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Dedekind sums and signatures of intersection forms*

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The classical Dedekind sums $s(a,c)$ are defined for a pair of integers (a,c) with $c \neq 0$ by

$$s(a,c) = \frac{1}{4|c|} \sum_{r(c)}{}' \cot \pi\left(\frac{ar}{c}\right) \cot \pi\left(\frac{r}{c}\right)$$

(where $\sum'$ means to drop the meaningless terms). Many of their most important properties are best understood in terms of the Rademacher function $\phi : \Gamma \to \frac{1}{3}\mathbb{Z}$ on $\Gamma = \mathrm{SL}_2\mathbb{Z}$,

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{3c} - 4\,\mathrm{sign}(c)\, s(a,c) \quad \text{resp.} \quad \frac{b}{3d} \tag{1}$$

according to whether $c \neq 0$ or $c = 0$. For instance, for $A_i \in \Gamma$, $i = 1,2,3$ with $A_1A_2A_3 = 1$, we have

$$\phi(A_1) + \phi(A_2) + \phi(A_3) = -\,\mathrm{sign}(c_1c_2c_3)\,, \qquad A_i = \begin{pmatrix} * & * \\ c_i & * \end{pmatrix}, \tag{2}$$

which shows that ϕ is almost a homomorphism on Γ. The expression $\mathrm{sign}(c_1c_2c_3)$ in this equation, viewed as a function of A_1 and A_2, defines an integer valued 2-cocycle on Γ, called the **area** cocycle. Conversely, since $H^1(\Gamma, \mathbb{Q}) = 0$, the area cocycle determines the Rademacher function uniquely. In addition to ϕ, we consider in this paper two closely related cousins of ϕ. The first one is given by

$$\varphi(A) = -\,\phi(A) + \nu(A)\,,$$
$$\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \mathrm{sign}(b)\,, & \text{if } c = 0 \text{ and } a = 1 \\ \mathrm{sign}(c(a+d-2)) & \text{otherwise}\,. \end{cases} \tag{3}$$

It was introduced by Meyer [Me] who proved that the 2-cocycle σ on Γ defined by

$$\sigma(A_1, A_2) = \varphi(A_1) + \varphi(A_2) + \varphi(A_3)$$

* In memoriam Werner Meyer

is the so called **signature** cocycle [At, BG]. For an illuminating discussion of the area as well as the signature cocycle, we refer to the paper of Kirby and Melvin [KM] in this journal. The second cousin of ϕ depends on an auxiliary parameter $x \in \mathbb{R}^2\backslash\{0\}$, and is defined by

$$\begin{aligned} \psi(A)(x) &= \phi(A) + \varrho(A)(x), \\ \varrho\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) &= \operatorname{sign}(cx_2(cx_1 - ax_2)). \end{aligned} \tag{4}$$

It was shown in [S2] that ψ satisfies the relation

$$\psi(A_1A_2)(x) = \psi(A_1)(x) + \psi(A_2)(A_1^{-1}x).$$

In other words, ψ is a 1-cocycle on Γ with values in the Γ-module of functions $f:\mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ equipped with the Γ-action $(Af)(x) = f(A^{-1}x)$. As we will see in a moment, this cocycle is very useful for exploring the connection between Dedekind sums and the signature of certain intersection matrices.

To this end let $b_0, b_1, \ldots, b_n$ be a sequence of arbitrary integers. Then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & b_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & b_n \end{pmatrix}$$

is clearly an element of Γ and, conversely, every element of Γ can be written so in infinitely many ways. An initial pair of real numbers α_{-1}, α_0 determines a sequence α_k by $\alpha_{k+1} = b_k\alpha_k - \alpha_{k-1}$, $k = 0, 1, \ldots, n$. If α_{-1}, α_0 are not both zero, then the same is true for two successive α_{k-1}, α_k, and the cocycle property of ψ implies

$$\begin{aligned} \psi(A)\begin{pmatrix} -\alpha_0 \\ \alpha_{-1} \end{pmatrix} &= \sum_{k=0}^{n} \psi\begin{pmatrix} 0 & -1 \\ 1 & b_k \end{pmatrix}\begin{pmatrix} -\alpha_k \\ \alpha_{k-1} \end{pmatrix} \\ &= \sum_{k=0}^{n} \left(\frac{b_k}{3} - \operatorname{sign}(\alpha_{k-1}\alpha_k)\right) \end{aligned} \tag{5}$$

using (4) and (1). We assume now that all α_k are nonzero and let $w_k = \alpha_{k-1}/\alpha_k$,

$$I = \begin{pmatrix} b_0 & 1 & & & \\ 1 & b_1 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & b_{n-1} & 1 \\ & & & 1 & b_n \end{pmatrix}, \quad M = \begin{pmatrix} b_0 & 1 & & & \\ 1 & b_1 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & b_{n-1} & 1 \\ & & & 1 & w_n \end{pmatrix}$$

where the (i, j) entry of I is 1 for $|i - j| = 1$, b_{i-1} for $i = j$ and 0 otherwise, while M is the same except that its (m, m) entry is w_n, $m = n + 1$.

Theorem 1. $$\psi(A)\begin{pmatrix} -\alpha_0 \\ \alpha_{-1} \end{pmatrix} = \frac{1}{3}\sum_{k=0}^{n} b_n - \operatorname{sign}(M).$$

By letting the continuous variable w_n in M approach b_n, we will deduce from this the following formula of Kirby and Melvin [KM].

Corollary. $$\phi(AJ) = \tfrac{1}{3}\operatorname{trace}(I) - \operatorname{sign}(I) \quad \textit{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. To prove the theorem, we have to show only

$$\operatorname{sign}(M) = \sum_{k=0}^{n} \operatorname{sign}(\alpha_{k-1}\alpha_k). \tag{6}$$

But this is easy since M is equivalent to the diagonal matrix W whose (i,i) entry is w_{i-1}, $1 \leqq i \leqq m$. Let

$$X_{ij} = \begin{Bmatrix} 1 & \text{if } i = j \\ 1/w_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{Bmatrix}, \qquad 1 \leqq i, j \leqq m.$$

Then

$$(XWX^t)_{ij} = \sum_{k=1}^{m} X_{ik} w_{k-1} X_{jk} = \begin{cases} w_{i-1} + 1/w_i & \text{if } i = j < m \\ w_n & \text{if } i = j = m \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

which shows that $XWX^t = M$ since $w_i + 1/w_{i+1} = (\alpha_{i-1} + \alpha_{i+1})/\alpha_i = b_i$. Moreover, since $\det(X) = 1$, this calculation yields also the useful result

$$\det(M) = \prod_{k=0}^{n} w_k = \alpha_{-1}/\alpha_n. \tag{7}$$

Proof of the corollary. Using (4) and $\phi(AJ) = \phi(A) - \operatorname{sign}(dc)$ (which follows from (2)), we see that the corollary holds iff

$$\operatorname{sign}(M) - \operatorname{sign}(I) = \operatorname{sign}(c\alpha_{-1}\alpha_n) - \operatorname{sign}(dc). \tag{8}$$

Letting $w_n \to b_n$ in (7) gives $\det(I) = d$. We consider the case $d \neq 0$ first. Then $\operatorname{sign}(M) = \operatorname{sign}(I)$ for w_n sufficiently close to b_n. This condition is satisfied for (α_n, α_{n+1}) close to $(1,0)$ or, equivalently, $(\alpha_{-1}, \alpha_0) = (d\alpha_n - c\alpha_{n+1}, a\alpha_{n+1} - b\alpha_n)$ close to $(d, -b)$. With this assumption, it is obvious that the right side in (8) is indeed zero. Now let $d = 0$. Then $\det(M) = w_0(\alpha_0/\alpha_n)$ has, as a function of $w_0 = \alpha_{-1}/\alpha_0$, a simple zero at $w_0 = 0$. Therefore,

$$\operatorname{sign}(M) - \operatorname{sign}(I) = \operatorname{sign}(w_0) = \operatorname{sign}(\alpha_{-1}\alpha_0)$$

for (α_{-1}, α_0) close to $(d, -b) = (0, \pm 1)$. Since $\operatorname{sign}(c\alpha_0) = \operatorname{sign}(\alpha_n) = 1$ in that case, (8) follows again and the corollary is proved. We remark in passing that Theorem 1 and Eq. (6) remain valid even if some of the α_k vanish provided α_n itself is nonzero. It should be also noted that all terms in Theorem 1 and the corollary change their sign individually under the involution $(\alpha_{-1}, \alpha_0) \to (\alpha_{-1}, -\alpha_0)$ and $b_k \to -b_k$ for all k.

The matrix I represents the intersection form of a certain class of 4-manifolds bounded by lens spaces [HN, KM]. Of interest is also the related class of 4-manifolds bounded by a torus bundle over the circle, since such a bundle is completely classified by its gluing matrix in Γ. In the case of a hyperbolic gluing matrix, these 4-manifolds arise naturally in number theory as closed neighbourhoods of cusps on Hilbert modular surfaces [Hi]. The corresponding intersection form is again determined by a finite sequence of integers b_k, but its matrix depends on the length of the sequence: Let

$$I_c = (2+b_0)\,, \qquad \begin{pmatrix} b_0 & 2 \\ 2 & b_1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} b_0 & 1 & & & 1 \\ 1 & b_1 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ 1 & & & 1 & b_n \end{pmatrix}$$

for $n = 0$, $n = 1$, $n \geqq 2$, respectively. Using the elementary matrices E_{ij} (whose only nonzero entry is the (i,j) entry which is 1), we can write $I_c = I + E_{1m} + E_{m1}$ in all three cases, $m = n+1$. Denote by I_c^- the matrix arising out of I_c by changing the sign of all the b_k's.

Theorem 2. $\varphi(A) = -\operatorname{sign}(I_c^-) - \frac{1}{3}\sum_{k=0}^{n} b_k\left(= \frac{1}{3}\operatorname{trace}(I_c^-) - \operatorname{sign}(I_c^-) \textit{ if } n \neq 0\right).$

As explained at the beginning, the function φ on the left is the unique 1-cochain on Γ whose coboundary is the signature cocycle σ. Contrary to the Rademacher function ϕ, this function does not transform in any simple way under the sign change $b_k \to -b_k$ which partly explains the necessity to introduce I_c^-. The theorem displays the well known fact [At, MS] that $\varphi(A)$ is a measure for the signature defect of the torus bundle over the circle associated to A.

Remark. Theorem 2 was originally proved in [S1], and was published without proof in [MS].

We consider first the case where A is hyperbolic ($|\operatorname{tr} A| > 2$). This case is of special interest because A has then two different real eigenvalues, say $\varepsilon, \varepsilon'$. In particular, we can choose for $(-\alpha_{n+1}, \alpha_n)$ a nonzero eigenvector of A which implies that the sequence of w_k's becomes periodic: $w_0 = w_{n+1}$ as it is easily seen from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\alpha_{n+1} \\ \alpha_n \end{pmatrix} = \begin{pmatrix} -\alpha_0 \\ \alpha_{-1} \end{pmatrix} = \varepsilon\begin{pmatrix} -\alpha_{n+1} \\ \alpha_n \end{pmatrix}. \tag{9}$$

Moreover, since all the w_k become quadratic irrationalities (in the field generated by the eigenvalues of A), they are all nonzero real numbers. Therefore, in order to deduce Theorem 2 from (5) in the hyperbolic case, we have to show only

$$\varphi(A) = -\psi(A)\begin{pmatrix} -\alpha_0 \\ \alpha_{-1} \end{pmatrix}, \qquad \operatorname{sign}(I_c^-) = -\sum_{k=0}^{n} \operatorname{sign}(w_k)\,.$$

The first equation follows from

$$\begin{aligned} \varrho(A) &= \operatorname{sign}(c\alpha_{-1}(-c\alpha_0 - a\alpha_{-1})) = -\operatorname{sign}(c\alpha_{-1}\alpha_n) \\ &= -\operatorname{sign}(c\varepsilon) = -\operatorname{sign}(c(a+d-2)) = -\nu(A) \end{aligned}$$

since $\varepsilon\varepsilon' = 1$, $\varepsilon + \varepsilon' = a + d$ and $|a+d| > 2$. To prove the second equality, it suffices to find a representation $I_c^- = -YWY^t$ with a non-singular matrix Y. It is the beauty of the hyperbolic case that such a representation does indeed exist. For $m > 1$ with $m = n + 1$, we have

$$Y_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ -1/w_i & \text{if } j \equiv i + 1 (m) \\ 0 & \text{otherwise} \end{array} \right\}, \qquad 1 \leqq i, j \leqq m,$$

$$-(YWY^t)_{ij} = -\sum_{k=1}^{m} Y_{ik} w_{k-1} Y_{jk} = \begin{cases} -w_{i-1} - 1/w_i & \text{if } i = j \\ e & \text{if } i - j \equiv \pm 1 (m) \\ 0 & \text{otherwise} \end{cases}$$

where $e = 1$ for $m > 2$, but $e = 2$ for $m = 2$. Note that the periodicity $w_m = w_0$ is absolutely crucial in this calculation. Y is not singular since

$$\det(Y) = 1 - \prod_{k=1}^{m} \left(-\frac{1}{w_k} \right) = 1 \pm \alpha_m / \alpha_0 = 1 \pm \varepsilon' \neq 0 .$$

Finally, in the case $m = 1$, we have $w_0 = \varepsilon$ and the desired factorization is

$$I_c^- = 2 - b_0 = 2 - \varepsilon - \varepsilon' = -(1 - \varepsilon') w_0 (1 - \varepsilon')$$

which proves the theorem in the hyperbolic case.

Unfortunately, this proof does not apply in the non-hyperbolic case. In order to treat the general case, we study for $n > 1$ the auxiliary matrix

$$N = \begin{pmatrix} -\alpha_0\alpha_1 & \alpha_0 & & & 1 \\ \alpha_0 & -b_1 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ 1 & & & 1 & -b_n \end{pmatrix}$$

whose entries are the same as in I_c^- except for $N_{11} = -\alpha_0\alpha_1$ and $N_{12} = N_{21} = \alpha_0$. In addition, we drop now all previous restrictions on α_{-1}, α_0 and assume only that α_0 is nonzero.

Lemma. $$\operatorname{sign}(N) = -\nu \begin{pmatrix} a/\alpha_0 & b/\alpha_0 \\ \alpha_n & \alpha_{n+1} \end{pmatrix} - \sum_{k=0}^{n-1} \operatorname{sign}(\alpha_k \alpha_{k+1}) .$$

The function ν in this equation is the same one as in the definition of φ. It is defined for an arbitrary matrix in $\mathrm{SL}_2\mathbb{R}$ by

$$\nu \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{cases} \operatorname{sign}(\beta), & \text{if } \gamma = 0 \text{ and } \alpha = 1 \\ \operatorname{sign}(\gamma(\alpha + \delta - 2)) & \text{otherwise} . \end{cases}$$

The matrix N is easily diagonalized using elementary row and column operations. if all α_k are nonzero, one gets for the diagonal elements $-\alpha_1\alpha_0$, $-\alpha_2/\alpha_1$, $-\alpha_3/\alpha_2, \dots, -\alpha_n/\alpha_{n-1}$, $\tilde{\alpha}_{n+1}/\alpha_n$ with

$$\tilde{\alpha}_{n+1} = \alpha_{n+1} - 2 + \alpha_n \sum_{k=0}^{n-1} \frac{1}{\alpha_k \alpha_{k+1}} = \alpha_{n+1} - 2 + \frac{a}{\alpha_0}$$

which proves the lemma in the generic case. As a corollary, one gets also the handy formula

$$\det(N) = (-1)^{n+1}(\alpha_0\alpha_{n+1} - 2\alpha_0 + a).$$

In general, the correction term ν enters the lemma as the sign of the last element on the diagonal after diagonalizing N by elementary row and column operations starting at the top.

Proof of Theorem 2. Let $\alpha_{-1} = 0$, $\alpha_0 = 1$. Then $\alpha_{n+1} = d$, $\alpha_n = c$ by (9), and, since $\alpha_1 = b_0$,

$$\operatorname{sign}(I_c^-) = -\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \sum_{k=0}^{n-1} \operatorname{sign}(\alpha_k\alpha_{k+1}),$$

$$\det(I_c^-) = (-1)^{n+1}(a + d - 2),$$

at first for $n > 1$, but this is easily checked also for $n = 0, 1$. On the other hand, by (4) and (5),

$$\varphi(A) = \nu(A) - \phi(A) = \nu(A) - \sum_{k=0}^{n}\left(\frac{b_k}{3} - \operatorname{sign}(\alpha_{k-1}\alpha_k)\right)$$

from which Theorem 2 follows at once.

Remark. The formula (3) for the correction term ν has an amusing history. The original formula in [Me] was in terms of the signature of a 2×2 matrix. This expression was simplified in [MS], but the result was still clumsy. A further compactification was attempted in [At], but unfortunately, the formula (6.17) given there is not quite correct; it corresponds to $\nu = \operatorname{sign}(ct(t^2 - 4))$, $t = a + d$, rather than $\nu = \operatorname{sign}(c(t - 2))$ if $c \neq 0$, and hence gives the wrong answer in the 3 cases $t = -2, -1, 0$. The simple formula for ν given in (3) was found by Kirby and Melvin [KM].

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