$. Since the signature is additive and one can see that $\operatorname{sign} Q_{\alpha\beta} = 0$, we deduce that we must have $\operatorname{sign} Q_M = \operatorname{sign} Q^{\perp}$.

We continue this procedure for $Q^{\perp}$, splitting off 2–dimensional pieces until there are no more classes α with $\iota\alpha \neq 0$ left. Then whatever is still there has to bound in W, and hence cannot contribute to the signature. Therefore $\operatorname{sign} Q_M = 0$. □

19. A reasoning analogous to the one we made earlier for Q_M applies to the intersection pairing $H_2(W;\mathbb{Z}) \times H_3(W,\partial W;\mathbb{Z}) \to \mathbb{Z}$. In particular, it is unimodular, and thus we have dual classes; since we work over $\mathbb{Q}$, the indivisibility of α is not required.

A consequence of this result is that, whenever two 4–manifolds can be linked by a cobordism, they must have the same signature. Indeed, if $\partial W = \overline{M} \cup N$, then $0 = \operatorname{sign}(\overline{M} \cup N) = -\operatorname{sign} M + \operatorname{sign} N$. That is:

Corollary. *If two manifolds are cobordant, then they have the same signature. Signature is a cobordism invariant.* □

The signature vanishes only for boundaries. A result quite more difficult to prove is the following:

Theorem (*V. Rokhlin*). *If a smooth oriented 4–manifold M has*

$$\operatorname{sign} Q_M = 0 ,$$

then there is a smooth oriented 5–manifold W such that $\partial W = M$.

> ***Idea of proof.*** A classic result of Whitney assures that any manifold X^m can be immersed in $\mathbb{R}^{2m-1}$; in particular, our M^4 can be immersed in $\mathbb{R}^7$. By performing various surgery modifications, we then arrange that M be cobordant to a 4–manifold M' that embeds in $\mathbb{R}^6$. Furthermore, a result of R. Thom[20] implies that M' must bound a 5–manifold W' inside $\mathbb{R}^6$. Attaching W' to the earlier cobordism from M to M' creates the needed W^5. A few more details for such a proof will be given in an inserted note on page 167. □

Therefore, the signature of M is zero if and only if M bounds. And hence:

Corollary (*Cobordisms and signatures*). *Two 4–manifolds have the same signature if and only if they are cobordant. Signature is the complete cobordism invariant.* □

A consequence is that, unlike h–cobordisms, simple cobordisms are not very interesting: *Every 4–manifold M is cobordant to a connected sum of $\mathbb{CP}^2$'s or of $\overline{\mathbb{CP}}^2$'s or to S^4.* Indeed, assume that $\operatorname{sign} M = m > 0$; then, since $\operatorname{sign} \mathbb{CP}^2 = 1$, it follows that M and $\# m\, \mathbb{CP}^2$ must be cobordant; if $m < 0$, use $\overline{\mathbb{CP}}^2$'s instead.

Simple examples of intersection forms

Since the first example of a 4–manifold that comes to mind, namely the sphere S^4, does not have any 2–homology, it has no intersection form worth mentioning. Thus, we move on:

20. The result was quoted back in footnote 3 on page 112.

The complex projective plane. The complex projective plane $\mathbb{CP}^2$ has intersection form

$$Q_{\mathbb{CP}^2} = [+1] \ .$$

Indeed, since $H_2(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}\{[\mathbb{CP}^1]\}$ where $[\mathbb{CP}^1]$ is the class of a projective line, and since two projective lines always meet in a point, the equality above follows.

The oppositely-oriented manifold $\overline{\mathbb{CP}}^2$ has

$$Q_{\overline{\mathbb{CP}}^2} = [-1] \ .$$

Sphere bundles. The manifold $\mathbb{S}^2 \times \mathbb{S}^2$ has intersection form

$$Q_{\mathbb{S}^2\times\mathbb{S}^2} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \ .$$

We will denote this matrix by H (from "hyperbolic plane").

Reversing orientation does not exhibit a new manifold: there exist orientation-preserving diffeomorphisms $\mathbb{S}^2 \times \mathbb{S}^2 \cong \overline{\mathbb{S}^2 \times \mathbb{S}^2}$, and they correspond algebraically to isomorphisms $H \approx -H$.

The twisted product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ is the unique nontrivial sphere-bundle[21] over $\mathbb{S}^2$. It is obtained by gluing two trivial patches (*hemisphere*) $\times\ \mathbb{S}^2$ along the equator of the base-sphere, using the identification of the $\mathbb{S}^2$–fibers that rotates them by 2π as we travel along the equator. The intersection form is

$$Q_{\mathbb{S}^2\tilde{\times}\mathbb{S}^2} = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \ .$$

A simple change of basis in $H_2(\mathbb{S}^2 \tilde{\times} \mathbb{S}^2;\ \mathbb{Z})$ exhibits the intersection form as

$$Q_{\mathbb{S}^2\tilde{\times}\mathbb{S}^2} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = [+1] \oplus [-1] \ .$$

Even more, it is not hard to argue that in fact we have a diffeomorphism[22]

$$\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \ ,$$

and so we have not really encountered anything essentially new.

21. Since an $\mathbb{S}^2$-bundle over $\mathbb{S}^2 = \mathbb{D}^2{}_1 \cup \mathbb{D}^2{}_2$ is described by an equatorial gluing map $\mathbb{S}^1 \to SO(3)$, and $\pi_1 SO(3) = \mathbb{Z}_2$, it follows that there are only two topologically-distinct sphere-bundles over a sphere.

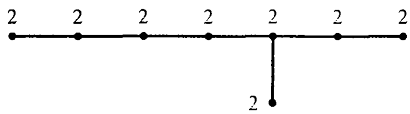
22. Quick argument: The equatorial gluing map $\mathbb{S}^1 \to SO(3)$ of $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ can be imagined as follows: as we travel along the equator of the base-sphere, it fixes the poles of the fiber-sphere and rotates the equator of the fiber-sphere by an angle increasing from 0 to 2π. Then these fiber-equators describe a circle-bundle of Euler number 1, which thus has to be the Hopf circle-bundle $\mathbb{S}^3 \to \mathbb{S}^2$. Hence the sphere-bundle is cut into two halves by a 3–sphere. Each of these halves is a disk-bundle of Euler number 1 and can therefore be identified with a neighborhood of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$, but the complement of such a neighborhood is just a 4–ball. Taking care of orientations yields the splitting $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Connected sums. Of course, through the use of connected sums we can build a lot of boring examples, such as $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# S^2 \times S^2$, whose intersection form is the sum $[+1] \oplus [-1] \oplus H$. (Incidentally, notice that this manifold has signature zero, and thus must be the boundary of some 5–manifold.)

The E_8–manifold. More interesting, though rather exotic, is Freedman's E_8–manifold $\mathcal{M}_{E_8} = P_{E_8} \cup_{\Sigma_P} \Delta$. This topological 4–manifold was built earlier[23] by plumbing on the E_8 diagram and capping with a fake 4–ball. Its intersection form can be read from the plumbing diagram to be

$$Q_{\mathcal{M}_{E_8}} = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} .$$

From now on, we will denote this matrix[24] by E_8, and succinctly write $Q_{\mathcal{M}} = E_8$. The E_8–manifold does not admit any smooth structures.[25]



3.10. The E_8 diagram, yet again

An alternative algebraic description of this most important E_8–form is the following: Consider the form $Q = [-1] \oplus 8\,[+1]$, with corresponding basis $\{e_0, e_1, \dots, e_8\}$. The vector $\kappa = 9e_0 + e_1 + \cdots + e_8$ has $\kappa \cdot \kappa = -1$; therefore its Q–orthogonal complement must be unimodular. This complement is the E_8–form. In particular, we have $E_8 \oplus [-1] \approx [-1] \oplus 8\,[+1]$.

Lemma. *The E_8–form is positive-definite, even, and of signature* 8.

Unexpectedly, proof. We will perform elementary operations on the rows and columns of the E_8–matrix. This will be fun.

---

23. See section 2.3 (page 86).

24. Various people have slightly different favorite choices for their E_8–matrix, for example, the negative of the above matrix. A brief discussion is contained in the end-notes of this chapter (page 137).

25. This is a consequence of Rokhlin's theorem, see section 4.4 (page 170) ahead.

First off, notice that these operations must be applied symmetrically, corresponding to changes of basis in $H_2(M;\mathbb{Z})$. That is to say, when for example we subtract $3/2$ times the first row from the third, we must afterwards also subtract $3/2$ times the first column from the third column. Indeed, since the matrix A of a bilinear form acts on $H_2 \times H_2$ by $(x,y) \mapsto x^tAy$, any elementary change of basis $I + \lambda E_{ij}$ on H_2 will transform A into $(I+\lambda E_{ji})A(I+\lambda E_{ij})$.

Denote by $(1),(2),(3),(4),(5),(6),(7),(8)$ the eight rows/columns of the E_8–matrix, and let us start: We write down the E_8–matrix, then subtract $1/2 \times (1)$ from (2):

$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

Subtract $2/3 \times (2)$ from (3), then subtract $3/4 \times (3)$ from (4):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

Subtract $4/5 \times (4)$ from (5), then subtract $1/2 \times (8)$ from (5):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 6/5 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & & 2 \end{bmatrix}.$$

Subtract $10/7 \times (5)$ from (6), then subtract $7/4 \times (6)$ from (7):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & & & \\ & & & & & 4/7 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & & & \\ & & & & & 4/7 & & \\ & & & & & & 1/4 & \\ & & & & & & & 2 \end{bmatrix}.$$

We have diagonalized E_8, and its signature is 8. It is positive-definite. Its determinant is $\det E_8 = 2 \cdot 3/2 \cdot 4/3 \cdot 5/4 \cdot 7/10 \cdot 4/7 \cdot 1/4 \cdot 2 = 1$ and hence E_8 is unimodular, as claimed. □

A few more examples. (1) *The intersection form of* $\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8}$ *is* $E_8 \oplus -E_8$. *Algebraically, we have* $E_8 \oplus -E_8 \approx \oplus 8\, H$ *through a suitable change of basis. As it turns out, this corresponds to an actual homeomorphism*[26]

$$\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8} \simeq \#8\, \mathbb{S}^2 \times \mathbb{S}^2 .$$

Hence the smooth manifold $\#8\, \mathbb{S}^2 \times \mathbb{S}^2$ *can be cut into two non-smoothable topological* 4*–manifolds, along a topologically-embedded* 3*–sphere.*

(2) *The intersection form of* $\mathcal{M}_{E_8} \# \overline{\mathbb{CP}}^2$ *is* $[-1] \oplus 8\,[+1]$, *same as the intersection form of* $\overline{\mathbb{CP}}^2 \# 8\, \mathbb{CP}^2$. *The two* 4*–manifolds, though, are* not *homeomorphic, and the manifold* $\mathcal{M}_{E_8} \# \overline{\mathbb{CP}}^2$ *does not admit any smooth structures.*[27]

(3) *The manifold* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8}$, *with intersection form* $E_8 \oplus E_8$, *is not smooth.*[28] *Neither is* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# \mathbb{S}^2 \times \mathbb{S}^2$, *nor is* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# 2\, \mathbb{S}^2 \times \mathbb{S}^2$. *However, suddenly* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# 3\, \mathbb{S}^2 \times \mathbb{S}^2$ does *admit smooth structures, and in what follows we will display such a smooth structure:*

3.3. Essential example: the *K*3 surface

A less exotic example (than the E_8–manifold) of a 4–manifold whose intersection form contains E_8's is the remarkable $K3$ complex surface that we build next:

The Kummer construction

Take the 4-torus

$$\mathbb{T}^4 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

and think of each $\mathbb{S}^1$–factor as the unit-circle inside $\mathbb{C}$. Consider the map

$$\sigma\colon \mathbb{T}^4 \to \mathbb{T}^4 \qquad \sigma(z_1, z_2, z_3, z_4) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$$

given by complex-conjugation in each circle-factor, as in figure 3.11 on the next page. The involution σ has exactly $16 = 2^4$ fixed points, and thus the quotient

$$\mathbb{T}^4 / \sigma$$

will have sixteen singular points where it will fail to be a manifold. Small neighborhoods of these singular points are cones[29] on $\mathbb{RP}^3$.

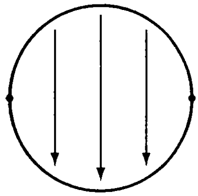
We wish to surger away these singular points of $\mathbb{T}^4/\sigma$ in order to obtain an actual 4–manifold. For that, we consider the *complex cotangent* bundle $T^*_{\mathbb{S}^2}$

26. This homeomorphism follows from Freedman's classification, see section 5.2 (page 239). A direct argument can also be made, starting with the observation that $\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8}$ is the boundary of $(\mathcal{M}_{E_8} \setminus \mathit{ball}) \times [0,1]$.

27. This follows, again, from Freedman's classification.

28. This is a consequence of Donaldson's theorem, section 5.3 (page 243).

29. Remember that the **cone** C_A of a space A is simply the result of taking $A \times [0,1]$ and collapsing $A \times 1$ to a single point (the "vertex").



3.11. Conjugation, acting on $\mathbb{S}^1$

of the 2–sphere. It is the 2–plane bundle over $\mathbb{S}^2$ with Euler number -2 (it has *opposite orientation*[30] to the tangent bundle $T_{\mathbb{S}^2}$, whose Euler number is $+2$). Its unit-disk subbundle $\mathbb{D}T^*_{\mathbb{S}^2}$ is a 4–manifold bounded by $\mathbb{RP}^3$.

Since a neighborhood of a singular point in $\mathbb{T}^4/\sigma$ has the same boundary as $\mathbb{D}T^*_{\mathbb{S}^2}$, we can cut the former out of $\mathbb{T}^4/\sigma$ and replace it by a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$. The result of this maneuver is essentially to remove the singular point and replace it with a sphere of self-intersection -2 (the zero-section of $\mathbb{D}T^*_{\mathbb{S}^2}$). We do this for all sixteen singular points.

Such a desingularization of $\mathbb{T}^4/\sigma$ yields a *simply-connected* smooth 4–manifold. This manifold admits a complex structure (thus it is a complex surface) and is called the ***K*3 surface.** The name comes from Kummer–Kähler–Kodaira.[31] The construction above is due to Kummer, which is why this manifold used to be known merely as the *Kummer surface.*

Homology. The $K3$ surface has homology $H_2(K3;\mathbb{Z}) = \oplus\, 22\mathbb{Z}$ (superficially, from 6 tori surviving from $\mathbb{T}^4$, plus the 16 desingularizing spheres). Its intersection form is

$$Q_{K3} = -\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \oplus -\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$
$$\oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

and clearly it is better kept abbreviated as

$$Q_{K3} = \oplus\, 2(-E_8) \oplus 3H\,.$$

30. For a discussion of orientations for complex-duals, see the end-notes of this chapter (page 134).

31. A. Weil wrote that, besides honoring Kummer, Kodaira and Kähler, the name "*K*3" was also chosen in relation to the famous K2 peak in the Himalayas: *"[Surfaces] ainsi nommées en l'honneur de Kummer, Kähler, Kodaira, et de la belle montagne K2 au Cachemire."*

Even if this manifold does not seem simple at all, it is in many ways as simple as it gets. We will see that $K3$ is indeed the *simplest*[32] simply-connected *smooth* 4–manifold that is not $\mathbb{S}^4$ nor a boring sum of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$'s.

The desingularization, revisited. Let us take a closer look at the desingularization of $\mathbb{T}^4/\sigma$ that created $K3$ and try to better visualize it.

Consider first a neighborhood inside $\mathbb{T}^4$ of a fixed point x_0 of σ. It is merely a 4–ball, which can be viewed as a cone over its boundary 3–sphere $\mathbb{S}^3$, with vertex at x_0. The action of σ on this cone can itself be viewed as being the cone[33] of the antipodal map $\mathbb{S}^3 \to \mathbb{S}^3$ (which sends w to $-w$). Therefore, the quotient of this neighborhood of x_0 by σ must be a cone on the quotient of $\mathbb{S}^3$ by the antipodal map, in other words, a cone on $\mathbb{RP}^3$.

Furthermore, $\mathbb{S}^3$ is fibrated by the Hopf map,[34] which makes it into a bundle with fiber $\mathbb{S}^1$ and base $\mathbb{S}^2$. Then its quotient $\mathbb{RP}^3$ inherits a structure of $\mathbb{RP}^1$–bundle over $\mathbb{S}^2$:

$$\begin{array}{ccccc} \mathbb{S}^1 & \subset & \mathbb{S}^3 & \longrightarrow & \mathbb{S}^2 \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & \| \\ \mathbb{RP}^1 & \subset & \mathbb{RP}^3 & \longrightarrow & \mathbb{S}^2. \end{array}$$

However, $\mathbb{RP}^1$ is simply a circle, so in fact we exhibited $\mathbb{RP}^3$ as an $\mathbb{S}^1$–bundle over $\mathbb{S}^2$.

Now let us look back at the neighborhood of a singular point of $\mathbb{T}^4/\sigma$. It is a cone on $\mathbb{RP}^3$, and we can think of it as being built by attaching a disk to each circle-fiber of $\mathbb{RP}^3$, and then identifying all their centers in order to obtain the vertex of the cone, the singular point. When we desingularize, we replace this cone-neighborhood in $\mathbb{T}^4/\sigma$ with a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$. This can be viewed simply as *not* identifying the centers of those disks attached to the fibers of $\mathbb{RP}^3$, but keeping them disjoint. The space of the circle-fibers of $\mathbb{RP}^3$ is the base $\mathbb{S}^2$ of the fibration. Thus the space of the attached disks is $\mathbb{S}^2$ as well, and thus their centers (now distinct) will draw a new 2–sphere, which replaced the singular point.

We can thus think of our desingularization as simply replacing each of the sixteen singular points of $\mathbb{T}^4/\sigma$ by a sphere with self-intersection -2.

32. We take "simple" to include "simple to describe". Smooth manifolds with simpler intersection forms already exist (*e.g.*, exotic $\#m\,\mathbb{S}^2 \times \mathbb{S}^2$'s, see page 553), and exotic $\mathbb{S}^4$'s could always appear.

33. Remember that the **cone** C_f of a map $f\colon A \to B$ is the function $C_f\colon C_A \to C_B$ defined by first extending $f\colon A \to B$ to $f \times id\colon A \times [0,1] \to B \times [0,1]$, then collapsing $A \times 1$ to a point and $B \times 1$ to another, with the the resulting function $C_f\colon C_A \to C_B$ sending vertex to vertex.

34. Remember that the **Hopf map** is defined to send a point $x \in \mathbb{S}^3 \subset \mathbb{C}^2$ to the point from $\mathbb{S}^2 = \mathbb{CP}^1$ that represents the complex line spanned by x inside $\mathbb{C}^2$. Topologically, the Hopf bundle $\mathbb{S}^3 \to \mathbb{S}^2$ is a circle-bundle of Euler class $+1$. Two distinct fibers will be two circles in $\mathbb{S}^3$ linked once (a so-called Hopf link, see figure 8.16 on page 318). The Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ represents the generator of $\pi_3\mathbb{S}^2 = \mathbb{Z}$.

Holomorphic construction

A complex geometer would construct the Kummer $K3$ in a way that visibly exhibits its complex structure. Specifically, she would start with $\mathbb{T}^4$ being a complex torus—for example the simplest such, the product of two copies of $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. Such a $\mathbb{T}^4$ comes equipped with complex coordinates (w_1, w_2), and the involution σ can be described as $\sigma(w_1, w_2) = (-w_1, -w_2)$ (which is obviously holomorphic).

As before, the action of σ has sixteen fixed points, but, before taking the quotient, the complex geometer will blow-up[35] $\mathbb{T}^4$ at these sixteen points. This has the result of replacing each fixed point of σ with a sphere of self-intersection -1 (a neighborhood of which looks like a neighborhood of $\overline{\mathbb{CP}}^1$ inside $\overline{\mathbb{CP}}^2$). The map σ can be extended across this blown-up 4–torus: since she replaced the fixed points of σ by spheres, she can extend σ across the new spheres simply as the identity, thus letting the whole spheres be fixed by the resulting σ.

Only now will the complex geometer take the quotient by σ of the blown-up 4–torus. The result is the $K3$ surface. The spheres of self-intersection -1 created when blowing-up the torus will project to the quotient $K3$ as themselves (they were fixed by σ), but their neighborhoods are doubly-covered through the action of σ; thus these spheres inside $K3$ have now self-intersection -2.

Many K3's. This is the place to note that a complex geometer will in fact see a *multitude* of $K3$ surfaces. Indeed, "$K3$" is not the name of *one* complex surface, but the name of a class of surfaces.[36] Any non-singular simply-connected complex surface with $c_1 = 0$ is a **$K3$ surface.**

For example, in the construction above, if we start with a different complex structure on $\mathbb{T}^4$ (from factoring $\mathbb{C}^2$ by a different lattice), then we will end up with a different $K3$ surface. All $K3$'s that result from such a construction are called **Kummer surfaces.** However, $K3$ surfaces can be built in many other ways. One example is the hypersurface of $\mathbb{CP}^3$ given by the homogeneous equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

(or any other smooth surface of degree 4). Another is the $E(2)$ elliptic surface that we will describe in chapter 8 (page 301).

This whole multitude of complex $K3$ surfaces, through the blinded eyes of the topologist, are just *one* smooth 4–manifold: any two $K3$'s are complex-deformations of each other, and thus are diffeomorphic. Hence, in this book we will carelessly be saying "*the* $K3$ surface".

35. For a discussion of blow-ups, see ahead section 7.1 (page 286).

36. For instance, the moduli space of all $K3$ surfaces has dimension 20.

$K3$ as an elliptic fibration

The $K3$ surface can be structured as a singular fibration over $\mathbb{S}^2$, with generic fiber a torus. A (singular) fibration by tori of a complex surface is called an **elliptic fibration** (because a torus in complex geometry is called an *elliptic curve*). A complex surface that admits an elliptic fibration is called an **elliptic surface.** The Kummer $K3$ is such an elliptic surface. Other examples of elliptic surfaces, as well as a different elliptic fibration on the $K3$ manifold, will be discussed later.[37]

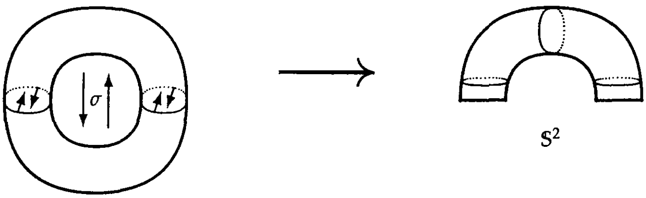
In any case, describing the elliptic fibration of $K3$ will help us better visualize this manifold. To exhibit it, we start with the projection

$$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

of $\mathbb{T}^4$ onto its first two factors. After taking the quotient by the action of σ, this projection descends to a map

$$\mathbb{T}^4/\sigma \longrightarrow \mathbb{T}^2/\sigma\,.$$

Its target $\mathbb{T}^2/\sigma$ is a non-singular sphere $\mathbb{S}^2$, as suggested in figure 3.12 (it seems like it has four singular points at the corners, but these are merely metric-singular, and can be smoothed over).



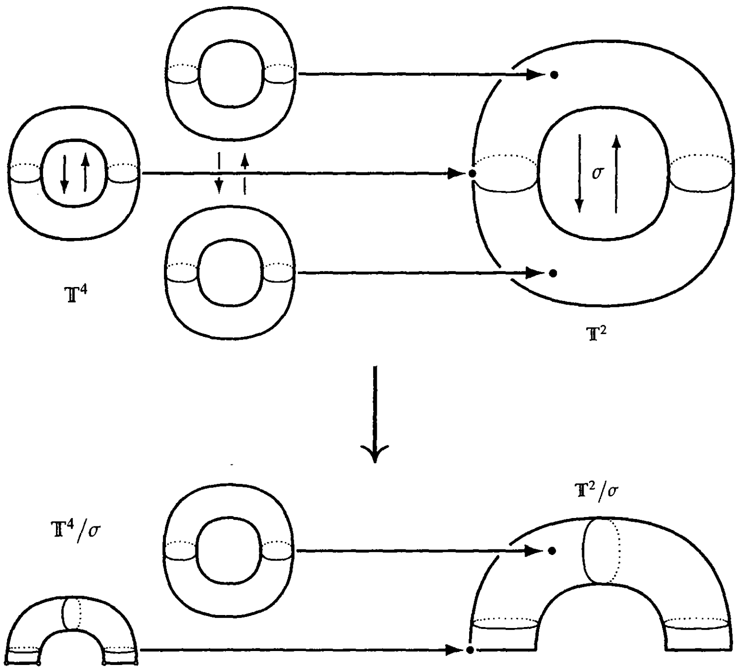
3.12. Obtaining the base sphere: $\mathbb{T}^2/\sigma = \mathbb{S}^2$

Aside from the corner-points of the base-sphere $\mathbb{T}^2/\sigma$, each of its other points comes from two distinct points (p, q) and $(\overline{p}, \overline{q})$ of $\mathbb{T}^2$ identified by σ. Thus, the fiber of the map $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ over a generic point appears from σ's identifying two distinct tori $p \times q \times \mathbb{S}^1 \times \mathbb{S}^1$ and $\overline{p} \times \overline{q} \times \mathbb{S}^1 \times \mathbb{S}^1$ from $\mathbb{T}^4$. The resulting fiber will itself be a torus. This is the generic fiber of $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$. See also figure 3.13 on the following page.

On the other hand, each of the four corner-points of the sphere $\mathbb{T}^2/\sigma$ comes from a single fixed point (p_0, q_0) of σ on $\mathbb{T}^2$. Thus, the fiber of $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ over such a corner appears from σ's sending a torus $p_0 \times q_0 \times \mathbb{S}^1 \times \mathbb{S}^1$ to itself. The quotient of this torus is again a cornered-sphere (just as before, in figure 3.12), but now its corners coincide with the sixteen global fixed points of σ on $\mathbb{T}^4$. In other words, each such sphere-fiber contains four

37. See chapter 8 (starting on page 301), which is devoted to these creatures.

of the sixteen singular points of the quotient $\mathbb{T}^4/\sigma$, points where the latter fails to be a manifold. See again figure 3.13.



3.13. The map $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ and its fibers

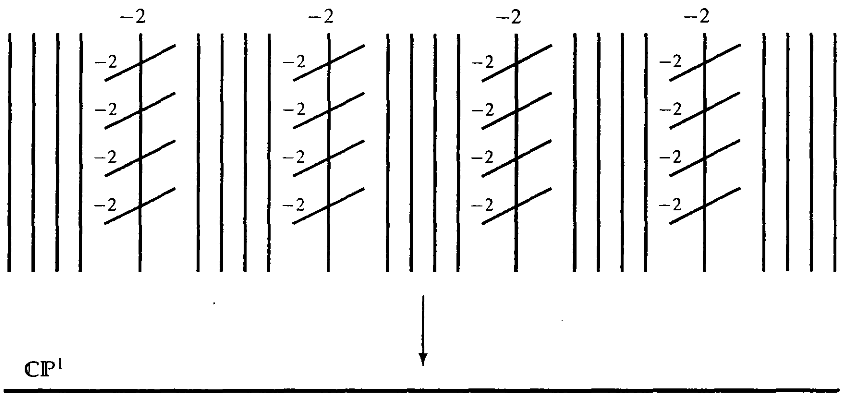
This might be a good moment to notice that $\mathbb{T}^4/\sigma$ is simply-connected. It fibrates over $\mathbb{S}^2$, which is simply-connected, and any loop in a generic torus fiber can be moved along to one of the singular sphere-fibers and contracted there. The desingularization of $\mathbb{T}^4/\sigma$ into $K3$ does not create any new loops, and therefore the $K3$ surface is, as claimed, simply-connected.

As explained before, we cut neighborhoods of the singular points out of $\mathbb{T}^4/\sigma$ and glue a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$ in their stead, thus replacing each singular point by a sphere; the result is the $K3$ surface. The projection $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ survives the desingularization as a map

$$K3 \longrightarrow \mathbb{S}^2 .$$

Indeed, since we only replaced sixteen points by sixteen spheres, we can send each of these spheres wherever the removed point used to go in $\mathbb{S}^2$.

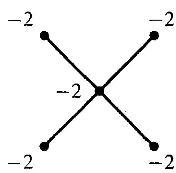
The generic fiber of $K3 \to \mathbb{S}^2$ is still a torus. However, there are now also four singular fibers, each made of five transversely-intersecting spheres: the old singular sphere-fiber of $\mathbb{T}^4/\sigma$, together with its four desingularizing spheres. A symbolic picture of this fibration is figure 3.14.



3.14. $K3$ as the Kummer elliptic fibration

Observe that the main sphere of the singular fiber must have self-intersection -2. This can be can argued as follows: Denote by S the main sphere of a singular fiber and by S_1, S_2, S_3, S_4 the desingularizing spheres. Recall how the main sphere S appeared from factoring by σ: doubly-covered by a torus. Imagine a moving generic torus-fiber F of $K3$ approaching our singular fiber: it will wrap around the main sphere twice, covering it. Also, the approaching fiber will extend to cover the desingularizing spheres once, and so in homology we have $F = 2S + S_1 + S_2 + S_3 + S_4$. We know that $F \cdot F = 0$ (since it is a fiber), and that each $S_k \cdot S_k = -2$; then one can compute that we must also have $S \cdot S = -2$.

Finally, note that a neighborhood of the singular fiber inside $K3$ can be obtained by plumbing five copies of $\mathbb{D}T^*_{S^2}$ following the diagram from figure 3.15.



3.15. Plumbing diagram for neighborhood of singular fiber

3.4. Notes

Note: Duals of complex bundles and orientations

The pretext for this note is to explain why the cotangent bundle $T^*_{S^2}$ (used earlier for building $K3$) has Euler class -2 rather than $+2$; that is to say, why $T^*_{S^2}$ and T_{S^2} have opposite orientations.

Let V be a real vector space, endowed with a complex structure. There are two ways to think of such a creature: (1) we can view V as a complex vector space, in other words, think of it as endowed with an action of the complex scalars $\mathbb{C} \times V \to V$ that makes V into a vector space over the field of complex numbers; or (2) we can view V as a real space endowed with an automorphism $J\colon V \to V$ with the property that $J \circ J = -id$. One should think of this J as a proxy for the multiplication by i. The two views are clearly equivalent, related by

$$J(v) = i \cdot v\,.$$

Nonetheless, they naturally lead to two different versions of a complex structure for the dual vector space.

The real version. Let us first discuss the case when we view V as a real vector space endowed with an anti-involution J. As a real vector space, the dual of V is

$$V^* = \mathrm{Hom}_{\mathbb{R}}(V;\mathbb{R})\,.$$

A vector space and its dual are isomorphic, but there is no natural choice of isomorphism. To fix a choice of such an isomorphism, we endow V with an auxiliary inner-product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}}$. Then V and V^* are naturally isomorphic through

$$V \xrightarrow{\approx} V^*\colon \qquad v \longmapsto v^* = \langle\,\cdot\,,v\rangle_{\mathbb{R}}\,.$$

If V is endowed with a complex structure J, then it is quite natural to restrict the choice of inner-product to those that are compatible with J. This means that we only choose inner-products that are invariant under J: we require that

$$\langle Jv, Jw\rangle_{\mathbb{R}} = \langle v,w\rangle_{\mathbb{R}}\,.$$

An immediate consequence is that we have $\langle Jv,\, w\rangle_{\mathbb{R}} = -\langle v,\, Jw\rangle_{\mathbb{R}}$.

We now wish to endow the dual V^* with a complex structure of its own. In other words, we want to define a natural anti-involution $J^*\colon V^* \to V^*$ induced by J. Since an isomorphism $V \approx V^*$ was already chosen, it makes sense now to simply transport J from V to V^* through that isomorphism. Namely, we define the complex structure J^* of V^* by

$$J^*(v^*) = (Jv)^*\,.$$

More explicitly, if $f \in V^*$ is given by $f(x) = \langle x, v\rangle_{\mathbb{R}}$ for some $v \in V$, then $(J^*f)(x) = \langle x, Jv\rangle_{\mathbb{R}}$. However, this means that $(J^*f)(x) = -\langle Jx, v\rangle_{\mathbb{R}}$, and so we have

$$J^*f = -f(J\cdot)\,.$$

Notice that we ended up with a formula that does *not* depend on the choice of inner-product. Hence we have defined a natural complex structure J^* on the real vector space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

The complex version. If, on the other hand, we think of the complex structure of V as an action of the complex scalars that makes V into a vector space $V_{\mathbb{C}}$ over the complex numbers, then a different notion of dual space comes to the fore. We must define the dual as

$$V_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})\,.$$

This vector space comes from birth equipped with a complex structure, namely

$$(i \cdot f)(x) = i\, f(x)$$

for every $f \in V_{\mathbb{C}}^*$. To better grasp what this $V_{\mathbb{C}}^*$ looks like, we will endow $V_{\mathbb{C}}$ with an auxiliary inner-product. The appropriate notion of inner-product for complex vector spaces is that of **Hermitian** inner-products. This differs from the usual inner products by the facts that it is complex-valued, and it is complex-linear in its first variable, but complex anti-linear in the second. We have $\langle\cdot,\cdot\rangle_{\mathbb{C}} : V \times V \to \mathbb{C}$ with $\langle zv, w\rangle_{\mathbb{C}} = z\langle v, w\rangle_{\mathbb{C}}$, but $\langle v, zw\rangle_{\mathbb{C}} = \bar{z}\langle v, w\rangle_{\mathbb{C}}$ for every[1] $z \in \mathbb{C}$.

Any Hermitian inner product can then be used to define a complex-isomorphism of $V_{\mathbb{C}}^*$, though not with $V_{\mathbb{C}}$, but with its **conjugate** vector space $\overline{V_{\mathbb{C}}}$. The latter is defined as being the real vector space V endowed with an action of complex scalars that is conjugate to that of $V_{\mathbb{C}}$. That is to say, in $\overline{V_{\mathbb{C}}}$ we have $i \cdot v = -iv$. The complex-isomorphism with the dual is:

$$\overline{V_{\mathbb{C}}} \xrightarrow{\approx} V_{\mathbb{C}}^* : \qquad v \longmapsto v^* = \langle\cdot, v\rangle_{\mathbb{C}}\,.$$

Notice that in the definition of v^* we must put v as the second entry in $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, so that v^* be a complex-linear function and thus indeed belong to $V_{\mathbb{C}}^*$.

If $f \in V_{\mathbb{C}}^*$ is given by $f(x) = \langle x, v\rangle_{\mathbb{C}}$ for some $v \in V$, then we have $(if)(x) = i\,f(x) = i\langle x, v\rangle_{\mathbb{C}} = \langle x, -iv\rangle_{\mathbb{C}}$. This means that we have

$$i \cdot v^* = (-iv)^*\,,$$

which shows that the complex-isomorphism above is indeed between the dual $V_{\mathbb{C}}^*$ and the *conjugate* vector space $\overline{V_{\mathbb{C}}}$.

Comparison. In review, if we view a complex vector space as (V, J), then its dual is (V^*, J^*) and the two are complex-isomorphic. If we view a complex vector space as $V_{\mathbb{C}}$, then its dual is $V_{\mathbb{C}}^*$, which is complex-isomorphic to $\overline{V_{\mathbb{C}}}$. To compare the two versions, it is enough to notice that $\overline{V_{\mathbb{C}}}$ translates simply as $(V, -J)$. Indeed, as *real* vector spaces (*i.e.*, ignoring the complex structures) V^* and $V_{\mathbb{C}}^*$ are

1. It is worth noticing that the concept of a real inner product compatible with a complex structure, and the concept of Hermitian inner product are equivalent: one can go from one to the other by using $\langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle_{\mathbb{R}} - i\langle iv, w\rangle_{\mathbb{R}}$ and $\langle v, w\rangle_{\mathbb{R}} = \mathrm{Re}\,\langle v, w\rangle_{\mathbb{C}}$.

naturally isomorphic. Specifically, the isomorphism $\operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}) \approx \operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ sends $f\colon V \to \mathbb{R}$ to the function $f_{\mathbb{C}}\colon V \to \mathbb{C}$ given by

$$f_{\mathbb{C}}(x) = \tfrac{1}{2}\big(f(x) - if(Jx)\big) .$$

The duals (V^*, J^*) and $V^*_{\mathbb{C}}$ thus differ not as real vector spaces, but because their complex structures are conjugate. This could be checked directly against the isomorphism above, or, in the simplifying presence of an inner-product, we could simply write:

$$J^*(v^*) = (iv)^* \qquad \text{and} \qquad i \cdot v^* = (-iv)^* .$$

Usage. We should emphasize that, while the "complex" version of dual is certainly the most often used, nonetheless both these versions are important.

As a typical example, consider a complex manifold X, which is endowed with a tangent bundle T_X and a cotangent bundle T^*_X. Owing to the complex structure of X, the tangent bundle has a natural complex structure on its fibers. The complex structure on T^*_X is *always* taken to be dual to the one on T_X in its "complex" version: as complex bundles, we have $T^*_X \approx \overline{T_X}$. In general for vector bundles with complex structures, the dual is usually taken to be the "complex" dual.

The "real" version of dual is also used in complex geometry. Thinking now of the complex structure of T_X as $J\colon T_X \to T_X$, we let it induce its own dual complex structure J^* on T^*_X. We then extend J^* by linearity to the complexified vector space $T^*_X \otimes_{\mathbb{R}} \mathbb{C}$. The advantage of such an extension is that now J^* has eigenvalues $\pm i$, and thus splits the bundle $T^*_X \otimes \mathbb{C}$ into its $\pm i$–eigenbundles as

$$T^*_X \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1} ,$$

and hence separates complex-valued 1-forms on X into type $(1,0)$ and type $(0,1)$. This is simply a splitting into complex-linear and complex-anti-linear parts: indeed $J^*(\alpha) = -i\alpha$ if and only if $\alpha(Jx) = +i\alpha(x)$, and then $\alpha \in \Lambda^{1,0}$.

The advantage of using J lies in part with clarity of notation: for a complex-valued creature, J will denote the complex action on its arguments (living on X), while i denotes the complex action on its values (living in $\mathbb{C}$).

More on complex-valued forms. *Every complex-valued function $f\colon X \to \mathbb{C}$ has its differential $df \in \Gamma(T^*_X \otimes \mathbb{C})$ split into its $(1,0)$-part $\partial f \in \Gamma(\Lambda^{1,0})$ and its $(0,1)$-part $\bar\partial f \in \Gamma(\Lambda^{0,1})$. Hence, $\bar\partial f = 0$ means that f's derivative is complex-linear, $df(Jx) = i\,df$, and thus that f is* holomorphic.

By using local real coordinates $(x_1, y_1, \dots, x_m, y_m)$ on X such that $z_k = x_k + iy_k$ are local complex coordinates on X, we can define $dz_k = dx_k + i\,dy_k$ and $d\bar z_k = dx_k - i\,dy_k$, and write $\Lambda^{1,0} = \mathbb{C}\{dz_1, \dots, dz_m\}$ and $\Lambda^{0,1} = \mathbb{C}\{d\bar z_1, \dots, d\bar z_m\}$. Indeed, $J^*(d\bar z_k) = +i\,d\bar z_k$.

The split $\Lambda^1 \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$ further leads to a splitting of all complex-valued forms into (p,q)-types, as in $\Lambda^k \otimes \mathbb{C} = \Lambda^{k,0} \oplus \Lambda^{k-1,1} \oplus \cdots \oplus \Lambda^{1,k-1} \oplus \Lambda^{0,k}$. Specifically, $\Lambda^{p,q}$ is made of all complex-valued forms that can be written using p of the dz_k's and q of the $d\bar z_k$'s. For example, $\Lambda^{2,0}$ contains all complex-bilinear 2-forms.

The exterior differential $d\colon \Gamma(\Lambda^k) \to \Gamma(\Lambda^{k+1})$ splits, after complexification, as $d = \partial + \bar\partial$ with $\partial\colon \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p+1,q})$ and $\bar\partial\colon \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q+1})$. Since $\bar\partial\bar\partial = 0$, this can be used to define cohomology groups $H^{p,q}(X) = \operatorname{Ker}\bar\partial / \operatorname{Im}\bar\partial$ (called **Dolbeault cohomology***), which offer a cohomology splitting $H^k(X;\mathbb{C}) = H^{k,0}(X) \oplus H^{k-1,1}(X) \oplus \cdots \oplus H^{1,k-1}(X) \oplus H^{0,k}(X)$, with $H^{p,q}(X) \approx \overline{H^{q,p}(X)}$; further, if X is Kähler, then the Hodge duality operator[2] $*$ will take*

2. The Hodge operator will be recalled in section 9.3 (page 350).

(p,q)-forms to $(m-q, m-p)$-forms, and lead into complex Hodge theory, to just drop some names. Any complex geometry book will explain these topics properly, for example **P. Griffiths** *and* **J. Harris**'s ***Principles of algebraic geometry*** **[GH78, GH94]**; *we ourselves will make use of (p,q)-forms for some technical points later on.*[3] *Part of this topic will be explained in more detail in the end-notes of chapter 9 (page 365).*

Orientations. Every vector space with a complex structure (defined either way) is naturally oriented by any basis like $\{e_1, ie_1, \ldots, e_k, ie_k\}$ (or $\{e_1, Je_1, \ldots, e_k, Je_k\}$). Thus its dual vector space, getting a complex structure itself, will be naturally oriented as well. However, the choice of duality matters: if our vector space V is odd-dimensional (over $\mathbb{C}$), then the two versions of dual complex structure lead to *opposite* orientations of V's dual. Specifically, the real-isomorphism $V \approx V^*_{\mathbb{C}}$ reverses orientations, while $V \approx (V^*, J^*)$ preserves them.

For complex manifolds and their tangent/cotangent bundles, as we mentioned above, one uses the "complex" version of duality. Therefore, for a complex curve C (for example, S^2) we have that the tangent bundle T_C and the cotangent bundle T^*_C, while isomorphic as real bundles, are naturally oriented by opposite orientations. In particular, the tangent bundle T_{S^2} is the plane bundle of Euler class $+2$, while the cotangent bundle $T^*_{S^2}$ is the plane bundle with Euler class -2.

For a complex surface M (for example, $K3$), the tangent and cotangent bundles do not have opposite orientations. Nonetheless, their complex structures are conjugate, and this leads to phenomena like $c_1(T^*_M) = -c_1(T_M)$.

Note: Positive E_8, negative E_8

In some texts, the E_8-form is sometimes described by the matrix

$$E_8 \approx \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}.$$

Correspondingly, the negative-E_8-form is sometimes written

$$-E_8 \approx \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{bmatrix}.$$

These alternative matrices are in fact equivalent with the ones presented earlier, because one can always find an isomorphism between the two versions: simply change the sign of "every other" element of the basis. Then the self-intersections

3. In section 6.2 (page 278), the end-notes of chapter 9 (connections and holomorphic bundles, page 365) and the end-notes of chapter 10 (Seiberg–Witten on Kähler and symplectic, page 457).

are preserved, but, if done properly, the intersections between distinct elements will all change signs. Peek back at the E_8 diagram for inspiration.

Complex geometers always prefer to have $+1$'s off the diagonal (thinking in terms of complex submanifolds, which always intersect positively), and so they will write $-E_8$ in the version displayed above.

More than this, certain texts prefer to switch the names of the E_8– and negative–E_8–matrices. Since what we denote here by $-E_8$ appears quite more often than E_8, calling it E_8 does save some writing.

Pick your own favorites.

Bibliography

For strong and general results about representing homology classes by submanifolds, see **R. Thom**'s celebrated paper ***Quelques propriétés globales des variétés différentiables*** [Tho54] (the results were first announced in [Tho53a]).

For a definition of intersections directly in terms of cycles (not necessarily submanifolds), see **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** [GH78, GH94, sec 0.4]; there one can also find an intersection-based view of Poincaré duality. For the rigorous algebraic topology development of various products and pairings of co/homology, see **A. Dold**'s ***Lectures on algebraic topology*** [Dol80, Dol95]; the differential-forms approach is best culled from **R. Bott** and **L. Tu**'s ***Differential forms in algebraic topology*** [BT82].

Intersection forms can be defined in all dimensions $4k$, and their signature is an important invariant. For example, it is the main surgery obstruction in those dimensions, see for example **W. Browder**'s ***Surgery on simply-connected manifolds*** [Bro72].

That all 4–manifolds of zero signature bound was proved in **V. Rokhlin**'s ***New results in the theory of four-dimensional manifolds*** [Rok52], alongside his celebrated Rokhlin's theorem that we will discuss in the next chapter. A French translation of the paper can be read as [Rok86] from the volume ***À la recherche de la topologie perdue*** [GM86a]. A geometric proof of this result can be read from **R. Kirby**'s ***The topology of 4–manifolds*** [Kir89, ch VIII], and a slightly more complete outline will be presented on page 167 ahead. A different-flavored proof is contained in **R. Stong**'s ***Notes on cobordism theory*** [Sto68].

The E_8–manifold was defined, alongside the fake 4–balls Δ, in **M. Freedman**'s ***The topology of four-dimensional manifolds*** [Fre82]; see also, of course, **M. Freedman** and **F. Quinn**'s ***Topology of 4–manifolds*** [FQ90].

For the $K3$ surface, the algebro-geometric point-of-view is discussed at length in **W. Barth, C. Peters** and **A. Van de Ven**'s ***Compact complex surfaces*** [BPVdV84] (or the second edition [BHPVdV04], with **K. Hulek**). Also, inevitably, in **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** [GH78, GH94]. For a topological point-of-view on $K3$, see **R. Gompf** and **A. Stipsicz**'s ***4–Manifolds and Kirby calculus*** [GS99], or **R. Kirby**'s ***The topology of 4–manifolds*** [Kir89]. We will come back to the $K3$ surface ourselves in chapter 8 (starting on page 301), where we will discuss it alongside its elliptic-surface brethren.

Chapter 4

Intersection Forms and Topology

WE explore in what follows the topological ramifications of a 4–manifold having a certain intersection form. The results discussed are classical, such as Whitehead's theorem, Wall's theorems, and Rokhlin's theorem. All classification results are postponed until the next chapter.

We start by showing that the intersection form determines the homotopy type of a 4–manifold. This theorem of Whitehead is argued in two ways, once by using homotopy theory and once through a Pontryagin–Thom argument. The end-notes (page 230) contain a more general discussion of the Pontryagin–Thom technique.

In section 4.2 (page 149) we explain the results of C.T.C. Wall: first, if two smooth 4–manifolds are h–cobordant, then they become diffeomorphic after summing with enough copies of $\mathbb{S}^2 \times \mathbb{S}^2$; second, if two smooth 4–manifolds have the same intersection form, then they must be h–cobordant. Notice that this last result can be combined with M. Freedman's h–cobordism theorem to show that two smooth 4–manifolds with the same intersection forms must be homeomorphic.

In section 4.3 (page 160) we discuss the characteristic classes of the tangent bundle of a 4–manifold. Most important among these is the second Stiefel–Whitney class $w_2(T_M)$. Its vanishing is equivalent, on one hand, to the intersection form being even, and on the other hand, to the existence of a spin structure on M. Various definitions of spin structures and related concepts are explained in the end-notes, and we refer to their introduction on page 173 for an outline of their contents.

Section 4.4 discusses the integral lifts of $w_2(T_M)$, called characteristic elements. These always exist, and their self-intersections are congruent modulo 8 to the signature of M. A striking result of Rokhlin's states that if $w_2(T_M)$ vanishes and M is smooth, then the signature of M is not merely a multiple of 8, but of 16; the consequences of this fact pervade all of topology. For us, an immediate consequence is that E_8 can never be the intersection form of a smooth simply-connected 4–manifold.

Finally, we should also mention that the end-notes contain a discussion of the theory of smooth structures on topological manifolds of high dimensions (page 207).

4.1. Whitehead's theorem and homotopy type

It is obvious that, if two 4–manifolds are homotopy-equivalent, then their intersection forms must be isomorphic. A first hint of the overwhelming importance that intersection forms have for 4–dimensional topology comes from the following converse:

Whitehead's Theorem. *Two simply-connected 4–manifolds are homotopy-equivalent if and only if their intersection forms are isomorphic.*

The result as stated was proved by J. Milnor, based on J.H.C. Whitehead's work. The rest of this section is devoted to a proof of this result.[1]

Start of the proof. Take a simply-connected 4–manifold M: it has homology only in dimensions 0, 2 and 4. Therefore, by Hurewicz's theorem,

$$\pi_2(M) \approx H_2(M;\mathbb{Z}) .$$

Since M is simply-connected, the latter has no torsion and thus is isomorphic to some $\oplus\, m\,\mathbb{Z}$. Hence the isomorphism $\pi_2 \approx H_2$ can be realized by a map[2]

$$f\colon S^2 \vee \cdots \vee S^2 \longrightarrow M .$$

Such f induces an isomorphism on 2–homology, and thus on all homology groups but the fourth.

To remedy this defect, we can cut out a small 4–ball from M and thus annihilate its H_4. The remainder, denoted by M°, is now homotopy-equivalent to $S^2 \vee \cdots \vee S^2$: Indeed, the map f can be easily arranged to avoid the missing 4–ball, and it then induces an isomorphism of the whole homologies of

1. The next section starts on page 149.

2. Remember that $A \vee B$ is obtained by identifying a random point of A with a random point of B. (One can realize $A \vee B$ as $A \times b \cup a \times B$ inside $A \times B$.) Thus, $S^2 \vee \cdots \vee S^2$ is a bunch of spheres with exactly one point in common; it is called a **bouquet** of spheres.

the two spaces. Invoking a celebrated result of Whitehead[3] implies that f is in fact a homotopy equivalence

$$M^\circ \sim \mathbb{S}^2 \vee \cdots \vee \mathbb{S}^2 .$$

Since M can be reconstructed by gluing the 4–ball back to M°, we deduce that the homotopy type of M can equivalently be obtained from $\bigvee m\, \mathbb{S}^2$ by gluing a 4-ball $\mathbb{D}^4$ to it:

$$M \sim \bigvee m\, \mathbb{S}^2 \cup_\varphi \mathbb{D}^4 .$$

The attachment of the ball is made through some suitable map

$$\varphi\colon \partial \mathbb{D}^4 \longrightarrow \bigvee m\, \mathbb{S}^2 .$$

In conclusion, the homotopy type of M is completely determined by the homotopy class of this φ; this class should be viewed as an element of $\pi_3(\bigvee m\, \mathbb{S}^2)$.

To prove Whitehead's theorem, we need only show that the homotopy class of φ is completely determined by the intersection form of M. This can be seen in two ways, an algebro-topologic argument and a more geometric (but longer) argument. We present both of them:

Homotopy-theoretic argument

For the following proof, the reader is assumed to have a friendly relationship with algebraic topology; if not, skip to the alternative argument.

At the outset, it is worth noticing that, through the homotopy equivalence $M \sim \bigvee m\, \mathbb{S}^2 \cup_\varphi \mathbb{D}^4$, the fundamental class $[M] \in H_4(M;\mathbb{Z})$ corresponds to the class of the attached 4–ball $\mathbb{D}^4$; indeed, since the latter has its boundary entirely contained in the 2–skeleton $\bigvee m\, \mathbb{S}^2$, it represents a 4–cycle.

Think of each $\mathbb{S}^2$ as a copy of $\mathbb{CP}^1$ inside $\mathbb{CP}^\infty$. Then embed

$$\mathbb{S}^2 \vee \cdots \vee \mathbb{S}^2 \subset \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty ,$$

and consider the exact homotopy sequence

$$\pi_4(\times m\, \mathbb{CP}^\infty) \to \pi_4(\times m\, \mathbb{CP}^\infty, \bigvee m\, \mathbb{S}^2) \to \pi_3(\bigvee m\, \mathbb{S}^2) \to \pi_3(\times m\, \mathbb{CP}^\infty) .$$

Since $\mathbb{CP}^\infty$ is an Eilenberg–MacLane $K(\mathbb{Z},2)$–space, the only non-zero homotopy group of $\times m\, \mathbb{CP}^\infty$ is π_2, and thus the above sequence exhibits an isomorphism

$$\pi_4(\times m\, \mathbb{CP}^\infty, \bigvee m\, \mathbb{S}^2) \approx \pi_3(\bigvee m\, \mathbb{S}^2) .$$

3. The statement is: *If between two simply-connected CW–complexes there exists a map that induces isomorphisms on all homology groups, then this map must be a homotopy equivalence.* Note that an abstract isomorphism of homologies is not sufficient.

The above π_4 is made of maps $\mathbb{D}^4 \to \times m\,\mathbb{CP}^\infty$ that take $\partial\,\mathbb{D}^4$ to $\vee m\,\mathbb{S}^2$. The isomorphism associates to $\varphi\colon \partial\,\mathbb{D}^4 \to \vee m\,\mathbb{S}^2$ in π_3 the class of any of its extensions

$$\widetilde{\varphi}\colon \mathbb{D}^4 \longrightarrow \times m\,\mathbb{CP}^\infty\ .$$

Further, since the inclusion $\vee m\,\mathbb{S}^2 \subset \times m\,\mathbb{CP}^\infty$ induces an isomorphism on π_2's, a different portion of the same homotopy exact sequence implies that both π_2 and π_3 of the pair $(\times m\,\mathbb{CP}^\infty,\ \vee m\,\mathbb{S}^2)$ must vanish. Therefore, Hurewicz's theorem shows that we have a natural identification

$$\pi_4(\times m\,\mathbb{CP}^\infty,\ \vee m\,\mathbb{S}^2) \ \approx\ H_4(\times m\,\mathbb{CP}^\infty,\ \vee m\,\mathbb{S}^2;\ \mathbb{Z})\ .$$

Through this identification, the class of $\widetilde{\varphi}$ from π_4 is sent to the class

$$\widetilde{\varphi}_*[\mathbb{D}^4] \in H_4(\times m\,\mathbb{CP}^\infty,\ \vee m\,\mathbb{S}^2;\ \mathbb{Z})\ ,$$

where $\widetilde{\varphi}_*$ is the morphism induced on homology by the map $\widetilde{\varphi}$.

Moreover, since both H_4 and H_3 of $\vee m\,\mathbb{S}^2$ vanish, the homology exact sequence makes appear the isomorphism

$$H_4(\times m\,\mathbb{CP}^\infty,\ \vee m\,\mathbb{S}^2;\ \mathbb{Z}) \ \approx\ H_4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z})\ .$$

For example, since $\widetilde{\varphi}_*[\mathbb{D}^4]$ represents a 4–class and its boundary is included in the 2–skeleton of $\times m\,\mathbb{CP}^\infty$, it follows that $\widetilde{\varphi}_*[\mathbb{D}^4]$ can be viewed as a 4–cycle directly in $H_4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z})$.

Owing to the lack of torsion, we also have a natural duality

$$H^4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z}) \ =\ \operatorname{Hom}\bigl(H_4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z}),\ \mathbb{Z}\bigr)\ .$$

This shows that, in order to determine $\widetilde{\varphi}_*[\mathbb{D}^4]$ in H_4, it is enough to evaluate all classes from H^4 on it. In other words, the class $\varphi \in \pi_3(\vee m\,\mathbb{S}^2)$ (and thus the homotopy type of M) are completely determined by the set of values $\alpha_k(\widetilde{\varphi}_*[\mathbb{D}^4])$ for some basis $\{\alpha_k\}_k$ of $H^4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z})$.

Such a basis can be immediately obtained by cupping the classes dual to each $\mathbb{S}^2$, that is to say, we have

$$H^4(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z}) \ =\ \mathbb{Z}\{\omega_i \cup \omega_j\}_{i,j}\ ,$$

where ω_k denotes the 2–class dual to $\mathbb{CP}^1$ inside the k^{th} copy of $\mathbb{CP}^\infty$. Furthermore, since

$$H^2(\times m\,\mathbb{CP}^\infty;\ \mathbb{Z}) \approx H^2(\vee m\,\mathbb{S}^2;\ \mathbb{Z}) \approx H^2(M^\circ;\ \mathbb{Z}) \approx H^2(M;\mathbb{Z})\ ,$$

we see that each class ω_k of $\times m\,\mathbb{CP}^\infty$ can in fact be viewed as a 2–class w_k of M itself.

Specifically, the inclusion $\iota\colon \vee m\,\mathbb{S}^2 \subset \times m\,\mathbb{CP}^\infty$ extends by $\widetilde{\varphi}$ to the map

$$M \sim \vee m\,\mathbb{S}^2 \cup_\varphi \mathbb{D}^4 \xrightarrow{\ \iota+\widetilde{\varphi}\ } \times m\,\mathbb{CP}^\infty\ .$$

The w_k's appear as the pull-backs $w_k = (\iota + \widetilde{\varphi})^* \omega_k$ and make up a basis of $H^2(M;\mathbb{Z})$.

Evaluating $\omega_i \cup \omega_j$ on $\widetilde{\varphi}_*[\mathbb{D}^4]$ inside $\times m\,\mathbb{CP}^\infty$ yields the same result as pulling ω_i and ω_j back to M, cupping there, and then evaluating on $[\mathbb{D}^4]$:

$$\begin{aligned}(\omega_i \cup \omega_j)\big(\widetilde{\varphi}_*[\mathbb{D}^4]\big) &= \big((\iota+\widetilde{\varphi})^*(\omega_i \cup \omega_j)\big)[\mathbb{D}^4] \\ &= \big((\iota+\widetilde{\varphi})^*\omega_i\big) \cup \big((\iota+\widetilde{\varphi})^*\omega_j\big)[\mathbb{D}^4] \\ &= (w_i \cup w_k)[\mathbb{D}^4]\,.\end{aligned}$$

However, as we noticed at the outset, the class $[\mathbb{D}^4]$ coincides with the fundamental class $[M]$ of M, and hence

$$(w_i \cup w_k)[\mathbb{D}^4] = Q_M(w_i, w_k)\,.$$

Since $\{w_1, \ldots, w_m\}$ is a basis in $H^2(M;\mathbb{Z})$, we deduce that the set of values $Q_M(w_i, w_k)$ fills-up a complete matrix for the intersection form Q_M of M.

On the other hand, as we have argued, by staying in $\times m\,\mathbb{CP}^\infty$ and evaluating all the $\omega_i \cup \omega_j$'s on $\widetilde{\varphi}_*[\mathbb{D}^4]$ we fully determine the class of φ in $\pi_3\big(\vee m\,\mathbb{S}^2\big)$ and thus fix the homotopy type of M.

This concludes one proof of Whitehead's theorem. □

Pontryagin–Thom argument

We have seen that the homotopy type of M can be represented as the result of gluing a 4–ball $\mathbb{D}^4$ to a bouquet of spheres $\mathbb{S}^2 \vee \cdots \vee \mathbb{S}^2$ by using some map $\varphi\colon \partial\,\mathbb{D}^4 \to \vee m\,\mathbb{S}^2$. Thus, the homotopy type of M corresponds to the homotopy class of φ. We need to argue that φ is determined by the intersection form of M.

A geometric way of seeing how the intersection form Q_M determines the attaching map

$$\varphi\colon \mathbb{S}^3 \longrightarrow \vee m\,\mathbb{S}^2$$

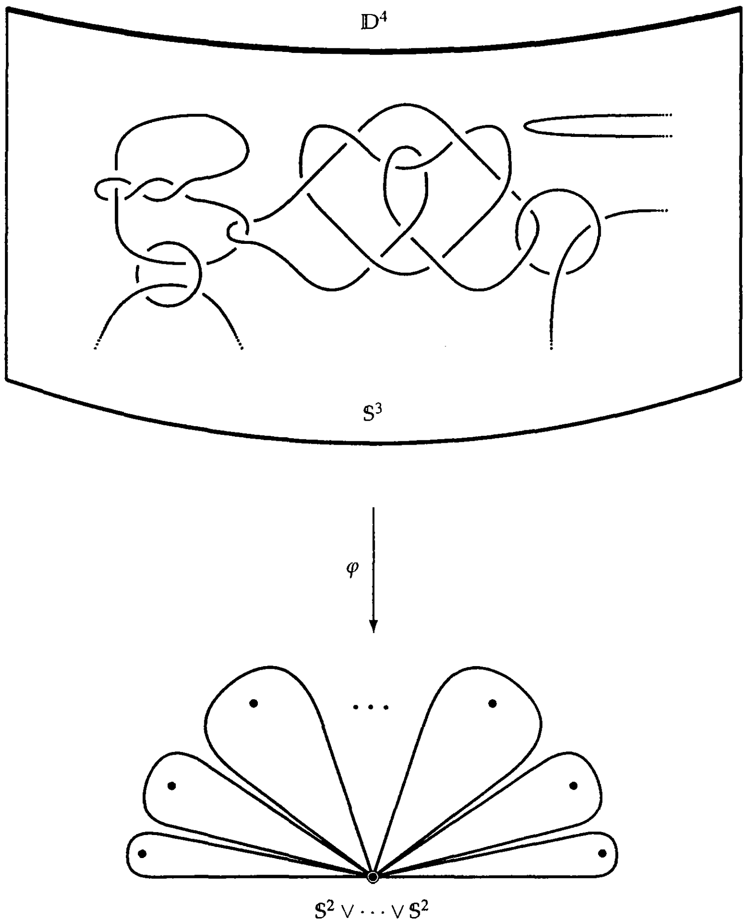
comes from what is known as the *Pontryagin–Thom construction.* The latter technique will be detailed in more generality in the end-notes of this chapter (page 230).

The framed link. Pick some points $p_1, \ldots, p_m$, one from each 2–sphere of $\vee m\,\mathbb{S}^2$. Arrange by a small homotopy that φ be transverse to these points. Also, wiggle φ until each pre-image $\varphi^{-1}[p_k]$ is connected.[4] Then each $L_k = \varphi^{-1}[p_k]$ is an embedded circle in $\mathbb{S}^3$ (a knot), and so the union

$$L = L_1 \cup \cdots \cup L_m$$

is a link in $\mathbb{S}^3$, as suggested in figure 4.1 on the following page.

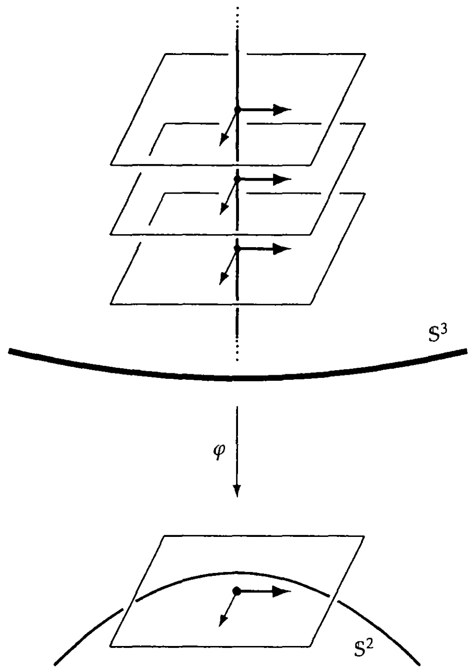
4. If "wiggle" is not convincing, read from the end-notes of this chapter (page 230).



4.1. Framed link, from attaching a 4–ball to $\mathbb{S}^2 \vee \cdots \vee \mathbb{S}^2$

The way this link L appears out of the map φ endows it with an extra bit of structure, namely a *framing:* For each L_k, embed its normal bundle $N_{L_k/\mathbb{S}^3}$ as a subbundle of $T_{\mathbb{S}^3}$ over L_k. Since φ is transverse to p_k and can be assumed to be differentiable all around L_k, it follows that $d\varphi\colon T_{\mathbb{S}^3}|_{L_k} \longrightarrow T_{\mathbb{S}^2}|_{p_k}$ restricts to a map $N_{L_k/\mathbb{S}^3} \to T_{\mathbb{S}^2}|_{p_k}$ that is an isomorphism on fibers, see figure 4.2 on the next page. The effect is that the normal bundle $N_{L_K/\mathbb{S}^3}$ is thus trivialized. Such a trivialization of the normal bundle of L_k is called a **framing** of the knot L_k. Doing this for each p_k results in a **framed link** $L = L_1 \cup \cdots \cup L_m$. Also notice that each component of the link gains a natural orientation.[5]

5. We have $T_{\mathbb{S}^3}|_{L_k} = T_{L_k} \oplus N_{L_k/\mathbb{S}^3}$; since $\mathbb{S}^3$ is oriented and $N_{L_k/\mathbb{S}^3}$ lifts an orientation from $\mathbb{S}^2$ (at the same time with the framing), this induces an orientation of T_{L_k}.



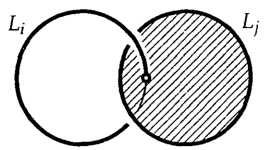
4.2. Pulling-back a framing

The linking matrix. We now focus on some simple numerical data that is expressed by our L. On one hand, for every two components L_i and L_j, we have the **linking numbers**

$$\mathrm{lk}(L_i, L_j) .$$

This integer measures how many times L_i twists around L_j.

More rigorously, one chooses in S^3 *an oriented surface* F_j *bounded by*[6] L_j *and counts the intersection number of* F_j *with* L_i *in* S^3, *as in figure 4.3. The linking number does not depend on the choice of* F_j *and is symmetric on link components:* $\mathrm{lk}(L_i, L_j) = \mathrm{lk}(L_j, L_i)$.

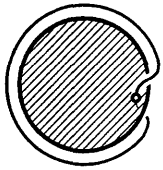


4.3. Linking number of two knots

We also have the self-linkings numbers $\mathrm{lk}(L_k, L_k)$, induced from the framing. These count the twists of the trivialization of L_k's normal bundle.

6. Such a surface always exists and is called an (orientable) **Seifert surface** for L_j; we will say a bit more in a second.

The self-linking number can be defined by picking some section of N_{L_k/S^3} that follows the trivialization of N_{L_k/S^3} given by the framing, then thinking of that section as drawing a parallel copy L'_k of L_k in S^3, and finally setting $\mathrm{lk}(L_k, L_k)$ to equal the linking number $\mathrm{lk}(L'_k, L_k)$ of L_k with this parallel copy, as suggested in figure 4.4. In our context, this self-linking number can also be defined directly: since $L_k = \varphi^{-1}[p_k]$, pick a point p'_k close to p_k, and define $\mathrm{lk}(L_k, L_k) = \mathrm{lk}(\varphi^{-1}[p_k],\, \varphi^{-1}[p'_k])$.



4.4. Self-linking number of a framed knot

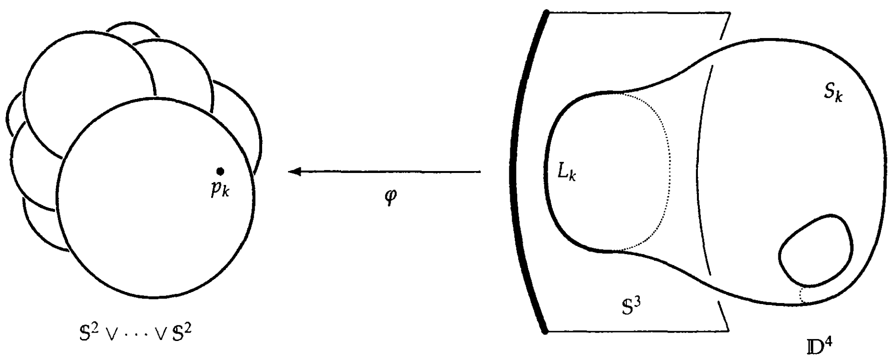
All these self/linking numbers can be fit together into a matrix

$$\left[\,\mathrm{lk}(L_i,\, L_j)\,\right]_{i,j}\,,$$

which is called the **linking matrix** of the framed link L.

On one hand, it turns out that this linking matrix is exactly the matrix of the intersection form of M, as we will argue shortly. On the other hand, a Pontryagin–Thom framed-bordism argument[7] can be used to show that the homotopy class of φ is entirely determined by this linking matrix.

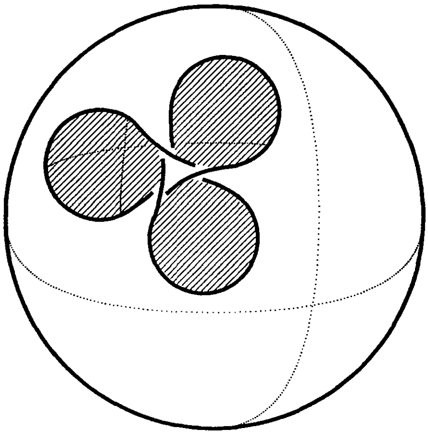
The intersection form. To see that the linking matrix of L indeed governs intersections in M, start by choosing for each L_k an oriented surface S_k inside $\mathbb{D}^4$ that is bounded by L_k, as in figure 4.5.



4.5. Building intersections out of a link.

7. See the end-notes of this chapter (page 230).

Such S_k's exist because, as we mentioned before, every knot K in $\mathbb{R}^3$ bounds an orientable surface that is bounded by K, called a **Seifert surface** *for K. (If not convinced, draw a knot, then try to draw its Seifert surface.*[8] *Take a peek at figure 4.6 for inspiration. In any case, this is merely a particular case of the general fact that homologically-trivial codimension–2 submanifolds must bound codimension–1 submanifolds.) To get the S_k's above, one can start with Seifert surfaces in $\mathbb{S}^3$ for each L_k, then push their interiors into $\mathbb{D}^4$.*



4.6. A Seifert surface for the trefoil knot

The fundamental fact to notice is that $\operatorname{lk}(L_i, L_j)$ is in fact the intersection number $S_i \cdot S_j$ of the corresponding surfaces in $\mathbb{D}^4$:

$$\operatorname{lk}(L_i, L_j) = S_i \cdot S_j .$$

See figure 4.7 on the following page for an argument.

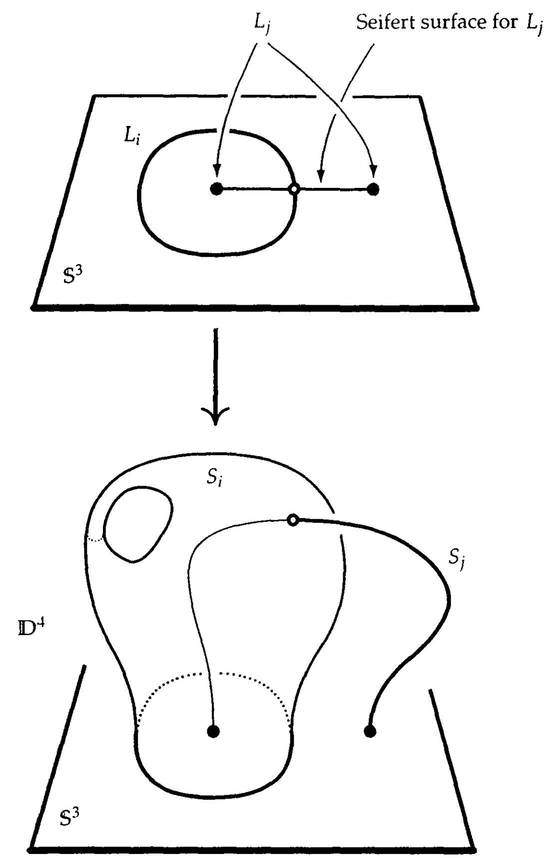
Therefore, when rebuilding the homotopy type of M through attaching $\mathbb{D}^4$ to $\bigvee m\, \mathbb{S}^2$ *via* the map φ, each S_k has its boundary L_k collapsed to the point p_k, and thus creates a closed surface S_k^*. Since the intersection numbers $S_i^* \cdot S_j^*$ in (the homotopy type of) M are exactly $\operatorname{lk}(L_i, L_j)$, we conclude that the linking matrix captures part of the intersection form of M.

To conclude the proof, all we need to do is argue that the intersections of the S_k^*'s in fact exhaust the whole intersection form of M. In other words, we need to argue that the S_k^*'s represent a basis for $H_2(M; \mathbb{Z})$. For this, recall that the homology $H_2(M; \mathbb{Z})$ was generated by the classes of the spheres of $\bigvee m\, \mathbb{S}^2$. The classes S_k^* intersect the classes of those spheres exactly once. Since the intersection form of M is unimodular, this implies that the S_k^*'s make up the dual basis[9] to the basis exhibited by the spheres of $\bigvee m\, \mathbb{S}^2$.

This concludes the alternative proof of Whitehead's theorem. □

8. Be careful to not draw a non-orientable surface.

9. Two classes α and β were called dual to each other if $\alpha \cdot \beta = 1$; see back on page 117.



4.7. Linking numbers are intersection numbers of bounded surfaces

Example. Let us conclude the discussion of Whitehead's theorem with a very simple example. If we take $\varphi\colon \mathbb{S}^3 \to \mathbb{S}^2$ to be the Hopf map,[10] then its link is the unknot[11] with framing $+1$, and the homotopy type obtained by attaching $\mathbb{D}^4$ to $\mathbb{S}^2$ using this φ is none other than $\mathbb{CP}^2$'s.

Upside-down handle diagrams. *In a certain sense, the whole procedure from the above proof is an upside-down version of a handle decomposition: the framed link L is nothing but a Kirby diagram[12] for attaching 2-handles to $\mathbb{D}^4$. The closing of S_k into S_k^* by collapsing L_k to p_k is homotopy-equivalent to gluing along L_k a disk with center p_k: the core of a 2-handle. Then the framings can be used to thicken this disk to an actual 2-handle and eventually transform the whole procedure from gluing $\mathbb{D}^4$ to $\vee m\,\mathbb{S}^2$ into attaching 2-handles to $\mathbb{D}^4$ along the link L in $\partial\,\mathbb{D}^4$.*

10. The Hopf map was recalled back in footnote 34 on page 129.

11. A knot K is called the **unknot** if it is trivial, or not knotted. Specifically, this means that K bounds some embedded disk.

12. Kirbv diagrams were explained back in the end-notes of chapter 2 (page 91).

However, the framed link L is just one of many Kirby diagrams that can be obtained through homotopies of φ. The intersection form (i.e., the homotopy class of φ) is far from determining precisely the shape of this link. Most of these links will not even lead to constructions that close-up to a smooth closed 4-manifold. (They always close-up as topological *4-manifolds by using Freedman's fake 4-balls, since if one starts with a unimodular matrix, then the resulting boundary will be a homology 3-sphere.[13]) The framed link L is just one of many diagrams for a handle decomposition of a creature homotopy-equivalent to M, but rarely of M itself.*

4.2. Wall's theorems and h-cobordisms

We will now present a series of results due to C.T.C. Wall, which culminates with the statement that, if two *smooth* simply-connected 4-manifolds have isomorphic intersection forms, then they are not merely homotopy-equivalent, but in fact are h-cobordant. Combining this with Freedman's topological h-cobordism theorem will yield immediately that, if two smooth simply-connected 4-manifold have the same intersection form, then they must be homeomorphic.

Sum-stabilizations

Two smooth 4-manifolds M and N are often h-cobordant without being diffeomorphic. To obtain a diffeomorphism, we can first "stabilize" the manifolds. A **sum-stabilization**[14] of a 4-manifold means connect-summing with copies of $S^2 \times S^2$. The world of smooth 4-manifolds considered up to such stabilizations is considerably simplified:

Wall's Theorem on Stabilizations. *If M and N are smooth, simply-connected and h-cobordant, then there is an integer k such that we have a diffeomorphism*

$$M \# kS^2 \times S^2 \cong N \# kS^2 \times S^2 .$$

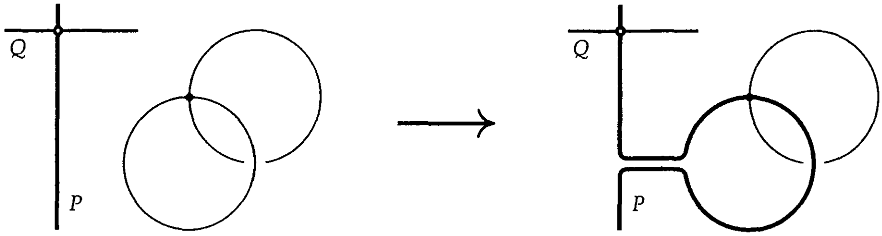
Proof. Adding $S^2 \times S^2$'s essentially allows us to go through with the h-cobordism theorem's program. This is owing to the fact that the new spheres can be used to undo unwanted intersections of surfaces, such as self-intersections of immersed Whitney disks.

Imagine that two surfaces P and Q have an intersection point that we want to be rid of. First, since $S^2 \times S^2$ contains two spheres meeting in exactly one point, we can join P with one such sphere by using a thin tube, as in figure 4.8 on the next page; the result is that P is now

13. This last fact will be proved in the the end-notes of the next chapter (page 261).

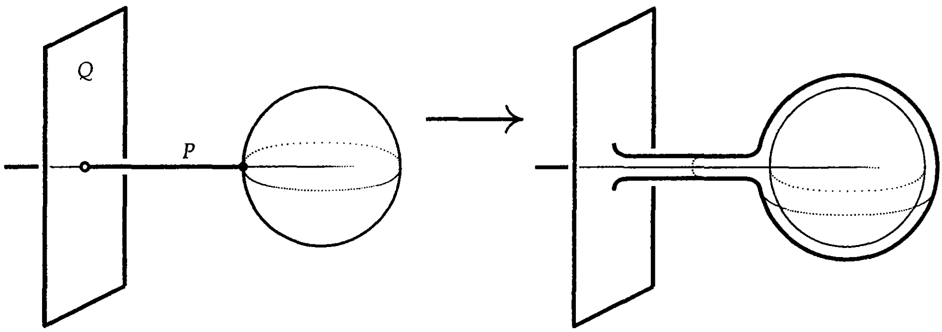
14. The name "stabilization" is in tune with, for example, stable properties of vector bundles—those preserved after adding trivial bundles; or stable homotopy groups—the part preserved after suspensions.

meeting the *other* sphere in exactly one point. (A sphere meeting a surface P in exactly one point is sometimes called a *transverse sphere* for P.)



4.8. Joining a sphere

Second, we pick a path in P from the intersection point with Q to the intersection point with the transverse sphere. Then, using a thin tube following this chosen path, we can connect Q to a parallel copy of the sphere, as in figure 4.9. The intersection point of P and Q has vanished.



4.9. Eliminating an intersection by sliding over a sphere

Notice that none of these maneuvers changed the genus of either P or Q. Thus, one can use this procedure to eliminate self-intersections of immersed Whitney disks and proceed with the h–cobordism program.

Finally, for dealing with the framing obstruction for the Whitney trick in dimension 4, which was observed back in the end-notes of chapter 1 (page 57), one can connect-sum the Whitney disk with the diagonal or anti-diagonal sphere[15] *of an extra* $S^2 \times S^2$, *which changes the framing of the disk by* ± 2. *Since having intersection points of opposite signs guarantees that the framing of a Whitney disk is even, summing with enough such diagonal spheres achieves the vanishing of the framing, and hence allows us to proceed with the Whitney trick.*

15. The diagonal sphere in $S^2 \times S^2$ is the image of the embedding $S^2 \to S^2 \times S^2 \colon x \mapsto (x, x)$ and has self-intersection $+2$. The anti-diagonal sphere is the image of $x \mapsto (x, -x)$, with self-intersection -2.

With luck, a same $\mathbb{S}^2 \times \mathbb{S}^2$-term could be used for eliminating several (if not all) intersections.[16] If not, add more. □

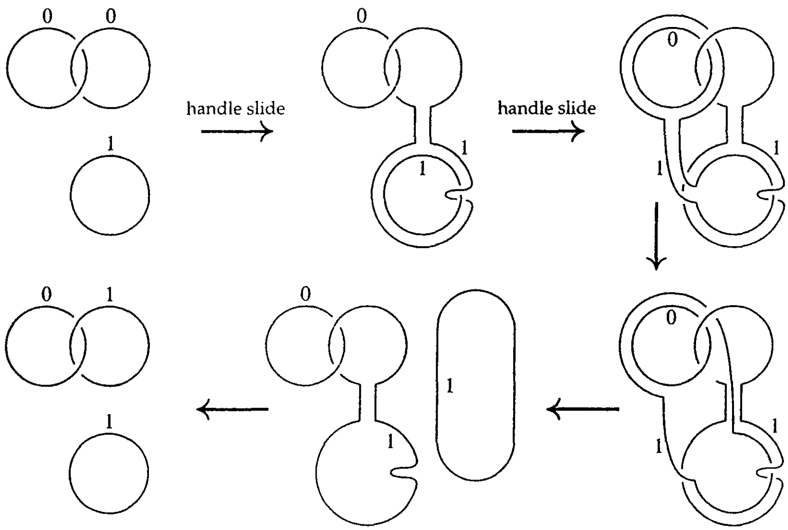
An alternative argument (more economical with $\mathbb{S}^2 \times \mathbb{S}^2$-terms) will be encountered on page 157, in the middle of the proof of Wall's theorem on h-cobordisms.

Of course $\mathbb{S}^2 \times \mathbb{S}^2$ is not the only summand that can be used with similar effects as above. One might imagine that, for example, the twisted product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ would work just as well. However, on one hand, summing with $\mathbb{S}^2 \times \mathbb{S}^2$'s preserves the parity and signature of M, which is usually desirable; and, on the other hand, in many cases summing with $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ is nothing different, since one can prove directly that:

Lemma. *If M^4 has odd intersection form, then there is a diffeomorphism*

$$M \# \mathbb{S}^2 \times \mathbb{S}^2 \cong M \# \mathbb{S}^2 \tilde{\times} \mathbb{S}^2 .$$

Idea of proof. Consider the simple case when M is $\mathbb{CP}^2$. For brevity, we use Kirby calculus, as outlined in the end-notes of chapter 2 (page 91). Then, after two handle slides and a bit of clean-up, it is done, as shown in figure 4.10. For the general case, one would slide over some odd-framed handle of M, then use similar tricks to untangle and separate $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ from M. □



4.10. Proof that $\mathbb{CP}^2 \# \mathbb{S}^2 \times \mathbb{S}^2 \cong \mathbb{CP}^2 \# \mathbb{S}^2 \tilde{\times} \mathbb{S}^2$

16. It is worth noting that in all *known* cases summing with just *one* copy of $\mathbb{S}^2 \times \mathbb{S}^2$ is enough. Currently, there are no devices able to detect cases when more than one copy of $\mathbb{S}^2 \times \mathbb{S}^2$ would be necessary.

Automorphisms of the intersection form

Wall also investigated algebraic automorphisms of intersection forms, and the question of their realizability by self-diffeomorphisms of an underlying 4–manifold.

Algebraic automorphisms. Let us consider for a moment the intersection form as an abstract algebraic creature, a symmetric bilinear unimodular form

$$Q\colon Z \times Z \longrightarrow \mathbb{Z}\,,$$

defined on some finitely-generated free $\mathbb{Z}$–module Z. An automorphism of Q is a self-isomorphism $\varphi\colon Z \approx Z$ that preserves the values of Q; that is to say, $Q(x,y) = Q(\varphi x,\, \varphi y)$.

The **divisibility** of an element x of Z is the greatest integer d such that x can be written as $x = dx_0$ for some non-zero $x_0 \in Z$. An element of divisibility 1 is called **indivisible.**

An element $\underline{w}$ of a $\mathbb{Z}$–module endowed with a symmetric bilinear unimodular form Q is called **characteristic** if it satisfies

$$Q(\underline{w}, x) = Q(x,x) \pmod 2$$

for all $x \in Z$. Notice that, if Q is even, then the divisibility of any characteristic element must be even; further, if Q is even, then $\underline{w} = 0$ is characteristic. An element is called **ordinary** if it is not characteristic. Whether some $x \in Z$ is characteristic or ordinary is called the **type** of x.

Wall's Theorem on Automorphisms. *If* $\operatorname{rank} Q - |\operatorname{sign} Q| \geq 4$, *then, given any two elements x', $x'' \in Z$ with the same divisibility, self-intersection and type, there must exist an automorphism φ of Q so that $\varphi(x') = x''$.* □

Since $\operatorname{rank} Q - \operatorname{sign} Q$ *is always even, the condition* $\operatorname{rank} Q - |\operatorname{sign} Q| \geq 4$ *only excludes definite forms (when* $\operatorname{sign} Q = \pm \operatorname{rank} Q$*) and forms with* $\operatorname{rank} Q - |\operatorname{sign} Q| = \pm 2$ *(which Wall calls* near-definite*). As we will see later,*[17] *the only excluded forms are H and $[+1] \oplus m\,[-1]$ and $[-1] \oplus m\,[+1]$ and all definite forms. Further, as far as smooth 4–dimensional topology is concerned, the only relevant definite forms are*[18] $\oplus\, m\,[+1]$ *and* $\oplus\, m\,[-1]$.

The characteristic elements of an intersection form will continue to play an important role and will be visited again in section 4.4 (page 168) ahead.

17. From Serre's classification of indefinite forms; see section 5.1 (page 238).

18. This follows from Donaldson's theorem; see section 5.3 (page 243) ahead.

Automorphisms and diffeomorphisms. It is obvious that any self-diffeomorphism of a 4-manifold induces an automorphism of its intersection form. The converse is true, but only after stabilizing once:

Wall's Theorem on Diffeomorphisms. *Let M be a smooth simply-connected 4-manifold with Q_M indefinite.*[19] *Then any automorphism of the intersection form of $M \# \mathbb{S}^2 \times \mathbb{S}^2$ can be realized by a self-diffeomorphism of $M \# \mathbb{S}^2 \times \mathbb{S}^2$.*

Idea of the proof. One identifies a concrete set of generators for the group of automorphisms of $Q_M \oplus H$, then one shows directly that each of these generators corresponds to a self-diffeomorphism. □

Topological heaven. *It should no longer come as a surprise that, if we weaken to the realm of topological 4-manifolds, stabilization is no longer necessary:*

Theorem (*M. Freedman*). *Any automorphism of Q_M can be realized by a self-homeomorphism of M, unique up to isotopy.* □

Of course, the smooth version of such a result fails.[20]

Self-diffeomorphism from spheres. *For amusement, we briefly mention a couple of examples of self-diffeomorphisms of a 4-manifold. These are built around an embedded sphere S of self-intersection*[21] *± 1 or ± 2, and act on homology by $[S] \mapsto -[S]$ and by fixing the Q-complement of $[S]$; in other words, they act as reflections on the homology lattice. Of course, finding such spheres is an endeavor in itself and often they do not exist.*[22]

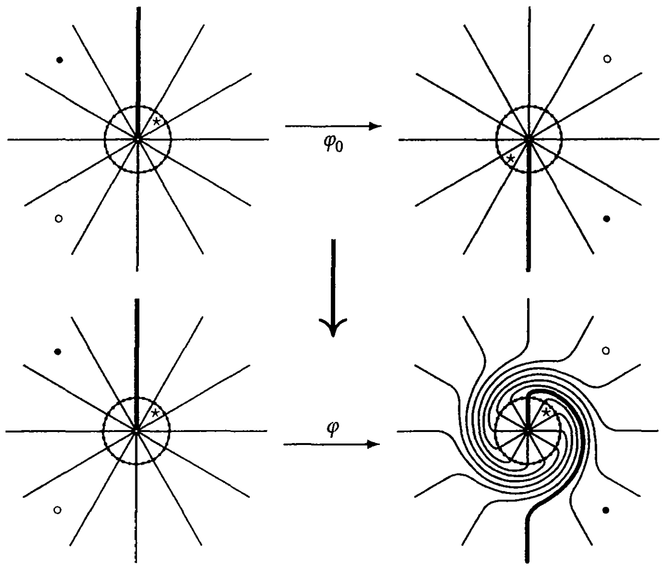
Reflection on a (± 1)-sphere. A neighborhood of a $(+1)$-sphere S in M is diffeomorphic to a neighborhood of $\mathbb{CP}^1$ in $\mathbb{CP}^2$, and furthermore $M = M' \# \mathbb{CP}^2$, with S appearing as $\mathbb{CP}^1$ in $\mathbb{CP}^2$. Our diffeomorphism acts on $\mathbb{CP}^2$ and fixes M'. We take coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{CP}^2$ and consider the complex conjugation $\varphi_0 \colon \mathbb{CP}^2 \to \mathbb{CP}^2$, with $\varphi_0[z_0 : z_1 : z_2] = [\bar{z}_1 : \bar{z}_2 : \bar{z}_2]$. Away from the projective line $\mathbb{CP}^1 = \{z_0 = 0\}$, on $\mathbb{CP}^2 \setminus \mathbb{CP}^1 = \mathbb{C}^2$, this conjugation acts as $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$, or, in real coordinates, $(x_1, y_1, x_2, y_2) \mapsto (x_1, -y_1, x_2, -y_2)$. We pick a small 4-ball $\mathbb{D}^4$ around $0 \in \mathbb{C}^2$ and modify φ_0 as we approach $\mathbb{D}^4$ by increasingly rotating the (y_1, y_2)-plane by an angle growing from 0 to π, until φ_0 becomes the identity on all $\mathbb{D}^4$; see figure 4.11 on the following page. We have built a self-diffeomorphism φ of $\mathbb{CP}^2$ that flips $\mathbb{CP}^1$ but fixes a small 4-ball $\mathbb{D}^4$. If we think of $M = M' \# \mathbb{CP}^2$ as being built by cutting out $\mathbb{D}^4$ from $\mathbb{CP}^2$, then φ extends from $\mathbb{CP}^2$ to the whole M by the identity. (For a (-1)-sphere, reverse orientations.)

19. Requiring that the intersection form of a *smooth* 4-manifold be indefinite is not a strong restriction, since in fact the only excluded forms are $\oplus m\,[\pm 1]$; see section 5.3 (page 243) ahead.

20. For example, a simple obstruction is that any automorphism of Q_M that can be realized by diffeomorphisms must send Seiberg–Witten basic classes to basic classes (for these notions, see chapter 10, starting on page 375 ahead), but even that in general is not sufficient.

21. For the extent of this inserted note, we will call such spheres (± 1)- and (± 2)-spheres.

22. Nonetheless, recall that we did identify twenty (-2)-spheres inside the $K3$ surface, see page 133.

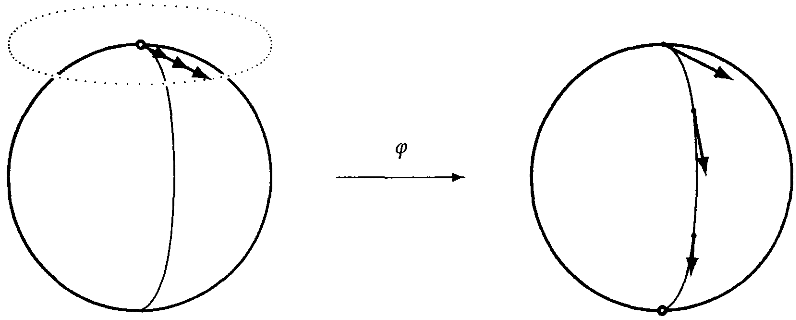


4.11. Modification toward reflection on a (-1)-sphere

Reflection on a (± 2)*-sphere. A neighborhood of a* $(+2)$*-sphere* S *in* M *is diffeomorphic to the unit-disk bundle* $\mathbb{D}T_{S^2}$. *We think of* $\mathbb{D}T_{S^2}$ *as* $\{(v,w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |v| = 1,\ |w| \leq 1,\ v \perp w\}$ *and define a self-diffeomorphism* $\varphi \colon \mathbb{D}T_{S^2} \to \mathbb{D}T_{S^2}$ *by*

$$\varphi(v,w) = \begin{cases} \left(\cos\vartheta \cdot v \ + \ \sin\vartheta \cdot \frac{1}{|w|} w\,,\ \cos\vartheta \cdot w \ + \ \sin\vartheta \cdot |w|\,(-v)\right) & \textit{if } w \neq 0 \\ (-v, 0) & \textit{if } w = 0 \end{cases}$$

with $\vartheta = (1 - |w|)\pi$. *Specifically, each tangent vector* w *determines a great circle in* S^2 *and we slide* w *along this circle by a distance depending on* $|w|$: *the shorter* w *is, the more we travel; see figure 4.12. The resulting* φ *restricts as the antipodal map on the sphere* $S = \{(v;0)\}$, *but as the identity on* $\partial\,\mathbb{D}T_{S^2}$ *and thus can be extended by the identity to the rest of* M, *yielding a self-diffeomorphism* φ *of* M. *(For a* (-2)*-sphere, reverse orientations.)*



4.12. Reflection on a $(+2)$-sphere

Wall's theorem on diffeomorphisms plays an essential role in proving the fundamental result that we present next.

Intersection forms and h-cobordisms

Going quite further than Whitehead's theorem, C.T.C. Wall proved that two smooth manifolds with the same intersection form are more than merely homotopy-equivalent:

Wall's Theorem on h-Cobordisms. *If M and N are smooth, simply-connected, and have isomorphic intersection forms, then M and N must be h-cobordant.*

If we combine with the earlier theorem on stabilizations, this yields:

Corollary. *If M and N are smooth, simply-connected, and have the same intersection form, then there is an integer k such that we have a diffeomorphism*

$$M \# kS^2 \times S^2 \;\cong\; N \# kS^2 \times S^2 . \qquad \square$$

On the other hand, if we combine the above theorem on h-cobordisms with M. Freedman's topological h-cobordism theorem, then we deduce the following most remarkable result:

Corollary (*M. Freedman*). *If two smooth simply-connected 4-manifolds have isomorphic intersection forms, then they must be homeomorphic.* $\square$

This came almost twenty years after Wall's results. Even today the attempt to strengthen the above to diffeomorphisms does not get farther than the preceding direct combination of Wall's old results.

Because of this striking consequence, in what follows we will present a fairly complete proof of Wall's theorem on h-cobordisms; it will take the rest of this section.[23]

Proof of Wall's theorem on h-cobordisms

Since M and N have the same signature, $\overline{M} \cup N$ has signature zero, and thus it must bound some 5-manifold; in other words, there is some oriented W^5 that establishes a cobordism between M and N.

The proof of the theorem consists in modifying this W (without changing its boundary) until it becomes simply-connected and homologically-trivial, in other words, until it becomes an h-cobordism from M to N.

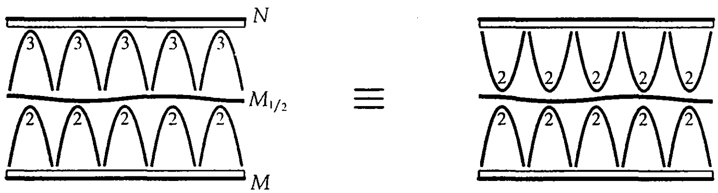
23. The next section starts on page 160.

Kill the fundamental group. The first step is to modify W^5 to make it simply-connected. We choose a set of generating loops $\ell_1, \ldots, \ell_n$ for $\pi_1(W)$, realized as disjointly embedded circles. We will add disks to kill these homotopy classes. Specifically, for each ℓ_k we take a tubular neighborhood $S^1 \times \mathbb{D}^4$ of ℓ_k and cut it out. This leaves a hole with boundary $S^1 \times S^3$, which we fill by gluing-in a copy of $\mathbb{D}^2 \times S^3$. In the resulting 5–manifold, the class of ℓ_k is trivial. Repeating for all ℓ_k's yields a new cobordism between M and N, still denoted by W, that is simply-connected.

Divide and conquer. Choose now a handle decomposition of W^5. Since W is connected, we can cancel all 0– and 5–handles. Further, since W is simply-connected, all its 1–handles can be traded for 3–handles, and, upside-down, all 4–handles for 2–handles. We end up with a handle decomposition of W that only contains 2– and 3–handles, and view W as

$$W^5 = M^4 \times [0,\varepsilon] \cup \{2\text{–handles}\} \cup \{3\text{–handles}\} \cup N^4 \times [-\varepsilon,0] \, ,$$

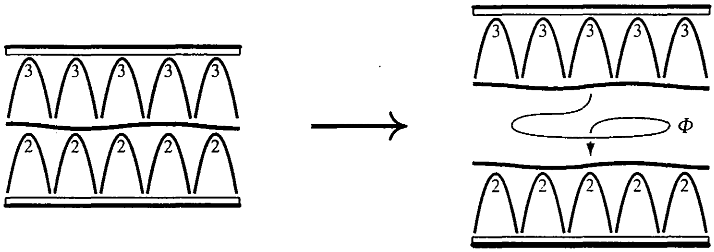
which we split into the two obvious halves: on one side, M and the 2–handles, on the other, N and the 3–handles, as on the left of figure 4.13. Looking upside-down at the upper half of W, instead of seeing the 3–handles as glued to the lower half, we can view them as 2–handles glued upwards to $N \times [-\varepsilon,0]$.



4.13. The two halves of a simply-connected cobordism

Hence the middle level $M_{1/2}$, in between the 2– and the 3–handles, is a 4–manifold that can be obtained either from M by adding regular 2–handles attached downwards, or from N by adding upside-down 2–handles attached upwards.

The strategy for the remainder of the proof is the following: We will cut W into its two halves, then glue them back after twisting by a suitable self-diffeomorphism Φ of $M_{1/2}$, as in figure 4.14 on the next page. This cut-and-reglue procedure will create a new cobordism from M to N. If we choose the right diffeomorphism Φ, then the 3–handles from the upper half will cancel algebraically the 2–handles from the lower half. This means that the newly created cobordism between M and N will have no homology relative to its boundaries, and so will indeed be an h–cobordism from M to N.



4.14. Modifying a cobordism into an h-cobordism

On the frontier. Let us first clarify the shape of $M_{1/2}$. Think of it as obtained from M after adding the 2-handles of W.

A 5-dimensional 2-handle is a copy of $\mathbb{D}^2 \times \mathbb{D}^3$, to be attached by gluing $\mathbb{S}^1 \times \mathbb{D}^3$ to M^4. To attach such a 2-handle to M, we need to specify where the attaching circle $\mathbb{S}^1 \times 0$ is being sent, but a circle in a 4-manifold is isotopic to any other embedded circle. We also need to specify how the "thickening" of the attaching circle is to be glued to M. Since[24] $\pi_1 SO(3) = \mathbb{Z}_2$, there are only two ways of doing that, depending on whether the 3-disk $\mathbb{D}^3$ in M twists an even or an odd number of times around the attaching circle.[25] Therefore, to fully describe $M_{1/2}$ all we need is to specify how many "odd" and how many "even" 2-handles are to be attached.

Attaching a 2-handle $\mathbb{D}^2 \times \mathbb{D}^3$ deletes a copy of $\mathbb{S}^1 \times \mathbb{D}^3$ from M and, as a step toward $M_{1/2}$, replaces it with a copy of $\mathbb{D}^2 \times \mathbb{S}^2$. On one hand, if the 2-handle is even, then the disk $\mathbb{D}^2$ from $\mathbb{D}^2 \times \mathbb{S}^2$ can be closed to a 2-sphere of self-intersection 0: unite the disk with a small Seifert disk of the attaching circle in M; the self-intersection of such a Seifert disk in M is the same with the framing modulo 2 (compare with page 148 earlier). Hence, the result of adding this even 2-handle is the same as connect-summing with $\mathbb{S}^2 \times \mathbb{S}^2$. On the other hand, if the 2-handle is odd, then the disk closes to a sphere of self-intersection $+1$, and one can see that attaching it is the same as connect-summing with $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$. In conclusion, we have

$$M_{1/2} = M^4 \# m' \mathbb{S}^2 \times \mathbb{S}^2 \# m'' \mathbb{S}^2 \tilde{\times} \mathbb{S}^2 .$$

We will assume in the sequel that no $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$-terms are present.

No twists, and a proof of Wall's theorem on stabilizations. The assumption that there are no $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$-summands can be argued quite rigorously:

24. Think of $SO(3)$ as the space of all oriented orthonormal frames in $\mathbb{R}^3$. Thus, $\pi_1 SO(3)$ will measure how many distinct trivializations of the 3-plane bundle $\mathbb{S}^1 \times \mathbb{R}^3$ exist. Some comments on $\pi_1 SO(m)$ will be made in the the end-notes of this chapter (page 177).

25. Contrast this with what happens when, instead of building a 5-manifold as above, we build a 4-manifold. The framing for attaching a 2-handle is then determined by an element of $\pi_1 SO(2) = \mathbb{Z}$, an integer.

On one hand, if the intersection form of M is odd, then adding $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ or adding $\mathbb{S}^2 \times \mathbb{S}^2$ produces the same result, as we mentioned a bit earlier.[26]

On the other hand, if the intersection form of M is even, then a deeper result shows that $\overline{M} \cup N$ can be safely assumed to bound a 5–manifold that does not contain any odd handles.[27] This odd-less manifold should then be the one used as our W right back from the start of the argument.

By the way, if we accept that we can indeed avoid $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$–summands, then we have stumbled upon another proof for Wall's theorem on stabilizations: from the lower half of W we have $M_{1/2} = M \# m\mathbb{S}^2 \times \mathbb{S}^2$, while from the upper half we have $M_{1/2} = N \# m\mathbb{S}^2 \times \mathbb{S}^2$, since $M_{1/2}$ can also be obtained by attaching even 2–handles upwards to N. Therefore

$$M \# m\mathbb{S}^2 \times \mathbb{S}^2 \cong N \# m\mathbb{S}^2 \times \mathbb{S}^2 .$$

This was, in fact, C.T.C. Wall's original argument for this result.

In any case, getting back to proving Wall's theorem on h–cobordisms, in what follows we assume that we have $M_{1/2} = M^4 \# m\mathbb{S}^2 \times \mathbb{S}^2$.

Negotiating the reunification. We are trying to find a self-diffeomorphism Φ of $M_{1/2}$ such that, after re-gluing W through it, the homology of W disappears. In other words, we wish to arrange Φ so that the 3–handles from the upper half cancel *algebraically* the 2–handles of the lower half.

Whether a certain Φ is good or not for this purpose is entirely determined by the self-isomorphism Φ_* that Φ induces on the 2–homology of $M_{1/2}$. Therefore, for finding a good diffeomorphism Φ, we will proceed by reverse-engineering: we will determine a good algebraic automorphism

$$\tilde{\varphi}\colon H_2(M_{1/2}; \mathbb{Z}) \approx H_2(M_{1/2}; \mathbb{Z}) ,$$

preserving the intersection form of $M_{1/2}$, and then use Wall's earlier theorem on diffeomorphisms to claim that $\tilde{\varphi}$ can be realized as Φ_* of some self-diffeomorphism Φ of $M_{1/2}$. Wall's theorem on diffeomorphisms might require that we add an extra copy of $\mathbb{S}^2 \times \mathbb{S}^2$, but that can be achieved immediately by the creation in W^5 of a (geometrically) canceling pair of a 2– and a 3–handle—the trace of such a pair in $M_{1/2}$ is exactly the required extra $\mathbb{S}^2 \times \mathbb{S}^2$–summand.

Each $\mathbb{S}^2 \times \mathbb{S}^2$–summand in $M_{1/2}$ appears from a 2–handle $\mathbb{D}^2 \times \mathbb{D}^3$, attached to M along $\mathbb{S}^1 \times \mathbb{D}^3$. The belt sphere of this 2–handle is $0 \times \mathbb{S}^2$. The homological hole created by the addition of the 2–handle is represented by the

26. Back on page 151.

27. This result is due to V. Rokhlin, and states: *Any spin 4–manifold with zero signature must bound a spin 5–manifold.* For the concept of spin manifold, look ahead at section 4.3 (page 162); the result itself will be restated on page 165.

first sphere-factor of $S^2 \times S^2$ in $M_{1/2}$, while the belt sphere of the handle survives as the *second* factor of $S^2 \times S^2$ and is filled by the handle itself.

Looking now at the upper half of W^5, a 3-handle is a copy of $\mathbb{D}^3 \times \mathbb{D}^2$, attached to the lower half through $S^2 \times \mathbb{D}^2$. The attaching sphere of the 3-handle is $S^2 \times 0$. Therefore, if the 3-handle is to algebraically cancel a 2-handle from the lower half, then the attaching sphere $S^2 \times 0$ of the 3-handle must intersect the belt sphere $0 \times S^2$ of the 2-handle algebraically exactly once.[28] Indeed, in "handle homology", we would then have $\partial(3\text{-}handle) = (2\text{-}handle)$. (Intuitively, view the 3-handle as algebraically filling the homological hole $S^2 \times 1$ created by the 2-handle.)

Algebraization. To translate everything into algebra, we proceed as follows: We view $M_{1/2}$ as

$$M_{1/2} = M \# m S^2 \times S^2 ,$$

and we denote by α_k the class of $S^2 \times 1$ and by $\bar{\alpha}_k$ the class of $1 \times S^2$ in the k^{th} $S^2 \times S^2$-summand. The classes $\bar{\alpha}_k$ are the classes of the belt spheres of the lower 2-handles, and they bound in the lower cobordism. We write

$$H_2(M_{1/2}; \mathbb{Z}) = H_2(M; \mathbb{Z}) \oplus \mathbb{Z}\{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_m, \bar{\alpha}_m\} ,$$

with corresponding intersection form $Q_{M_{1/2}} = Q_M \oplus m H$.

Now we look at $M_{1/2}$ from upwards as

$$M_{1/2} = N \# m S^2 \times S^2 .$$

This decomposition is obtained by adding upside-down 2-handles to N in the upper half of W. For trivial algebraic reasons, the $S^2 \times S^2$-summands added to N are just as many as those added to M, but the respective summands in the two decompositions *do not* correspond by, say, a diffeomorphism (unless $M \cong N$).

Denote by β_k the class of $S^2 \times 0$ and by $\bar{\beta}_k$ the class of $0 \times S^2$ in the k^{th} $S^2 \times S^2$-summand of this latter splitting. The classes β_k are the classes of the attaching spheres of the upper 3-handles, and they bound in the upper cobordism. And we write

$$H_2(M_{1/2}; \mathbb{Z}) = H_2(N; \mathbb{Z}) \oplus \mathbb{Z}\{\beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m\} ,$$

with corresponding intersection form $Q_{M_{1/2}} = Q_N \oplus m H$.

A good self-diffeomorphism Φ of $M_{1/2}$ will be one that sends the class β_k onto α_k, thus guaranteeing that the attaching sphere β_k of each 3-handle has algebraic intersection $+1$ with the belt sphere $\bar{\alpha}_k$ of the corresponding 2-handle.

28. Requiring more, such as only one geometric intersection, *i.e.*, that $S^2 \times 0$ from the 3-handle be sent to $S^2 \times 0$ from the 2-handle, implies that these 3- and 2-handles cancel. However, if we could do that for all handles, we would end with a diffeomorphism $M \cong N$, which cannot happen in general.

The final dance. The hypothesis of this theorem states that the intersection forms of M and N are isomorphic. Denote by

$$\varphi\colon H_2(N;\mathbb{Z}) \approx H_2(M;\mathbb{Z})$$

such an intersections-preserving isomorphism. Then we can extend φ to

$$\widetilde{\varphi}\colon H_2(N;\mathbb{Z}) \oplus \mathbb{Z}\{\beta_1, \ldots, \overline{\beta}_m\} \approx H_2(M;\mathbb{Z}) \oplus \mathbb{Z}\{\alpha_1, \ldots, \overline{\alpha}_m\}$$

by setting

$$\widetilde{\varphi}(\beta_k) = \alpha_k \qquad \text{and} \qquad \widetilde{\varphi}(\overline{\beta}_k) = \overline{\alpha}_k\,.$$

This extended $\widetilde{\varphi}$ is easily seen to still preserve intersections. Therefore, by Wall's theorem on diffeomorphisms, there must exist an actual self-diffeomorphism Φ of $M_{1/2}$ that realizes $\widetilde{\varphi}$ as $\Phi_* = \widetilde{\varphi}$.

Then, if we cut our W^5 into its two halves and glue them back using this Φ, then the resulting cobordism will be simply-connected and with no 2–homology. That is to say, an h–cobordism between M and N. □

4.3. Intersection forms and characteristic classes

Time has come to comment on the other classical invariants of a 4–manifold, specifically on the characteristic classes of its tangent bundle. Only $w_2(T_M)$, $e(T_M)$ and $p_1(T_M)$ are actually relevant in this realm. After first reviewing these, we will relate them to intersection forms.

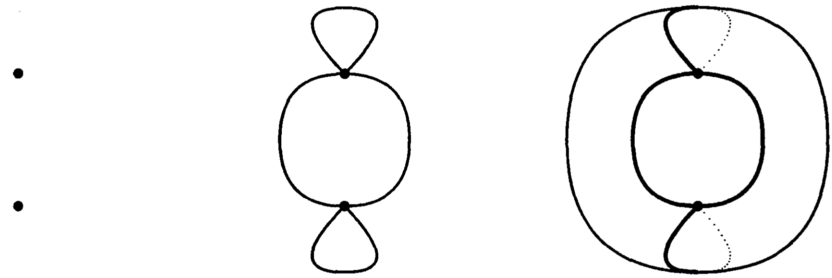
We start with the **Stiefel–Whitney classes**

$$w_k(T_M) \in H^k(M^4;\mathbb{Z}_2)\,.$$

The class $w_k(T_M)$ measures the obstruction to finding a field of $4-k+1$ linearly-independent vectors over the k–skeleton of M.

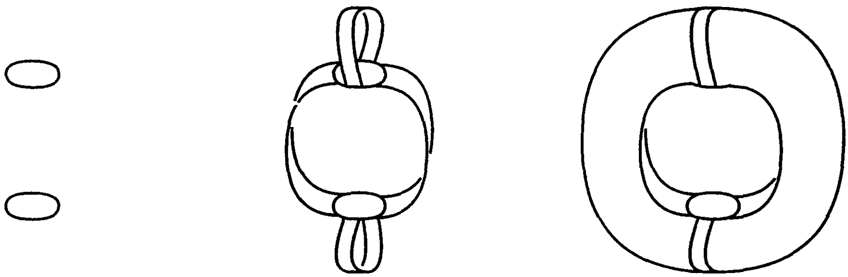
Skeleta. *Remember that, for a cellular complex, its* k–**skeleton** *is the union of all its cells of dimension* $\leq k$, *as in figure 4.15 on the facing page—similarly, for simplicial complexes (triangulations).*[29] *For a manifold* M, *one can also think in thickened terms and view the* k*–skeleton of* M *as the union of all the handles of order* $\leq k$, *in some handle decomposition of* M*; see figure 4.16 on the next page. Of course, the skeleta depend on the choice of cellular/handle decompositions.*

29. Simplices and triangulations are briefly recalled in footnote 5 on page 182 ahead.



0-skeleton 1-skeleton 2-skeleton

4.15. Skeleta of a torus, I: the cells



0-skeleton 1-skeleton 2-skeleton

4.16. Skeleta of a torus, II: the handles

Orientations and the first Stiefel–Whitney class

The class $w_1(T_M)$ measures the obstruction to finding a trivialization T_M over the 1–skeleton of M. It can be defined directly[30] by its values on embedded circles C in M, namely by setting

$$w_1(T_M) \cdot C = 0 \qquad \text{if and only if } T_M|_C \text{ is trivial;}$$
$$w_1(T_M) \cdot C = 1 \qquad \text{if and only if } T_M|_C \text{ is not trivial.}$$

Since a 4–plane bundle over a circle is either trivial or non-orientable, we observe that the first Stiefel–Whitney class merely detects orientation-reversing loops in M. Therefore w_1 is the obstruction to M being orientable.

Along these lines, it is not hard to see that an **orientation** of M is equivalent to a choice of trivialization of T_M over the 0–skeleton that can be extended over the 1–skeleton, considered up to homotopies.

Since we restricted our attention to oriented 4–manifolds, this class is not very interesting to us. Quite the opposite, though, can be said about the next Stiefel–Whitney class:

30. Since $H^1(M;\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}\big(H_1(M;\mathbb{Z}_2),\ \mathbb{Z}_2\big)$, nothing is lost.

Spin structures and the second Stiefel–Whitney class

The second Stiefel–Whitney class

$$w_2(T_M) \in H^2(M; \mathbb{Z}_2)$$

measures the obstruction to finding a 3–frame over the 2–skeleton. If w_1 was trivial and we picked an orientation of M, then by using this orientation we can complete any 3–frame to a 4–frame. Therefore we can say that, for oriented manifolds, $w_2(T_M)$ is the obstruction to trivializing T_M over the 2–skeleton[31] of M.

The origin of the $\mathbb{Z}_2$–coefficients of w_2 is in[32] $\pi_1 SO(4) = \mathbb{Z}_2$. The generator of the latter is any path of rotations of angles increasing from 0 to 2π; if the angle keeps further increasing to 4π, then the resulting loop will be null-homotopic in $SO(4)$. For trivializations of T_M, it is best to think of $SO(4)$ as the space of orienting orthonormal frames in $\mathbb{R}^4$. The class $w_2(T_M)$ is obtained by patching together local obstructions over each 2–cell D of M: a trivialization of T_M over the 1–skeleton induces a map $\varphi\colon \partial D \to SO(4)$; the trivialization extends across D if and only if φ extends over D, in other words, if φ represent the trivial element of $\pi_1 SO(4)$.

Displaying $w_2(T_M)$ as a cochain. *Given a random trivialization of T_M over the 1–skeleton of M, we can define a cellular cochain ϑ for $w_2(T_M)$ by assigning $1 \in \mathbb{Z}_2$ to any 2–cell D across which the chosen trivialization cannot be extended. This cochain will be trivial if and only if the trivialization extends over the 2–skeleton. Of course, one can try to go back and change the trivialization over the 1–skeleton, then check again. It turns out that all such changes modify our cellular cochain ϑ by the addition of a coboundary. Further, our cochain turns out to be a cocycle. Therefore, the existence of a trivialization that extends is equivalent to the cohomology class of ϑ being trivial.[33] (Observe that such a discussion can very well be carried out with 2–handles instead of 2–cells; the cocycle above assigns to each 2–handle the framing coefficient[34] modulo 2 of its attaching circle.)*

Look at surfaces. Since "2–skeleton" might not be your friendliest of notions, we can also rely upon

Lemma. *The second Stiefel–Whitney class $w_2(T_M) \in H^2(M; \mathbb{Z}_2)$ is the obstruction to trivializing T_M over the oriented surfaces embedded in M.*

---

31. Keep in mind that, the manifold being oriented, T_M can already be trivialized over the 1–skeleton.

32. The group $SO(4)$ is the group of orientation-preserving isometries of $\mathbb{R}^4$, *i.e.*, its group of rotations.

33. This is obstruction theory and is better explained in the end-notes of this chapter (page 197).

34. Compare also with Kirby calculus, in the end-notes of chapter 2 (page 91).

Proof. On one hand, we have $H^2(M;\mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}_2}(H_2(M;\mathbb{Z}_2), \mathbb{Z}_2)$, and thus w_2 is completely determined by its values $w_2 \cdot x$ on all modulo 2 classes $x \in H_2(M;\mathbb{Z}_2)$. On the other hand, when $H_1(M;\mathbb{Z})$ has no 2–torsion (for example when M is simply-connected), we further have that $H_2(M;\mathbb{Z}_2) = H_2(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$, or, in other words, classes in $H_2(M;\mathbb{Z}_2)$ are just modulo 2 reductions of integral classes from $H_2(M;\mathbb{Z})$. Therefore w_2 is completely determined by its values $w_2 \cdot S$ on the oriented surfaces S of M. Furthermore, $w_2(T_M) \cdot S = w_2(T_M|_S)$ is precisely the obstruction to trivializing T_M over S. □

Thus, when M is simply-connected, we can define $w_2(T_M)$ directly by

$$w_2(T_M) \cdot S = 0 \qquad \text{if and only if } T_M|_S \text{ is trivial,}$$
$$w_2(T_M) \cdot S = 1 \qquad \text{if and only if } T_M|_S \text{ is not trivial,}$$

for each oriented surface S embedded in M.

Look at self-intersections. By using the obvious splitting of T_M over any surface S as $T_M|_S = T_S \oplus N_{S/M}$, we compute

$$\begin{aligned} w_2(T_M) \cdot S &= w_2(T_M|_S) \\ &= w_2(T_S \oplus N_{S/M}) \\ &= w_2(T_S) + w_2(N_{S/M}) + w_1(T_S) \cdot w_1(N_{S/M}) \,. \end{aligned}$$

Since both T_S and $N_{S/M}$ are orientable, the last term vanishes. More, since $w_2(T_S)$ is the modulo 2 reduction of the Euler class $\chi(S) = 2 - 2\,\text{genus}(S)$, the first term on the right vanishes as well. We are left with $w_2(N_{S/M})$, which is the modulo 2 reduction of $e(N_{S/M})$. The latter measures the self-intersection of S in M. We have proved:

Wu's Formula. *For all oriented surfaces S embedded in M, we have:*

$$w_2(T_M) \cdot S = S \cdot S \pmod 2 \,. \qquad \square$$

This is the 4–dimensional case of the general Wu formula.[35] A verbose but more concrete alternative proof will appear on page 168 in the next section.

A nice consequence of Wu's formula is:

Corollary. *If $w_2(T_M) = 0$, then the intersection form of M is even.* □

The converse is true whenever $H_1(M;\mathbb{Z})$ has no 2–torsion.

---

35. Wu's formula is a general statement about Stiefel–Whitney classes; see for example **J. Milnor** and **J. Stasheff**'s ***Characteristic classes*** [MS74].

Spin structures. Since $w_2(T_M)$ is the obstruction to trivializing T_M over the 2–skeleton of M, in the spirit of the earlier re-definition of orientations, we can define the concept of spin structure:

A **spin structure** on M is a choice of trivialization of T_M over the 1–skeleton that can be extended over the 2–skeleton, considered up to homotopies. Various alternative ways of defining spin structures and related matters are contained in the end-notes of this chapter.[36] A manifold endowed with a spin structure is called a **spin manifold.**[37]

Then we can state that $w_2(T_M) = 0$ if and only if M admits a spin structure. The simplest examples of spin 4–manifolds are S^4, $S^2 \times S^2$, and the $K3$ surface. In general:

Corollary (*Spin structures and even forms*). *Any 4–manifold without 2–torsion, for example simply-connected, admits spin structures if and only if its intersection form is even.* □

Action of $H^1(M;\mathbb{Z}_2)$ on spin structures. *Let $\mathfrak{s}$ be a spin structure on M, described by a trivialization of T_M over the 1–skeleton of M (for some fixed triangulation of M). Choose a class $\alpha \in H^1(M;\mathbb{Z}_2)$ and represent it by its dual unoriented 3–submanifold Y_α in M. Arrange that Y_α does not touch any vertex of M's triangulation and is transverse to all its edges. Then one can define a new spin structure $\alpha \cdot \mathfrak{s}$ on M by twisting $\mathfrak{s}$'s trivialization over each edge ε that meets Y_α through the addition of a 2π–twist each time ε meets Y_α. For every loop ℓ in the 1–skeleton that bounds a 2–simplex D, the intersection of Y_α and D occurs along arcs linking the intersections points of ℓ and Y_α; therefore there must be an even number of such intersection points, and so the trivialization offered by $\alpha \cdot \mathfrak{s}$ along ℓ differs from $\mathfrak{s}$'s by an even number of 2π–twists; hence the trivialization of $\alpha \cdot \mathfrak{s}$ still extends across D—it is indeed a spin structure.*

The resulting action of $H^1(M;\mathbb{Z}_2)$ on the set of all spin structures of M is free and transitive.[38] *Therefore, after fixing a spin structure on M, this action establishes a bijective correspondence between the elements of $H^1(M;\mathbb{Z}_2)$ and the set of all spin structures on M (the correspondence depends on the choice of "base" spin structure). In particular, if M is simply-connected and has $w_2(T_M) = 0$, then M admits a unique spin structure.*

---

36. For the more usual, differential-geometric definition, see the end-notes of this chapter (page 174); see also section 10.2 (page 383) ahead. A homotopy-theoretic definition is presented in the end-notes of this chapter (page 204).

37. Often, one calls "spin manifold" any manifold that admits a spin structure, even if no specific structure has been chosen, instead of more honestly naming it, for example, "spinnable manifold".

38. The action of a group G on a set S is called **transitive** if for every two elements s' and s'' of S there is some $g \in G$ so that $g \cdot s' = s''$. The action is called **free** if we can have $g \cdot s = s$ for some $s \in S$ only when $g = 1$.

Signatures and bounding spin-manifolds. In the context of spin structures, an important result is the spin version of the bounding theorem from section 3.2 (page 123). The latter stated that all zero-signature 4–manifolds must bound some oriented 5–manifold. For spin 4–manifolds, the following refinement is true:

Theorem (*V. Rokhlin*). *If a closed 4–manifold M is endowed with a spin structure and has*

$$\operatorname{sign} Q_M = 0 ,$$

then there exists a spin 5–manifold W^5 that is bounded by M so that the spin structure of W induces the spin structure of M. □

Spin structures on 5–manifolds are defined exactly as for manifolds of dimension 4: they are trivializations of T_W over the 1–skeleton that extend over the 2–skeleton.[39] A spin structure on W^5 induces a spin structure on ∂W by using an outward-pointing trivialization of the normal bundle $N_{\partial W/W}$ to obtain a trivialization of $T_{\partial W}$ over its 1–skeleton, *etc.*

In particular, it follows that:

Corollary (*Spin cobordism*). *If two spin 4–manifolds M and N have the same signature, then they can be linked by a cobordism W^5 that is a spin 5–manifold, and its spin structure induces on M and N their respective spin structures.* □

Notice that we have already relied on this result in the proof of Wall's theorem on h–cobordisms (page 157).

Third Stiefel–Whitney class

The third Stiefel–Whitney class $w_2(T_M) \in H^3(M; \mathbb{Z}_2)$ turns out to be rather uninteresting:

On one hand, if M is orientable and admits spin structures, equivalently if both $w_1(T_M)$ and $w_2(T_M)$ vanish, then $w_3(T_M)$ must vanish as well. Indeed, any spin structure offers a trivialization of T_M over the 2–skeleton, and since the group $\pi_2 SO(4)$ is trivial, this trivialization can always be extended across the whole 3–skeleton[40] of M.

39. More geometrically, a 5–manifold W admits spin structures if and only if every surface embedded in W has trivial normal bundle. As we saw, a 4–manifold M admits spin structures if and only if every surface embedded in M has normal bundle of even Euler class.

40. Indeed, think of $SO(4)$ as the space of orthonormal frames in $\mathbb{R}^4$. Take a 3–cell E with $T_M|_{\partial E}$ trivialized. The trivialization determines a map $\partial E \to SO(4)$, which, since $\pi_2 SO(4) = 0$, must be null-homotopic and thus extend to a map $E \to SO(4)$; but the latter is just a trivialization of $T_M|_E$. The relation between the $\pi_k SO(m)$'s and w_k's is probably best viewed under the light of the concepts presented in the end-notes of this chapter, on page 197 and page 204.

Since we can always choose handle decompositions of M with exactly one 4–handle and then shrink that 4–handle toward a point, we deduce that every spin 4–manifold M has T_M trivial over $M \setminus \{point\}$; such manifolds are called **almost-parallelizable.**[41]

In general, the values of $w_3(T_M) \in H^3(M; \mathbb{Z}_2)$ do not matter—they are determined by the other characteristic classes of M, as will become clear a bit ahead, from the Dold-Whitney theorem.

The Euler class

The fourth and last Stiefel–Whitney class $w_4(T_M) \in H^4(M; \mathbb{Z}_2)$ is not the only remaining obstruction to trivializing T_M over the whole M. In fact, if M is oriented, then $w_4(T_M)$ can be refined to the integral **Euler class**

$$e(T_M) \in H^4(M; \mathbb{Z}) = \mathbb{Z} .$$

The Euler class counts the self-intersections of M, viewed as the zero-section inside the manifold T_M. Equivalently, it counts the zeros of a generic vector field on M, and we have $e(T_M) = \chi(M)$. If $e(T_M) = 0$, then T_M admits a nowhere-zero section. Clearly though, all simply-connected 4–manifolds have $e(T_M) = 2 + \operatorname{rank} Q_M$ and hence $e(T_M) > 0$.

Signatures and the Pontryagin class

Another relevant class is the **Pontryagin class**

$$p_1(T_M) \in H^4(M; \mathbb{Z}) = \mathbb{Z} .$$

It is defined in terms of Chern classes as $p_1(T_M) = -c_2(T_M \otimes \mathbb{C})$ and can be interpreted as the obstruction to finding three $\mathbb{C}$–linearly-independent global sections in $T_M \otimes \mathbb{C}$.

More obscurely, the Pontryagin number also coincides with -3 times the algebraic count of triple-points of a generic immersion[42] $M^4 \to \mathbb{R}^6$.

On a 4–manifold the Pontryagin class is completely determined by its intersection form, owing to the 4–dimensional instance of F. Hirzebruch's celebrated signature theorem:

Hirzebruch's Signature Theorem. *For every closed 4–manifold M we have*

$$p_1(T_M) = 3 \operatorname{sign} Q_M . \qquad \square$$

41. A manifold is called **parallelizable** if its tangent bundle is trivial over the whole manifold. An example of parallelizable 4–manifold is $S^1 \times S^3$; there are no simply-connected examples.

42. See **R. Herbert**'s ***Multiple points of immersed manifolds*** [Her81]; also proved in **R. Kirby**'s ***The topology of 4–manifolds*** [Kir89, ch IV].

Signatures and bounding manifolds, revisited. *We quoted earlier*[43] *the fact that, if a 4-manifold has vanishing signature, then it must bound an oriented 5-manifold. A proof of that statement can be assembled by using the signature theorem, together with the above interpretation of p_1 in terms of triple-points of immersions.*

First, one builds an immersion of M into $\mathbb{R}^6$ (by using immersion theory, it is enough to build a candidate for the normal bundle of the immersed M inside $\mathbb{R}^6$, and thus the problem is reduced to a characteristic class computation). Such an immersion will have double-points, forming surfaces in M, and will have isolated triple-points. Since $3 \operatorname{sign} Q_M = p_1(M)$, and the latter is an algebraic count of these triple-points, we conclude that the triple-points cancel algebraically. Furthermore, there is a modification of M inside $\mathbb{R}^6$ that geometrically eliminates all these triple-points[44] *and changes M merely by a cobordism inside $\mathbb{R}^6$. After that, the double points can be eliminated without obstruction (think of our method for eliminating double-points of surfaces in 4-space*[45] *and cross with $\mathbb{R}^2$), and this further changes M by a cobordism inside $\mathbb{R}^6$. We end up with a 4-manifold embedded in $\mathbb{R}^6$. Since the result is homologically-trivial and embedded, it must bound a 5-manifold W inside*[46] *$\mathbb{R}^6$. Putting together the cobordisms used to modify M with this last 5-manifold yields a filling 5-manifold for our initial 4-manifold.*[47]

That's it, the bundle is done

The above-mentioned characteristic classes completely determine T_M as a vector bundle. In fact, only w_2, e and p_1 are needed:

Dold–Whitney Theorem. *If two oriented 4–plane bundles over an oriented 4–manifold have the same second Stiefel–Whitney class w_2, Pontryagin class p_1 and Euler class e, then they must be isomorphic.* □

All these three characteristic classes can be related to intersection forms. In review, by using the partial Betti numbers $b_2^{\pm}$ we can write, for every simply-connected 4-manifold M,

$$
\begin{aligned}
e(T_M) &= b_2^+(M) + b_2^-(M) + 2\,, \\
p_1(T_M) &= b_2^+(M) - b_2^-(M)\,,
\end{aligned}
$$

and recall that $w_2(T_M)$ vanishes exactly when Q_M is even.

---

43. See back in section 3.2 (page 123).

44. Somewhat in the spirit of figure 11.7 on page 486.

45. Look back at figure 3.1 on page 113.

46. Owing to a general result of R. Thom, stated back in footnote 3 on page 112.

47. See **R. Kirby**'s ***The topology of 4-manifolds*** [Kir89, ch VIII] for the full argument.

4.4. Rokhlin's theorem and characteristic elements

We continue the story of the second Stiefel–Whitney class $w_2(T_M)$, but this time by focusing on the integral classes that reduce to it. Afterwards, we state a fundamental theorem for topology in general, namely Rokhlin's theorem: a smooth spin 4–manifold can only have a multiple of 16 as its signature.

Characteristic elements of the intersection form

We defined $w_2(T_M) \in H^2(M; \mathbb{Z}_2)$ as the obstruction to trivializing T_M over the 2–skeleton of M. We now look at representations of the class $w_2(T_M)$ by oriented surfaces and integral classes.

Make it a surface. Assume that $w_2(T_M)$ can be realized as an *oriented* surface Σ embedded in M. In other words, assume that $[\Sigma] \in H_2(M; \mathbb{Z})$ is (Poincaré-dual to) an integral lift $\underline{w}$ of the class w_2. Such a surface Σ with

$$\Sigma = w_2(T_M) \pmod 2$$

is called a **characteristic surface** of M, while its class $\underline{w} \in H_2(M; \mathbb{Z})$ is called a **characteristic element.**[48] Characteristic elements are certainly not unique: just add to such a $\underline{w}$ any even class 2γ to obtain another integral lift of w_2. Remember that we encountered characteristic elements before, in Wall's theorem on the automorphisms of an intersection form.[49]

Wu, again. Take now a random surface S in M. The obstruction to trivializing T_M over S is then given by $w_2(T_M) \cdot S \pmod 2$ or, in other words, by $\Sigma \cdot S \pmod 2$. We have already seen that this coincides modulo 2 with the self-intersection $S \cdot S$, but we prove it once again using a slightly different argument.

Wu's Formula. *Let M be a simply-connected 4–manifold. An oriented surface Σ is characteristic if and only if*

$$\Sigma \cdot S = S \cdot S \pmod 2$$

for all oriented surfaces S inside M.

Proof. Let $\tau \in \Gamma(T_S)$ be a vector field tangent to S, and let $\nu \in \Gamma(N_{S/M})$ be a field normal to S. If τ and ν are generic, then they are zero only at isolated points of S. Arrange that τ and ν are never zero at a same point of S. Pick a vector field τ^* complementary to τ in T_S, so

48. Another customary name is *characteristic class*, but we will use "characteristic element" throughout, to avoid any chance of confusion with characteristic classes of the tangent bundle.

49. See back in section 4.2 (page 152).

that τ^* is zero only at the zeros[50] of τ. Also pick a complement ν^* to ν in $N_{S/M}$ that is zero only at the zeros of ν. Then the vector field $\tau^* + \nu^*$ is nowhere-zero on S. The 3–frame $\{\tau, \nu, \tau^* + \nu^*\}$ can be completed to a full 4–frame of T_M, well-defined on S away from the zeros of τ and the zeros of ν.

Against extending this frame across the remaining points of S lies a $\mathbb{Z}_2$–obstruction: indeed, a neighborhood of a singularity is a copy of $\mathbb{D}^2 \setminus 0$, and the frame-field around 0 defines a map $f\colon \mathbb{S}^1 \to SO(4)$; the frame-field can be extended across 0 if and only if f is homotopically-trivial in $\pi_1 SO(4) = \mathbb{Z}_2$. It is not hard to argue that the obstructions at various singularities can be added together,[51] and thus yield a global $\mathbb{Z}_2$–obstruction to extending the frame-field over the whole surface S. Since $\tau^* + \nu^*$ is nowhere-zero, this obstruction comes entirely from the zeros of τ and ν.

Since τ and ν were chosen generic, their zeros are simple, and thus the obstruction can be computed as

$$\text{obstruction} = \#\{\text{zeros of } \tau\} + \#\{\text{zeros of } \nu\} \pmod 2 .$$

However, the number of zeros of a tangent vector field like τ is equivalent modulo 2 to $\chi(S)$, which is always even and thus disappears from the above formula. We are left with the number of zeros of the normal vector field ν, which is equivalent modulo 2 to $S \cdot S$. In conclusion,

$$\text{obstruction} = S \cdot S \pmod 2 .$$

However, the same obstruction can also be seen to be $w_2(T_M|_S) = w_2(T_M) \cdot S = \Sigma \cdot S \pmod 2$, and this concludes the proof. □

It might be amusing to look back at page 163 and compare the two proofs that relate w_2 to self-intersections—the version above is essentially just a more concrete version of the computations made there.

In any case, the property that $w_2 \cdot x = x \cdot x \pmod 2$ for all $x \in H^2(M;\mathbb{Z})$ completely determines the class $w_2(T_M)$ inside $H^2(M;\mathbb{Z}_2)$. In particular, if we find an integral class $\underline{w} \in H_2(M;\mathbb{Z})$ satisfying

$$\underline{w} \cdot x = x \cdot x \pmod 2 ,$$

then the modulo 2 reduction of $\underline{w}$ must be $w_2(T_M)$: we have found a *characteristic element* of the intersection form.

---

50. For example, pick a complex structure on T_S and define $\tau^* = i\tau$.

51. For example, by using an argument similar to the classic Poincaré–Hopf theorem on indices of vector fields: if the sum of indices is zero, then there is a nowhere-zero vector field. Here, since τ and ν are generic, the indices are ± 1; further, since we are dealing with a 4–plane bundle over a surface, the sum of indices only matters modulo 2.

They do exist. Characteristic elements (and hence characteristic surfaces) exist in all 4–manifolds:

Lemma. *On every 4–manifold M, there always exist integral classes $\underline{w}$ such that*

$$\underline{w} \cdot x = x \cdot x \pmod 2$$

for all $x \in H_2(M;\mathbb{Z})$.

> ***Proof.*** This is a purely algebraic argument. Let $Q\colon Z \times Z \to \mathbb{Z}$ be a symmetric bilinear unimodular form, defined over a free $\mathbb{Z}$–module Z. We can build its modulo 2 reduction by taking $Z'' = Z/2Z$ and $Q'' = Q \pmod 2$. We obtain a symmetric $\mathbb{Z}_2$–bilinear unimodular form
>
> $$Q''\colon Z'' \times Z'' \longrightarrow \mathbb{Z}_2 \,.$$
>
> The unimodularity of Q'' over $\mathbb{Z}_2$ translates as the following property: for every $\mathbb{Z}_2$–linear function $f\colon Z'' \to \mathbb{Z}_2$ there must be some element $x_f \in Z''$ so that $f(\,\cdot\,) = Q''(x_f, \,\cdot\,)$. However, since $(a+b)\cdot(a+b) = a \cdot a + b \cdot b + 2a \cdot b \equiv a \cdot a + b \cdot b \pmod 2$, we notice that the correspondence $x \longmapsto Q''(x,x)$ is additive, and thus is $\mathbb{Z}_2$–linear. Therefore there must exist an element $w'' \in Z''$ so that $Q''(x,x) = Q''(w'',x)$; in other words, we have
>
> $$w'' \cdot x = x \cdot x \pmod 2 \qquad \text{for all } x \in Z'' \,.$$
>
> Since the element $w'' \in Z'' = Z/2Z$ represents a coset of Z, there must be integral elements $\underline{w} \in Z$ whose modulo 2 reduction is w''. In other words, there always exist characteristic elements for Q, *i.e.*, elements $\underline{w} \in Z$ with $\underline{w} \cdot x = x \cdot x \pmod 2$ for all $x \in Z$. □

The existence of integral lifts of $w_2(T_M)$ is important also because of spin$^{\mathbb{C}}$ structures (complexified spin structures). As we will see later[52] the existence of $\underline{w}$'s is equivalent to the existence of spin$^{\mathbb{C}}$ structures on M; the latter will play an essential role in Seiberg–Witten theory.

Rokhlin's theorem

First, an algebraic argument shows that:

Van der Blij's Lemma. *For every characteristic element $\underline{w}$ we must have*

$$\operatorname{sign} Q_M = \underline{w} \cdot \underline{w} \pmod 8 \,. \qquad \square$$

We prove this statement in the end-notes of the next chapter (page 263).[53]

In particular, it follows that every spin manifold (for which we can always pick $\underline{w} = 0$) must have signature multiple of 8. Surprisingly, more is true:

52. In section 10.2 (page 382).

53. The reason for this postponement is not the difficulty of the argument, but merely its reliance on the classification of algebraic forms, which is discussed in the next chapter.

Rokhlin's Theorem. *If M^4 is smooth and has $w_2(T_M) = 0$, then its intersection form must have*

$$\operatorname{sign} Q_M = 0 \pmod{16} . \qquad \square$$

In part for reasons of space, proofs of this theorem are exiled to the endnotes of chapter 11 (one proof starting on page 507, another starting on page 521).

__Three's company.__ Notice that we have already encountered several statements due to V. Rokhlin: one from page 123 (about zero-signature manifolds bounding), one from a few pages back (about zero-signature spin-manifolds spin-bounding), and the one right above.[54] *In this volume, only the last result will be called "Rokhlin's theorem".*

Smooth exclusions. A first consequence of Rokhlin's theorem is that E_8 can never be the intersection form of a smooth simply-connected 4–manifold: indeed, E_8 is an even form with signature 8. In particular it follows that, as we claimed earlier, the E_8–manifold $\mathcal{M}_{E_8}$ does not admit any smooth structures at all.

Historically, we should note that, even though it was clear from Rokhlin's theorem that the E_8–form would never appear as the intersection form of a smooth 4–manifold, it was not known until Freedman's work that the E_8–form does nonetheless appear as the intersection form of a topological 4–manifold. Indeed, recall[55] *that the definition of $\mathcal{M}_{E_8}$ involves Freedman's contractible Δ's, whose construction in turn needs Freedman's major result on Casson handles.*

More generally, since E_8 has signature 8 and H has signature 0, we deduce:

Corollary. *If M is smooth and has no 2–torsion, for example when M is simply-connected, and its intersection form is*

$$Q_M = \oplus \pm m\, E_8 \oplus n\, H ,$$

then m must be even. $\square$

As we will see shortly, *all* even indefinite intersection forms do in fact fall under the jurisdiction of this corollary.

We should note that the absence of 2–torsion is essential: the complex Enriques surface (doubly-covered by $K3$) has intersection form $-E_8 \oplus H$ but fundamental group $\pi_1 = \mathbb{Z}_2$; its 2–torsion allows the intersection form to be even without w_2 vanishing, and hence Rokhlin's theorem does not apply.

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54. Furthermore, all three results appeared in the same *four-pages*-long paper, ***New results in the theory of four-dimensional manifolds*** [Rok52].

55. From section 2.3 (page 86).

It is also worth noting the fact that, for the thirty years between Rokhlin's and Donaldson's work, *no* new methods of excluding intersection forms from the smooth realm were discovered. Indeed, Rokhlin in the 1950s excluded E_8 from ever being the intersection form of a smooth 4–manifold, but the form $E_8 \oplus E_8$ was only excluded by Donaldson in the 1980s.

Other consequences. Rokhlin's theorem is a fundamental result in topology. Its consequences extend quite far, as we will comment in the various notes at the end of this chapter. For example, Rokhlin's theorem sends its tentacles into dimension 3 (the Rokhlin invariant, defined in the end-note on page 224), as well as into high dimensions (the Kirby–Siebenmann invariant, governing whether a topological manifold admits smooth structures, see the end-note on page 207); the theorem is essentially equivalent to the fact that for big n we have $\pi_{n+3} S^n = \mathbb{Z}_{24}$ instead of $\mathbb{Z}_{12}$.

Rokhlin's theorem also admits generalizations in dimension 4, such as:

Corollary (*M. Kervaire & J. Milnor*). *Let M be any smooth 4–manifold. If Σ is a characteristic sphere in M, then we must have:*

$$\operatorname{sign} M = \Sigma \cdot \Sigma \pmod{16} . \qquad \square$$

This last result was put to use for determining which characteristic elements cannot be represented by embedded spheres, and a fuller discussion will be carried through in section 11.1 (page 482).

An even further generalization of Rokhlin's theorem, due to M. Freedman and R. Kirby, is the formula

$$\operatorname{sign} M = \Sigma \cdot \Sigma + 8 \operatorname{Arf}(M, \Sigma) \pmod{16} ,$$

involving general characteristic surfaces Σ and needing a correction term $\operatorname{Arf}(M, \Sigma)$, with values in $\mathbb{Z}_2$ and depending only on the homology class of Σ. This last statement will be fully explained and proved in the end-notes[56] of chapter 11. Since the Freedman–Kirby formula will be proved from scratch, in particular it will offer a complete proof of Rokhlin's theorem. If one wishes so, one can skip ahead and read it right now.[57]

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56. Statement and heuristics starting on page 502 and detailed proof starting on page 507. An alternative spin-flavored proof starts on page 521.

57. It is recommended, though, to first visit with the end-notes of chapter 10 (the characteristic cobordism group, page 427) and the end-notes of chapter 11 (the Arf invariant, page 501). This late placement of the proof of Rokhlin's theorem owes more to reasons of space organization of this volume, than to logical structure.

4.5. Notes

Introduction

Half of the following notes can be viewed as comments on the concept of spin structure. Part of this emphasis can be justified by the foundational role that their complex cousins—spin$^{\mathbb{C}}$ structures—play in the definition of the Seiberg-Witten invariants that we will encounter in chapter 10. Another (non-disjoint) half of the notes can be viewed as comments on Rokhlin's theorem.

In the main text we defined spin structures as extendable trivializations. The more usual definition is in terms of a reduction of the structure group of T_M to the group $Spin(4)$. The first note (page 174) is devoted to explaining this definition. For this purpose, the concept of cocycle defining a vector bundle is first introduced. The note ends with a comment on the non-spin case and with the definition of principal bundles and their relation to spin structures.

The second note (page 181) contains a hands-on proof that the two definitions of spin structures are indeed equivalent. It is a direct argument involving triangulations and cover spaces, and was included owing to its absence from the standard literature.

The third note (page 189) develops the concept of cocycle for a bundle in its natural context: Čech cohomology. We develop this notion just enough to encompass bundle cocycles, but not general sheaf-cohomology. This leads in particular to concrete representations of the Chern class of a complex line bundle and of the second Stiefel–Whitney class of an oriented vector bundle, together with its relation to spin structures.

The fourth note (page 197) is a quick presentation of obstruction theory for bundles; this is a method for encoding the obstacles to building a section of a fiber bundle into suitable cohomology classes. To this is added, in the fifth note (page 204), the concept of classifying spaces for G–bundles. Besides relating these to spin structures and $w_2(T_M)$, both obstruction theory and classifying spaces are needed in the subsequent note.

The sixth note (page 207) presents the theory of endowing topological manifolds with smooth structures, as developed among others by S. Cairns, J. Munkres, J. Milnor, M. Hirsch, B. Mazur, R. Kirby, and L. Siebenmann. For this, tangent bundles for topological manifolds are defined. In dimensions at least 5, a suitable reduction of their structure group (a smoothing of the bundle) can be integrated to a smooth structure on the manifold itself. The obstacles toward this group reduction are investigated using classifying spaces and obstruction theory, and lead to the Kirby–Siebenmann invariant as primary obstruction, as well as to higher obstructions. This theory is weak in dimension 4, but the Kirby–Siebenmann invariant is still defined, and we conclude the note (page 221) by commenting on its 4–dimensional behavior, its strong relation to Rokhlin's theorem, and with a nod toward exotic $\mathbb{R}^4$'s.

We should mention that this note on smoothing theory is a node in the parallel threads of this volume. Inwards, it is a far-reaching consequence of Rokhlin's theorem; a full understanding of it is helped by reading the earlier note on exotic spheres, at the end of chapter 2 (page 97), and the notes ahead on obstruction theory (page 197) and on classifying spaces (page 204). Outwards, it underlies Freedman's classification to be presented in the next chapter. It offers the right contrasting background for the results on smooth 4–manifolds that come from gauge theory, starting with Donaldson's theorem in section 5.3 (page 243) and passing through the exotic $\mathbb{R}^4$'s of section 5.4 (page 250); and it further motivates the Freedman–Kirby generalized Rokhlin theorem to be explained at the end of chapter 11 (page 502).

The seventh note (page 224) presents briefly the Rokhlin invariant of 3–manifolds that appears as a consequence of Rokhlin's theorem. Along the way, the Novikov additivity of signatures for 4–manifolds glued along their boundaries is stated.

The eighth note (page 227) presents the groups that appear by considering two manifolds equivalent if they are cobordant. The oriented cobordism group and the spin cobordism group are displayed.

The ninth note (page 230) explains the Pontryagin–Thom construction. This technique was already used during the geometric proof of Whitehead's theorem and is placed here in its proper place, as a framed cobordism theory. Relations with homotopy groups of spheres are outlined.

Finally, on page 234 are gathered the usual end-of-chapter bibliographical comments. The next chapter starts on page 237; for the sake of continuity the reader is strongly recommended to skip all these notes at a first reading and resume reading there.

Note: Spin structures, the structure group definition

The customary definition of a spin structure is in terms of the *Spin* group, namely as reduction of the structure group of T_M from $SO(4)$ to its simply-connected double-cover $Spin(4)$. In this note we discuss this definition. The equivalence with the definition presented in the main text will be detailed in the next note (page 181). The structure group approach will also be taken up in section 10.2 (page 382), where we will present spin$^{\mathbb{C}}$ structures in order to define the Seiberg–Witten invariants.

Describing vector bundles by using cocycles. A **vector bundle** E of rank k over X^m (also called a k–**plane bundle** over X) is an open $(m+k)$–manifold E together with a map $p\colon E \to X$ such that its fibers $p^{-1}[x]$ are vector spaces isomorphic to $\mathbb{R}^k$, and p locally looks like projections $U \times \mathbb{R}^k \to U$. In other words, there is an open covering $\{U_\alpha\}$ of X and an atlas of maps

$$\{\varphi_\alpha\colon p^{-1}[U_\alpha] \cong U_\alpha \times \mathbb{R}^k\}\ ,$$

with $\mathrm{pr}_1 \circ \varphi_\alpha = p$, and so that the overlaps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are described by

$$(x,\ w) \longmapsto (x,\ g_{\alpha\beta}(x)\cdot w)$$

for some suitable change-of-coordinates functions[1]

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL(k)\ ,$$

thus ensuring that the $\mathbb{R}^k$–factors are identified linearly.

The maps $g_{\alpha\beta}$ are in fact all that is needed to describe E: One can just glue-up E from trivial patches $U_\alpha \times \mathbb{R}^k$ by identifying (x, w_α) from $U_\alpha \times \mathbb{R}^k$ with (x, w_β) from $U_\beta \times \mathbb{R}^k$ whenever $w_\alpha = g_{\alpha\beta}(x)\cdot w_\beta$.

For an open covering $\{U_\alpha\}$ of X together with a random collection of maps

$$\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL(k)\}$$

to actually define a k–plane bundle, certain simple compatibility relations need to be satisfied. These are:

$$g_{\alpha\alpha}(x) = id\ , \qquad g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1}\ , \qquad g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)\cdot g_{\beta\gamma}(x)\ .$$

These three can be contracted into just one condition:

$$g_{\alpha\beta}(x)\cdot g_{\beta\gamma}(x)\cdot g_{\gamma\alpha}(x) = id\ .$$

The latter is called the **cocycle condition.** Any collection $\{U_\alpha, g_{\alpha\beta}\}$ satisfying it will be called a **cocycle.** (The name of "cocycle" comes from Čech cohomology; this setting will be detailed in the note on page 189 ahead.)

As a simple example of cocycle defining a bundle, if $\{\Phi_\alpha\colon U_\alpha \simeq U'_\alpha \subset \mathbb{R}^m\}$ is an atlas of charts for the smooth manifold X^m, then the cocycle

$$g_{\alpha\beta}(x) = d(\Phi_\alpha \circ \Phi_\beta^{-1})|_x\ ,$$

made from the derivatives of the overlaps, defines the tangent bundle T_X of X.

Sections. Given a section $s\colon X \to E$ of some bundle $E \to X$, we can use the charts $\{\varphi_\alpha\colon E|_{U_\alpha} \approx U_\alpha \times \mathbb{R}^k\}$ to express s in coordinates. We obtain a collection of maps $\{s_\alpha\colon U_\alpha \to \mathbb{R}^k\}$ given by $s_\alpha = \varphi_\alpha \circ s$. The various local maps s_α are compatible through the relations

$$s_\alpha(x) = g_{\alpha\beta}(x)\cdot s_\beta(x)\ .$$

Conversely, in terms of cocycles alone, given a set of maps $\{s_\alpha\colon U_\alpha \to \mathbb{R}^k\}$, if they satisfy the above compatibility with some cocycle $\{g_{\alpha\beta}\}$, then they define a section in the vector bundle described by $\{g_{\alpha\beta}\}$.

Morphisms. Bundle morphisms can be described in terms of cocycles as well. Consider two bundles $E' \to X$ and $E'' \to X$ with fibers $\mathbb{R}^m$ and $\mathbb{R}^n$, both over a same base X endowed with

1. In case one finds the notations $GL(m)$ and $SO(m)$ somewhat obscure, they are reviewed later, in section 9.2 (page 333).

a covering $\{U_\alpha\}$. Let E' be described by charts $\{\varphi'_\alpha\}$ and E'' by $\{\varphi''_\alpha\}$, inducing corresponding cocycles $\{g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(m)\}$ and $\{g''_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n)\}$. Consider any linear bundle morphism $f\colon E' \to E''$, covering the identity $X \to X$. The morphism f can be expressed as a collection of maps $\{f_\alpha\colon U_\alpha \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)\}$ obtained by writing f in coordinates through the formulae $\varphi''_\alpha(f(w)) = f_\alpha(x)\cdot\varphi'_\alpha(w)$ for all $w \in E'$ and $x = p(w) \in X$. These f_α's satisfy the relations

$$f_\alpha(x)\cdot g'_{\alpha\beta}(x) = g''_{\alpha\beta}(x)\cdot f_\beta(x)\ .$$

Conversely, in terms of cocycles alone, given a set of maps $\{f_\alpha\colon U_\alpha \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)\}$, if they satisfy the above compatibility with some cocycles $\{g'_{\alpha\beta}\}$ and $\{g''_{\alpha\beta}\}$, then they must define a bundle morphism from the bundle defined by $\{g'_{\alpha\beta}\}$ to the one defined by $\{g''_{\alpha\beta}\}$.

Two $GL(k)$–valued cocycles $\{g'_{\alpha\beta}\}$ and $\{g''_{\alpha\beta}\}$, associated to a same covering $\{U_\alpha\}$, describe the same bundle (up to isomorphisms) if and only if there exists a collection of maps $\{f_\alpha : U_\alpha \to GL(k)\}$ such that

$$g''_{\alpha\beta}(x) = f_\alpha(x)\cdot g'_{\alpha\beta}(x)\cdot f_\beta(x)^{-1}\ .$$

Indeed, these f_α's are just a description in local coordinates of a vector-bundle isomorphism between the bundles defined by $\{g'_{\alpha\beta}\}'$ and $\{g''_{\alpha\beta}\}$.

By ignoring the underlying vector bundles, we will say directly that two cocycles $\{g'_{\alpha\beta}\}$ and $\{g''_{\alpha\beta}\}$ are **isomorphic** whenever they can be linked with f_α's as above.

For comparing two cocycles $\{g'_{\alpha'\beta'}\}$ and $\{g''_{\alpha''\beta''}\}$ associated to two different coverings $\{U'_{\alpha'}\}$ and $\{U''_{\alpha''}\}$ of M, we can first move to the common subdivision $\{U'_{\alpha'} \cap U''_{\beta''}\}$, then proceed as above.

Keep in mind that any bundle over a contractible set must be trivial, and thus, if one starts with a covering $\{U_\alpha\}$ of X by, say, disks, then such a covering can alone be used to describe *all* bundles over X.

Reductions of structure groups. Let E be a k–plane bundle, and let G be some subgroup of $GL(k)$. If we manage to describe E using a G–valued cocycle $g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, then we say that we have **reduced the structure group** of E from $GL(k)$ to its subgroup G.

This notion can also be described in terms of cocycles alone: Given some cocycle $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k)$, we say that we reduced its structure group to G if we can find a G–valued cocycle $g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ so that $\{g'_{\alpha\beta}\}$ is isomorphic to $\{g_{\alpha\beta}\}$.

For example, every vector bundle E can be endowed with a fiber-metric (*i.e.*, an inner product in each fiber, varying smoothly from fiber to fiber). Then, by restricting our choice of charts $\varphi_\alpha\colon E|_{U_\alpha} \approx U_\alpha \times \mathbb{R}^k$ to those φ_α's that establish *isometries* between the fibers of E and $\mathbb{R}^k$ (with its standard inner product), we are led to a description of E by an $O(k)$–valued cocycle

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow O(k)\ .$$

We then say that a fiber-metric has reduced the structure group of E from $GL(k)$ to its subgroup $O(k)$.

If our bundle is orientable and we choose an orientation, then, by further restricting the φ_α's to those providing *orientation-preserving* isometries from the fibers of E to $\mathbb{R}^k$, we obtain a $SO(k)$–valued cocycle for E

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow SO(k)\ .$$

We say that an orientation has further reduced the structure group of E from $O(k)$ to its subgroup $SO(k)$.

A spin structure on E can itself be described as a further "reduction" of the structure group of E from $SO(k)$ to the group $Spin(k)$. However, since $Spin(k)$ is not a subgroup of $GL(k)$, this "reduction" has to be developed abstractly, at the level of cocycles and not directly on the vector bundles.

Definition of a spin structure. While the notion of spin structure can be developed for general vector bundles E, for concreteness in what follows we will restrict to the case of the tangent bundle of a 4–manifold. The extension to the general case should be obvious enough.

Start with an oriented 4–manifold M and pick a random Riemannian metric on it. This reduces the structure group of T_M to $SO(4)$, and thus T_M can be described by an $SO(4)$–valued cocycle $\{U_\alpha, g_{\alpha\beta}\}$ with

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow SO(4) .$$

The group $SO(4)$ is connected, but has fundamental group

$$\pi_1 SO(4) = \mathbb{Z}_2 .$$

This fundamental group is generated by a path of rotations of angles increasing from 0 to 2π. On the other hand, if one keeps rotating until reaching 4π, then the resulting loop in $SO(4)$ will be null-homotopic; this can be observed in figure 4.17 on the following page, if properly interpreted. In conclusion, a loop $\ell\colon S^1 \to SO(4)$ is homotopically-trivial if and only if it twists $\mathbb{R}^4$ by an even multiple of 2π, and nontrivial if it twists by an odd multiple.

The fundamental group is unfolded in $SO(4)$'s universal cover, specifically in the Lie group

$$Spin(4) ,$$

which double-covers[2] $SO(4)$.

Ledger. One can think of the Spin group as a method for bookkeeping 2π–rotations: Consider a random loop $\ell\colon [0,1] \to SO(4)$, with $\ell(0) = \ell(1)$. On one hand, if ℓ is homotopically-trivial, then it can be lifted to a loop $\tilde{\ell}$ in $Spin(4)$, with $\tilde{\ell}(0) = \tilde{\ell}(1)$. On the other hand, if ℓ describes a rotation of 2π, then it can only be lifted to an open path with $\tilde{\ell}(0) = -\tilde{\ell}(1)$.

A **spin structure** on M is defined as a lift of the $SO(4)$–cocycle $\{g_{\alpha\beta}\}$ of T_M to a $Spin(4)$–valued cocycle, considered up to isomorphisms. Specifically, given the $SO(4)$–cocycle

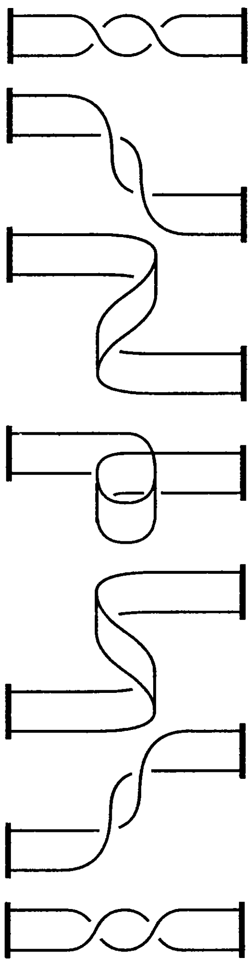
$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow SO(4)$$

of T_M, we lift these maps against the projection $Spin(4) \to SO(4)$ to get maps[3]

$$\tilde{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow Spin(4) .$$

2. As a bit of help in visualizing $Spin(4) \to SO(4)$ with its $\pi_1 SO(4) = \mathbb{Z}_2$, one can invoke for a moment the thought of $S^2 \to \mathbb{RP}^2$. Or, even better, of $S^3 \to \mathbb{RP}^3$. "Better", because in fact $S^3 = Spin(3)$ and $\mathbb{RP}^3 = SO(3)$. In dimension 4, we have $Spin(4) = S^3 \times S^3$ and $SO(4) = S^3 \times S^3/\pm 1$.

3. Such a lift is always possible: choose the covering $\{U_\alpha\}$ so that all $U_\alpha \cap U_\beta$'s are simply-connected.



4.17. $\pi_1 SO(n) = \mathbb{Z}_2$ (when $n \geq 3$)

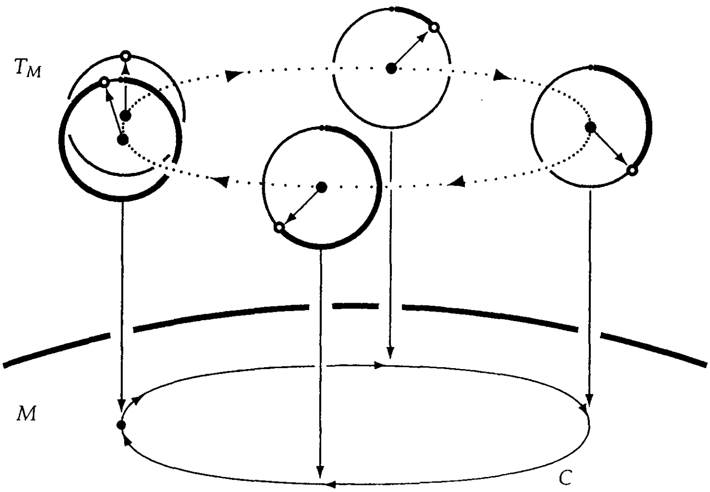
The problem is that, since $Spin(4) \to SO(4)$ is a double-cover, on triple-intersections $U_\alpha \cap U_\beta \cap U_\gamma$ such lifts *a priori* satisfy merely the conditions

$$\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = \pm\, id\ .$$

The appearance of an actual minus-sign makes $\{\tilde{g}_{\alpha\beta}\}$ fail from being a cocycle.

Hence, the manifold M is said to admit spin structures if and only if one can find a good $SO(4)$–cocycle $\{U_\alpha,\, g_{\alpha\beta}\}$ of T_M that can be lifted to $Spin(4)$–valued maps $\{U_\alpha,\, \tilde{g}_{\alpha\beta}\}$ for which *no* minus-signs appears in the equality above, and which thus make up a $Spin(4)$–cocycle.

No oddities. Intuitively, a $Spin(4)$–valued cocycle $\{\tilde{g}_{\alpha\beta}\}$ for T_M exists if and only if odd multiples of 2π can be avoided when gluing up T_M. Explicitly, take a circle C bounding a disk in M and imagine that there are a few locally-trivialized patches $U_\alpha \times \mathbb{R}^4$ of T_M covering C that, when matched up, describe a rotation of 2π when travelling along C (see figure 4.18 on the next page). Then, since these patches describe the nontrivial class in $\pi_1 SO(4) = \mathbb{Z}_2$, they and their



4.18. A non-extendable trivialization of T_M over the circle C

gluing maps $g_{\alpha\beta}$ cannot be used toward lifting to a $Spin(4)$-cocycle. This will be made more clear later.

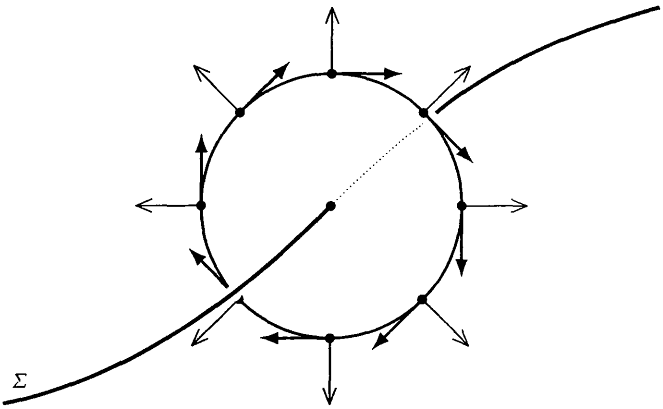
Homotopic simplifications. *Choosing an orientation on M reduces the structure group of T_M from the disconnected group $O(4)$ to the connected group $SO(4)$. Choosing a spin structure on M reduces the structure group of T_M to the simply-connected group $Spin(4)$. This process of homotopy-simplification of the structure group ends here. We already have $\pi_2 SO(4) = 0$ (and thus $\pi_2 Spin(4) = 0$). Further asking of a Lie group G to have $\pi_3 G = 0$ would force G to be contractible, and thus the bundle to be topologically trivial.*

In the remainder of this note, we will comment on what happens when M does not admit spin structures and explain the principal bundle point-of-view on spin structures. The latter will help us argue in the next note (page 181) that the two definitions of spin structures, the one with cocycles and the one with trivializations, are indeed equivalent. The third note (page 189) will develop bundle cocycles in their natural habitat, Čech cohomology. The fourth note (page 197) will present a smattering of obstruction theory and apply it to spin structures, while the fifth note (page 204) will present the homotopy-theoretic point-of-view on spin structures. Some consequences of the cocycle definition of spin structures (spinor bundles, Dirac operators) will be outlined in section 10.2 (page 382), as a quick prelude to the introduction of spin$^{\mathbb{C}}$ structures. The standard reference for spin structures is **B. Lawson** and **M-L. Michelson**'s ***Spin geometry*** **[LM89]**.

When not spinnable. The existence of a spin structure is equivalent to the vanishing of $w_2(T_M)$. We wish to note what happens when no spin structures exist, that is, when $w_2(T_M) \neq 0$. In the cocycle point-of-view, this means that every $Spin(4)$-valued maps $\{\widetilde{g}_{\alpha\beta}\}$, lifted from the $SO(4)$-cocycle of T_M, must have triples α, β, γ with $U_\alpha \cap U_\beta \cap U_\gamma$ non-empty and such that $\widetilde{g}_{\alpha\beta}(x) \cdot \widetilde{g}_{\beta\gamma}(x) \cdot \widetilde{g}_{\gamma\alpha}(x) = -id$.

We pick an integral lift $\underline{w} \in H^2(M;\mathbb{Z})$ of $w_2(T_M)$ and represent $\underline{w}$ by an embedded oriented surface Σ in M. Since the characteristic surface Σ is the incarnation of the obstruction to the existence of a spin structure on M, there exist spin structures away from Σ, on the complement $M \setminus \Sigma$. None of these outside spin structures can be extended across Σ. (In terms of cocycles, we can arrange that the failing triples α, β, γ occur when and only when we go around Σ.)

In the trivializations point-of-view, such an outside spin structure on $M \setminus \Sigma$ offers a trivialization of T_M over the 1–skeleton, which restricts to a trivialization of T_M over small circles surrounding Σ (*e.g.*, fibers of the normal circle-bundle $\mathbb{S}N_{\Sigma/M}$ of Σ in M). Since the outside spin structure cannot extend across Σ, it follows that the trivialization of T_M over each such circle around Σ must describe a twist of 2π, as in figure 4.19. In the note ahead on Čech cohomology (page 196), this description will be made rigorous by using a concrete representation of $w_2(T_M)$.



4.19. Outside spin structure, not extending across a characteristic surface Σ

Principal bundle point-of-view. For any group G, a **principal G–bundle** is a locally-trivial fiber bundle with fiber G and structure group G. In other words, a principal G–bundle over X is a space $\mathcal{P}_G$ together with a projection map $p\colon \mathcal{P}_G \to X$ so that there is some covering $\{U_\alpha\}$ of X and maps $\varphi_\alpha\colon p^{-1}[U_\alpha] \cong U_\alpha \times G$, with $\mathrm{pr}_1 \circ \varphi_\alpha = p$ and so that the overlaps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are described by formulae $(x, \gamma) \mapsto (x, \overline{g}_{\alpha\beta} \cdot \gamma)$ for suitable functions $\overline{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, acting on G by multiplication. Hence $\mathcal{P}_G \to X$ can be obtained by gluing trivial pieces $U_\alpha \times G \to U_\alpha$ using the G–cocycle $\{\overline{g}_{\alpha\beta}\}$, identifying $(x, \gamma_\alpha) \in U_\alpha \times G$ with $(x, \gamma_\beta) \in U_\beta \times G$ if and only if $\gamma_\alpha = \overline{g}_{\alpha\beta}(x) \cdot \gamma_\beta$.

Notice that, unlike a vector bundle, a principal G–bundle does not admit any global sections, unless it is trivial.[4]

Bundle of frames. For example, the $SO(4)$–valued cocycle $\{g_{\alpha\beta}\}$ of T_M acts directly on the group $SO(4)$ itself. Then, by gluing trivial pieces $U_\alpha \times SO(4)$, one obtains from $\{g_{\alpha\beta}\}$ a principal $SO(4)$–bundle

$$\mathcal{P}_{SO(4)} \to M\,.$$

4. The fibers of $\mathcal{P}_G$ may look like G, and G itself acts on them, but they are merely "affine" copies of G, without, for example, a specified identity element. A global section in $\mathcal{P}_G$ can be viewed as offering a coherent choice of identity element, and thus yields an isomorphism $\mathcal{P}_G \approx X \times G$.

The bundle $\mathcal{P}_{SO(4)}$ depends only on T_M, not on the particular choice of $SO(4)$–cocycle $\{g_{\alpha\beta}\}$. Geometrically, one should think of $\mathcal{P}_{SO(4)} \to M$ as the bundle of orienting orthonormal frames of T_M.

A local section $\tau\colon U \to \mathcal{P}_{SO(4)}$ is a frame-field in T_M over U. It is thus equivalent to a trivialization of T_M over U. In particular, a trivialization of T_M over the 1–skeleton $M|_1$ of M is the same as a section $M|_1 \to \mathcal{P}_{SO(4)}$. The trivialization is extendable over the 2–skeleton $M|_2$ if and only if the corresponding section of $\mathcal{P}_{SO(4)}$ can be extended across $M|_2$.

Spin structures. Assume now that the $SO(4)$–cocycle $\{g_{\alpha\beta}\}$ lifts to some $Spin(4)$–valued maps $\{\tilde{g}_{\alpha\beta}\}$ that satisfy the cocycle condition. Then we can use this lifted cocycle to glue a principal $Spin(4)$–bundle

$$\mathcal{P}_{Spin(4)} \to M$$

from trivial pieces $U_\alpha \times Spin(4)$. More, the double-cover $Spin(4) \to SO(4)$ defines fiber-to-fiber a natural map $\mathcal{P}_{Spin(4)} \to \mathcal{P}_{SO(4)}$, fitting in the diagram

$$\begin{array}{ccccc} Spin(4) & \subset & \mathcal{P}_{Spin(4)} & \longrightarrow & M \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & \| \\ SO(4) & \subset & \mathcal{P}_{SO(4)} & \longrightarrow & M\,. \end{array}$$

The map $\mathcal{P}_{Spin(4)} \to \mathcal{P}_{SO(4)}$ is itself a double-cover of $\mathcal{P}_{SO(4)}$.

A spin structure can thus be redefined as a principal $Spin(4)$–bundle $\mathcal{P}_{Spin(4)}$ that double-covers the bundle $\mathcal{P}_{SO(4)}$ (and fits in the diagram above).

Note: Equivalence of the definitions of a spin structure

In what follows, we will prove hands-on the equivalence between defining spin structures as extendable trivializations of T_M and defining them as lifted $Spin(4)$–cocycles. Reading the preceding note is, obviously, a requisite.

Of course, more streamlined arguments exist. (Here is the best one: both the existence of an extendable trivialization and of a $Spin(4)$–cocycle are equivalent with the vanishing of $w_2(T_M)$; the end.) Nonetheless, in what follows we favor a concrete approach, which is rather expensive; we choose to present it here owing to its absence from the literature.

Our argument is rather long and involves some careful play with triangulations, principal bundles and double-covers, but the basic idea is pretty straightforward: let $E \to \mathbb{D}^2$ be a vector bundle over a disk, with fiber $\mathbb{R}^4$. Since $\mathbb{D}^2$ is contractible, E must be trivial; for definiteness fix a reference trivialization $E \approx \mathbb{D}^2 \times \mathbb{R}^4$. Consider some other random trivialization $\varphi\colon E|_{S^1} \approx S^1 \times \mathbb{R}^4$ over the boundary of the base. Think of φ as a field of frames in E over $\partial\mathbb{D}^2$, that is to say, as a map $\varphi_f\colon S^1 \to SO(4)$. The trivialization φ will extend across all $\mathbb{D}^2$ if and only if the frame-field φ_f can be extended over $\mathbb{D}^2$. That happens if and only if the loop φ_f in $SO(4)$ is homotopically-trivial, that is to say, if and only if

$\varphi_f\colon \mathbb{S}^1 \to SO(4)$ can be lifted to a closed loop $\widetilde{\varphi}_f\colon \mathbb{S}^1 \to Spin(4)$ (and not to an open path $\widetilde{\varphi}_f\colon [0,1] \to Spin(4)$, with $\widetilde{\varphi}_f(0) = -\widetilde{\varphi}_f(1)$).

Throughout this note, assume that M has been triangulated, in other words, exhibited as a simplicial complex.[5] Denote by $M|_1$ the 1–skeleton of M, by $M|_2$ the 2–skeleton, and so on. Further, for any bundle E over M, denote by $E|_k$ the restriction of E to the k–skeleton of M (and not the k–skeleton of the manifold E).

From cocycles to trivializations. Assume first that a $SO(4)$–cocycle $\{g_{\alpha\beta}\}$ of T_M lifts to some maps $\{\widetilde{g}_{\alpha\beta}\}$ that actually satisfy the cocycle condition. Then a corresponding principal $Spin(4)$–bundle $\mathcal{P}_{Spin(4)}$ is well-defined. We will show that the existence of the bundle $\mathcal{P}_{Spin(4)}$ implies that T_M can be trivialized over the 2–skeleton $M|_2$. Specifically, we will show that the frame-bundle $\mathcal{P}_{SO(4)}$ admits a section over $M|_2$. For that, we define a section $\widetilde{\tau}$ of $\mathcal{P}_{Spin(4)}$ over $M|_2$ and project it to a section of $\mathcal{P}_{SO(4)}$. The section $\widetilde{\tau}$ is defined using a simplex-by-simplex construction.[6]

We start with the vertices of M and define each $\widetilde{\tau}(vertex)$ in some random manner as an element of $\mathcal{P}_{Spin(4)}$ in the fiber above it.

Any edge ε of M is contractible, and thus $\mathcal{P}_{Spin(4)}|_\varepsilon$ is trivial. Choose some trivialization $\mathcal{P}_{Spin(4)}|_\varepsilon \approx \varepsilon \times Spin(4)$. The section $\widetilde{\tau}$ is already defined at the endpoints (vertices) of ε. By looking through the trivialization, we see that the fact that $Spin(4)$ is connected implies that $\widetilde{\tau}$ can always be extended over ε, and thus eventually across the whole 1–skeleton $M|_1$.

There remain the 2–simplices. Any 2–simplex D is contractible and thus $\mathcal{P}_{Spin(4)}|_D$ can be trivialized as $D \times Spin(4)$. The section $\widetilde{\tau}$ is already defined over the edges that make up the boundary ∂D. Looking through the trivialization and using that $Spin(4)$ is simply-connected allows us to extend $\widetilde{\tau}$ over D, and eventually across the whole 2–skeleton $M|_2$.

The resulting section $\widetilde{\tau}\colon M|_2 \to \mathcal{P}_{Spin(4)}$ can be projected through the double-cover $\mathcal{P}_{Spin(4)} \to \mathcal{P}_{SO(4)}$ to a section $\tau\colon M|_2 \to \mathcal{P}_{SO(4)}$. The latter is a field of frames in T_M that trivializes T_M over $M|_2$.

Notice that, since we have $\pi_2 SO(4) = 0$ (and thus $\pi_2 Spin(4) = 0$), a bit more can be done: the section $\widetilde{\tau}$ of $\mathcal{P}_{Spin(4)}$ can be further extended across the 3–skeleton of M, yielding a trivialization of T_M over $M|_3$, which can be viewed as a trivialization over $M \setminus \{point\}$.

5. A **triangulation** is a decomposition of M into simplices. A 0–simplex, or *vertex*, is a point. A 1–simplex, or *edge*, is a copy of $[0,1]$; its faces are its endpoint-vertices. A 2–simplex is a triangle (interior included); its faces are its three edges. A 3–simplex is a tetrahedron (interior included); its faces are the obvious four 2–simplices. A 4–simplex is whatever you want to call what follows; its faces are 3–simplices. If a simplex is part of a triangulation, then all its faces must be simplices of the triangulation. All simplices of a triangulation of M must be embedded in M and must either have exactly a whole sub-simplex (= face, or face-of-face, or…) in common with another simplex or be disjoint from it. In short, a triangulation of M means making M look like a polyhedron with simple "triangular" faces.

6. This simplex-by-simplex method is just a most simple application of the method of *obstruction theory,* which will be explained in generality in the note on page 197 ahead. If you do not like the word "simplex", you can substitute "handle" or "cell" throughout.

Uniqueness. It is worth noting that the trivialization τ of $T_M|_1$ that we obtained above is uniquely determined, up to homotopies, by the spin structure $\mathcal{P}_{Spin(4)}$. Indeed, assume two random sections $\tilde{\tau}'$ and $\tilde{\tau}''$ of $\mathcal{P}_{Spin(4)}$ are given over $M|_1$. We will define a homotopy between them over the 1-skeleton of M. For that, we view a homotopy as a section of the bundle $\mathcal{P}_{Spin(4)} \times [0,1] \longrightarrow M \times [0,1]$ that limits to $\tilde{\tau}'$ over $M \times 0$ and to $\tilde{\tau}''$ over $M \times 1$. Since $\tilde{\tau}'$ and $\tilde{\tau}''$ are given, such a section is already defined over the vertices of $M \times [0,1]$. It can be extended across the edges connecting $M \times 0$ with $M \times 1$, using as above that $Spin(4)$ is connected. Then it can be extended over the 2-simplices of $M \times [0,1]$ by using that $\pi_1 Spin(4) = 0$. Thus, we have defined a homotopy between $\tilde{\tau}'$ and $\tilde{\tau}''$ over $M|_1$. This descends to a homotopy between the induced trivializations τ' and τ'' of T_M, proving uniqueness.

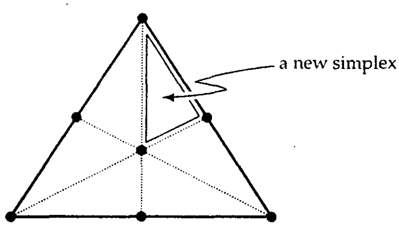
In conclusion, a spin structure defined *via* cocycles determines an extendable trivialization of $T_M|_1$, unique up to homotopies.

From trivializations to cocycles: Preparation. The converse argument involves a rather cumbersome setup that will allow us to link 1-skeletons and trivializations to cocycles and their lifts. It will take the rest of this note (through page 189).

Assume that M has been endowed with a fixed triangulation $\mathscr{T}$. For definiteness, fix a Riemannian metric on M. We will prove that any trivialization of $T_M|_1$ that extends across $M|_2$ defines a $Spin(4)$-cocycle for T_M.

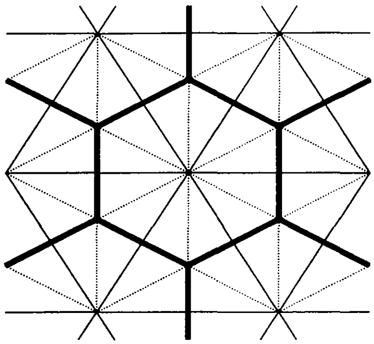
First, remember that any triangulation $\mathscr{T}$ admits a *dual cellular decomposition* $\mathscr{T}^*$.

Given a triangulation $\mathscr{T}$ of M^4, its **dual cellular decomposition** *$\mathscr{T}^*$ is obtained by taking the barycentric subdivision[7] $\mathscr{T}'$ of $\mathscr{T}$, then, for each $(4-k)$-simplex Δ_α of $\mathscr{T}$, defining its dual k-cell Δ_α^* in $\mathscr{T}^*$ by taking the union of all k-simplices of $\mathscr{T}'$ that touch the barycenter of Δ_α. For example, the vertices of $\mathscr{T}^*$ are the barycenters of the 4-simplices of $\mathscr{T}$, the 1-cells of $\mathscr{T}^*$ are arcs normal to the 3-simplices of $\mathscr{T}$ (and link the vertices of $\mathscr{T}^*$), while the 4-cells of $\mathscr{T}^*$ are neighborhoods of the vertices of $\mathscr{T}$. See figure 4.21 on the following page. The dual cellular decomposition is an especially nice cellular decomposition, in that it fails from being a triangulation only by using more general "polygonal" cells rather than just "triangular" simplices; otherwise, all cells are embedded, etc. (On the side, note that dual cellular decompositions can be used to offer a nice visualization of Poincaré duality.)*



4.20. Barycentric subdivision of a 2-simplex

7. The **barycenter** of a simplex Δ is simply a canonical center for it. The barycenter of a vertex is the vertex itself. The **barycentric subdivision** $\mathscr{T}'$ of $\mathscr{T}$ is obtained by taking as new k-simplices every *join* of the barycenter of an old k-simplex of $\mathscr{T}$ with the barycenter of a face and the barycenter of a face of that face and... For example, a 2-simplex in $\mathscr{T}'$ is the triangle that appears by joining the barycenter of a triangle of $\mathscr{T}$ with the center of one of its edges and with the vertex at one end of that edge. See figure 4.20. The **join** of two subsets A and B of $\mathbb{R}^n$ is the union of all segments that start in A and end in B.



4.21. Cellular decomposition dual to a triangulation

Since we have to deal with trivializations of T_M over the 1–skeleton $M|_1$ and their extendability over the 2–skeleton $M|_2$, we will only use cocycles $\{U_\alpha, g_{\alpha\beta}\}$ of T_M that are nicely compatible with the chosen triangulation $\mathscr{T}$ of M.

Namely, we will take the U_α's to be small neighborhoods of the 4–cells Δ^*_α of the dual decomposition $\mathscr{T}^*$ of M. The 4–cell Δ^*_α is a closed set surrounding a vertex v_α and touching the barycenters of all 4–simplices that contain v_α. In particular, each edge ε of $\mathscr{T}$ links the center of U_α with the center of U_β and passes through the overlap $U_\alpha \cap U_\beta$. The latter intersection is just a small neighborhood of the 3–cell (dual to ε) that Δ^*_α and Δ^*_β have in common.

Since each U_α is contractible, $T_M|_{U_\alpha}$ is trivial. Using the Riemannian metric of M, we choose trivializations

$$\varphi_\alpha \colon T_M|_{U_\alpha} \approx U_\alpha \times \mathbb{R}^4$$

that are isometries on the fibers. We compare these trivializations over $U_\alpha \cap U_\beta$ and obtain transition maps

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow SO(4) \qquad \text{with} \qquad \varphi_\alpha = g_{\alpha\beta} \cdot \varphi_\beta \,.$$

These will be the cocycles $\{U_\alpha, g_{\alpha\beta}\}$ of T_M that we will consider. Notice that these cocycles depend essentially only on the choice of trivializations φ_α over the U_α's.

Trivializations and partial Spin–bundles. Given any trivialization

$$\Theta \colon T_M|_1 \approx M|_1 \times \mathbb{R}^4$$

of T_M over the 1–skeleton of M, we express Θ in coordinates with respect to the charts $\varphi_\alpha \colon T_M|_{U_\alpha} \approx U_\alpha \times \mathbb{R}^4$. Namely, we describe Θ by a collection of $SO(4)$–valued maps τ_α, defined on the part of the 1–skeleton of M that is included in U_α, which we denote by $U_\alpha|_1$ (see figure 4.22 on the next page).

Specifically, the maps

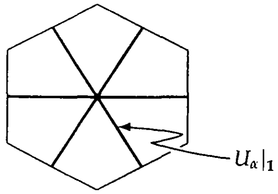
$$\tau_\alpha \colon U_\alpha|_1 \longrightarrow SO(4)$$

are defined by the equations $\tau_\alpha(x) \cdot w = \varphi_\alpha(\Theta^{-1}(x, w))$ and will satisfy compatibility relations

$$\tau_\alpha = g_{\alpha\beta} \cdot \tau_\beta \,.$$

An alternative view of the τ_α's is as defining a section

$$\tau \colon M|_1 \longrightarrow \mathcal{P}_{SO(4)}|_1 \,,$$



4.22. Open set U_α, and the 1-skeleton of M

corresponding to the frame-field induced by the trivialization Θ.

Consider a random lift of the maps $\tau_\alpha \colon U_\alpha|_1 \to SO(4)$ to some maps

$$\tilde{\tau}_\alpha \colon U_\alpha|_1 \longrightarrow Spin(4) .$$

Given such a collection $\{\tilde{\tau}_\alpha\}$, we can correspondingly choose lifts

$$\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow Spin(4)$$

of the $g_{\alpha\beta}$'s in such manner as to fit the various $\tilde{\tau}_\alpha$'s, namely so that

$$\tilde{\tau}_\alpha = \tilde{g}_{\alpha\beta} \cdot \tilde{\tau}_\beta .$$

Since this fitting amounts merely to a choice of sign for each $\tilde{g}_{\alpha\beta}$ and owing to the special shape of our covering $\{U_\alpha\}$, such a lift can always be made.

Of course, $\{\tilde{g}_{\alpha\beta}\}$ is most likely not a cocycle. Whether it is or not depends only on the τ_α's, not on the random lifts $\tilde{\tau}_\alpha$. To see this, consider two random lifts $\{\tilde{\tau}'_\alpha\}$ and $\{\tilde{\tau}''_\alpha\}$. They can differ at most by a collection of signs $\varepsilon_\alpha \in \mathbb{Z}_2 = \{-1, +1\}$ with $\tilde{\tau}''_\alpha = \varepsilon_\alpha \tilde{\tau}'_\alpha$. The corresponding transition maps are then related by $\tilde{g}''_{\alpha\beta} = \varepsilon_\alpha \varepsilon_\beta \tilde{g}'_{\alpha\beta}$. Clearly, we have $\tilde{g}''_{\alpha\beta} \cdot \tilde{g}''_{\beta\gamma} \cdot \tilde{g}''_{\gamma\alpha} = +1$ if and only if $\tilde{g}'_{\alpha\beta} \cdot \tilde{g}'_{\beta\gamma} \cdot \tilde{g}'_{\gamma\alpha} = +1$. In particular, when one choice of $\tilde{\tau}_\alpha$'s leads to a cocycle, then so will any other choice, and the various choices lead to isomorphic cocycles, *i.e.*, a unique spin structure.

By definition, the maps $\tilde{g}_{\alpha\beta}$ satisfy $\tilde{g}_{\alpha\beta} = \tilde{g}_{\beta\alpha}^{-1}$. Therefore, if we avoid all triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$, then the lifts $\tilde{g}_{\alpha\beta}$ can be used to define a principal $Spin(4)$–bundle away from the $U_\alpha \cap U_\beta \cap U_\gamma$'s. In particular, we get a bundle

$$\mathcal{P}_{Spin(4)}|_1$$

well-defined over the 1–skeleton of M.

Of course, $\mathcal{P}_{Spin(4)}|_1$ is a double-cover of $\mathcal{P}_{SO(4)}|_1$, built fiberwise from the projection $Spin(4) \to SO(4)$. Furthermore, the maps $\tilde{\tau}_\alpha$ can be viewed as defining a section $\tilde{\tau} \colon M|_1 \to \mathcal{P}_{Spin(4)}|_1$.

Trivial versus nontrivial covers. Since the bundle $\mathcal{P}_{Spin(4)}|_1$ defined above is a principal bundle, having a section $\tilde{\tau}$ implies that it is a trivial bundle over $M|_1$. Nonetheless, it can project in a nontrivial way onto $\mathcal{P}_{SO(4)}|_1$. In what follows we will investigate how this nontriviality can be detected. Since $\mathcal{P}_{Spin(4)}|_1 \to \mathcal{P}_{SO(4)}|_1$ is a cover projection, fundamental groups will play a prominent role in the argument.

Restrict to the boundary ∂D of a fixed 2–simplex D. Since both $\mathcal{P}_{SO(4)}|_{\partial D}$ and $\mathcal{P}_{Spin(4)}|_{\partial D}$ admit sections τ and $\tilde{\tau}$, they are trivial, and thus $\mathcal{P}_{SO(4)}|_{\partial D} \approx \partial D \times SO(4)$ and $\mathcal{P}_{Spin(4)}|_{\partial D} \approx \partial D \times Spin(4)$. Therefore

$$\pi_1\big(\mathcal{P}_{SO(4)}|_{\partial D}\big) = \mathbb{Z} \oplus \mathbb{Z}_2 \qquad \text{and} \qquad \pi_1\big(\mathcal{P}_{Spin(4)}|_{\partial D}\big) = \mathbb{Z}\,.$$

Denote by d the double-cover map

$$d\colon \mathcal{P}_{Spin(4)}|_{\partial D} \longrightarrow \mathcal{P}_{SO(4)}|_{\partial D}\,,$$

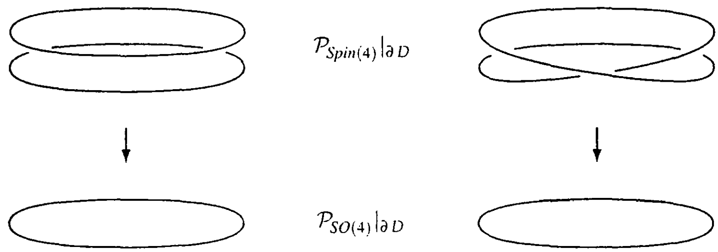
fitting in the diagram

$$\begin{array}{ccc} \mathcal{P}_{Spin(4)}|_{\partial D} & \xrightarrow[d]{} & \mathcal{P}_{SO(4)}|_{\partial D} \\ \downarrow & & \downarrow \\ \partial D & = & \partial D \end{array} \qquad \text{or, on } \pi_1\text{'s:} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow[d_*]{} & \mathbb{Z} \oplus \mathbb{Z}_2 \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ \mathbb{Z} & = & \mathbb{Z} \end{array}\,.$$

Being a cover map, d's induced morphism d_* must be injective. We deduce that there are only two choices: either

$$d_*(1) = 1 \oplus 0 \qquad \text{or} \qquad d_*(1) = 1 \oplus 1 \quad \in \mathbb{Z} \oplus \mathbb{Z}_2\,.$$

The case $d_*(1) = 1 \oplus 0$ corresponds to the case when the cover $\mathcal{P}_{Spin(4)}|_{\partial D} \to \mathcal{P}_{SO(4)}|_{\partial D}$ is trivial, while $d_*(1) = 1 \oplus 1$ happens when the fiber of $\mathcal{P}_{Spin(4)}|_{\partial D}$ twists once as we go around ∂D, as suggested in figure[8] 4.23.



4.23. Trivial and nontrivial covers

To better visualize how this can happen, consider the trivial bundles $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{RP}^3$ over $\mathbb{S}^1$. There are two possible double-cover projections d of $\mathbb{S}^1 \times \mathbb{S}^3$ onto $\mathbb{S}^1 \times \mathbb{RP}^3$ that both commute with the bundle projections and hence fit in a diagram

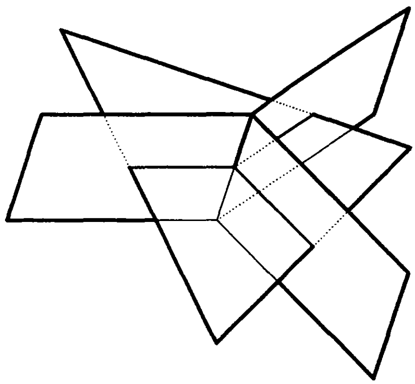
$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^3 & \xrightarrow[d]{} & \mathbb{S}^1 \times \mathbb{RP}^3 \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & = & \mathbb{S}^1\,. \end{array}$$

One possible double-cover is the obvious one, the product of the identity on $\mathbb{S}^1$ with the double-cover $\mathbb{S}^3 \to \mathbb{RP}^3$. The other can be seen as follows: start with $[0,1] \times \mathbb{S}^3$ and glue the ends $0 \times \mathbb{S}^3$ and $1 \times \mathbb{S}^3$ using the antipodal map on $\mathbb{S}^3$; project each $\mathbb{S}^3$ to $\mathbb{RP}^3$ to get a double-cover of $\mathbb{S}^1 \times \mathbb{RP}^3$. However, since the antipodal of $\mathbb{S}^3$ is homotopic to the identity, what we glued is still $\mathbb{S}^1 \times \mathbb{S}^3$. The first map has $d_*(1) = 1 \oplus 0$, while the second has $d_*(1) = 1 \oplus 1$. In fact, this example is pretty close to our concerns, since $\mathbb{S}^3 = Spin(3)$ and $\mathbb{RP}^3 = SO(3)$.

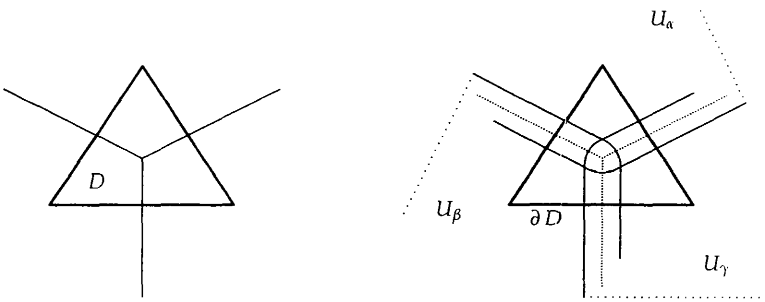
8. Owing to dimension-reduction, figure 4.23 is misleading: on both sides, the space $\mathcal{P}_{Spin(4)}|_{\partial D}$ should be the same trivial bundle over ∂D.

Detecting nontriviality with cocycle candidates. The two cases $d_*(1) = 1 \oplus 0$ and $d_*(1) = 1 \oplus 1$ are detected both by the lifted $Spin(4)$–valued maps $\widetilde{g}_{\alpha\beta}$ and by the section $\widetilde{\tau}$ of $\mathcal{P}_{Spin(4)}|_1$. We start with the $\widetilde{g}_{\alpha\beta}$'s.

Since D is a 2–simplex of M, it is surrounded by three of the open sets from our covering, say U_α, U_β and U_γ, with the center of D right in the middle of $U_\alpha \cap U_\beta \cap U_\gamma$, as suggested in figures 4.24 and 4.25.



4.24. Set-up for equivalence argument, I



4.25. Set-up for equivalence argument, II

We claim that, for the indices α, β, γ around D, we have

$$\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha} = +1$$

if and only if the cover $\mathcal{P}_{Spin(4)}|_{\partial D} \to \mathcal{P}_{SO(4)}|_{\partial D}$ is trivial, that is to say, if and only if $d_*(1) = 1 \oplus 0$.

Assume first that the product of the $\widetilde{g}_{\alpha\beta}$'s around D is $+1$. Then the $\widetilde{g}_{\alpha\beta}$'s can safely be used to extend $\mathcal{P}_{Spin(4)}|_{\partial D}$ over D as a bundle $\mathcal{P}_{Spin(4)}|_D$, fitting in

$$\begin{array}{ccc} \mathcal{P}_{Spin(4)}|_{\partial D} & \subset & \mathcal{P}_{Spin(4)}|_D \\ {\scriptstyle d}\big\downarrow & & \big\downarrow \\ \mathcal{P}_{SO(4)}|_{\partial D} & \subset & \mathcal{P}_{SO(4)}|_D \end{array} \qquad \text{or, on } \pi_1\text{'s:} \qquad \begin{array}{ccc} \mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle d_*}\big\downarrow & & \big\downarrow \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \xrightarrow{\ \mathrm{pr}_2\ } & \mathbb{Z}_2 \end{array} .$$

Thus the only possibility for d_* is

$$d_*(1) = 1 \oplus 0 \,.$$

Conversely, assume that $d_*(1) = 1 \oplus 0$. Then $\mathcal{P}_{Spin(4)}|_{\partial D} \to \mathcal{P}_{SO(4)}|_{\partial D}$ must be the trivial double-cover, with

$$\mathcal{P}_{Spin(4)}|_{\partial D} \approx \mathcal{P}_{SO(4)}|_{\partial D} \times \{-1, +1\} \,.$$

Therefore it can be extended to a double-cover $\mathcal{P}$ of $\mathcal{P}_{SO(4)}$ across the whole D, with $\mathcal{P}|_{\partial D} = \mathcal{P}_{Spin(4)}|_{\partial D}$. Such a double-cover, when projected down to D, can only have as fibers copies of $Spin(4)$. Moreover, since $\mathcal{P}$ projects to $\mathcal{P}_{SO(4)}|_D$, its cocycle must project to the cocycle $g_{\alpha\beta}$ of $\mathcal{P}_{SO(4)}$. Further, since $\mathcal{P} \to D$ is glued over ∂D by the $\widetilde{g}_{\alpha\beta}$'s, it must be that it is glued over the whole D by the $\widetilde{g}_{\alpha\beta}$'s. This in particular implies that the $\widetilde{g}_{\alpha\beta}$'s, since they glue an actual bundle over D, must be a genuine cocycle over D, and thus

$$\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha} = +1 \,.$$

In conclusion, $\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha} = +1$ if and only if $d_*(1) = 1 \oplus 0$.

Detecting nontriviality with trivializations. Now we will see how to distinguish between the two cases $d_*(1) = 1 \oplus 0$ and $d_*(1) = 1 \oplus 1$ by using the trivialization $\Theta\colon T_M|_1 \approx M|_1 \times \mathbb{R}^4$.

The trivialization Θ expresses itself through the section τ of $\mathcal{P}_{SO(4)}|_1$, with local coordinates $\tau_\alpha\colon U_\alpha \to SO(4)$. Recall that we chose random lifts $\widetilde{\tau}_\alpha\colon U_\alpha \to Spin(4)$ and then picked the maps $\widetilde{g}_{\alpha\beta}$ in such manner as to ensure that the $\widetilde{\tau}_\alpha$'s would define a section in the partial $Spin(4)$–bundle $\mathcal{P}_{Spin(4)}|_1$ that is glued by the $\widetilde{g}_{\alpha\beta}$'s.

Over the boundary ∂D, we have the diagram

$$\begin{array}{ccc} \mathcal{P}_{Spin(4)}|_{\partial D} & \xrightarrow[d]{} & \mathcal{P}_{SO(4)}|_{\partial D} \\ {\scriptstyle\widetilde{\tau}}\uparrow & & \uparrow{\scriptstyle\tau} \\ \partial D & = & \partial D \end{array} \qquad \text{or, on } \pi_1\text{'s:} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow[d_*]{} & \mathbb{Z} \oplus \mathbb{Z}_2 \\ {\scriptstyle\widetilde{\tau}_*}\uparrow & & \uparrow{\scriptstyle\tau_*} \\ \mathbb{Z} & = & \mathbb{Z} \end{array} \,.$$

Since from commuting we must have that $\tau_*(1) = d_*(1)$, it follows that either $\tau_*(1) = 1 \oplus 0$ or $\tau_*(1) = 1 \oplus 1$.

Trivialize $\mathcal{P}_{SO(4)}$ over D as $D \times SO(4)$ and use the inclusion

$$\mathcal{P}_{SO(4)}|_{\partial D} \subset \mathcal{P}_{SO(4)}|_D \approx D \times SO(4)$$

to obtain from $\tau\colon \partial D \to \mathcal{P}_{SO(4)}|_{\partial D}$ a map $\tau_0\colon \partial D \to SO(4)$. Then the section τ of $\mathcal{P}_{SO(4)}|_{\partial D}$ can be extended to a section of $\mathcal{P}_{SO(4)}$ over all D if and only if the induced map $\tau_0\colon \partial D \to SO(4)$ is homotopically-trivial. In other words, if and only if we have $\tau_*(1) = 1 \oplus 0$ and not $1 \oplus 1$.

In conclusion, the trivialization Θ of T_M over the 1–skeleton can be extended over the 2–simplex D if and only if $d_*(1) = 1 \oplus 0$.

Final twirl. Gathering our toys, we notice that we have proved the statement:

Given a trivialization Θ of $T_M|_1$, it can be extended over a 2–simplex D surrounded by the open sets U_α, U_β, U_γ if and only if $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = +1$.

In particular, if Θ is a trivialization of T_M over the 1–skeleton that extends across the whole 2–skeleton, then it can be used to define lifted maps $\{\tilde{g}_{\alpha\beta}\}$ that will constitute a $Spin(4)$–cocycle.

The proof is concluded: an extendable trivialization defines a unique $Spin(4)$–cocycle, up to isomorphisms.

Note: Bundles, cocycles, and Čech cohomology

In this note we describe the Čech cohomology of a manifold, with constant coefficients in an Abelian group G. Then we extend this concept, on one hand, to non-Abelian groups and, on the other hand, to non-constant coefficients. (We will not take the next step of defining the general cohomology of a sheaf.)

This will enable us to present a cocycle defining a bundle as a Čech cocycle that defines a cohomology class in $\check{H}^1(M; C^\infty GL(k))$. Consequently, $\check{H}^1(M; C^\infty GL(k))$ can be viewed as the set of all k–plane bundles over M, up to isomorphisms. This approach will allow us to get concrete descriptions of a few characteristic classes and will be used to touch upon the obstruction and uniqueness of spin structures on M.

Čech cohomology. One should think of Čech cohomology as a cohomology theory that uses open coverings and the way their open sets assemble (intersect) patching-up the manifold M, in order to detect the topology of M.

Let $\{U_\alpha\}$ be a covering of M by open sets, and G an Abelian group. We consider collections of G–valued functions defined on intersections of the U_α's. Pick an integer n and choose a set of *locally-constant* functions

$$\varphi = \{\varphi_{\alpha_0\ldots\alpha_n} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \longrightarrow G\},$$

each defined on the intersection of $n+1$ of the open sets U_α. This collection is called a **Čech n–cochain** with values in G. We denote by

$$\check{C}^n(\{U_\alpha\}; G)$$

the Abelian group of all such Čech n–cochains.

The **coboundary** operator $\delta \colon \check{C}^n \to \check{C}^{n+1}$ sends each φ to an $(n+1)$–cochain $\delta\varphi$, a set of functions defined on intersections of $n+2$ of the U_α's, each described as an alternating sum of restrictions of φ's. Namely, we set

$$(\delta\varphi)_{\alpha_0\ldots\alpha_{n+1}} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_{n+1}} \longrightarrow G$$
$$(\delta\varphi)_{\alpha_0\ldots\alpha_{n+1}}(x) = \sum (-1)^k \varphi_{\alpha_0\ldots\widehat{\alpha}_k\ldots\alpha_{n+1}}(x)$$

(where $\widehat{\alpha}_k$ signals the removal of α_k).

If an n–cochain φ has $\delta\varphi = 0$, then φ is called a **Čech cocycle.** If $\varphi = \delta\alpha$ for some $(n-1)$–chain α, then φ is called a **Čech coboundary.** The **Čech cohomology group**

$\check{H}^*(\{U_\alpha\};G)$ of the covering $\{U_\alpha\}$ of M is then defined in the usual fashion, as cocycles modulo coboundaries:

$$\check{H}^n(\{U_\alpha\};G) = \{\varphi \in \check{C}^n(\{U_\alpha\};G) \mid \delta\varphi = 0\} \Big/ \{\delta\alpha \mid \alpha \in \check{C}^{n-1}(\{U_\alpha\};G)\}.$$

A priori these groups depend on the chosen open covering $\{U_\alpha\}$. Eliminating this dependence, the **Čech cohomology group** of M is defined as the direct limit

$$\check{H}^*(M;G) = \varinjlim \check{H}^*(\{U_\alpha\},G),$$

taken over refinements of the open cover.

If M is a manifold, then $\check{H}^*(M;G)$ coincides with the usual singular cohomology of M, as we will prove directly in an instant.

Taking the direct limit is rather unpleasant, and it is almost never done. Indeed, it is enough to consider a fine enough covering, for example a covering $\{U_\alpha\}$ of M by contractible open sets, with all intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ contractible as well.[9] For such coverings we have $\check{H}^*(M;G) = \check{H}^*(\{U_\alpha\};G)$.

Simple examples. The group $\check{H}^0(\{U_\alpha\};G)$ comes from 0–cocycles, that is to say, from collections $\varphi = \{\varphi_\alpha : U_\alpha \to G\}$ of locally-constant functions defined on the U_α's and satisfying $\delta\varphi = 0$. In this case, the cocycle condition is

$$\delta\varphi = 0 \qquad \Longleftrightarrow \qquad \varphi_\alpha = \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta,$$

and therefore immediately

$$\check{H}^0(\{U_\alpha\};G) = \{\text{locally-constant functions } M \to G\}.$$

Hence $\check{H}^0$ detects the components of M: if M is connected, then $\check{H}^0(M;G) = G$.

The first group $\check{H}^1(\{U_\alpha\};G)$ comes from 1–cocycles, that is to say, from families $\varphi = \{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ satisfying $\delta\varphi = 0$, where

$$\delta\varphi = 0 \qquad \Longleftrightarrow \qquad \varphi_{\alpha\gamma} = \varphi_{\alpha\beta} + \varphi_{\beta\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

In particular, notice that a 1–cocycle must satisfy the skew-symmetry $\varphi_{\alpha\beta} = -\varphi_{\beta\alpha}$. These 1–cocycles yield cohomology classes in $\check{H}^0(\{U_\alpha\};G)$ by considering them up to the addition of a coboundary. That is, for any two cocycles φ' and φ'', we have:

$$[\varphi'] = [\varphi''] \quad \text{in } \check{H}^1 \qquad \Longleftrightarrow \qquad \varphi'_{\alpha\beta} = \varphi''_{\alpha\beta} + f_\alpha - f_\beta$$

for some 0–cochain $f = \{f_\alpha : U_\alpha \to G\}$.

And the usual suspects. We now prove directly that nothing new is obtained:

Lemma. *If X is a simplicial complex (e.g., a triangulated manifold), then*

$$\check{H}^*(X;G) = H^*(X;G),$$

where on the right we have the simplicial cohomology of X.

9. A typical geometric method for building such coverings is to pick a Riemannian metric on M and choose *geodesically convex* open sets for the U_α's. A more topological method would use a triangulation of M and take the U_α's to be the *stars* of the vertices of M.

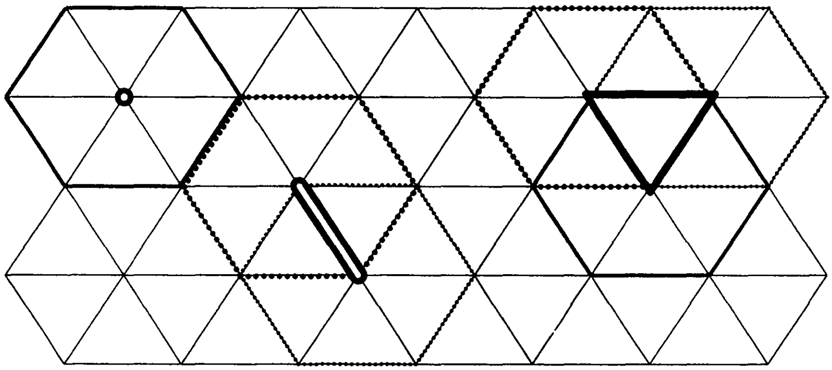
Proof. For every vertex v of X we define its **star**, denoted by $\mathrm{star}(v)$, as the union of all simplices of X that contain v. List the vertices of X as $\{v_\alpha\}$ and define the open sets U_α as

$$U_\alpha = \text{interior of } \mathrm{star}(v_\alpha) \;.$$

Then we have that

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \varnothing \quad \text{if and only if} \quad v_{\alpha_0}, \ldots, v_{\alpha_n} \text{ span an } n\text{-simplex.}$$

See also figure 4.26.



4.26. Linking Čech cochains with simplicial cochains

Each of these intersections is connected, and therefore every Čech n-cochain φ is constant on it. Thus, a Čech n-cochain φ simply assigns to every n-simplex $\langle v_{\alpha_0}, \ldots, v_{\alpha_n} \rangle$ of X an element $\varphi_{\alpha_0 \ldots \alpha_n}$ of G, and hence corresponds bijectively to a simplicial n-cochain.

Finally, it is not hard to check that the Čech and simplicial coboundary operators correspond perfectly, and thus

$$\check{H}^*(\{U_\alpha\}; G) = H^*(X; G) \;.$$

Going to the limit with the coverings is not a problem, *e.g.*, by using subdivisions of the simplicial complex. □

Even though nothing new appears at the outset, Čech theory admits a remarkable extension from coefficients in a group to coefficients in a presheaf and leads to the sheaf cohomology that is essential in complex geometry. We will not fully pursue that avenue, but the reader is encouraged to consult **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** **[GH78, GH94]**.

Another remarkable extension of the theory is to non-commutative groups:

Non-commutative Čech cohomology. One should notice that the whole cohomology apparatus depends on G being Abelian, and thus the extension to the non-Abelian case will have serious restrictions. Namely, $H^1(M; G)$ ceases to be a group and $H^2(M; G)$ ceases to be altogether. However, since vector bundles are glued using non-commutative groups such as $GL(k)$, $SO(k)$, $U(k)$, we do need to pursue this direction. Thus, let G be a *non-Abelian* group, written *multiplicatively*. We can define Čech cochains just as before. However, when it comes to defining the coboundary operator, we need to be careful.

We are only interested in $H^1(M;G)$, so let $\varphi = \{\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G\}$ be a G–valued 1–cochain. Switching from additive to multiplicative writing, we write

$$(\delta\varphi)_{\alpha\beta\gamma} = \varphi_{\alpha\beta} \cdot \varphi_{\beta\gamma} \cdot \varphi_{\gamma\alpha} \, .$$

A 1–cocycle must then be any φ with $(\delta\varphi)_{\alpha\beta\gamma} = 1$ for every α, β, γ. In particular, every cocycle has $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$.

Now let $f = \{f_\alpha\colon U_\alpha \to G\}$ be a 0–cochain. Its coboundary is, naturally,

$$(\delta f)_{\alpha\beta} = f_\alpha \cdot f_\beta^{-1} \, .$$

Nonetheless, when it comes to defining when two 1–cochains φ' and φ'' are cohomologous, that is, when φ' and φ'' are considered to differ by δf, the non-commutativity of G makes essential a specific choice of order. The right one is:

$$[\varphi'] = [\varphi''] \qquad \Longleftrightarrow \qquad \varphi'_{\alpha\beta} = f_\alpha \cdot \varphi''_{\alpha\beta} \cdot f_\beta^{-1} \, .$$

Then we can define in the usual manner the Čech cohomology *set* $\check{H}^1(\{U_\alpha\}\colon G)$ of the covering $\{U_\alpha\}$, and thereafter its limit $\check{H}^1(M;G) = \varinjlim \check{H}^1(\{U_\alpha\};G)$. Since the coboundaries cannot be expected to make up a normal subgroup of the cocycles, this $\check{H}^1$ is not a group, but merely a set with a distinguished element, the class of the trivial cocycle given by $\mathbf{1}_{\alpha\beta} = 1$.

The similarities with the cocycles that glue bundles should be obvious by now. Nonetheless, to fully engulf that case we need to extend the notion of cochain a bit to allow for non-locally-constant functions.

Non-constant cochains. We extend the notion of cochain. Namely, given a *topological* group G and a covering $\{U_\alpha\}$ of M, we define a *continuous* n–cochain $\varphi = \{\varphi_{\alpha_0\ldots\alpha_n}\}$ as a collection of *continuous* functions

$$\varphi_{\alpha_0\ldots\alpha_n}\colon U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \longrightarrow G \, .$$

The rest of the theory flows just as before and leads to what one should properly call the *Čech cohomology with coefficients in the sheaf of continuous G–valued functions*, and denote it by something like

$$\check{H}^*(M; C^0(G)) \, .$$

Notice that, if G is a *discrete* group (such as $\mathbb{Z}$), then the cochains will be forced to be locally-constant, and so in particular $\check{H}^*(M; C^0(\mathbb{Z})) = \check{H}^*(M;\mathbb{Z})$.

Assuming that M is a smooth manifold and G is a *Lie group*, we can further require the cochains to be made of *smooth* functions, thus leading to the *Čech cohomology with coefficients in the sheaf of smooth G–valued functions*,

$$\check{H}^*(M; C^\infty(G)) \, .$$

It is important to note that, if one merely chooses G to be the additive groups $\mathbb{R}$ or $\mathbb{C}$, then nothing much happens, since it is proved that $\check{H}^n(M; C^\infty(\mathbb{R})) = 0$ for every $n \geq 1$.

Finally, note in passing that, if M and G happen to be complex manifolds, then we can require the cocycles to be holomorphic. This leads to the Čech cohomology with coefficients in the sheaf of holomorphic G-valued bundles, denoted by $\check{H}^*(M; \mathcal{O}(G))$. If one then takes G to be the additive group $\mathbb{C}$, then $\check{H}^n(M; \mathcal{O}(\mathbb{C}))$—usually denoted by $H^n(M; \mathcal{O})$—is very much nontrivial, and plays an essential role in complex geometry.

A further generalization of Čech cohomology allows, in a sense, for the coefficient-group G to vary from point to point, and that leads to **sheaf cohomology**, but not in this volume. For ramifications in complex geometry, see **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** **[GH78, GH94]**. For algebraic topology applications, see **R. Godement's** ***Topologie algébrique et théorie des faisceaux*** **[God58, God73]**. For topological use in combination with differential forms, see **R. Bott** and L. Tu's ***Differential forms in algebraic topology*** **[BT82]**.

Finally, we reached the bundles. We now combine the two extensions above, allowing both non-commutative groups and non-constant cochains. Assume that G is a subgroup of $GL(k)$. Then

$$\check{H}^1(M; C^\infty(G))$$

is the set of all k-plane bundles with structure group G, up to isomorphisms. Its distinguished element $[\{\mathbf{1}_{\alpha\beta}\}]$ is the trivial bundle $M \times \mathbb{R}^k \to M$.

To convince ourselves, let us notice that a class in $\check{H}^1(M; C^\infty(G))$ is determined by a G-valued 1-cochain

$$\{g_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow G\}$$

that is coclosed, meaning that we must have $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$. Two such cocycles g' and g'' define a same class if they differ by a coboundary, that is to say,

$$[g'] = [g''] \qquad \Longleftrightarrow \qquad g'_{\alpha\beta} = f_\alpha \cdot g''_{\alpha\beta} \cdot f_\beta^{-1}$$

for some collection $\{f_\alpha \colon U_\alpha \to G\}$. However, this defines nothing but a smooth vector bundle, unique up to isomorphisms and with structure group G, as was explained back on page 176.

More generally, for any group G the set $\check{H}^1(M; C^\infty(G))$ is the set of all principal G-bundles, with distinguished element $M \times G \to M$.

Let us now look at a few examples:

Complex line bundles. Since any complex-line bundle can be endowed with a Hermitian metric, which reduces its structure group from $GL_{\mathbb{C}}(1)$ to $U(1) = \mathbb{S}^1$, it becomes clear that

$$H^1(M; C^\infty(\mathbb{S}^1))$$

is the set of all *smooth*[10] complex-line bundles on M. Since $\mathbb{S}^1$ is Abelian, the set $H^1(M; C^\infty(\mathbb{S}^1))$ turns out to be a group; its operation corresponds to tensor products of line bundles.

Further, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ fits into the exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{e^{2\pi\cdot}} \mathbb{S}^1 \longrightarrow 0$$

10. For *holomorphic* line bundles on a complex manifold M, one would look at $\check{H}^1(M; \mathcal{O}(\mathbb{C}^*))$, usually denoted by $H^1(M, \mathcal{O}^*)$.

(with the groups $\mathbb{Z}$ and $\mathbb{R}$ written additively, but $\mathbb{S}^1$ written multiplicatively). This short exact sequence leads, as usual, to a long exact sequence in cohomology, part of which is:

$$\cdots \longrightarrow \check{H}^1(M;\, C^\infty(\mathbb{R})) \longrightarrow \check{H}^1(M;\, C^\infty(\mathbb{S}^1)) \longrightarrow$$
$$\longrightarrow \check{H}^2(M;\, C^\infty(\mathbb{Z})) \longrightarrow \check{H}^2(M;\, C^\infty(\mathbb{R})) \longrightarrow \cdots$$

Since $\check{H}^n(M; C^\infty(\mathbb{R})) = 0$ and $\check{H}^n(M; C^\infty(\mathbb{Z})) = H^n(M;\mathbb{Z})$, exactness provides an isomorphism

$$0 \longrightarrow \check{H}^1(M;\, C^\infty(\mathbb{S}^1)) \xrightarrow[\approx]{c_1} H^2(M;\mathbb{Z}) \longrightarrow 0\,.$$

In terms of bundles, this isomorphism is established by sending a line bundle L to its first Chern class:

$$L \longmapsto c_1(L)\,.$$

In particular, this proves (again) that every 2–class of M can be represented by a smooth complex-line bundle on M, and thus (by taking the zero-locus of a generic section) by a surface embedded in M.

Čech cocycle for Chern. By explicitly following the isomorphism $\check{H}^1(M;\, C^\infty(\mathbb{S}^1)) \approx \check{H}^2(M;\mathbb{Z})$, we obtain a concrete description of a cocycle for $c_1(L)$: Let L be a complex-line bundle, defined by a cocycle $\{g_{\alpha\beta}\}$ with values in $\mathbb{S}^1$. Lift each map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{S}^1$ to some map $\vartheta_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}$ so that $g_{\alpha\beta}(x) = e^{2\pi\,\vartheta_{\alpha\beta}(x)}$ and $\vartheta_{\alpha\beta} = -\vartheta_{\beta\alpha}$. The cocycle condition $g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha} = 1$ only lifts to $\vartheta_{\alpha\beta} + \vartheta_{\beta\gamma} + \vartheta_{\gamma\alpha} \in \mathbb{Z}$. Then define the Čech 2–cocycle $\{c_{\alpha\beta\gamma}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}\}$ by setting

$$c_{\alpha\beta\gamma} = \vartheta_{\alpha\beta} + \vartheta_{\beta\gamma} + \vartheta_{\gamma\alpha}\,.$$

Its cohomology class is $c_1(L) \in H^2(M;\mathbb{Z})$. This exhibits $c_1(L)$ as essentially a cohomological bookkeeping of the 2π rotations used while building L. (For that matter, so is $w_2(L)$, but only modulo 2.)

Orientable vector bundles. Since every k–plane bundle can be endowed with a fiber metric, the set

$$\check{H}^1(M;\, C^\infty O(k))$$

is still the set of all k–plane bundles on M. A vector bundle is *orientable* if its structure group can be reduced to $SO(k)$. The exact sequence

$$0 \longrightarrow SO(k) \longrightarrow O(k) \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 0$$

(with $\mathbb{Z}_2 = \{-1, +1\}$ written multiplicatively) leads to an exact sequence of sets[11]

$$\cdots \longrightarrow H^0(M;\mathbb{Z}_2) \longrightarrow \check{H}^1(M;\, C^\infty SO(k)) \longrightarrow$$
$$\longrightarrow \check{H}^1(M;\, C^\infty O(k)) \xrightarrow{w_1} H^1(M;\mathbb{Z}_2)\,.$$

The map denoted w_1 is the assignment of the first Stiefel–Whitney class

$$E \longmapsto w_1(E)\,.$$

By exactness, if a bundle $E \in \check{H}^1(M;\, C^\infty O(n))$ has $w_1(E) = 0$, then E must come from $\check{H}^1(M;\, C^\infty SO(n))$, that is to say, E can be oriented. If a bundle is orientable, then its various orientations are all classified by the elements of $H^0(M;\mathbb{Z}_2)$.

11. An exact sequence of sets (each with a distinguished element) means that the image of one map coincides with the preimage of the distinguished element through the next map.

Čech cocycle for $w_1(E)$. Specifically, if E is defined by the $O(k)$-cocycle $\{g_{\alpha\beta}\}$, then $w_1(E) \in \check{H}^1(M;\mathbb{Z}_2)$ is determined by the $\mathbb{Z}_2$-valued Čech 1-cocycle $\{\det g_{\alpha\beta}\}$.

Spin structures. An oriented k-plane bundle (with k at least 3) is said to admit a *spin structure* if its $SO(k)$-cocycle lifts through the double-cover $Spin(k) \to SO(k)$ to a $Spin(k)$-valued cocycle. The exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(k) \longrightarrow SO(k) \longrightarrow 0$$

(with $\mathbb{Z}_2 = \{-1,+1\}$ written multiplicatively) leads to an exact sequence

$$\cdots \longrightarrow H^1(M;\mathbb{Z}_2) \longrightarrow \check{H}^1(M;\, C^\infty Spin(k)) \longrightarrow$$
$$\longrightarrow \check{H}^1(M;\, C^\infty SO(k)) \xrightarrow{w_2} H^2(M;\mathbb{Z}_2)\,.$$

The map w_2 above simply ascribes the second Stiefel–Whitney class

$$E \longmapsto w_2(E)\,.$$

By exactness, if a bundle $E \in \check{H}^1(M;\, C^\infty SO(k))$ has $w_2(E) = 0$, then E must come from a $Spin(k)$-cocycle from $\check{H}^1(M;\, C^\infty Spin(k))$. Further, the spin structures on a bundle E with $w_2(E) = 0$ are classified by $H^1(M;\mathbb{Z}_2)$.

Čech cocycle for $w_2(E)$. Let $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to SO(k)\}$ be a cocycle for an oriented bundle E. Assuming that the $U_\alpha \cap U_\beta$'s are all simply-connected, we can always lift the maps $g_{\alpha\beta}$ to maps

$$\widetilde{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow Spin(k)$$

with $\widetilde{g}_{\alpha\beta} = \widetilde{g}_{\beta\alpha}^{-1}$. The product $\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha}$ will take values in $\mathbb{Z}_2 = \{-1,+1\}$. We can then define a $\mathbb{Z}_2$-valued Čech 2-cochain $\{w_{\alpha\beta\gamma}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2\}$ by setting

$$w_{\alpha\beta\gamma} = \widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha}\,.$$

Clearly, the cochain $\{w_{\alpha\beta\gamma}\}$ measures the failure of the $\widetilde{g}_{\alpha\beta}$'s to define a spin structure on E. Moreover, $\{w_{\alpha\beta\gamma}\}$ is a cocycle: indeed, it is not hard to check that

$$(\delta w)_{\alpha\beta\gamma\delta} = (\widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\delta} \cdot \widetilde{g}_{\delta\beta}) \cdot (\widetilde{g}_{\alpha\gamma} \cdot \widetilde{g}_{\gamma\delta} \cdot \widetilde{g}_{\delta\alpha}) \cdot (\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\gamma} \cdot \widetilde{g}_{\gamma\alpha})$$

is constantly $+1$. The cocycle $\{w_{\alpha\beta\gamma}\}$ represents in Čech cohomology the second Stiefel–Whitney class of E:

$$w_2(E) = [w_{\alpha\beta\gamma}]\,.$$

This can be argued indirectly by using the fact that the vanishing of both $w_2(E)$ and of $[w_{\alpha\beta\gamma}]$ are equivalent to the existence of a spin structure on $E \to X$. Indeed, if $[w_{\alpha\beta\gamma}] = 0$, that means that $w_{\alpha\beta\gamma}$ is a coboundary. In other words, there must be a $\mathbb{Z}_2$-valued 1-cochain $\{\varepsilon_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{Z}_2\}$ so that $w_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta} \cdot \varepsilon_{\beta\gamma} \cdot \varepsilon_{\gamma\alpha}$. However, that implies that $(\varepsilon_{\alpha\beta} \cdot \widetilde{g}_{\alpha\beta}) \cdot (\varepsilon_{\beta\gamma} \cdot \widetilde{g}_{\beta\gamma}) \cdot (\varepsilon_{\gamma\alpha} \cdot \widetilde{g}_{\gamma\alpha}) = +1$ or, in other words, that the $\varepsilon_{\alpha\beta}$'s represent the *corrections* needed to make the $\widetilde{g}_{\alpha\beta}$'s into a genuine $Spin(4)$-cocycle. Thus, $[w_{\alpha\beta\gamma}] = 0$ if and only if E admits a spin structure.

Simplicial cocycle for $w_2(E)$. Passing the identity

$$w_2(E) = [w_{\alpha\beta\gamma}]$$

through the isomorphisms between Čech and simplicial cohomology exhibited earlier, leads to the uncovering of a simplicial cocycle ϑ for $w_2(E)$:

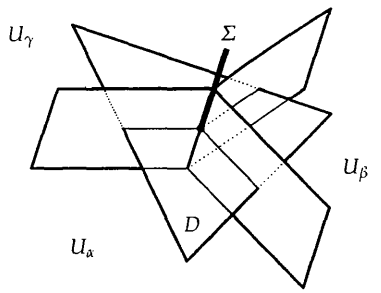
Triangulate the base X and use for all cocycles the covering $U_\alpha = \text{star}(v_\alpha)$ corresponding to the vertices v_α of X. Then a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ is non-empty if and only if it corresponds to a 2–simplex $\langle v_\alpha, v_\beta, v_\gamma \rangle$ (and in that case the interior of $\langle v_\alpha, v_\beta, v_\gamma \rangle$ is included in $U_\alpha \cap U_\beta \cap U_\gamma$).

Choose a random lift of the $SO(4)$–cocycle $\{g_{\alpha\beta}\}$ of E to some set of $Spin(4)$–valued maps $\{\tilde{g}_{\alpha\beta}\}$. Then the simplicial 2–cocycle ϑ for $w_2(E)$ is defined by assigning to every 2–simplex $\langle v_\alpha, v_\beta, v_\gamma \rangle$ the $\mathbb{Z}_2$–value of the product $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha}$.

Switching from writing $\mathbb{Z}_2 = \{-1, +1\}$ multiplicatively to the more familiar additive writing $\mathbb{Z}_2 = \{0, 1\}$, we translate to having ϑ assign 0 to D if and only if $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = +1$, and assign 1 if and only if $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = -1$.

Around a characteristic surface. Let us focus on the case of 4–manifolds M and their tangent bundles T_M. Using the above description of a simplicial cocycle ϑ for $w_2(T_M)$, we can imagine a characteristic surface of M as a surface that manages to cross an odd number of times exactly those 2–simplices that ϑ assigns to 1.

An even better way to see this is probably in the slightly different setting used in the proof of equivalence of the spin structure definitions (preceding note, page 181), as is recalled in figure 4.27. Recall that in that case the U_α's were small neighborhoods of the 4–cells dual to the vertices of M.

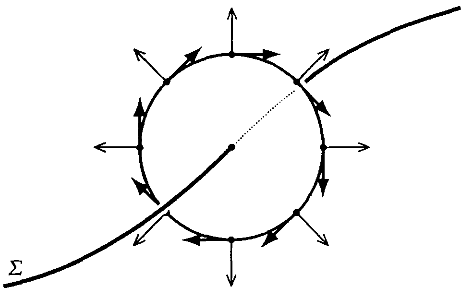


4.27. Drawing a characteristic surface

Assume now that D is a 2–simplex, surrounded by the open sets U_α, U_β, U_γ, with $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = -1$. Then the 2–cell Σ dual to D is part of a simplicial chain that describes a (modulo 2) homology class Poincaré-dual to $w_2(T_M)$.

With a bit of luck in choosing the lifts $\tilde{g}_{\alpha\beta}$, the union of all these distinguished dual 2–cells will make up an actual (unoriented) embedded surface in M ("luck" is needed, because *a priori* there might be problems at the vertices). With a bit more luck, the surface Σ will be orientable, in which case it represents an integral homology class dual to $w_2(T_M)$, and thus is deserving of the name "characteristic surface".

This picture also has the advantage of exhibiting a characteristic surface Σ as surrounded by 2π–twists of T_M, as was mentioned earlier (page 179) and is recalled here through figure 4.28. Away from Σ, the maps $\tilde{g}_{\alpha\beta}$ are a genuine cocycle and thus define a spin structure on the complement $M \setminus \Sigma$; clearly, this spin structure on $M \setminus \Sigma$ cannot be extended across Σ.



4.28. Outside spin structure, not extending across a characteristic surface Σ

Note: Obstruction theory

In this note, we give a short presentation of obstruction theory. On one hand, this will shed light on several constructions already seen in this chapter. On the other hand these techniques will be needed in the note on page 207 ahead, where the theory of smooth structures on topological manifolds is explained.

Obstruction theory deals with the problem of existence and uniqueness of sections of fiber bundles, encoding it into cohomology classes with coefficients in the homotopy groups of the fiber. At the outset, the case of a vector bundle E is uninteresting, since there always exist sections. However, obstruction theory can be applied to bundles associated to E, such as its sphere bundle SE (uncovering the obstruction to the existence of a nowhere-zero section in E), or the bundle $\mathcal{P}_{SO(E)}$ of frames in E (uncovering obstructions to the existence of a global frame-field in E, that is, obstructions to trivializing E), or bundles of partial frames—the resulting obstructions turn out to be the usual characteristic classes of E. In particular, from this note we will gain yet another point-of-view on the characteristic classes of a 4–manifold.

The argument to follow has two main components, each propelling the other: on one hand, defining things through cell-by-cell extensions and climbing from each k–skeleton to the $(k+1)$–skeleton; on the other hand, meshing the issue of extending sections with the issue of their uniqueness up to homotopy.

Set-up. A **fiber bundle** E with fiber F over a manifold X is a space E and a map $p\colon E \to X$ so that X is covered by open sets U over which the restriction of p to $p^{-1}[U]$ looks like the projection $U \times F \to U$.

Assume that that the fiber F is connected; furthermore, assume that F's *first non-trivial* homotopy group[12] is $\pi_m(F)$. (If $m = 1$, assume further that $\pi_1(F)$ is Abelian.)

Moreover, choose a random cellular decomposition[13] of X. We denote by $X|_k$ the k–skeleton of X, and by $E|_k$ the restriction of E to $X|_k$ (not the k–skeleton of the space E). Also, let $\sigma|_k$ denote the restriction of σ to $X|_k$.

Free ride, up to the m–skeleton. We try to define a section σ of E by defining it over the vertices of X, then try to extend σ over the 1–skeleton of X, then over the 2–skeleton, and so on, cell-by-cell. This plan proceeds without problems until we attempt to extend from the m–skeleton across the $(m+1)$–skeleton.

Indeed, to reach the m–skeleton, we start by defining $\sigma(vertex)$ in any random way. Then, assuming σ was already defined over the k–skeleton of X, we try to extend σ across the $(k+1)$–cells of X: For every $(k+1)$–cell C, we notice that the restriction $E|_C$ is trivial (since C is contractible)[14] and hence $E|_C \approx C \times F$. Our σ, already defined on the k–sphere ∂C, induces a map $\partial C \to F$. Then $\sigma|_{\partial C}$ can be extended across C if and only if the induced map $\partial C \to F$ is homotopically-trivial. However, as long as $k \leq m-1$, we have $\pi_k(F) = 0$ and thus every map $\partial C \to F$ can be extended over C, and hence so can σ. Therefore, we can always define sections σ over the m–skeleton of X.

Uniqueness so far. Let us investigate for a second the dependence (up to homotopy) of the resulting $\sigma|_m$ on the choices made along the way; again, we split the problem in stages between the k– and $(k+1)$–skeletons.

Take k and assume that σ is fixed over $X|_k$, then let σ' and σ'' be two extensions of σ from $X|_k$ across $X|_{k+1}$. A homotopy between $\sigma'|_{k+1}$ and $\sigma''|_{k+1}$ means a section Φ in the product-bundle $p \times id\colon E \times [0,1] \to X \times [0,1]$, defined over $(X|_{k+1}) \times [0,1]$ and limiting to σ' on $X \times 0$ and to σ'' on $X \times 1$.

We choose the obvious cellular decomposition of $X \times [0,1]$ induced from the chosen decomposition of X, with each j–cell C of X creating two j–cells $C \times 0$ and $C \times 1$ in $X \times [0,1]$, and a $(j+1)$–cell $C \times [0,1]$.

Certainly Φ must be defined to be $\sigma' \times 0$ on $(X|_{k+1}) \times 0$ and to be $\sigma'' \times 1$ on $(X|_{k+1}) \times 1$. Furthermore, since σ' and σ'' were taken to coincide over the k–skeleton of X, it follows that, for every j–cell C of X with $j \leq k$, we have $\sigma'|_C = \sigma''|_C$. We can then extend Φ across the $(j+1)$–cell $C \times [0,1]$ simply as $\sigma \times id$. Therefore, for fully extending Φ across $(X|_{k+1}) \times [0,1]$, we need only extend Φ across those

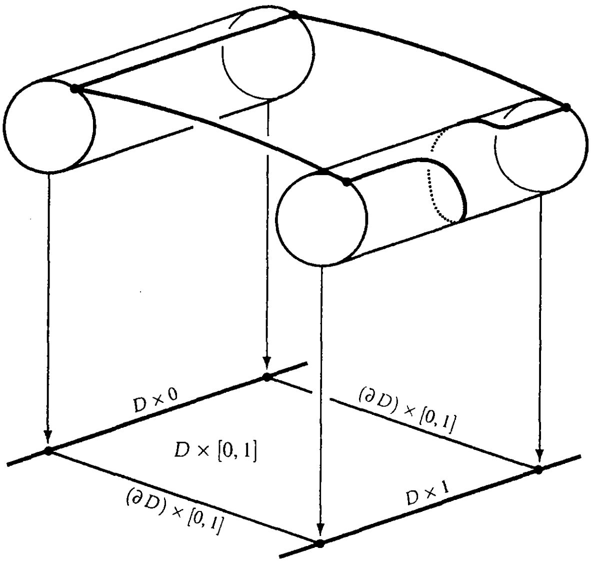
---

12. Remember that the **homotopy group** $\pi_k(A)$ is the set of all homotopy-classes of maps $S^k \to A$, with a suitable group operation. An element $f \in \pi_k(F)$ is trivial if and only if $f\colon S^k \to A$ can be extended to a map $\tilde{f}\colon \mathbb{D}^{k+1} \to A$. Whenever k is at least 2, the group $\pi_k(A)$ is Abelian.

13. Handle decompositions would work just as well. Just substitute the word "handle" for "cell" in all that follows. Or one could use a triangulation of X (as recalled back in footnote 5 on page 182) and substitute "simplex" for "cell" throughout.

14. Technically, since the cell C is not necessarily embedded along ∂C, one should view $E|_C$ as the pull-back $\iota^* E$, where $\iota\colon C \to X$ is the "inclusion" of the cell in X.

$(k+2)$–cells of $X \times [0,1]$ that are of shape $D \times [0,1]$ for some $(k+1)$–cell D of X. Compare with figure 4.29.



4.29. Toward a homotopy between two sections

Notice that Φ is already defined over the whole boundary $\partial(D \times [0,1])$. Thus, Φ restricted to the $(k+1)$–sphere $\partial(D \times [0,1])$ determines an element of $\pi_{k+1}(F)$. It follows that Φ extends across $D \times [0,1]$ if and only if the class of $\Phi|_{\partial(D\times[0,1])}$ in $\pi_{k+1}(F)$ is trivial.

Therefore, since all homotopy groups of F were assumed trivial up to dimension m, it follows that the extension of σ up to the $(m-1)$–skeleton of X must be unique up to homotopy. However, when we extend σ from the $(m-1)$–skeleton across the m–skeleton, the various options can differ over each m–cell by elements of $\pi_m(F)$. We will come back to this issue.

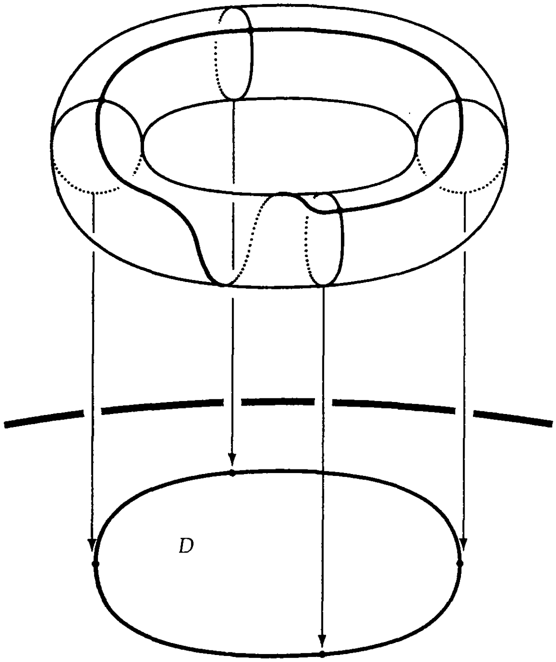
Across the $(m+1)$–cells: obstruction cocycles. In any case, our fibre bundle $E \to X$ admits a section σ defined over the m–skeleton of the base. When attempting to extend σ from the m–skeleton across the $(m+1)$–skeleton, obstructions appear. Indeed, if D is a $(m+1)$–cell, then $\sigma|_{\partial D}$ might describe a nontrivial element in $\pi_m(F)$, and then our σ will not extend across D. Compare with figure 4.30 on the following page.

To measure this, we define the function

$$\vartheta_\sigma\colon \{\,(m+1)\text{–cells of } X\,\} \longrightarrow \pi_m(F) \qquad D \longmapsto [\sigma|_{\partial D}]\,,$$

which takes an $(m+1)$–cell D to the element of $\pi_m(F)$ that is determined by $\sigma|_{\partial D}$ through some random trivialization $E|_D \approx D \times F$. We can then extend ϑ_σ by linearity, and think of it as a $\pi_m(F)$–valued[15] cellular $(m+1)$–cochain on X.

15. In truth, the twists of our fiber bundle $E \to X$ might twist the way the π_m's of the various fibers can be assembled together. Thus, to get a well-defined map ϑ, one must in fact use *twisted coefficients*



4.30. Obstruction to extending a section

This cochain ϑ_σ is in fact *coclosed.* Indeed, on every $(m+2)$–cell B, we have

$$(\delta\vartheta_\sigma)(B) = \vartheta_\sigma(\partial B) = [\sigma|_{\partial\partial B}] = 0\,,$$

where ∂ denotes taking the homological boundary and we use that $\partial\partial = 0$. We call ϑ_σ the **obstruction cocycle** of σ. Our chosen section σ will extend over the $(m+1)$–skeleton if and only if $\vartheta_\sigma = 0$.

Even when the cocycle ϑ_σ happens to be nontrivial, we can still try to go back and change the way σ was defined over the m–skeleton of X, and maybe the new version σ' will have $\vartheta_{\sigma'} = 0$ and hence extend. We need to revisit the issue of uniqueness of sections of $E|_m$:

Uniqueness, revisited: difference cochains. Given any two sections σ' and σ'' of E defined over the m–skeleton, they cannot differ homotopically over the $(m-1)$–skeleton. Therefore there must exist a homotopy κ between $\sigma'|_{m-1}$ and $\sigma''|_{m-1}$.

We try to extend this κ to a homotopy Φ between $\sigma'|_m$ and $\sigma''|_m$. As before, we view Φ as a partial section of $E \times [0,1] \to X \times [0,1]$ and set Φ to be $\sigma'|_m \times 0$ on $(X|_m) \times 0$ and $\sigma''|_m \times 1$ on $(X|_m) \times 1$, and thereafter extend it across $(X|_{m-1}) \times [0,1]$ by spreading κ over it, thus linking $\sigma'|_{m-1} \times 0$ with $\sigma''|_{m-1} \times 1$.

To extend this to a full homotopy between $\sigma'|_m$ and $\sigma''|_m$, we need only extend Φ across every $(m+1)$–cell $C \times [0,1]$ that corresponds to some m–cell C of X. The

(better known as *local coefficients*) that twist $\pi_m(F)$ by the action of $\pi_1(X)$ on the fibers of F. Let us assume that X is *simply-connected* and move on as if nothing happened…

homotopic difference between σ' and σ'' can be encoded in the obstruction to this extension, namely in the function

$$d(\sigma'_\kappa\sigma'') : \{m\text{–cells of } X\} \longrightarrow \pi_m(F) \qquad C \longmapsto [\Phi|_{\partial(C\times[0,1])}] \ .$$

This $d(\sigma'_\kappa\sigma'')$ can be extended by linearity and defines a $\pi_m(F)$–valued[16] cellular m–cochain on X. It is called the **difference cochain** of σ' and σ''.

The homotopy κ between $\sigma'|_{m-1}$ and $\sigma''|_{m-1}$ can be extended to a full homotopy between σ' and σ'' if and only if $d(\sigma'_\kappa\sigma'')$ is identically-zero. However, a different choice of homotopy $\kappa\colon \sigma'|_{m-1} \sim \sigma''|_{m-1}$ might be a better choice toward obtaining a homotopy between $\sigma'|_m$ and $\sigma''|_m$. We will come back to this issue.

Back to obstruction cocycles: primary obstructions. We return to the extension problem, to see how different choices of sections over $X|_m$ influence our chances of extension across $X|_{m+1}$. Let σ' and σ'' be two sections of $E|_m$ and choose a random homotopy κ between $\sigma'|_{m-1}$ and $\sigma''|_{m-1}$. Consider the bundle $E \times [0,1] \to X \times [0,1]$ and denote by $\sigma'_\kappa\sigma''$ its section defined as $\sigma' \times 0$ over $X|_m \times 0$, as $\sigma'' \times 1$ over $X|_m \times 1$, and as κ over $(X|_{m-1}) \times [0,1]$.

Notice that this section $\sigma'_\kappa\sigma''$ is defined over the whole m–skeleton of the base $X \times [0,1]$. We can therefore define its obstruction cocycle $\vartheta_{\sigma'_\kappa\sigma''}$. We observe that this cocycle is made of three distinct parts: (1) the obstruction to extending $\sigma' \times 0$ across the $(m+1)$–cells $D \times 0$ of $X \times 0$, which can be identified with $\vartheta_{\sigma'}(D)$; (2) the obstruction to extending $\sigma'' \times 1$ across the $(m+1)$–cells $D \times 1$ of $X \times 1$, which can be identified with $\vartheta_{\sigma''}(D)$; finally, (3) the obstruction to extending κ across the $(m+1)$–cells of shape $C \times [0,1]$, which can be identified with $d(\sigma'_\kappa\sigma'')(C)$.

Take any $(m+1)$–cell D of X and consider the $(m+2)$–cell $D \times [0,1]$ of $X \times [0,1]$. Apply the above decomposition of $\vartheta_{\sigma'_\kappa\sigma''}$ to $\partial(D \times [0,1])$. On one hand, since $\vartheta_{\sigma'_\kappa\sigma''}$ is a cocycle, it must vanish on every boundary and in particular on $\partial(D \times [0,1])$. On the other hand, we have $\partial(D \times [0,1]) = D\times 1 \cup D\times 0 \cup (\partial D)\times[0,1]$, and we can evaluate the parts of $\vartheta_{\sigma'_\kappa\sigma''}$ on these pieces. We end up with $\vartheta_{\sigma'}(D)$, $\vartheta_{\sigma''}(D)$, and $d(\sigma'_\kappa\sigma'')(\partial D)$. Gathering up and keeping track of signs, we obtain the equality $\vartheta_{\sigma'}(D) - \vartheta_{\sigma''}(D) = d(\sigma'_\kappa\sigma'')(\partial D)$, which translates to

$$\vartheta_{\sigma'} - \vartheta_{\sigma''} = \delta\, d(\sigma'_\kappa\sigma'') \ .$$

The conclusion is that different choices of sections in $E|_m$ change the corresponding obstruction cochain by a *coboundary.* It follows that the obstruction cocycle determines a well-defined cohomology class

$$[\vartheta_\sigma] \in H^{m+1}(X; \pi_m(F)) \ .$$

This class depends only on the bundle $E \to X$ and *not* on the choice of section σ. Moreover, this class is trivial if and only if there is some m–cochain d such that $\vartheta_\sigma = \delta d$. In that case, we can change σ over the m–skeleton of X to a section σ' with $d(\sigma\sigma') = d$, and then the new σ' will have obstruction cocycle $\vartheta_{\sigma'} = 0$: it will extend across $X|_{m+1}$.

16. Again, in general one needs twisted coefficients.

In conclusion, $E \to X$ admits sections over the $(m+1)$–skeleton of X if and only if the class $[\vartheta]$ vanishes. We call this class the **primary obstruction**[17] of $E \to X$.

Back to uniqueness: difference cocycles. If the primary obstruction $[\vartheta]$ vanishes, then conceivably there exist several distinct sections of $E|_m$ that extend across the $(m+1)$–skeleton of X.

Assume that $\sigma'|_m$ and $\sigma''|_m$ are two such extendable sections of $E|_m$ and take κ to be some homotopy between $\sigma'|_{m-1}$ and $\sigma''|_{m-1}$. We have $\vartheta_{\sigma'} - \vartheta_{\sigma''} = \delta\, d(\sigma'_\kappa\sigma'')$, but since both σ' and σ'' were assumed extendable, their obstruction cocycles vanish, and thus

$$\delta\, d(\sigma'_\kappa\sigma'') = 0\,.$$

In other words, the difference cochain is now in fact a *cocycle*.

Further, the difference cochain $d(\sigma'_\kappa\sigma'')$ can in this case be understood as representing the whole obstruction cocycle $\vartheta_{\sigma'_\kappa\sigma''}$ of the section $\sigma'_\kappa\sigma''$ across the $(m+1)$–skeleton of $X \times [0,1]$. We can then apply the previous results about obstruction cocycles to this $d(\sigma'_\kappa\sigma'')$. It follows that changing the choice of homotopy $\kappa\colon \sigma'|_{m-1} \sim \sigma''|_{m-1}$ merely modifies $d(\sigma'_\kappa\sigma'')$ by the addition of a coboundary. Therefore, the difference cocycle itself determines a well-defined cohomology class

$$\left[d(\sigma'_\kappa\sigma'')\right] \;\in\; H^m(X;\, \pi_m(F))\,.$$

This class depends only on the extendable sections σ' and σ'' and *not* on the choice of homotopy κ. Furthermore, if $[d(\sigma'\sigma'')] = 0$, then there exists a choice of κ that can be extended to a full homotopy Φ between $\sigma'|_m$ and $\sigma''|_m$.

Conclusion. For every fiber bundle $E \to X$ whose fiber F has its first nontrivial homotopy group in dimension m, the primary obstruction

$$[\vartheta] \in H^{m+1}(X;\, \pi_m(F))$$

vanishes if and only if there are sections of $E \to X$ defined over the m–skeleton of X that extend across the $(m+1)$–skeleton.

Moreover, if $[\vartheta] = 0$ and one chooses some extendable section σ, then all other sections σ' of $E|_m$ that extend across $X|_{m+1}$ are classified up to homotopy by the elements of

$$H^m(X;\, \pi_m(F))$$

via their corresponding difference classes $[d(\sigma\sigma')]$

Application: trivializing the tangent bundle. We will now apply the method of obstruction theory to the problem of trivializing the tangent bundle T_M of an oriented 4–manifold M. Since a trivialization of T_M over some subset U of M is equivalent to a field of frames over U, the problem becomes one of finding sections in the bundle of frames $\mathcal{P}_{SO(4)}$ of T_M.

The fiber of $\mathcal{P}_{SO(4)}$ is the Lie group $SO(4)$, which is connected and has

$$\pi_1 SO(4) = \mathbb{Z}_2\,, \qquad \pi_2 SO(4) = 0\,, \qquad \pi_3 SO(4) = \mathbb{Z} \oplus \mathbb{Z}\,.$$

17. "Primary", because the project can conceivably be continued by attempting to further extend across higher skeleta, until we exhaust all X.

Therefore, when applying the obstruction theory method, we first encounter a primary obstruction in $H^2(M;\, \pi_1 SO(4))$. This obstruction class is none other than the Stiefel–Whitney class

$$w_2(T_M) \in H^2(M; \mathbb{Z}_2)\,.$$

Hence, if $w_2(T_M) = 0$, then $\mathcal{P}_{SO(4)}$ admits a section over the 2–skeleton of M, in other words, T_M can be trivialized over $M|_2$. Two such sections of $\mathcal{P}_{SO(4)}$ over $M|_2$ differ by difference cocycles from $H^1(M; \mathbb{Z}_2)$. Such trivializations of T_M over $M|_2$ are, of course, spin structures on M.

Assuming that $w_2(T_M)$ vanished and we did choose a section of $\mathcal{P}_{SO(4)}$ over $M|_2$, we can now try to further extend it over M. Since $\pi_2 SO(4) = 0$, extending across the 3–skeleton encounters no problems. The next significant obstruction appears in $H^4(M;\, \pi_3 SO(4))$, and it can be identified as the pair

$$\big(e(T_M),\, p_1(T_M)\big) \in H^4(M;\, \mathbb{Z} \oplus \mathbb{Z})\,,$$

made from the Euler class $e(T_M)$ and the Pontryagin class $p_1(T_M)$.

The Euler class appears as the obstruction to extending a nowhere-zero vector field over all M, that is to say, $e(T_M)$ is the primary obstruction to defining a section in the 3–sphere bundle $\mathbb{S}T_M$ of T_M; thus, it belongs to $H^4(M;\, \pi_3 \mathbb{S}^3)$.

That the pair $(e,\, p_1)$ fully catches the secondary obstruction can be argued directly by computing characteristic classes of 4–plane bundles over $\mathbb{S}^4$ that are built using equatorial gluing maps from $\pi_3 SO(4)$; an exposition can be found in **[Sco03]**.

If, besides $w_2(T_M)$ being trivial, we also have that both $e(T_M)$ and $p_1(T_M)$ vanish, then the tangent bundle T_M can be completely trivialized, $T_M \approx M \times \mathbb{R}^4$. This happens for example with $\mathbb{S}^1 \times \mathbb{S}^3$, but never for simply-connected 4–manifolds.

Similar results apply for general oriented 4–plane bundles over 4–manifolds. In particular, notice that moving along these lines one can quickly obtain a proof of the Dold–Whitney theorem (stated on page 167).

Application: characteristic classes. The obstruction-theoretic approach was in fact the one initially used by E. Stiefel and H. Whitney when they discovered characteristic classes.

Given a vector bundle $E \to X$ with fiber $\mathbb{R}^k$, consider the Stiefel bundle $\mathcal{V}_j(E) \to X$ of all j–frames in E. Then the corresponding primary obstruction $[\vartheta_j]$ of $\mathcal{V}_j(E)$ appears in H^{k-j+1} and determines the Stiefel–Whitney classes by[18]

$$w_{k-j+1}(E) = \begin{cases} [\vartheta_j] & \text{if } k-j+1 \text{ is even and } < k \\ [\vartheta_j] \pmod 2 & \text{if } k-j+1 \text{ is odd, or } j=1 \end{cases} \in H^{k-j+1}(X; \mathbb{Z}_2)\,.$$

Thus, each class w_{k-j+1} reveals itself as an obstruction to defining a field of j–frames in E over the $(k-j+1)$–skeleton of X.

18. The modulo 2 reduction in the following formula is done because in those cases ϑ appears at the outset with twisted $\mathbb{Z}$–coefficients. Still, if we know all the w_j's, no information is lost through this reduction.

Finally, if $E \to X$ is oriented, then for $j = 1$ the full obstruction of $\mathcal{V}_1(E) = \mathbb{S}E$ is caught by the Euler class

$$e(E) = [\vartheta_1] \in H^k(X, \mathbb{Z}) ,$$

which measures the obstruction to defining a nowhere-zero section of E over the k–skeleton of X.

A similar approach can be used for Chern classes.

References. Classic obstruction theory, including a description of Stiefel–Whitney and Chern classes, is presented in **N. Steenrod's *The topology of fibre bundles* [Ste51, Ste99,** part III], and is still the best introduction. For a recent discussion of obstruction theory, see for example **J. Davis** and **P. Kirk's *Lecture notes in algebraic topology* [DK01]**.

We will use obstruction theory again in the note on page 207 ahead, where we will explore the obstructions to the existence of smooth structures on topological manifolds.

Note: Classifying spaces and spin structures

In what follows, we will define spin structures in terms of maps to classifying spaces. We will start by saying a few words about the spaces $\mathscr{B}G$ that classify all fiber bundles with structure group G, then describe a spin structure on a bundle $E \to X$ as the lift of its classifying map $X \to \mathscr{B}SO(m)$ to a map $X \to \mathscr{B}Spin(m)$.

Part of this note will be better understood if one first reads the preceding note (starting on page 197) on obstruction theory.

This and the preceding note can be viewed both as a continuation of the survey of spin structures from earlier notes, and as preparing the ground for the theory of smoothing topological manifolds that will be explained in the next note (starting with page 207).

Fiber bundles and classifying spaces. A (locally-trivial) **fiber bundle** E with fiber F over a manifold X is a space E and a map $p\colon E \to X$ so that X is covered by open sets $\{U_\alpha\}$ and over each U_α the restriction of p to $p^{-1}[U_\alpha]$ looks like the projection $U_\alpha \times F \to U_\alpha$.

The fiber bundle E is said to have **structure group** G, or is called a G**–bundle**, if over every overlap $U_\alpha \cap U_\beta$ the two trivializations $p^{-1}[U_\alpha] \simeq U_\alpha \times F$ and $p^{-1}[U_\beta] \simeq U_\beta \times F$ are related by a self-homeomorphism of $(U_\alpha \cap U_\beta) \times F$ acting by $(x, f) \longmapsto (x, g_{\alpha\beta}(x) \cdot f)$, where $g_{\alpha\beta}$ is a map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ and G is a group acting on F by homeomorphisms.

For every topological group G there exists a space $\mathscr{B}G$, called the **classifying space** of G, and for every fiber F on which G acts there exists a G–bundle

$$\mathscr{E}_F G \longrightarrow \mathscr{B}G$$

with fiber F, called the **universal bundle** of fiber F and group G. The spaces $\mathscr{B}G$ and $\mathscr{E}_F G$ are unique up to homotopy-equivalence. The reason for the names "classifying" and "universal" is that that all G–bundles over any X are classified by the homotopy classes of maps $X \to \mathscr{B}G$.

This means that for every G–bundle $E \to X$ with fiber F there must exist a map $\xi\colon X \to \mathscr{B}G$ so that E is isomorphic to the pull-back through ξ of the universal bundle $\mathscr{E}_F G \to \mathscr{B}G$; in other words, ξ can be covered by a bundle morphism $\overline{\xi}$, fitting in the diagram

$$\begin{array}{ccc} E & \xrightarrow{\overline{\xi}} & \mathscr{E}_F G \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{\xi} & \mathscr{B}G\,, \end{array}$$

so that $\overline{\xi}$ is a G–homeomorphism on the fibers. Further, two bundles E' and E'' are isomorphic if and only if their corresponding maps $\xi', \xi''\colon X \to \mathscr{B}G$ are homotopic. In brief, the set of all G–bundles can be identified with the set $[X, \mathscr{B}G]$ of homotopy classes of maps $X \to \mathscr{B}G$.

Construction. *The classifying space $\mathscr{B}G$ can be built as follows: Take G and start joining[19] it to itself, building $G * G$, then $G * G * G$, then $G * G * G * G$, then... In the limit, we obtain the space $\mathscr{E}G = G * G * G * \cdots$. The group G acts freely on $\mathscr{E}G$, and we can then build the quotient space $\mathscr{B}G = \mathscr{E}G/G$, which is the classifying space of G. The bundle $\mathscr{E}G \to \mathscr{B}G$ is the universal bundle that classifies all principal G–bundles. To get the universal bundle for some other fiber F, pick some cocycle for $\mathscr{E}G \to \mathscr{B}G$, let it act on F, and glue $\mathscr{E}_F G$ with it. More generally, if $\mathcal{E}$ is any* contractible *space on which G acts freely, then the map $\mathcal{E} \to \mathcal{E}/G$ is a principal G–bundle, and in fact, up to homotopy equivalence, $\mathcal{E} \to \mathcal{E}/G$ coincides with $\mathscr{E}G \to \mathscr{B}G$. This construction is due to* **J. Milnor**'s ***Construction of universal bundles. II*** **[Mil56a]**.

Vector bundles. A vector bundle of fiber $\mathbb{R}^k$ over X is a fiber bundle with group $GL(k)$. Then its classifying space can be determined to be

$$\mathscr{B}GL(k) = \mathscr{G}r_k(\mathbb{R}^\infty)\,,$$

i.e., the Graßmann space of all k–planes in $\mathbb{R}^\infty$, defined as $\varinjlim \mathscr{G}r_k(\mathbb{R}^m)$ when $m \to \infty$. The universal bundle $\mathscr{E}_{\mathbb{R}^k} GL(k)$ is the vector bundle over $\mathscr{B}GL(k)$ that has as fiber over a point $P \in \mathscr{G}r_k(\mathbb{R}^\infty)$ the actual k–plane P. Intuitively, all twists and turns that a fiber of a vector bundle over X might have can be retrieved by positions of k–planes in $\mathbb{R}^\infty$, and a description of these positions yields the classifying map. More rigorously, one can show that for every bundle $E \to X$ there exists a bundle $F \to X$ so that $E \oplus F \approx X \times \mathbb{R}^N$, and thus the fibers of E can be identified with k–planes in $\mathbb{R}^N$. For example, if X has dimension m, then one can use $N = m + k + 1$ and $\mathscr{G}r_k(\mathbb{R}^{m+k+1})$ instead of the full $\mathscr{G}r_k(\mathbb{R}^\infty)$.

A similar approach works for complex bundles and shows that $\mathscr{B}GL_{\mathbb{C}}(k)$ is the complex Graßmannian $\mathscr{G}r_k(\mathbb{C}^\infty)$. In particular, complex-line bundles are classified by maps to $\mathscr{B}GL_{\mathbb{C}}(1) = \mathbb{CP}^\infty$. For line bundles on 4–manifolds, it is enough to consider maps to $\mathbb{CP}^2$.

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19. The **join** $A * B$ of two spaces A and B is defined as follows: take $A \times B \times [0,1]$, then collapse $A \times 0$ to a point and $B \times 1$ to another point. The join is easiest to visualize if we imagine both A and B as embedded in general position in some high-dimensional $\mathbb{R}^N$; then $A * B$ is the union of all straight segments starting in A and ending in B. For example, the join of two 1–simplices (segments) will be a 3–simplex (a tetrahedron).

Metrics. The group $GL(k)$ is homotopy-equivalent to[20] $O(k)$. Since the whole theory is homotopy-flavored, it follows that $\mathscr{B}GL(k)$ and $\mathscr{B}O(k)$ are homotopy-equivalent, and thus a $GL(k)$-bundle is the same thing as an $O(k)$-bundle. In down-to-earth terms, this simply means that every vector bundle admits a fiber-metric.

Orientations. If the vector bundle is oriented, then its structure group can be further refined from $O(k)$ to $SO(k)$. In terms of classifying spaces, the inclusion $SO(k) \subset O(k)$ induces a map[21]

$$S\colon \mathscr{B}SO(k) \to \mathscr{B}O(k) .$$

Finding an orientation for a bundle E is the same as finding a lift of its classifying map $\xi\colon X \to \mathscr{B}O(k)$ to a map $\xi^s : X \to \mathscr{B}SO(k)$, fitting in

$$\begin{array}{ccc} X & \xrightarrow{\ \xi^s\ } & \mathscr{B}SO(k) \\ \| & & \big\downarrow S \\ X & \xrightarrow{\ \xi\ } & \mathscr{B}O(k) . \end{array}$$

The map $S\colon \mathscr{B}SO(k) \to \mathscr{B}O(k)$ is itself a fiber bundle with fiber $O(k)/SO(k) = \mathbb{Z}_2$. We can pull this bundle back over X by using the map $\xi\colon X \to \mathscr{B}O(k)$, and hence transform the problem of finding a lifted map ξ^s into the problem of finding a global section in the pulled-back bundle $\xi^*S \to X$ from

$$\begin{array}{ccc} \xi^*S & \longrightarrow & \mathscr{B}SO(k) \\ \big\downarrow & & \big\downarrow S \\ X & \xrightarrow{\ \xi\ } & \mathscr{B}O(k) . \end{array}$$

The fiber of $\xi^*S \to X$ is of course still $\mathbb{Z}_2$.

The obstruction to the existence of a section in ξ^*S can then be attacked by obstruction theory, similar to the outline from the preceding note.[22] This yields as unique obstruction the first Stiefel–Whitney class

$$w_1(E) \in H^1(X; \mathbb{Z}_2) .$$

If one such section (*i.e.*, an orientation of E) is chosen, then all other sections, up to homotopy, are classified by the elements of $H^0(X; \mathbb{Z}_2)$; in other words, you can change the orientation on each connected component of X.

Spin structures. The group $SO(k)$ is double-covered by the Lie group $Spin(k)$, and the double-cover projection $Spin(k) \to SO(k)$ induces a map of classifying spaces

$$Sp\colon \mathscr{B}Spin(k) \to \mathscr{B}SO(k) .$$

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20. Indeed, if we think of a matrix $A \in GL(k)$ as a frame in $\mathbb{R}^k$, then we can apply the Gram–Schmidt procedure to split A as a product $A = T \cdot R$ of an upper-triangular matrix T and an orthogonal matrix $R \in O(k)$; since all upper-triangular matrices make up a contractible space, the claim follows.

21. Notice that $\mathscr{B}SO(k)$ can be represented as the Graßmannnian of all *oriented* k-planes inside $\mathbb{R}^\infty$.

22. A rather special case of obstruction theory, since one plays with $\pi_0(\textit{fiber})$.

This map is a fiber bundle. Its fiber is denoted $SO(k)/Spin(k)$ and it is an Eilenberg–Maclane $K(\mathbb{Z}_2, 1)$–space. This means that $\pi_1(SO(k)/Spin(k)) = \mathbb{Z}_2$ is its only non-zero homotopy group.[23]

A **spin structure** on an oriented bundle E is the same as a lift of its classifying map $\zeta\colon X \to \mathscr{B}SO(k)$ to a map $\zeta^{sp}\colon X \to \mathscr{B}Spin(k)$, made against the map $Sp\colon \mathscr{B}Spin(k) \to \mathscr{B}SO(k)$. Equivalently, by pulling back over X,

$$\begin{array}{ccc} \zeta^* Sp & \longrightarrow & \mathscr{B}Spin(k) \\ \downarrow & & \downarrow{\scriptstyle Sp} \\ X & \xrightarrow{\ \zeta\ } & \mathscr{B}SO(k)\,, \end{array}$$

we see that a spin structure on E is the same as a global section of the bundle $\zeta^* Sp \to X$, whose fiber is $SO(k)/Spin(k)$.

After applying obstruction theory to this setting, it turns out that the unique obstruction to the existence of such a section is the second Stiefel–Whitney class

$$w_2(E) \in H^2(X; \mathbb{Z}_2)\,.$$

Characteristic classes. *It is worth noting that, avoiding any obstruction theory, the characteristic classes of a vector bundle $E \to X$ can be viewed directly as pull-backs of cohomology classes of the appropriate classifying space. Indeed, we have isomorphisms between the cohomology rings of the $\mathscr{B}G$'s and polynomial rings generated by the various characteristic classes (endowed with suitable degrees). Specifically,*

$$\begin{aligned} H^*(\mathscr{B}O(k); \mathbb{Z}_2) &= \mathbb{Z}_2[w_1, w_2, \ldots, w_k] \\ H^*(\mathscr{B}SO(k); \mathbb{Z}_2) &= \mathbb{Z}_2[w_2, \ldots, w_k] \\ H^*(\mathscr{B}SO(2i); \mathbb{Z}) &= \mathbb{Z}[p_1, p_2, \ldots, p_{i-1}, e] \\ H^*(\mathscr{B}SO(2i+1); \mathbb{Z}[1/2]) &= \mathbb{Z}[1/2][p_1, p_2, \ldots, p_i] \\ H^*(\mathscr{B}U(k); \mathbb{Z}) &= \mathbb{Z}[c_1, c_2, \ldots, c_k], \end{aligned}$$

where $w_j \in H^j$ is the j^{th} Stiefel–Whitney class of the corresponding universal bundle $\mathscr{E}_{\mathbb{R}^k}$, while $p_j \in H^{4j}$ is its m^{th} Pontryagin class, $e \in H^{2i}$ is the Euler class, and $c_j \in H^{2j}$ is the j^{th} Chern class of the universal complex bundle $\mathscr{E}_{\mathbb{C}^k}$. The difference between the $SO(2i)$ and $SO(2i+1)$ cases is owing to the fact that in the first case $e \cup e = p_i$, while in the second $e = 0$; further, the ring $\mathbb{Z}[1/2]$ is needed to kill the 2-torsion (and $\mathbb{Q}$ or $\mathbb{R}$ could be used instead). Indeed, remember that $p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C})$, but that the classes $c_{2j+1}(E \otimes \mathbb{C})$, which are all of order 2, escape. See also **D. Husemoller***'s* ***Fibre bundles*** **[Hus66, Hus94,** ch 17**]**

Note: Topological manifolds and smoothings

In what follows, we will outline the theory of topological manifolds and of their smooth structures. The theory works best in dimensions 5 or more, where it offers complete answers on the existence and classification of smooth structures on topological manifolds. The theory is quite weaker in dimension 4, but it is still relevant.

Requisites for understanding this note are the two previous notes, namely the one on page 197, where the rudiments of obstruction theory were presented, and the

23. Since $Spin(k) \to SO(k)$ is a cover map, we have $\pi_m(Spin(k)) = \pi_m(SO(k))$ for all $m > 2$.

one on page 204, where general fiber bundles and their classifying spaces were explained. The groups of homotopy spheres Θ_m, described in the end-notes of chapter 2.5 (page 97), will also make an appearance. On the other hand, if one skips the paragraphs on smoothing bundles, then one merely needs the simple definition of a general fiber bundle, which can be read from the beginning of the note on page 204.

Historically, at first the realm of purely topological manifolds and pure homeomorphisms seemed unapproachable, so mathematicians attacked the gap between smooth and *piecewise-linear (PL) manifolds,* meaning manifolds structured by a nice triangulation (where "nice" means that the link[24] of every vertex is required to be simplicially-homeomorphic to a standard polyhedral sphere; such triangulated manifolds are also called *combinatorial manifolds*). Success with smoothing PL manifolds started with S. Cairns and continued with M. Hirsch and B. Mazur, which completely elucidated the gap between PL and smooth. The door on smoothing topological manifolds was opened by J. Milnor, who introduced the right concept of tangent bundle for a topological manifold. Finally R. Kirby breached the barrier toward the study of topological manifolds, and together with L. Siebenmann clarified the gap between topological and PL manifolds. See also the bibliographical comments on page 67 at the end of chapter 1, as well as the references ahead on page 219.

Since we are focused on 4–manifolds while the gap between PL and smooth manifolds only starts to make its presence felt in dimension 7, in our presentation below we will shortcut the PL level and discuss smoothing theory only in terms of the gap between topological and smooth manifolds.

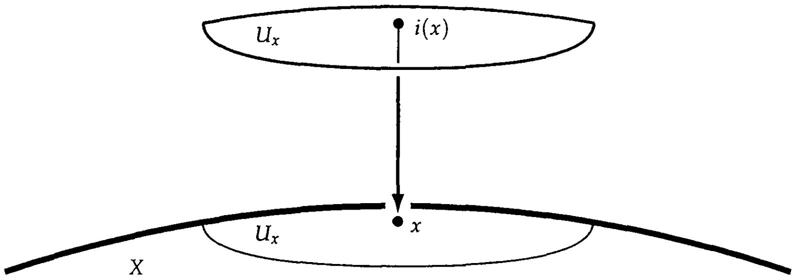
Tangent bundles for topological manifolds. Remember that a topological manifold of dimension m is merely a separable metrizable topological space that locally looks like $\mathbb{R}^m$; in other words, X is covered by open sets U that are homeomorphic to $\mathbb{R}^m$.

For smooth manifolds, one of the most useful objects used in their study is the tangent bundle, which gives the infinitesimal image of the manifold and thus approximates its structure by simpler spaces. A suitable analogue for topological manifolds can only prove useful.

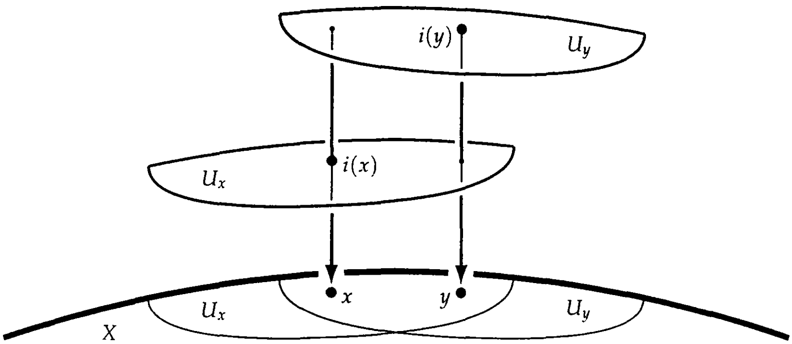
A first idea would be to pick for each $x \in X$ a small open neighborhood U_x homeomorphic to $\mathbb{R}^m$ and consider it as the fiber of T_X at x, as in figure 4.31 on the next page. Parts of nearby such fibers would get identified just as the corresponding open sets in X: the fiber U_x over x and the fiber U_y over y have their common part $U_x \cap U_y$ identifiable, as suggested in figure 4.32 on the facing page.

Such a tangent "bundle" has fiber $\mathbb{R}^m$ and has an obvious "zero section" i sending $x \in X$ to $x \in U_x = T_X|_x$. This creature is not a bundle: neighboring fibers cannot be identified with each other, since only parts of them overlap. However, it is conceivable that, by restricting to smaller neighborhoods of the zero-section

24. Think of the **link** of a vertex v essentially as the (simplicial) boundary of a small simplicial neighborhood of v. Specifically, take all simplices σ that contain v and take the faces of σ that do not touch v; the union of all such faces makes up the link of v.



4.31. Building a tangent bundle, I



4.32. Building a tangent bundle, II

and deforming our structure by homotopies, one would end-up with a genuine fiber bundle, with fiber $\mathbb{R}^m$. If, for the resulting bundle, we take care to identify each fiber above x with $\mathbb{R}^m$ in such manner that x corresponds to 0, then the structure group of the bundle would be the group of all self-homeomorphisms $\varphi\colon \mathbb{R}^m \simeq \mathbb{R}^m$ that fix the origin, $\varphi(0) = 0$. Let us denote this group by

$$TOP(m) \; .$$

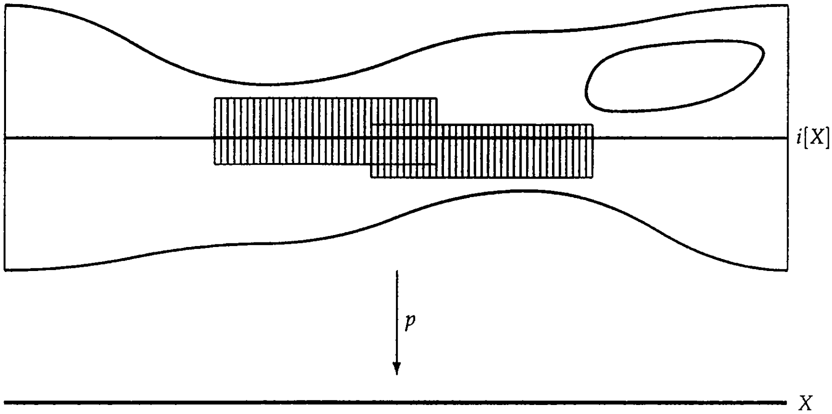
Thus, our proposed tangent structure appears to induce a $TOP(m)$–bundle.

The only real problem with such an approach is that the construction does not appear canonical, since the choice of neighborhoods/fibers U_x is random. It is important that each topological manifold have a *canonical* tangent bundle T_X. In order to achieve this, the main observation is that what really matters is what happens around x—whatever U_x has been chosen to be, the most important part of $T_X|_x$ is the immediate neighborhood of $x \in T_X|_x$ and how it relates to its neighboring fibers. Thus, one should consider, instead of the whole U_x's, just their *germs* at x. This idea was concretized in **J. Milnor**'s notion of a *microbundle*, which he introduced in ***Microbundles*** **[Mil64]**.

Microbundles and the topological tangent bundle. A k–**microbundle** ξ on X is a configuration

$$\xi\colon\; X \xrightarrow{\;i\;} E \xrightarrow{\;p\;} X \,,$$

made of a topological space E (called the *total space*), together with two maps, $i\colon X \to E$ (called the *zero section*) and $p\colon E \to X$ (called the *projection*). These are required to satisfy two properties: (1) i must behave like a section, so we have $p \circ i = id$; and (2) $E \to X$ must be *locally trivial, i.e.*, for every $x \in X$, there is a neighborhood V_x of $i(x)$ in E such that $p|_{V_x} \colon V_x \to M$ looks like a projection $U \times \mathbb{R}^k \to U$. Notice that, as suggested in figure 4.33, nothing is required far from $i[X]$ or on the overlaps of the various local "charts": only parts of the fibers match.



4.33. A microbundle

You should think of a microbundle as a fiber bundle in which all that matters is what happens around the zero section, or as a vector bundle in which we are focused near the zero-section and all requirements of linearity have been dropped. Indeed, microbundles behave pretty much like vector bundles: they can be pulled-back, direct sums are defined, *etc.* We leave such amusements to the reader.

Two k–microbundles $\zeta' \colon X \xrightarrow{i'} E' \xrightarrow{p'} X$ and $\zeta'' \colon X \xrightarrow{i''} E'' \xrightarrow{p''} X$ are called **isomorphic** if there are neighborhoods W' of $i'[X]$ in E' and W'' of $i''[X]$ in E'' and a homeomorphism $\varphi \colon W' \simeq W''$ fitting in the diagram

$$\begin{array}{ccccc} X & \xrightarrow[i']{} & W' & \xrightarrow[p']{} & X \\ \| & & \varphi \downarrow \simeq & & \| \\ X & \xrightarrow{i''} & W'' & \xrightarrow{p''} & X\,. \end{array}$$

Of course, any actual bundle with fiber $\mathbb{R}^k$ is a k–microbundle, and two isomorphic fiber bundles are also isomorphic as microbundles.

Further, inside every microbundle actually lies a genuine bundle:

Kister–Mazur Theorem. *For every k–microbundle $X \xrightarrow{i} E \xrightarrow{p} X$ there is a neighborhood W of $i[X]$ in E such that $p|_W \colon W \to X$ is a locally-trivial fiber bundle with fiber $\mathbb{R}^k$ and zero-section i. The contained fiber bundle is unique up to isomorphism, and even up to isotopy.*

Idea of proof. The crux of the argument is J. Kister's result that the space of topological *embeddings* $\mathbb{R}^k \to \mathbb{R}^k$ that fix the origin can be deformation-retracted to the space of *homeomorphisms* $\mathbb{R}^k \simeq \mathbb{R}^k$ that fix the origin. Thus, the partly-matching "charts" of a microbundle can be reduced and deformed to get a small global genuine bundle. See **J. Kister**'s ***Microbundles are fibre bundles*** [Kis64]. □

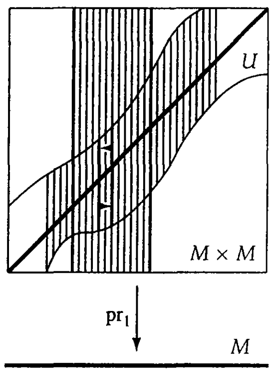
Thus, to every k–microbundle is associated a canonical fiber bundle with group $TOP(k)$ and fiber $\mathbb{R}^k$. Microbundles have the advantage that they are easy to describe. Thus, if we define a canonical *tangent microbundle* for a topological m–manifold, then we can pass it through Kister–Mazur to obtain a *canonical* tangent bundle, with structure group $TOP(m)$.

The **tangent microbundle** of a topological manifold X is defined simply as

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{pr}_1} X,$$

where Δ is the diagonal map $x \mapsto (x, x)$ and pr_1 is the projection $(x, y) \mapsto x$.

Close to the diagonal $\Delta[X]$, the fibers of pr_1 are just copies of neighborhoods of points in X. They are stacked next to each other according to their position in X: indeed, $z' \in \mathrm{pr}_1^{-1}[x']$ and $z'' \in \mathrm{pr}_1^{-1}[x'']$ are close to each other in $X \times X$ if and only if $\mathrm{pr}_2(z')$ and $\mathrm{pr}_2(z'')$ are close to each other in X. See also figure 4.34.



4.34. The tangent microbundle

We can then define the **topological tangent bundle**

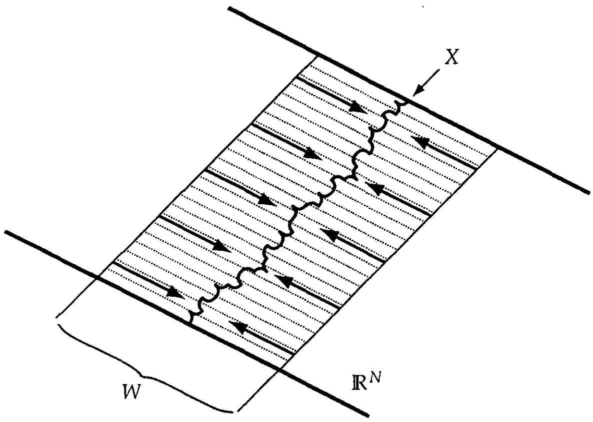
$$T_X^{top}$$

of the topological m–manifold X to be the $TOP(m)$–bundle contained inside the tangent microbundle of X. One can prove that, if X happens to be a smooth manifold and hence is endowed with a tangent vector bundle T_X, then T_X and T_X^{top} are isomorphic fiber bundles.

Using the topological tangent bundle for smoothing. Start with a topological m–manifold X. Embed X^m into some large[25] $\mathbb{R}^N$ and choose a neighborhood W of

25. To build an embedding of a topological manifold in some $\mathbb{R}^N$, the easiest way is as follows: When X is compact, cover X^m by open sets $U_1, \ldots, U_n$, each homeomorphic to an open subset of $\mathbb{R}^m$ through embeddings $f_k \colon U_k \subset \mathbb{R}^m$; extend each f_k to a continuous maps $\tilde{f}_k \colon M \to \mathbb{R}^m$, then gather all of them together to get an embedding $(\tilde{f}_1, \ldots, \tilde{f}_n) \colon M \to \mathbb{R}^{mn}$. In general, by dimension theory one can find an open covering $\{U_\alpha\}$ of X so that at every point of X no more than $m+1$ of the U_α's meet; eventually one gets an embedding in $\mathbb{R}^{m(m+1)}$.

X in $\mathbb{R}^N$ that *retracts* to X, *i.e.*, for which there is map $r\colon W \to X$ so that $r|_X = id$. See figure 4.35.



4.35. X embedded as a Euclidean neighborhood retract

Build the tangent bundle T_X^{top} of X and then use the retraction r to pull it back over the whole W; denote the total space of the result by $r^*T_X^{top}$.

$$\begin{array}{ccc} r^*T_X^{top} & \longrightarrow & T_X^{top} \\ \downarrow & & \downarrow \\ W & \xrightarrow{\ r\ } & X\,. \end{array}$$

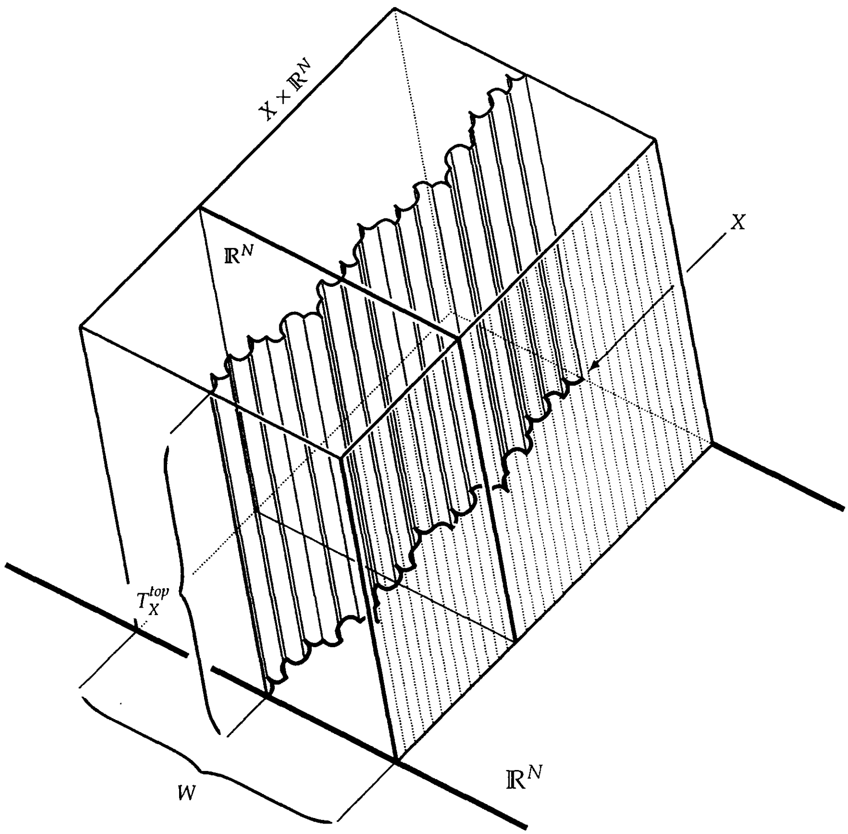
The total space of this pulled-back bundle can also be viewed as sitting on top of X, through the composition $r^*T_X^{top} \longrightarrow W \xrightarrow{r} X$; in reverse, X can be embedded in $r^*T_X^{top}$ through the composition of the inclusion $X \subset W$ with the zero section of the bundle $r^*T_X^{top} \to W$. We have the following remarkable property:

Lemma. *The space $r^*T_X^{top}$ is homeomorphic to $X \times \mathbb{R}^N$, with $X \subset r^*T_X^{top}$ corresponding to $X \times 0 \subset X \times \mathbb{R}^N$.*

Idea of proof. As a first approximation, think in terms of vector bundles: Assume that X was a smooth manifold, and T_X its tangent bundle. Smoothly embed X in $\mathbb{R}^N$, then choose a tubular neighborhood $W \approx N_{X/\mathbb{R}^N}$, which retracts to X through the bundle projection $r\colon N_{X/\mathbb{R}^N} \to X$. Then $r^*T_X \to X$ is isomorphic to the bundle $T_X \oplus N_{X/\mathbb{R}^N} = T_{\mathbb{R}^N}|_X = X \times \mathbb{R}^N$.

We can use a similar argument for our lemma if we start with a better W. Namely, we could start with an embedding of X into a large-enough $\mathbb{R}^N$, so that X admits a topological normal bundle $N_{X/\mathbb{R}^N}^{top}$ in $\mathbb{R}^N$, and take W to be the total space of $N_{X/\mathbb{R}^N}^{top}$ and r be its projection. □

Microbundle proof. *Without choosing a nice W and getting involved with topological normal bundles, one can also use a general argument, which is easiest to state in terms of microbundles: Consider T_X^{top} as the m-microbundle $X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{pr}_1} X$. The pull-back over W has total space*



4.36. Smoothing $X \times \mathbb{R}^N$ by using the tangent bundle of X

$r^*T_X^{top} = \{(w, r(w), x) \in W \times X \times X\}$, projection $p\colon r^*T_X^{top} \to W$, $p(w, r(w), x) = w$, and zero-section $i\colon W \to r^*T_X^{top}$, $i(w) = (w, r(w), r(w))$. First, notice that the total space $r^*T_X^{top}$ is naturally homeomorphic to $X \times W$, by sending $(w, r(w), x)$ to (x, w). This homeomorphism $r^*T_X^{top} \simeq X \times W$ sends $i[X]$ to $\Delta[X]$. Then, by translating the inclusion $X \times W \subset X \times \mathbb{R}^N$ through the map $X \times \mathbb{R}^N \to X \times \mathbb{R}^N\colon (x, v) \mapsto (x, v - x)$, we obtain an embedding of $r^*T_X^{top}$ into $X \times \mathbb{R}^N$ that sends $i[X]$ to $X \times 0$. While this is a bit less than the statement of the lemma, all further developments could be slightly modified to be happy with this version.

Owing to this lemma, if we manage to make the total space $r^*T_X^{top}$ into a *smooth manifold*, then that means that we have endowed $X \times \mathbb{R}^N$ with a smooth structure. We would be a bit closer to smoothing X itself.

As mentioned, the tangent bundle T_X^{top} is a fiber bundle over X^m with fiber $\mathbb{R}^m$ and structure group $TOP(m)$. Denote now by

$$DIFF(m)$$

the group of diffeomorphisms $\varphi\colon \mathbb{R}^m \cong \mathbb{R}^m$ with $\varphi(0) = 0$. If we could reduce the structure group of T_X^{top} from $TOP(m)$ to $DIFF(m)$, then the pull-back $r^*T_X^{top}$ would be a bundle over W whose fibers are glued by smooth maps from $DIFF(m)$. Since W is open in $\mathbb{R}^N$, it is itself a smooth manifold. The base being smooth and the fibers being smoothly-matched, it follows that the total space of $r^* T_X^{top} \to W$ would itself be a smooth manifold. However, $r^* T_X^{top}$ is homeomorphic to $X \times \mathbb{R}^N$, and therefore the latter inherits a smooth structure.

Milnor's Smoothing Theorem. *Let X be a topological m–manifold. If its tangent bundle T_X^{top} admits a $DIFF(m)$–structure, then, for N big enough, $X \times \mathbb{R}^N$ must admit a smooth structure.* □

This was proved[26] by **J. Milnor's *Microbundles*** **[Mil64]**, first announced in ***Topological manifolds and smooth manifolds*** **[Mil63c]**.

We postpone the investigation of the existence of $DIFF(m)$–structures on T_X^{top} for later. In the mean time, let us see how to get rid of the $\mathbb{R}^N$–factor, so that we may end up with a smooth structure on X itself.

Structures on products and products of structures. The following results are due to R. Kirby and L. Siebenmann. The first statement below is analogous to the Cairns–Hirsch theorem, which dealt with the PL case.

Product Structure Theorem. *Let X be a topological m–manifold, with m at least 5. If $X \times \mathbb{R}^N$ admits a smooth structure, then this structure must be isotopic to a product smooth structure on $X \times \mathbb{R}^N$, coming from a smoothing of X crossed with the standard smooth structure on $\mathbb{R}^N$.* □

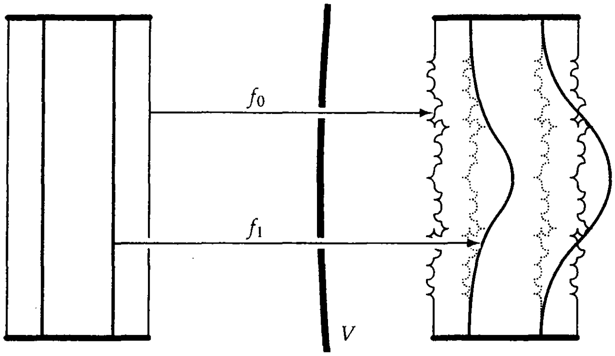
Note that the isotopy conclusion above is stronger than a mere diffeomorphism between the two smooth structures on $X \times \mathbb{R}^N$.

Isotopies of smoothings. For convenience, call ξ the given smooth structure on $X \times \mathbb{R}^N$, by ζ the resulting smooth structure on X, and by $\zeta \times std$ the product structure on $X \times \mathbb{R}^N$. The existence of an isotopy between ξ and $\zeta \times std$ means two things: First, that ξ and $\zeta \times std$ are concordant, meaning that there exists a smooth structure on $(X \times \mathbb{R}^N) \times [0,1]$ that coincides with ξ near $(X \times \mathbb{R}^N) \times 0$ and with $\zeta \times std$ near $(X \times \mathbb{R}^N) \times 1$. Second, that there is a smooth map $h\colon (X \times \mathbb{R}^N) \times [0,1] \longrightarrow (X \times \mathbb{R}^N,\ \xi)$ so that each slice $h_t = h(\,\cdot\,,t)\colon X \times \mathbb{R}^N \times \{t\} \xrightarrow{\cong} X \times \mathbb{R}^N$ is a diffeomorphism onto $X \times \mathbb{R}^N$ smoothed by ξ. Thus, h_0 is the identity map from $(X \times [0,1],\ \xi)$ to itself, while h_1 is a diffeomorphism from $(X \times [0,1],\ \zeta \times std)$ to $(X \times [0,1],\ \xi)$, and h_t is the isotopy between them.

Notice the dimensional restriction $m \geq 5$ that appears in the statement of the theorem. Its appearance is owing to the inevitable reliance of the proofs on the h–cobordism theorem (and its non-simply-connected cousin, the s–cobordism theorem). This is what prevents smoothing theory from fully applying to 4–dimensional manifolds.

Proving the product theorem. The essential tool for proving the product structure theorem is the following handle-smoothing technique: Assume we have a smooth manifold V^m and a smooth embedding of a thickened sphere $S^{k-1} \times [0,\varepsilon) \times \mathbb{R}^{m-k} \subset V^m$ (think of $S^{k-1} \times [0,\varepsilon)$ as a collar on S^{k-1} in $\mathbb{D}^k$). Further assume that this smooth embedding can be extended as a topological embedding $f_0\colon \mathbb{D}^k \times \mathbb{R}^{m-k} \subset V^m$ of an open k–handle into V. We say that the handle f_0 can be **smoothed** *in V if there is an isotopy f_t between f_0 and a map f_1 that restricts to a smooth embedding of the closed k–handle $f_1\colon \mathbb{D}^k \times \mathbb{D}^{m-k} \subset V^m$, and so that f_t fixes f_0 outside a compact neighborhood of $\mathbb{D}^k \times \mathbb{D}^{m-k}$. See figure 4.37 on the facing page.*

26. Proved before the discovery of the Kister–Mazur theorem.



4.37. Smoothing a handle

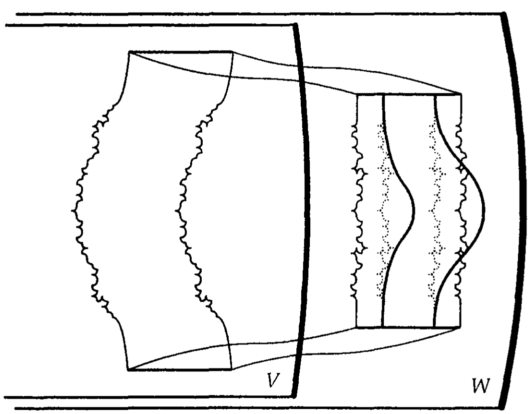
It turns out that the property of a handle f_0 to be smoothable is invariant under concordance:

Handle Smoothing Theorem. *Let f_0 be an open k-handle in V^m as above, and let W^m be a smooth manifold (of same dimension) containing V^m. Assume that there is an isotopy $F\colon \mathbb{D}^k \times \mathbb{R}^{m-k} \times [0,1] \longrightarrow W$ so that $F(\cdot, 0) = f_0$, that F moves the attaching sphere smoothly, and that $F(\cdot, 1)$ is a handle in W that can be smoothed in W. If $m \geq 5$, then f_0 itself can be smoothed inside V.* □

See also figure 4.38. This theorem is due to **R. Kirby** *and* **L. Siebenmann**, *see* ***Foundational essays on topological manifolds, smoothings, and triangulations*** [KS77]. *An essential ingredient for proving this handle smoothing theorem is, of course, the h-cobordism theorem. A consequence of it is the following stability property:*

Corollary. *Let $f_0\colon \mathbb{D}^k \times \mathbb{R}^{m-k} \subset V^m$ be some open k-handle as above and assume that $m \geq 5$. If the product-handle $f_0 \times id\colon \mathbb{D}^k \times \mathbb{R}^{m-k} \times \mathbb{R} \subset V \times \mathbb{R}$ can be smoothed inside $V \times \mathbb{R}$, then f_0 itself can be smoothed inside V.*

The proof of the product structure theorem then uses a chart-by-chart induction. Since each chart $\Phi\colon U \simeq U' \subset \mathbb{R}^m$ endows U with a smooth structure, this means that in each chart we can use handle decompositions, with handles that are then smoothed and made to fit on the overlaps of the charts.[27]



4.38. Handle smoothing theorem

27. It is worth noticing how, even when investigating purely topological manifolds, it is the differential world that offers the local tools, which are then extended by careful patching and matching.

In conclusion, by combining Milnor's smoothing theorem with the Kirby–Siebenmann product structure theorem, we obtain:

Corollary. *Let X^m be a topological m–manifold, with m at least 5. If its topological tangent bundle T_X^{top} admits a DIFF(m)–structure, then X must itself admit a smooth structure.* □

In other words, we are able to "integrate" an infinitesimal differentiable structure on the tangent bundle to a differentiable structure on the manifold X itself.

It is now time to see what obstructions appear when trying to smooth the tangent bundle of a topological manifold X^m:

Smoothing bundles: the setting. The topological tangent bundle T_X^{top} is a bundle with fiber $\mathbb{R}^m$ and group $TOP(m)$; we wish to reduce its structure group to $DIFF(m)$. The method of choice will be obstruction theory, applied to classifying spaces. Thus, for a better understanding of the following, it is recommended to first read the earlier notes (on page 197 and on page 204).

At the outset, we should remark that the group $DIFF(m)$ of self-diffeomorphisms of $\mathbb{R}^m$ fixing the origin is homotopy-equivalent with the more familiar group $GL(m)$ of invertible matrices. Indeed, if $\varphi_1 \colon \mathbb{R}^m \cong \mathbb{R}^m$ is a diffeomorphism with $\varphi_1(0) = 0$, then the Alexander isotopy $\varphi_t(x) = \frac{1}{t}\varphi(tx)$ provides a deformation of φ_1 to $\varphi_0 = d\varphi_1|_0 \in GL(m)$, and thus contracts $DIFF(m)$ to $GL(m)$. This implies that a fiber bundle with structure group $DIFF(m)$ is nothing but a vector bundle. Therefore, to reduce the structure group of the tangent bundle T_X^{top} from $TOP(m)$ to $DIFF(m)$ means merely to organize T_X^{top} as a vector bundle.

The group $TOP(m)$ has a classifying space denoted by $\mathscr{B}TOP(m)$. As a consequence, the tangent bundle T_X^{top} is described by a classifying map

$$\tau \colon X \longrightarrow \mathscr{B}TOP(M)\,.$$

The group $DIFF(m)$ has a classifying space[28] $\mathscr{B}DIFF(m)$. The natural inclusion $DIFF(m) \subset TOP(m)$ induces a fibration

$$\mathscr{S} \colon \mathscr{B}DIFF(m) \longrightarrow \mathscr{B}TOP(m)$$

with fiber $TOP(m)/DIFF(m)$. Then endowing the tangent bundle T_X^{top} with a $DIFF(m)$–structure is the same as lifting the classifying map τ to a map $\tau^{sm} \colon X \to \mathscr{B}DIFF(m)$ that fits in

$$\begin{array}{ccc} X & \xrightarrow[\tau^{sm}]{} & \mathscr{B}DIFF(m) \\ \| & & \downarrow{\scriptstyle\mathscr{S}} \\ X & \xrightarrow{\tau} & \mathscr{B}TOP(m)\,. \end{array}$$

We can pull the fibration $\mathscr{S} \colon \mathscr{B}DIFF(m) \longrightarrow \mathscr{B}TOP(m)$ back over X as

$$\begin{array}{ccc} \tau^*\mathscr{S} & \longrightarrow & \mathscr{B}DIFF(m) \\ \downarrow & & \downarrow{\scriptstyle\mathscr{S}} \\ X & \xrightarrow{\tau} & \mathscr{B}TOP(m)\,, \end{array}$$

and then smoothing T_X^{top} is equivalent to finding a section in this pulled-back fibration $\tau^*\mathscr{S}$. The fiber of $\tau^*\mathscr{S} \to X$ is $TOP(m)/DIFF(m)$.

28. $\mathscr{B}DIFF(m)$ is the same (homotopy-equivalent) with $\mathscr{B}GL(m) = \mathscr{B}O(m) = \mathscr{G}r_m(\mathbb{R}^\infty)$.

Think of all this as a setting on which to use obstruction theory. We start with a random smoothing of the tangent bundle over the vertices of some cellular decomposition of X, viewed as a section of $\tau^*\mathscr{S}$ over the 0–skeleton of X. We then strive to extend this section cell-by-cell across all X. When extending from the k–skeleton of X across the $(k+1)$–skeleton of X, obstructions appear in

$$H^{k+1}(X;\ \pi_k(TOP(m)/DIFF(m)))\ .$$

Further, if a given section σ of $\tau^*\mathscr{S}$ over the k–skeleton is extendable across the $(k+1)$–skeleton, then the elements of

$$H^k(X;\ \pi_k(TOP(m)/DIFF(m)))$$

classify up to homotopy all other sections over the k–skeleton that are extendable across the $(k+1)$–skeleton and are homotopic to σ over the $(k-1)$–skeleton.

In terms of smooth structures on T_X^{top} or, equivalently when $m \geq 5$, in terms of the induced smooth structures on X^m, any homotopy of a section of $\tau^*\mathscr{S}$ corresponds to a *concordance* of smooth structures on X. Two smooth structures ζ' and ζ'' on X are called **concordant** if there is a smooth structure on $X \times [0,1]$ that is ζ' on $X \times 0$ and is ζ'' on $X \times 1$. Keep in mind that smooth structures can be diffeomorphic without being concordant; simple examples come from manifolds that do not admit orientation-reversing diffeomorphisms. (Furthermore, in high-dimensions concordance implies isotopy.)

Hence, obstruction theory can be used to clarify the existence and classification up to concordance of smooth structures on topological manifolds of dimension at least 5. Of course, in order to effectively put obstruction theory to work, we need to determine the homotopy groups of the fiber $TOP(m)/DIFF(m)$.

Smoothing bundles: computing the homotopy groups. This paragraph is rather dense and very sketchy. It can be safely skipped; the next paragraph starts on page 220.

High homotopy. Let us apply the above obstruction theory setting to the case of the sphere S^n. Since the topological manifold S^n admits smooth structures, no obstructions appear. Further, the only non-zero classifying cohomology group $H^k(X;\pi_k)$ appears when $k = n$, in which case we have

$$H^n(S^n;\ \pi_n(TOP(n)/DIFF(n))) \;=\; \pi_n(TOP(n)/DIFF(n))\ .$$

Therefore, for $n \geq 5$ we have

$$\{\text{smooth structures on } S^n\} \;\approx\; \pi_n(TOP(n)/DIFF(n))$$

(smooth structures considered up to concordance). That is to say:

Lemma. *When $n \geq 5$, we have $\pi_n(TOP(n)/DIFF(n)) = \Theta_n$, where Θ_n denotes the group of homotopy n–spheres.* □

The groups Θ_n have been presented in the end-notes of chapter 2 (page 97). They are defined as the set of all smooth homotopy n–spheres, considered up to h–cobordisms and with addition given by connected sums. We have seen that, when $n \geq 5$, the set Θ_n can be understood as the group of concordance classes of smooth structures on S^n; hence we could call Θ_n "the group of exotic n–spheres". These groups can be computed using surgery methods. All groups Θ_n are finite, and the first nontrivial one is $\Theta_7 = \mathbb{Z}_{28}$.

Further, after using stabilizations $TOP(k) \subset TOP(k+1)$, we are led to:

Theorem. *For all n and m with $5 \leq n \leq m+1$, we have*

$$\pi_n\big(TOP(m)/DIFF(m)\big) = \Theta_n . \qquad \square$$

This theorem follows from the delicate result that, for $n \leq m+1$ and $m \geq 4$, we have

$$\pi_n\big(TOP(m+1)/DIFF(m+1),\ TOP(m)/DIFF(m)\big) = 0 .$$

The cases when $m \geq 5$ were proved in **R. Kirby** *and* **L. Siebenmann***'s* ***Foundational essays on topological manifolds, smoothings, and triangulations*** **[KS77]**. *The cases when $m = 4$ were cleared in* **F. Quinn***'s* ***Ends of maps. III. Dimensions 4 and 5*** **[Qui82]** *for $n \leq 3$; in his* ***Isotopy of 4–manifolds*** **[Qui86]** *for $n = 5$; and in* **R. Lashof** *and* **L. Taylor***'s* ***Smoothing theory and Freedman's work on four-manifolds*** **[LT84]** *for $n = 3, 4$.*

Low homotopy. We now need to compute the low-dimensional homotopy groups of $TOP/DIFF$. For $n \geq 5$, we have used $\mathbb{S}^n$ to evaluate π_n. For $n \leq 4$, we can instead increase the dimension of $\mathbb{S}^n$ by thickening it to $\mathbb{S}^n \times \mathbb{R}^k$ such that $n + k \geq 5$. Then, after using stabilizations, we have

$$\{\text{smooth structures on } \mathbb{S}^n \times \mathbb{R}^k\} \approx \pi_n\big(TOP(m)/DIFF(m)\big)$$

for all $m \geq 4$. However, smooth structures on the open manifold $\mathbb{S}^n \times \mathbb{R}^k$ are hard to approach directly. Instead, one considers smooth structures on $\mathbb{S}^n \times \mathbb{T}^k$. On one hand, by climbing the universal cover $\mathbb{R}^k \to \mathbb{T}^k$, it is clear that each smooth structure on $\mathbb{S}^n \times \mathbb{T}^k$ induces a smooth structure on $\mathbb{S}^n \times \mathbb{R}^k$.

The fundamental fact is that, conversely, the smooth structures on $\mathbb{S}^n \times \mathbb{R}^k$ correspond to smooth structures on $\mathbb{S}^n \times \mathbb{T}^k$, more precisely to *homotopy smooth structures* on $\mathbb{S}^n \times \mathbb{T}^k$. A **homotopy smooth structure** on a topological m–manifold X^m is a homotopy equivalence $X^m \sim V^m$ with some smooth m–manifold V^m (same dimension).

This converse is a consequence of the celebrated torus unfurling trick *of* **R. Kirby**, *which first appeared in* ***Stable homeomorphisms and the annulus conjecture*** **[Kir69]**, *and was used in our context in* **R. Kirby** *and* **L. Siebenmann***'s* ***On the triangulation of manifolds and the Hauptvermutung*** **[KS69]** *(see also* ***Foundational essays...*** **[KS77]***).*

When $n + k \leq 6$, the homotopy smooth structures on $\mathbb{S}^n \times \mathbb{T}^k$ (thought of as smooth structures on $\mathbb{D}^n \times \mathbb{T}^k$ relative to the boundary) are known by surgery theory to be classified by the elements of $H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2)$. Thus, for $n \geq 4$ there is only one homotopy smooth structure on $\mathbb{S}^n \times \mathbb{T}^k$, the standard one. For $n \leq 2$, all structures are known to be finitely-covered by the standard one (and thus can be standardized after climbing a finite cover of $\mathbb{T}^k$). Finally, for $n = 3$ there is at most one structure that is not covered by the standard one. Therefore the conclusion is that, for all small n not 3, we have

$$\pi_n\big(TOP(m)/DIFF(m)\big) = 0$$

and, moreover, that $\pi_3\big(TOP(m)/DIFF(m)\big)$ has either one or two elements.

As was first noticed by L. Siebenmann, it turns out that π_3 cannot be trivial, and hence

$$\pi_3(TOP(m)/DIFF(m)) = \mathbb{Z}_2 .$$

If one accepts everything else that was claimed above, then, for proving this nontriviality of π_3, we need only exhibit *one* topological manifold of dimension less than 7 that does not admit any smooth structures.

In dimension 4, Freedman's E_8–manifold $\mathcal{M}_{E_8}$ is an example, as follows from Rokhlin's theorem. For dimensions higher than 4, we also have:

Lemma. *The topological manifold $\mathcal{M}_{E_8} \times \mathbb{S}^k$ does not admit any smooth structures.*

Proof. Assume that $\mathcal{M}_{E_8} \times \mathbb{S}^k$ admits a smooth structure. Then, by writing $\mathbb{S}^k = \mathbb{R}^k \cup \{\infty\}$, we obtain a smooth structure on $\mathcal{M}_{E_8} \times \mathbb{R}^k$. We apply the product structure theorem and deduce that $\mathcal{M}_{E_8} \times \mathbb{R}$ admits a smooth structure. Consider the projection map $\mathrm{pr}_2 \colon \mathcal{M}_{E_8} \times \mathbb{R} \to \mathbb{R}$. Then, since $\mathcal{M}_{E_8} \times \mathbb{R}$ is smooth, we can perturb pr_2 over $\mathcal{M}_{E_8} \times (0,\infty)$ so that it becomes smooth over $\mathcal{M}_{E_8} \times (\varepsilon,\infty)$ but remains unchanged on $\mathcal{M}_{E_8} \times (-\infty,0)$. Pick a positive regular value $c > \varepsilon$ of pr_2; then $\mathrm{pr}_2^{-1}[c]$ is a smooth 4–manifold. Since $\mathcal{M}_{E_8}$ has $w_2 = 0$, so must $\mathrm{pr}_2^{-1}[c]$. However, $\mathrm{pr}_2^{-1}[-1] = \mathcal{M}_{E_8}$, and hence the 5–manifold $\mathrm{pr}_2^{-1}[-1,c]$ is a cobordism between $\mathcal{M}_{E_8}$ and the smooth manifold $\mathrm{pr}_2^{-1}[c]$. Since signatures are cobordism-invariants, it follows that the smooth 4–manifold $\mathrm{pr}_2^{-1}[c]$ has signature 8, but $w_2 = 0$. This, of course, is forbidden by Rokhlin's theorem. □

The manifolds $\mathcal{M}_{E_8} \times \mathbb{S}^k$ do not admit PL structures either. More important, notice the fundamental role that Rokhlin's theorem plays[29] in the nontriviality of $\pi_3(TOP(m)/DIFF(m))$.

What was omitted. *A more detailed discussion would of course have taken into account the intermediate piecewise-linear level between smooth and topological, and infinite stabilizations.*

Stabilization means considering everything up to adding trivial bundles. This embeds $TOP(m)$ into $TOP(m+1)$ and in the limit yields the group $TOP = \varinjlim TOP(m)$, with its own classifying space $\mathscr{B}TOP$. Similarly, $DIFF(m)$ stabilizes to $DIFF = \varinjlim DIFF(m)$, with classifying space $\mathscr{B}DIFF$. The group of piecewise-linear self-homeomorphisms of $\mathbb{R}^m$ that fix 0 is denoted by $PL(m)$, stabilizing to PL and with classifying space $\mathscr{B}PL$. The inclusions $TOP \subset PL \subset DIFF$ lead to fibrations $\mathscr{B}PL \to \mathscr{B}TOP$ and $\mathscr{B}DIFF \to \mathscr{B}PL$, with corresponding fibers TOP/PL and $PL/DIFF$.

Between smooth and PL: The study of the smooth/PL gap was attacked by **S. Cairns** *in* ***The manifold smoothing problem*** **[Cai61]**. *Then* **R. Thom***'s* ***Des variétés triangulées aux variétés différentiables*** **[Tho60]** *suggested that the smoothing problem should admit a setting in terms of obstruction theory. A natural simplex-by-simplex obstruction theory was developed by* **J. Munkres***'* ***Obstructions to the smoothing of piecewise-differentiable homeomorphisms*** **[Mun59, Mun60b]** *(see also his* **[Mun64]** *and* **[Mun65]***). A different obstruction theory was outlined in* **M. Hirsch***'s* ***Obstruction theories for smoothing manifolds and maps*** **[Hir63]***, and also proved a product structure theorem for the smooth/PL gap. Then appeared* **J. Milnor***'s* ***Microbundles*** **[Mil64]**. *All this led to an obstruction theory based on the classifying spaces $\mathscr{B}DIFF$ and $\mathscr{B}PL$, developed by* **M. Hirsch** *and* **B. Mazur** *and eventually published in the volume* ***Smoothings of***

29. Of course, R. Kirby and L. Siebenmann's result that $\pi_3(TOP/DIFF) = \mathbb{Z}_2$ was proved before M. Freedman built the fake 4–balls that are used in the construction of $\mathcal{M}_{E_8}$. Nonetheless, their examples also rest upon Rokhlin's theorem.

piecewise-linear manifolds **[HM74]**. *(A quick comparison of Munkres' and Hirsch–Mazur's approaches can be read from* **J. Munkres'** ***Concordance of differentiable structures—two approaches*** **[Mun67]**.*)*

The passing of the smooth/PL gap depends on the fiber $DIFF/PL$, which has homotopy groups

$$\pi_n(DIFF/PL) = 0 \quad \text{for all } n \leq 6, \qquad \text{and} \qquad \pi_n(DIFF/PL) = \Theta_n \quad \text{for all } n \geq 5 \ .$$

For proving the triviality of π_n in low-dimensions, the cases $n = 1,2$ are boring, the case $n = 3$ follows from **J. Munkres***'s* ***Differentiable isotopies on the 2-sphere*** **[Mun60a]** *and* **S. Smale***'s* ***Diffeomorphisms of the 2-sphere*** **[Sma59]**. *The case $n = 4$ was proved by* **J. Cerf***'s series of papers* ***La nullité de $\pi_0(\mathrm{Diff}\, S^3)$*** **[Cer64]**, *later published in the volume* ***Sur les difféomorphismes de la sphère de dimension trois*** **($\Gamma_4 = 0$)** **[Cer68a]**. *The cases $n = 5,6$ follow from the computations of Θ_n in* **M. Kervaire** *and* **J. Milnor***'s* ***Groups of homotopy spheres*** **[KM63]**.

Thus, the first non-zero homotopy group of $DIFF/PL$ is $\pi_7 = \mathbb{Z}_{28}$, coming from Milnor's exotic 7-spheres; geometrically, this first group corresponds to the existence of PL 8-manifolds that cannot be smoothed; an example is the 8-dimensional topological manifold $\mathcal{M}^8_{E_8}$ built by E_8-plumbing eight copies of $\mathbb{D}T_{S^4}$ and capping with an 8-disk, see back on page 98. In general all $\mathcal{M}^{4k}_{E_8}$'s are PL and non-smoothable.

Between PL and topological: For the study of topological manifolds, some important steps along the way were **B. Mazur***'s* ***On embeddings of spheres*** [Maz59, Maz61], *followed by* **M. Brown***'s* ***A proof of the generalized Schoenflies theorem*** [Bro60], *then* **A. Černavskiĭ***'s* ***Local contractibility of the group of homeomorphisms of a manifold*** [Čer68b, Čer69]. *Then came* ***R. Kirby****'s already mentioned torus unfurling trick, in* ***Stable homeomorphisms and the annulus conjecture*** **[Kir69]**, *which was then put to work together with L. Siebenmann.*

The passing of the PL/topological gap is governed by the fiber TOP/PL. The latter was shown to be an Eilenberg–Maclane $K(\mathbb{Z}_2; 3)$-space, that is to say,

$$\pi_3(TOP/PL) = \mathbb{Z}_2 \qquad \text{and} \qquad \pi_n(TOP/PL) = 0 \quad \text{for all } n \neq 3 \ .$$

This can be read from **R. Kirby** *and* **L. Siebenmann***'s* ***Foundational essays on topological manifolds, smoothings, and triangulations*** **[KS77]**. *Examples of topological $(4+k)$-manifolds that do not admit any PL structure are all $\mathcal{M}_{E_8} \times S^k$ and $\mathcal{M}_{E_8} \times \mathbb{T}^k$. A recent exposition of the PL/topological gap can also be read from* **Y. Rudyak***'s* ***Piecewise linear structures on topological manifolds*** **[Rud01]**.

The evaluation of the homotopy groups of TOP/PL rests upon the determination of all homotopy PL structures on $S^n \times \mathbb{T}^k$ (viewed as structures on $\mathbb{D}^n \times \mathbb{T}^k$ relative to the boundary). These were cleared using surgery by **A. Casson**, *then by* **W-c. Hsiang** *and* **J. Shaneson***'s* ***Fake tori, the annulus conjecture, and the conjectures of Kirby*** [HS69], *based on the surgery techniques developed by* **C.T.C Wall***'s* ***On homotopy tori and the annulus theorem*** **[Wal69b]** *(see also* ***Surgery on compact manifolds*** **[Wal70, Wal99, ch 15]***).*

Smoothing bundles: the Kirby–Siebenmann invariant. Reviewing the results outlined in the preceding paragraph, we can now state:

Theorem. *For every n and m with $5 \leq n \leq m+1$, we have:*

$$\pi_n\big(TOP(m)/DIFF(m)\big) = 0 \quad \text{for } 3 \neq n \leq 6$$
$$\pi_3\big(TOP(m)/DIFF(m)\big) = \mathbb{Z}_2$$
$$\pi_n\big(TOP(m)/DIFF(m)\big) = \Theta_n \quad \text{for } n \geq 5$$

where Θ_n is the group of homotopy n-spheres.[30] □

30. For those who skipped the preceding paragraphs: The groups of homotopy spheres Θ_n have been presented in the end-notes of chapter 2 (page 97). They can be defined for $n \geq 5$ as the set of smooth

We can now apply obstruction theory to study smoothings of topological manifolds of dimension at least 5, *via* smoothings of their topological tangent bundles.

Since the first dimension with a nontrivial homotopy group is $n = 3$, it follows that the primary obstruction to endowing the topological tangent bundle of X with a $DIFF(m)$–structure appears as a class in $H^4(X;\mathbb{Z}_2)$. It is called the **Kirby–Siebenmann invariant** and is denoted by

$$\mathrm{ks}(X) \in H^4(M;\mathbb{Z}_2) .$$

The existence of this first obstruction rests upon Rokhlin's theorem. Further, the difference cocycles are elements of $H^3(X;\mathbb{Z}_2)$.

Past dimension 7, higher obstructions appear from $H^{n+1}(X;\Theta_n)$, the first one from $H^8(X;\mathbb{Z}_{28})$. Higher difference cocycles live in $H^n(X;\Theta_n)$, the first ones in $H^7(X;\mathbb{Z}_{28})$.

Bringing in the intermediate PL level, we should say: The Kirby–Siebenmann invariant $\mathrm{ks}(X) \in H^4(X;\mathbb{Z}_2)$ *is the complete obstruction to endowing a topological manifold* X^m *of dimension* $m \geq 5$ *with a PL structure. If such a structure exists, all other PL structures are classified up to concordance (and thus isotopy) by* $H^3(X;\mathbb{Z}_2)$. *The higher obstructions from* $H^{n+1}(X;\Theta_n)$ *govern the possibility of endowing a PL manifold* X^m *with a smooth structure and do not appear until* $m = 8$. *Notice also that every PL 7–manifold admits exactly 28 distinct smooth structures, up to concordance.*

Since $\mathbb{Z}_2$ and all the Θ_n's are finite, a consequence is that any topological manifold of dimension *not* 4 admits at most *finitely-many* distinct smooth structures.[31]

Another consequence of the theory is that, for all $m \geq 5$, any topological manifold homeomorphic to $\mathbb{R}^m$ admits a unique smooth structure. Since the cases $m \leq 3$ are similar, this leaves $\mathbb{R}^4$ as the only possible support of exotic structures.

Conclusion. If the Kirby-Siebenmann invariant $\mathrm{ks}(X)$ vanishes and $m \leq 7$, then the tangent bundle of X^m admits a $DIFF(m)$–structure. If moreover $m \geq 5$, then this bundle structure can be integrated to a smooth structure on X itself. For example, all simply-connected topological 5–manifolds admit smooth structures.[32]

Moreover, if X admits some smooth structure, then all other smooth structures on X are classified (up to concordance/isotopy, *via* difference cocycles) by the elements of $H^3(X;\mathbb{Z}_2)$. Starting with dimension 8, beside $\mathrm{ks}(X)$ appear higher obstructions to the existence of smooth structures, living in the groups $H^{n+1}(X;\Theta_n)$.

***The case of dimension* 4.** The Kirby–Siebenmann invariant can certainly still be defined in dimension 4. However, lacking the power of the (smooth) h–cobordism theorem behind it, it mainly has negation power.

For a topological 4–manifold M, the Kirby–Siebenmann invariant

$$\mathrm{ks}(M) \in H^4(M;\mathbb{Z}_2)$$

---

structures on S^n considered up to concordance, with addition defined by connected sums; all groups Θ_n are finite, and the first nontrivial one is $\Theta_7 = \mathbb{Z}_{28}$.

31. The cases of dimension 2 and 3 being handled, of course, separately.

32. Since $H^4(X^5;\mathbb{Z}_2) = H_1(X^5,\mathbb{Z}) = 0$.

is simply a $\mathbb{Z}_2$–valued invariant: it is either 0 or 1. Its value is strongly related to Rokhlin's theorem (and its generalizations). Specifically, $\mathrm{ks}(M)$ detects whether a smooth structure on M is prohibited by Rokhlin's or not.

Evaluating the Kirby–Siebenmann invariant. Let M be any topological 4–manifold with no 2–torsion in $H_1(M;\mathbb{Z})$ and with *even* intersection form Q_M (such a manifold can safely be called a "spin manifold"). We have:

$$\mathrm{ks}(M) = \tfrac{1}{8}\,\mathrm{sign}\, M \pmod 2 .$$

In particular, $\mathrm{ks}(\mathcal{M}_{E_8}) = 1$.

More generally, regardless of the parity of Q_M, if a characteristic element of M can be represented by a topologically embedded *sphere* Σ, then we have

$$\mathrm{ks}(M) = \tfrac{1}{8}(\mathrm{sign}\, M - \Sigma\cdot\Sigma) \pmod 2 .$$

This is related to the Kervaire–Milnor generalization of Rokhlin's theorem.

Finally, *via* the Freedman–Kirby generalization of Rokhlin's theorem, we have, for every topological 4–manifold M with an embedded characteristic surface Σ,

$$\mathrm{ks}(M) = \tfrac{1}{8}(\mathrm{sign}\, M - \Sigma\cdot\Sigma) + \mathrm{Arf}(M,\Sigma) \pmod 2 ,$$

where $\mathrm{Arf}(M,\Sigma)$ is a $\mathbb{Z}_2$–invariant that measures the obstruction to representing Σ by a sphere, and depends only on the homology class of Σ. The Freedman–Kirby theorem will be discussed and proved in the end-notes of chapter 11 (page 502); it is readable anytime.

When Kirby–Siebenmann vanishes. If M admits a smooth structure, then $\mathrm{ks}(M) = 0$. The converse is false: if $\mathrm{ks}(M) = 0$, then M might still not admit any smooth structures. Such examples were uncovered starting with Donaldson's work[33] and they are not rare. Nonetheless, if $\mathrm{ks}(M) = 0$, then the 5–manifolds $M \times \mathbb{R}$ or $M \times S^1$ do admit smooth structures. Further, without increasing dimension, if $\mathrm{ks}(M) = 0$, then for m big enough the stabilization $M \# m\, S^2 \times S^2$ must admit a smooth structure.

On the other hand, it was proved that all *open* 4–manifolds can be smoothed. In particular, any closed 4–manifold M can be endowed with a smooth structure off a point.

In the case when $\mathrm{ks}(M) = 0$, then, since $M \# m\, S^2 \times S^2$ can be smoothed, such a smoothing-off-points for M can be made in a controlled fashion:

Theorem (F. Quinn). *If M is a topological 4–manifold with $\mathrm{ks}(M) = 0$, then there is a finite set of points $p_1, \dots, p_m$ in M and a smooth structure on*

$$M \setminus \{p_1, \dots, p_m\} ,$$

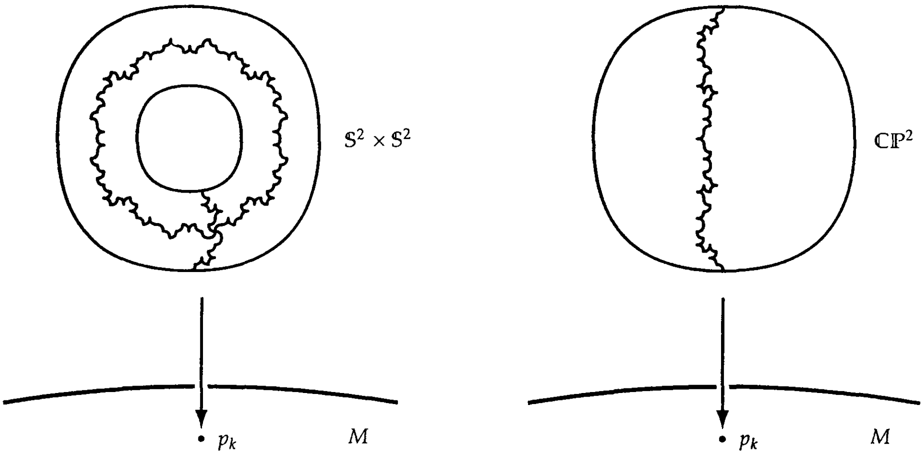
such that, for each k, on one hand there is a neighborhood U_k of p_k in M, and on the other hand there is a self-homeomorphism $\varphi_k\colon S^2 \times S^2 \simeq S^2 \times S^2$ (isotopic to the identity), a neighborhood U_k' of $h_k[S^2 \vee S^2]$ in $S^2 \times S^2$; and we have a diffeomorphism

$$U_k \setminus p_k \cong U_k' \setminus \varphi_k[S^2 \vee S^2] .$$

33. See ahead section 5.3 (page 243).

In other words, the complement of each p_k is locally smoothed like the complement of a displacement of $\mathbb{S}^2 \vee \mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{S}^2$. □

See the left side of figure 4.39. This result was proved in **F. Quinn**'s ***Smooth structures on 4–manifolds*** [Qui84] and can also be read from **M. Freedman** and **F. Quinn**'s ***Topology of 4–manifolds*** [FQ90].



4.39. Almost-smoothing a 4–manifold with $\text{ks}(M) = 0$

Since $\mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{CP}^2 = \# 2\,\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, the theorem can immediately be restated by instead using displacements of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ and diffeomorphisms

$$U_k \setminus p_k \cong U_k' \setminus \varphi_k[\mathbb{CP}^1]$$

(some of which which could reverse orientations). See the right side of figure 4.39, and also think in analogy with blow-ups of complex manifolds.[34]

A fundamental remark to be made in this context is that both $\mathbb{S}^2 \times \mathbb{S}^2 \setminus \varphi_k[\mathbb{S}^2 \vee \mathbb{S}^2]$ and $\mathbb{CP}^2 \setminus \varphi_k[\mathbb{CP}^1]$ are open smooth 4–manifolds that are homeomorphic to $\mathbb{R}^4$. This implies that, if M has $\text{ks}(M) = 0$ but is not smoothable, then these open manifolds must exhibit non-standard smooth structures on $\mathbb{R}^4$. In other words, they must be **exotic $\mathbb{R}^4$'s**. This, in part, explains why the discovery of exotic $\mathbb{R}^4$'s had to wait for Donaldson's work.[35] Exotic $\mathbb{R}^4$'s will be discussed in section 5.4 (page 250) ahead.

When Kirby–Siebenmann does not vanish. If $\text{ks}(M) = 1$, then M does not admit any smooth structures. If $\text{ks}(M) = 1$, then stabilizations do not help: $\text{ks}(M \# m\,\mathbb{S}^2 \times \mathbb{S}^2)$ will still be 1, and all the $M \# m\,\mathbb{S}^2 \times \mathbb{S}^2$'s will be non-smoothable. Indeed, the Kirby–Siebenmann invariant is nicely additive:

$$\text{ks}(M \cup_\partial N) = \text{ks}(M) + \text{ks}(N)\,.$$

34. Blow-ups are described in section 7.1 (page 286) ahead.

35. Of course, it also had to wait for A. Casson's and M. Freedman's work. Nonetheless, one can still ask whether the existence of exotic $\mathbb{R}^4$'s can be obtained as a consequence of Rokhlin's theorem while avoiding Donaldson's theory or equivalents. No.

In particular $\mathrm{ks}(M \# N) = \mathrm{ks}(M) + \mathrm{ks}(N)$, and so, if $\mathrm{ks}(M) = 1$, then $\mathrm{ks}(M \# m S^2 \times S^2) = 1$. Another important property to note is that the Kirby–Siebenmann invariant is unchanged by cobordisms.[36]

The invariant ks misses most of the wildness of dimension 4: for example, the Kirby–Siebenmann invariant of $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8}$ vanishes; but the latter has intersection form $E_8 \oplus E_8$, which is excluded from the smooth realm by the results of Donaldson: Kirby–Siebenmann's does not see what Rokhlin's does not exclude.

Note: The Rokhlin invariant of 3–manifolds

The Rokhlin theorem has major consequences beyond dimension 4. As we have seen in the preceding note (starting on page 207), in high-dimensions it is fundamentally implied in the non-existence of smooth structures on topological manifolds. In dimension 3, the Rokhlin theorem permits the definition of invariants for 3–manifolds, which are the topic of this note. The invariants for 3–manifolds are a $\mathbb{Z}_2$–invariant

$$\rho(\Sigma) \in \mathbb{Z}_2$$

for homology 3–spheres Σ, and a $\mathbb{Z}_{16}$–invariant

$$\mu(N) \in \mathbb{Z}_{16}$$

for 3–manifolds N endowed with spin structures.

Preparation: additivity of signatures. We have already seen that, if we connect-sum two 4–manifolds M and N, then we have $Q_{M\#N} = Q_M \oplus Q_N$, and as a consequence

$$\mathrm{sign}(M \# N) = \mathrm{sign}\, M + \mathrm{sign}\, N\,.$$

Intersection forms can also be defined for 4–manifolds with non-empty boundary, but they will not be unimodular unless the boundary is a homology sphere.[37] Then the additivity properties above are easy to prove for two manifolds M and N whose boundaries are a same homology sphere with *opposite* orientations: if we glue M and N along their boundaries, then $Q_{M \cup_\partial N} = Q_M \oplus Q_N$ and hence $\mathrm{sign}(M \cup_\partial N) = \mathrm{sign}\, M + \mathrm{sign}\, N$.

Examples. *For example, the 4–manifold[38] P_{E_8} has intersection form E_8 and signature 8. The manifold $P_{E_8} \cup_{\Sigma_P} \overline{P}_{E_8}$ is a closed 4–manifold with intersection form $E_8 \oplus -E_8 \approx \oplus 8H$ and signature 0. Because of signature-vanishing, we expect $P_{E_8} \cup_{\Sigma_P} \overline{P}_{E_8}$ to bound a 5–manifold, and indeed, it is the boundary of $P_{E_8} \times [0,1]$, as in figure 4.40 on the facing page. It turns out that $P_{E_8} \cup_{\Sigma_P} \overline{P}_{E_8}$ is none other than $\# 8 S^2 \times S^2$. (Notice that, since Σ_P does not have an orientation-reversing self-diffeomorphism, a manifold like $P_{E_8} \cup_{\Sigma_P} P_{E_8}$ does not exist.[39])*

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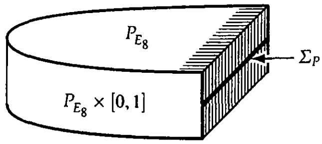
36. In fact, the topological cobordism group Ω_4^{top} of oriented topological 4–manifolds is $\Omega_4^{top} = \mathbb{Z} \oplus \mathbb{Z}_2$, with isomorphism given by $M \mapsto (\mathrm{sign}\, M, \mathrm{ks}(M))$. Cobordisms groups will be discussed in the note on page 227 ahead.

37. This will be fully proved in the end-notes of the next chapter (page 261).

38. Recall that P_{E_8} denotes the E_8–plumbing and is bounded by the Poincaré homology sphere Σ_P; see section 2.3 (page 86).

39. A roundabout argument: $P_{E_8} \cup_{\Sigma_P} P_{E_8}$ would be a *smooth* 4–manifold with definite intersection form $E_8 \oplus E_8$. However, that is excluded by Donaldson's theorem (see section 5.3, on page 243 ahead). Thus, this 4–manifold does not exist, and therefore Σ_P cannot admit an orientation-reversing self-diffeomorphism.



4.40. $P_{E_8} \cup_{\Sigma_P} \overline{P}_{E_8}$ is the boundary of $P_{E_8} \times [0,1]$

If two 4–manifolds have boundaries that are *not* homology spheres, then the additivity of the intersection forms ceases to hold. Nonetheless, signatures are still additive:

Novikov's Additivity Theorem. *Let M and N be two 4–manifolds with non-empty boundaries. Assume that their boundary 3–manifolds ∂M and ∂N admit an orientation-reversing diffeomorphism $\partial M \cong \overline{\partial N}$. Then the closed manifold $M \cup_\partial N$, built by identifying the boundaries ∂M and ∂N, has signature*

$$\operatorname{sign}(M \cup_\partial N) = \operatorname{sign} M + \operatorname{sign} N .$$

Outline of proof. Denote by Y^3 the (unoriented) boundaries of M and N as well as the resulting 3–submanifold in $M \cup_\partial N$. Take a random element $\alpha \in H_2(M \cup_\partial N)$, represented as surface transverse to Y. Then the intersection $\alpha \cap Y$ is a 1–cycle in Y.

On one hand, if $\alpha \cap Y$ is non-trivial in $H_1(Y^3; \mathbb{Z})$, then it admits a dual class $\beta \in H_2(Y; \mathbb{Q})$. (Notice that we must use rational coefficients, but that is no problem: signatures were defined by diagonalization over a field.) The class β can be included as a class in $M \cup_\partial N$. Since β in $M \cup_\partial N$ can be pushed off itself by using some nowhere-zero vector field normal to Y in $M \cup_\partial N$, it follows that $\beta \cdot \beta = 0$ in $M \cup_\partial N$. Therefore, the span of α and β in $H_2(M \cup_\partial N; \mathbb{Q})$ has intersection form

$$Q|_{\alpha\beta} = \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix} ,$$

whose signature is zero and thus does not contribute to $\operatorname{sign}(M \cup_\partial N)$.

On the other hand, if $\alpha \cap Y$ is homologically-trivial, then one shows, using a Mayer–Vietoris argument, that α must in fact be a sum $\alpha = \alpha_M + \alpha_N$ of classes from M and N. Therefore the contribution of α to the signature of $M \cup_\partial N$ is caught in $\operatorname{sign} M$ and $\operatorname{sign} N$. □

The complete proof can be found in **R. Kirby**'s ***The topology of 4–manifolds*** **[Kir89**, ch II].

If two 4–manifolds are glued on only parts of their boundaries, then the additivity of the signature ceases to hold. Nonetheless, there is a well-determined correction term, see C.T.C. Wall*'s* ***Non-additivity of the signature*** [Wal69a].

The Rokhlin invariant of homology 3–spheres. On 3–manifolds spin structures can be defined in the same way as on 4–manifolds. Since every 3–manifold N is parallelizable (*i.e.*, T_N is a trivial bundle), it admits spin structures. As in dimension 4, the group $H^1(N; \mathbb{Z}_2)$ acts transitively on the set of spin structures. In particular, if $H^1(N; \mathbb{Z}_2) = 0$, then N admits exactly *one* spin structure. Moreover, every spin 3–manifold N bounds a (smooth) spin 4–manifold M with the spin structure of M restricting to the chosen spin structure[40] of N.

Let Σ^3 be a homology 3–sphere. Let M be a smooth spin 4–manifold bounded by Σ. Being spin, the manifold M must have an *even* intersection form. Since Σ is a homology 3–sphere, the intersection form of M must be unimodular. Thus, using van der Blij's lemma, its signature must be a multiple of 8:

$$\operatorname{sign} M = 0 \pmod 8$$

(from the same algebraic argument[41] as for closed 4–manifolds). In other words, the residue of $\operatorname{sign} M$ modulo 16 is either 0 or 8.

We can then define the **Rokhlin invariant** of Σ by

$$\rho(\Sigma) = \tfrac{1}{8} \operatorname{sign} M \pmod 2 .$$

Due to Rokhlin's theorem, this is a well-defined invariant of Σ, which does not depend on the choice of the bounded 4–manifold M. Indeed, if Σ also bounds another spin 4–manifold M', then M and $\overline{M}'$ can be glued along Σ yielding a closed *spin* 4–manifold $M \cup_\Sigma \overline{M}'$, which must have

$$\operatorname{sign}(M \cup_\Sigma \overline{M}') = 0 \pmod{16} ,$$

and thus $\operatorname{sign} M - \operatorname{sign} M' = 0 \pmod{16}$.

For example, since it bounds P_{E_8} whose signature is 8, the Poincaré homology 3–sphere Σ_P must have $\rho(\Sigma_P) = 1$.

The Rokhlin invariant of $\mathbb{Z}_2$–homology 3–spheres. Assume now that the 3–manifold N is a $\mathbb{Z}_2$–homology sphere, *i.e.*, a closed 3–manifold with

$$H^1(N;\mathbb{Z}_2) = 0 .$$

Then N admits a unique spin structure. Pick some smooth spin 4–manifold M that is bounded by N, with compatible spin structures. The intersection form of M is still even, but no longer unimodular, and so the best we can do is define the **Rokhlin invariant** (or **μ–invariant**) of N by

$$\mu(N) = \operatorname{sign} M \pmod{16} .$$

A similar reasoning as above shows that it is well-defined, independent of M.

The Rokhlin invariant of spin 3–manifolds. Finally, if N is just a random closed 3–manifold, then we can choose a spin structure $\mathfrak{s}$ on N, find a spin 4–manifold M that is spin-bounded by N, and define the invariant

$$\mu(N) = \operatorname{sign} M \pmod{16} .$$

This is an invariant that depends on the chosen spin structure $\mathfrak{s}$.

Two easy properties of the Rokhlin invariants, in any of the above versions, are:

$$\mu(\overline{N}) = -\mu(N) \qquad \text{and} \qquad \mu(N' \# N'') = \mu(N') + \mu(N'') .$$

40. In the language of the next note (cobordism groups; page 227), we are saying that $\Omega_3^{Spin} = 0$.

41. For the proof of van der Blij's lemma, see the end-notes of the next chapter (page 263).

Vice-versa: proving Rokhlin's theorem from μ–invariants. The reason why the μ–invariant of a spin 3–manifold is a well-defined invariant modulo 16, rather than modulo 8, is Rokhlin's theorem. Surprisingly, one can also go in reverse: If one proves by other means that the μ–invariant is well-defined modulo 16, then from this fact one can deduce Rokhlin's theorem for 4–manifolds.

This brief and elegant proof of Rokhlin's theorem can be discovered hidden as an appendix to **R. Kirby** and **P. Melvin**'s ***The 3–manifold invariants of Witten and Reshetikhin–Turaev for*** $\mathfrak{sl}(2,\mathbb{C})$ **[KM91]**. Specifically, one starts with a presentation of the 3–manifold as a Kirby link diagram, then defines the μ–invariant in terms of that diagram and proves that it well-defined by using only Kirby calculus.[42]

References. The Rokhlin invariant first appeared, in a more general setting, in **J. Eells** and **N. Kuiper**'s *An invariant for certain smooth manifolds* [EK62]. Some early properties are explored in **F. Hirzebruch, W. Neumann** and **S. Koh**'s ***Differentiable manifolds and quadratic forms*** **[HNK71]**.

The Rokhlin invariant can be refined into the much more powerful *Casson invariant* of homology 3–spheres, to the exposition of which is devoted **S. Akbulut** and **J. McCarthy**'s ***Casson's invariant for oriented homology 3–spheres*** [AM90]. This was extended by **K. Walker** to an invariant of rational homology 3–spheres in ***An extension of Casson's invariant*** [Wal92], and then finally to general 3–manifolds in **C. Lescop**'s ***Global surgery formula for the Casson–Walker invariant*** [Les96]. A recent survey of such invariants is **N. Saveliev**'s ***Invariants for homology 3–spheres*** [Sav02]. In a different direction, the Casson invariant admits a gauge-theoretic interpretation in terms of Donaldson's instantons, as was noticed by **C. Taubes**' ***Casson's invariant and gauge theory*** [Tau90], and, even further, it is the Euler characteristic of an instanton-based homology theory built in **A. Floer**'s ***An instanton-invariant for 3–manifolds*** [Flo88]. However, all this is beyond the scope of the present volume.

Note: Cobordism groups

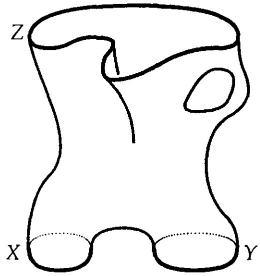
If we consider two m–manifolds as equivalent whenever there is a cobordism between them, then we separate manifolds into **cobordism classes**, and these can be organized as an Abelian group.

Oriented cobordism group. Consider the set of all oriented m–manifolds, together with the empty manifold $\varnothing$. Think of X^m and Y^m as equivalent if and only if they are cobordant, *i.e.*, if there is an oriented manifold W^{m+1} such that $\partial W = \overline{X} \cup Y$. The equivalence classes make up an Abelian group

$$\Omega_m^{SO}$$

called the **oriented cobordism group** in dimension m. Its addition comes from disjoint unions, $[X] + [Y] = [X \cup Y]$, as suggested in figure 4.41 on the next page.

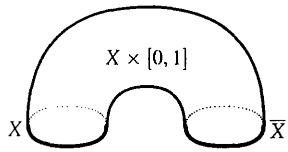
42. A quick overview of Kirby calculus was made in the end-notes of chapter 2 (page 91).



4.41. Cobordisms: $[X] + [Y] = [Z]$ in Ω_m^{SO}

The identity element in Ω_m^{SO} is given by $0 = [\varnothing]$. Any bounding m–manifold represents 0, and thus in particular the identity can also be represented by the m–sphere $\mathbb{S}^m$—since $\mathbb{S}^m$ bounds $\mathbb{D}^{m+1}$, we have $[\mathbb{S}^m] = [\varnothing]$.

The inverse in Ω_m^{SO} is given by reversing orientations: we have $-[X] = [\overline{X}]$, as argued in figure 4.42.

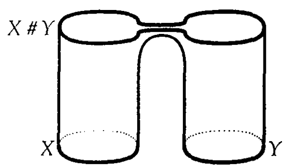


4.42. Cobordisms: $[X] + [\overline{X}] = 0$ in Ω_m^{SO}

It is worth noticing that $X \sqcup Y$ is always cobordant to $X \# Y$. This can be seen, for example, by using the boundary sum[43] $(X \times [0,1]) \natural (Y \times [0,1])$ as in figure 4.43. Thus, connected sum corresponds to addition in Ω_m^{SO}:

$$[X] + [Y] = [X \# Y] .$$

The diffeomorphisms $X \# \mathbb{S}^m = X$ reflect as $[X] + 0 = [X]$.



4.43. Cobordisms: $[X] + [Y] = [X \# Y]$ in Ω_m^{SO}

Cobordism ring. *Further, all the groups Ω_m^{SO} can in fact be put together to make up the* oriented cobordism ring Ω_*^{SO}, *with multiplication given by $[X] \cdot [Y] = [X \times Y]$, and unit the element $[+point] \in \Omega_0^{SO}$.*

As examples, it is easy to see that $\Omega_0^{SO} = \mathbb{Z}$, $\Omega_1^{SO} = 0$ and $\Omega_2^{SO} = 0$. It is a nontrivial result that $\Omega_3^{SO} = 0$. We have already mentioned that a 4–manifold is

43. Boundary sums were recalled back on page 13.

the boundary of some oriented 5–manifold if and only if its signature is zero. It follows that

$$\Omega_4^{SO} = \mathbb{Z}\,,$$

with isomorphism given by $[M] \mapsto \operatorname{sign} Q_M$. A generator of Ω_4^{SO} is $\mathbb{CP}^2$.

More cobordism groups are collected in table VI. The generator of Ω_5^{SO} is the manifold $\mathcal{Y}^5$ described by the equation[44] $(x_0 + x_1 + x_2)(y_0 + \cdots + y_4) = \varepsilon$ in $\mathbb{RP}^2 \times \mathbb{RP}^4$. The generators of Ω_8^{SO} are $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$. The generators of Ω_9^{SO} are $\mathcal{Y}^5 \times \mathbb{CP}^2$ and $\mathcal{Y}^9$, the latter being described by the equation $(x_0 + x_1 + x_2)(y_0 + \cdots + y_8) = \varepsilon$ in $\mathbb{RP}^2 \times \mathbb{RP}^8$. The generator of Ω_{10}^{SO} is $\mathcal{Y}^5 \times \mathcal{Y}^5$. The generator of Ω_{11}^{SO} is $\mathcal{Y}^{11}$, given by the equation $(x_0 + \cdots + x_4)(y_0 + \cdots + y_8) = \varepsilon$ in $\mathbb{RP}^4 \times \mathbb{RP}^8$. Keep in mind that Cartesian product organizes $\oplus\, \Omega_k^{SO}$ as a graded ring.

VI. Oriented cobordism groups

m	0	1	2	3	4	5	6	7	8	9	10	11
Ω_m^{SO}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Spin cobordism groups. The "*SO*" from the notation Ω_m^{SO} comes from the fact that an orientation of X^m is the same as a reduction of the structure group of T_X to $SO(m)$. The oriented cobordism group is not the only cobordism group—indeed, one can define a cobordism theory for most types of structure on manifolds.

In particular, the **spin cobordism group**

$$\Omega_m^{Spin}$$

is defined by starting with m–manifolds endowed with spin structures and considering X and Y as equivalent if and only if together they make up the boundary of a spin $(m+1)$–manifold W, with the spin structures on X and Y induced from the one on W.

In low-dimensions[45] we have $\Omega_1^{Spin} = \mathbb{Z}_2$, $\Omega_2^{Spin} = \mathbb{Z}_2$, and $\Omega_3^{Spin} = 0$. In dimension 4, we have

$$\Omega_4^{Spin} = \mathbb{Z}\,,$$

with isomorphism given by $[M] \mapsto \frac{1}{16} \operatorname{sign} Q_M$ (always an integer, by Rokhlin's theorem). The generator is the $K3$ surface.

More groups are collected in table VII. The generator of Ω_4^{Spin} is $K3$. The generators of Ω_8^{Spin} are $\mathbb{HP}^2$ and an 8–manifold $\mathcal{K}^8$ such that $\#4\mathcal{K}$ is spin cobordant to $K3 \times K3$.

VII. Spin cobordism groups

m	0	1	2	3	4	5	6	7	8
Ω_m^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}\oplus\mathbb{Z}$

44. The role of ε in the equation is merely to eliminate the singularities that would appear for $= 0$.

45. Defining spin structures for 1– and 2–manifolds requires first stabilization (because $\pi_1 SO(n)$ begins to be $\mathbb{Z}_2$ only for $n \geq 3$). Thus, for 1–manifolds C we will look at trivializations of $T_C \oplus \underline{\mathbb{R}}^2$, while for surfaces S, we look at $T_S \oplus \underline{\mathbb{R}}$. These low-dimensional spin structures and their cobordisms will be discussed in more detail in the end-notes of chapter 11 (page 521).

Uses. The application of such cobordism results usually follows this pattern: In order to prove a general statement about manifolds, first prove that it is invariant under cobordisms, then prove that the statement holds on the generators.

For example, the signature $\operatorname{sign} Q_M$ is an oriented-cobordism invariant, and such an argument is used in **M. Freedman** and **R. Kirby**'s ***A geometric proof of Rochlin's theorem*** **[FK78]** to prove Rokhlin's theorem; we will present two versions of that argument in the end-notes of chapter 11 (page 502 and page 521).

The most famous results first proved *via* cobordism arguments are Hirzebruch's signature theorem and the Atiyah–Singer index theorem.

References. Cobordism groups were first studied by **R. Thom**'s ***Variétés différentiables cobordantes*** [Tho53b] and fully detailed in his ***Quelques propriétés globales des variétés différentiables*** [Tho54]. That Ω_3^{SO} is trivial was proved in **A. Wallace**'s ***Modifications and cobounding manifolds*** **[Wal60]** or **R. Lickorish**'s ***A representation of orientable combinatorial** 3**–manifolds*** **[Lic62b]**. Both $\Omega_4^{SO} = 0$ and $\Omega_4^{Spin} = 0$ were first proved by **V. Rokhlin** in ***New results in the theory of four-dimensional manifolds*** [Rok52].

R. Kirby's ***The topology of** 4**–manifolds*** **[Kir89]** contains geometric proofs of the low-dimensional cobordism statements mentioned above. A general study of cobordisms can start with chapter 7 of **M. Hirsch**'s ***Differential topology*** **[Hir76, Hir94]**, then continue with **R. Stong**'s monograph ***Notes on cobordism theory*** **[Sto68]**.

As far as we are concerned, we will also encounter the spin$^{\mathbb{C}}$ cobordism group and the characteristic cobordism group, both discussed in the end-notes of chapter 10 (page 427); the two are in fact isomorphic. Also, in the note that follows, we will explore the framed version of cobordisms.

Note: The Pontryagin–Thom construction

In what follows, we will present the Pontryagin–Thom construction, which relates homotopies of maps to framed bordisms of submanifolds. An instance of this method was encountered in the proof of Whitehead's theorem,[46] and the following should shed some extra light on that argument. It is also of independent interest, since it adds geometric content to homotopy groups of spheres. In particular, it was during the pursuit of this method that Rokhlin discovered his celebrated theorem.

The construction. Let

$$f\colon X^{m+k} \to S^m$$

be any map, considered up to homotopy. Pick your favorite point p in S^m, then modify f slightly to make it smooth and transverse to p. The preimage $K = f^{-1}[p]$ is now a k–submanifold of X^{m+k}. Moreover, the differential $df\colon T_X \to T_{S^m}$ induces a map $df\colon N_{K/X} \to T_{S^m}|_p = \mathbb{R}^m$, which is an isomorphism on fibers and thus trivializes $N_{K/X}$. A submanifold together with a trivialization of its normal bundle is called a **framed submanifold.**

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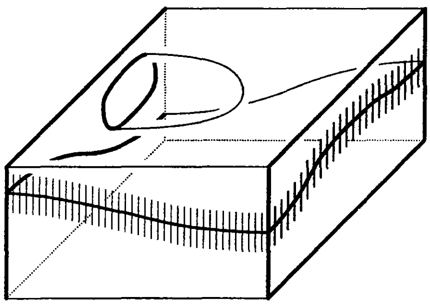
46. Back in section 4.1 (page 143).

In the reversed direction, let K be any k–submanifold of X^{m+k} with trivial normal bundle. Assume that a trivialization of its normal bundle $N_{K/X}$ has been chosen. This means that there is a projection $f\colon N_{K/X} \to \mathbb{R}^m$ that is an isomorphism on fibers. Think of f as defined on a tubular neighborhood $N_{K/X}$ of K in X and compactify its codomain $\mathbb{R}^m$ to $\mathbb{S}^m$ by adding a point ∞. Then $f\colon N_{K/X} \to \mathbb{S}^m$ can be extended on $X \setminus N_{K/X}$ simply by setting $f|_{X\setminus N_{K/X}} = \infty$, thus yielding a map $f\colon X^{m+k} \to \mathbb{S}^m$.

The correspondence $K \rightleftarrows f$ becomes bijective if we consider f only up to homotopies, and K only up to framed bordisms. Specifically, two k–submanifold K' and K'' of X^{m+k}, both with trivialized normal bundles, are called **framed bordant** if there exist both a $(k+1)$–submanifold $\widetilde{K}$ of $X \times [0,1]$ such that $\partial\widetilde{K} = \overline{K}' \times 0 \cup K'' \times 1$, and a trivialization of the normal m–plane bundle $N_{\widetilde{K}\,/\,X\times[0,1]}$ of $\widetilde{K}$ such that it induces the chosen trivializations of $N_{K'/X}$ and $N_{K''/X}$ when restricted to $\widetilde{K}$'s boundary. See figure 4.44.



4.44. A framed bordism

Lemma *(Pontryagin–Thom Construction). We have the bijection*

$$[X^{m+k}, \mathbb{S}^m] \approx \Omega_k^{framed}(X^{m+k}),$$

where the former denotes the set of homotopy classes of maps $X \to \mathbb{S}^m$, while the latter denotes the set of framed bordism classes of k–submanifolds of X.

> ***Sketch of proof.*** That $K \mapsto f \mapsto K$ is the identity is obvious. That $f_1 \mapsto K \mapsto f_0$ is the identity up to homotopy is shown by using the Alexander homotopy $f_t(x) = \frac{1}{t} f_1(tx)$ that links f_1 with $f_0 = df_1|_0$ (use coordinates on $\mathbb{S}^m = \mathbb{R}^m \cup \infty$ that set p at 0). Finally, apply the Pontryagin–Thom construction again to establish a correspondence between $(k+1)$–submanifolds of $X \times [0,1]$ and functions $X \times [0,1] \to \mathbb{S}^m$. Interpret the former as framed bordisms and the latter as homotopies. □

Lemma. *The bijection*

$$\pi_{m+k}\mathbb{S}^m \approx \Omega_k^{framed}(\mathbb{S}^{m+k})$$

is an isomorphism of groups. □

The group structure on the latter is the obvious bordism addition,

$$K' + K'' = K' \cup K'' \subset \mathbb{S}^m \# \mathbb{S}^m = \mathbb{S}^m .$$

Notice also that the bigger m becomes when compared to k, the less relevant the restriction to manifolds that embed in $\mathbb{S}^m$ becomes. In other words, the stable k-stem is given by abstract framed bordisms

$$\varinjlim_m \pi_{m+k}\,\mathbb{S}^m \approx \Omega_k^{framed} ,$$

where the latter is the cobordism group of k-manifolds endowed with a stable trivialization of their tangent bundle.[47]

Whitehead, revisited. Some claims made during the proof of Whitehead's theorem should now be clearer. First, going from $\pi_{m+k}(\mathbb{S}^m)$ to $\pi_{m+k}(\mathbb{S}^m \vee \cdots \vee \mathbb{S}^m)$ is trivial: just consider framed bordisms of several distinct (each maybe disconnected) submanifolds. After that, it is now obvious that any map $f\colon \mathbb{S}^{m+k} \to \mathbb{S}^m$ can be arranged to have $f^{-1}[p]$ connected (compare page 143), because it is easy to devise a framed bordism to a connected k-submanifold (connected sum inside $\mathbb{S}^m$ comes to mind). Similarly, the statement that the linking matrix of L determines the homotopy class of φ (page 146) can now be made rigorous, because the linking matrix is invariant under framed bordisms (allow the splitting of link components into disconnected pieces). It is in fact the only invariant, as will be suggested below.

References. The Pontryagin–Thom construction was created in the 1940s by **L. Pontryagin**, who used framed bordisms to compute homotopy groups of spheres, see his papers ***The homotopy group $\pi_{n+1}(K^n)$ $(n \geq 2)$ of dimension $n+1$ of a connected finite polyhedron K^n of arbitrary dimension, whose fundamental group and Betti groups of dimensions $2, \ldots, n-1$ are trivial*** [Pon49a], and ***Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one*** [Pon50], or the book **[Pon55]** translated as ***Smooth manifolds and their applications in homotopy theory*** **[Pon59]**.

Then, after the development by J.P. Serre of more powerful methods for computing homotopy groups,[48] **R. Thom** in ***Quelques propriétés globales des variétés différentiables*** [**Tho54**] went backwards and used computations of homotopy groups in order to compute cobordism groups.[49] Framed bordisms are explained in a friendly manner in **J. Milnor**'s ***Topology from the differentiable viewpoint*** **[Mil65b, Mil97]**, but see also **A. Kosinski**'s ***Differential manifolds*** [Kos93].

Application: homotopy groups of spheres. In what follows we will put to work the Pontryagin–Thom construction to offer geometric interpretations of certain simple homotopy groups of spheres. While this is how the homotopy groups below were first computed by L. Pontryagin and V. Rokhlin, the Pontryagin–Thom construction is a very weak method for evaluating homotopy groups when compared to Serre's later methods.

Lemma. $\pi_n\,\mathbb{S}^n = \mathbb{Z}$.

47. A **stable bundle** is a bundle considered up to additions of trivial bundles. A stable trivialization of the tangent bundle T_K means an isomorphism $T_K \oplus \underline{\mathbb{R}}^m \approx \underline{\mathbb{R}}^{m+k}$, corresponding to a virtual embedding in $\mathbb{S}^{m+k}$ with $N_{K/\mathbb{S}^{m+k}}$ trivialized as $K \times \mathbb{R}^m$.

48. See **J.P. Serre**'s *Homologie singulière des espaces fibrés. III. Applications homotopiques* [Ser51].

49. For a first taste of this approach, start with **M. Hirsch**'s ***Differential topology*** [Hir76, Hir94, ch 7].

Sketch of proof. Not that this is not clear for all sorts of reasons, but it can also be argued in terms of framed bordisms: $\Omega_0^{framed}(\mathbb{S}^n)$ contains framed points; the framing of a point $x \in \mathbb{S}^n$ is a trivialization of $T_{\mathbb{S}^n}|_x$ considered up to homotopy, in other words, an orientation of $T_M|_x$. Comparing it with the fixed orientation of $\mathbb{S}^n$ exhibits the elements of $\Omega_0^{framed}(\mathbb{S}^n)$ as points with signs. The isomorphism

$$\Omega_0^{framed}(\mathbb{S}^n) \approx \mathbb{Z}$$

is given simply by counting those points with signs. (Of course, on one hand this is just a very roundabout way of getting to the degree of a map $\mathbb{S}^n \to \mathbb{S}^n$; on the other hand, though, this is just the easiest instance of a pattern that we will see developing below.) □

Lemma. $\pi_3 \mathbb{S}^2 = \mathbb{Z}$, *and* $\pi_{n+1} \mathbb{S}^n = \mathbb{Z}_2$ *when* $n \geq 3$.

Outline of proof. For $\pi_3 \mathbb{S}^2$, we are looking at $\Omega_1^{framed}(\mathbb{S}^3)$, which contains framed links in $\mathbb{S}^3$. Each component of the link has a framing, determined by an integer, which can be added together to yield the isomorphism

$$\Omega_1^{framed}(\mathbb{S}^3) \approx \mathbb{Z} .$$

The framing is determined by an integer because we are talking about trivializations of 2–plane bundles over copies of $\mathbb{S}^1$, and $\pi_1 SO(2) = \mathbb{Z}$. As soon as the codimension increases, though, we have $\pi_1 SO(n) = \mathbb{Z}_2$ (detecting whether the bundle twists by an even or odd multiple of 2π), and thus

$$\Omega_1^{framed}(\mathbb{S}^{n+1}) \approx \mathbb{Z}_2 \quad \text{when } n \geq 3,$$

which concludes the argument. □

Lemma. $\pi_{n+2} \mathbb{S}^n = \mathbb{Z}_2$.

Outline of proof. Consider surfaces embedded in $\mathbb{S}^{n+2}$. Every surface S has a *skew*-symmetric bilinear unimodular intersection form on $H_1(S;\mathbb{Z})$, given by intersections of 1–cycles. It descends to an intersection form modulo 2 on $H_1(S;\mathbb{Z}_2)$.

Using the embedding of S in $\mathbb{S}^{n+2}$, we can define a **quadratic enhancement** q of the intersection forms, namely a map $q\colon H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ with

$$q(x+y) = q(x) + q(y) + x \cdot y \pmod 2 .$$

Such a q is defined as follows: represent $\ell \in H_1(S;\mathbb{Z}_2)$ by a circle embedded in S and consider the framing of $N_{S/\mathbb{S}^{n+2}}$ over ℓ: it is determined by a $\mathbb{Z}_2$–framing coefficient, and we define $q(\ell)$ to be that coefficient.

Any quadratic enhancement has an associated $\mathbb{Z}_2$–invariant, called its **Arf invariant**, which can be defined swiftly by setting

$$\mathrm{Arf}(q) = \sum q(e_k)\, q(\bar{e}_k)$$

for any choice of basis $\{e_1, \ldots, e_m, \bar{e}_1, \ldots, \bar{e}_m\}$ of $H_1(S;\mathbb{Z}_2)$ such that the only non-zero intersections are $e_k \cdot \bar{e}_k = 1$. A more thorough discussion of the algebra of the Arf invariant is made in the end-notes of chapter 11 (page 501).

In any case, the Arf invariant is the only framed bordism invariant and establishes the isomorphism

$$\Omega_2^{framed}(\mathbb{S}^n) \approx \mathbb{Z}_2 ,$$

and thus concludes the argument. □

All of the above computations are due to **L. Pontryagin** and can be read from his book **[Pon55]**, translated as ***Smooth manifolds and their applications in homotopy theory*** **[Pon59]**.

Finally, at the limits of Pontryagin–Thom's applicability, we have:

Theorem. $\pi_{n+3}\,\mathbb{S}^n = \mathbb{Z}_{24}$ *when* $n \geq 5$. □

This is already serious business and was first discovered by V. Rokhlin. While studying the problem of $\pi_{n+3}\,\mathbb{S}^n$ by using framed bordisms of 3–manifolds, V. Rokhlin first concluded that $\pi_{n+3}\,\mathbb{S}^n = \mathbb{Z}_{12}$. His mistake stemmed from thinking that a certain characteristic element in a 4–manifold could be represented by an embedded sphere. This was not the case, he corrected his mistake in ***New results in the theory of four-dimensional manifolds*** **[Rok52]**, and in the process discovered his theorem on the signature of almost-parallelizable 4–manifolds. The whole story can be followed in the volume ***À la recherche de la topologie perdue*** **[GM86a]**, edited by **L. Guillou** and **A. Marin,** with French translations of the relevant papers of Rokhlin, commentaries, *etc.*

For completeness, even though they were never obtained using the Pontryagin–Thom construction, we also state:

Theorem. $\pi_{n+4}\,\mathbb{S}^n = 0\,,\quad \pi_{n+5}\,\mathbb{S}^n = 0\,,\quad \pi_{n+6}\,\mathbb{S}^n = \mathbb{Z}_2$ *when n is big.* □

In particular it follows that $\Omega_4^{framed} = 0$. *This is not in contradiction with* $\Omega_4^{SO} = \mathbb{Z}$*, because not all 4–manifolds appear in* Ω_4^{framed}*, but only those that can be embedded in a sphere with trivial normal bundle, in other words, only those 4–manifolds M whose tangent bundle is stably-trivial, i.e.,* $T_M \oplus \mathbb{R}^n = \mathbb{R}^{n+4}$ *for some* n. *These M's have vanishing Pontryagin class, and thus vanishing signature.*

Bibliography

Whitehead's theorem as stated was proved in **J. Milnor**'s ***On simply connected 4–manifolds*** **[Mil58b]**, using results from **J.H.C. Whitehead**'s ***On simply connected, 4–dimensional polyhedra*** **[Whi49b]**. Results similar to Whitehead's were proved independently by **L. Pontryagin** in ***On the classification of four-dimensional manifolds*** **[Pon49b]**. The algebraic-topology argument we presented is the proof from **J. Milnor** and **D. Husemoller**'s ***Symmetric bilinear forms*** [MH73, sec V.1], while the long geometric argument involving the Pontryagin–Thom construction was taken from **R. Kirby**'s ***The topology of 4–manifolds*** **[Kir89**, ch II]. For background on knots and links, always look at **D. Rolfsen**'s ***Knots and links*** **[Rol76, Rol90, Rol03]**. Seifert surfaces were first introduced by **H. Seifert** in ***Über das Geschlecht von Knoten*** **[Sei35]**. The fundamental result that, if a map between simply-connected complexes induces isomorphisms on homology, then it is a homotopy equivalence (quoted in footnote 3 on page 141) was proved in such generality in **J.H.C.**

Whitehead's *Combinatorial homotopy* [Whi49a]; its proof can be found, for example, in **A. Hatcher's** ***Algebraic topology*** [Hat02, ch 4] or **E. Spanier's** ***Algebraic topology*** [Spa66, Spa81, ch 7].

C.T.C. Wall's theorem on algebraic automorphisms of intersection forms was published in ***On the orthogonal groups of unimodular quadratic forms*** [Wal62], and his work continued with the identification of generators in ***On the orthogonal groups of unimodular quadratic forms. II*** [Wal64a]. The theorem on diffeomorphisms is contained in ***Diffeomorphisms of 4–manifolds*** [Wal64b]. His theorems on stabilizations and h–cobordisms appeared in ***On simply-connected 4–manifolds*** [Wal64c]. The proof of the latter as outlined in this volume is from **R. Kirby's** ***The topology of 4–manifolds*** [Kir89, ch X].

Characteristic classes of vector bundles are masterfully described in **J. Milnor** and **J. Stasheff's** ***Characteristic classes*** [MS74]. Their chapter 12 presents the obstruction-theoretic view that we favored above. For the foundations of that view, one should look back at **N. Steenrod's** wonderful ***The topology of fibre bundles*** [Ste51, Ste99, part III]. Another standard reference for bundle theory in general is **D. Husemoller's** comprehensive ***Fibre bundles*** [Hus66, Hus94]. The Dold–Whitney theorem appeared in **A. Dold** and **H. Whitney's** ***Classification of oriented sphere bundles over a 4–complex*** [DW59]. The definition of spin structures as extensible trivialization is due to **J. Milnor's** ***Spin structures on manifolds*** [Mil63b].

Hirzebruch's signature theorem is a general statement about signatures and characteristic classes in all dimensions multiple of 4 and appeared in **F. Hirzebruch's** ***On Steenrod's reduced powers, the index of inertia, and the Todd genus*** [Hir53], then in the book ***Neue topologische Methoden in der algebraischen Geometrie*** [Hir56], which eventually became the famous monograph ***Topological methods in algebraic geometry*** [Hir66], last printed as [Hir95]. It is worth noting that the 4–dimensional case of the signature theorem was also proved in **V. Rokhlin's** ***New results in the theory of four-dimensional manifolds*** [Rok52]. The proof of the general case can be read in chapter 19 of **J. Milnor** and **J. Stasheff's** ***Characteristic classes*** [MS74].

Van der Blij's lemma appeared in **F. van der Blij's** ***An invariant of quadratic forms mod 8*** [vdB59].

A theorem of Rokhlin's. Rokhlin's theorem was published in a four-page paper, ***New results in the theory of four-dimensional manifolds*** [Rok52] (where it was also proved that if sign $= 0$, then the manifold bounds). It was translated in English in 1971. A French translation of this and three other remarkable papers of Rokhlin can be read as [Rok86] in the volume ***À la recherche de la topologie perdue*** [GM86a], edited by **L. Guillou** and **A. Marin**, where they are followed by a commentary [GM86b] that makes the dense style of Rokhlin easier to follow.

Rokhlin discovered his theorem by studying homotopy groups of spheres using the Pontryagin–Thom (framed-bordisms) approach, and by first mistakenly stating that $\pi_{n+3}\, S^n = \mathbb{Z}_{12}$; he then found his mistake, stated his theorem, and corrected to $\pi_{n+3}\, S^n = \mathbb{Z}_{24}$.

Most later proofs or textbook-treatments of Rokhlin's theorem actually deduce it from $\pi_{n+3} S^n = \mathbb{Z}_{24}$, with the latter fact obtained through the impressive machinery set up by **J.P. Serre**'s ***Homologie singulière des espaces fibrés. III. Applications homotopiques*** **[Ser51]** for computing homotopy groups, thus setting aside the direct geometric approach of Rokhlin's papers. For this homotopy-theoretic approach to proving Rokhlin's theorem, see **M. Kervaire** and **J. Milnor**'s ***Bernoulli numbers, homotopy groups, and a theorem of Rohlin*** [KM60].

Rokhlin's theorem was generalized successively by **M. Kervaire** and **J. Milnor** in ***On 2–spheres in 4–manifolds*** **[KM61]** (see also section 11.1, page 482), and further, along an unpublished outline of **A. Casson** from around 1975, by **M. Freedman** and **R. Kirby** in ***A geometric proof of Rochlin's theorem*** [FK78]. The latter statement and its proof from scratch (thus in particular proving Rokhlin's theorem as well) will be discussed in the end-notes of chapter 11, with a warm-up starting on page 502 and a detailed proof on page 507.[50] Alternative proofs of a similar flavor can be read in **L. Guillou** and **A. Marin**'s ***Une extension d'un théorème de Rohlin sur la signature*** [GM86c] and **Y. Matsumoto**'s ***An elementary proof of Rochlin's signature theorem and its extension[51] by Guillou and Marin*** [Mat86], both inside the same wonderful volume ***À la recherche de la topologie perdue*** **[GM86a]**. It has been reported that **V. Rokhlin** was himself long aware (1964) of these generalizations, but only published them in ***Proof of a conjecture of Gudkov*** **[Rok72]**, when he found an application.

Another version of the proof is found in **R. Kirby**'s ***The topology of 4–manifolds*** **[Kir89**, ch XI], where a nice streamlined argument with spin structures is used. This alternative proof is also explained in this volume, in the end-notes of chapter 11 (page 521).

A third and surprising proof of Rokhlin's theorem that starts with the μ–invariants of 3–manifolds can be read from the appendix of **R. Kirby** and **P. Melvin**'s ***The 3–manifold invariants of Witten and Reshetikhin–Turaev for*** $\mathfrak{sl}(2,\mathbb{C})$ **[KM91]**; it was briefly mentioned back on page 227.

50. The reason for the exile of the proof of Rokhlin's theorem to chapter 11 is mainly one of space: even though logically that proof would better fit with the present chapter, the current group of end-notes is already quite extensive.

51. The word "extension" from the last two titles refers to a refinement of the Kirby–Siebenmann formula from a $\mathbb{Z}_2$–equality to a $\mathbb{Z}_4$–equality, with the extra residues appearing only from non-orientable characteristic surfaces.