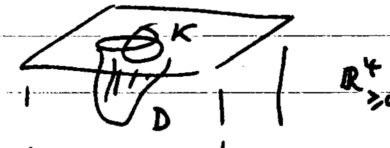
Peter Teichner

"4D Knot theory"

$S^1 \overset{K}{\hookrightarrow} \mathbb{R}^3$ is trivial if it bounds a disc.

For 4-D knot theory, want to see whether K is slice: bounding a disc in $\mathbb{R}^4_{\geq 0}$



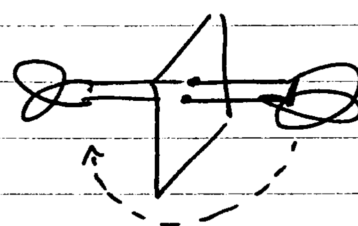
The terminology "slice" is due to Fox; it means that K is a slice of a 2-knot $S^2 \subseteq \mathbb{R}^4$ with a plane $\mathbb{R}^3$.

Why is this interesting?

1. Fox-Milnor : resolving PL singularities
2. Whitney trick
3. (Knots, #) / slice knots forms a group (abelian) - the knot concordance group.

eg. $K \# \overline{K}$ is slice

(- proves part 3.)

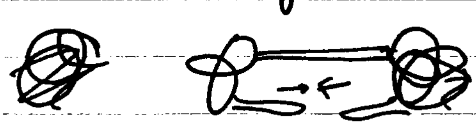


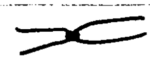
180° rotate about $\mathbb{R}^2 \subseteq \partial \mathbb{R}^4_{\geq 0}$ sweeps out a disc.

NB Really have oriented knots to make # well-defined: it is $K \# r\overline{K}$ which is slice. Thus, $r\overline{K}$ = inverse.

Alternative picture.

Build a slice disc by Morse theory:

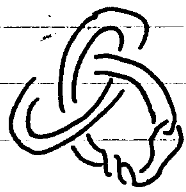


at crit time, touch the two fingers  & then afterwards get two circles; an unlink; isotopy apart & cap off.

Two discs, one saddle $\Rightarrow$ disc.

Fox & Milnor : in order to remove these locally, need the corresponding knots to be slice.
Then can insert the disc suitably.
(Or can coalesce the cusps via connect-sum)
So the knot concordance gp measures obstruction to smoothing PL maps.

NB topological local flatness gives different theory



Whitehead double of trefoil (0-framed)

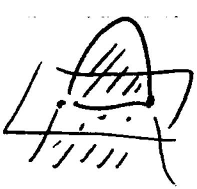
This is topologically - locally - flat - slice but NOT smoothly.
(Freedman - Gompf) (respectively)

The Whitney trick

Essential in classification of manifolds :
geometry $\longrightarrow$ algebra $\xrightarrow{\text{Whitney}}$ geometry
eg two submanifolds $A^p, B^q \subseteq \mathbb{R}^{p+q}$
have an intersection number $i(A,B) \in \mathbb{Z}$

Q: If $i(A,B) = 0$, can we arrange them to be disjoint ?

Whitney trick is : yes, if dimensions big enough, namely $p, q < p+q-2$, $p+q > 4$.
If $i=0$ then pair up the points of intersection ; have now a pair of arcs between them in A, B forming a circle in $\mathbb{R}^{p+q}$; get a disc it bounds which is embedded because $p+q > 4$. (No knotting)
(Other condition to make disc meet A, B only in its ∂)



Now push across the disc, using local flatness to go to

Corollary (Strong Whitney embedding theorem)

Any n-mfd embeds in $\mathbb{R}^{2n}$.

Pf Certainly into $\mathbb{R}^{2n+1}$ by gen pos.

But can immerse into $\mathbb{R}^{2n}$, make self-intersection 0 by introducing kinks; then apply the trick.

The $n=1,2$ cases work differently but $n \geq 3$ is by above methods.

The 4-dim case is critical: trying to bound a disc in 4-space is just like slicing a knot.

(Think of thickening the A, B to 4-mflds, drawing Whitney circle on ∂ of these, so asking for $K \subseteq \partial W^4$ to bound in W^4. A bit more general).

[Thoughts: · local R-move type char of slice? (saddle etc making disc is not good)

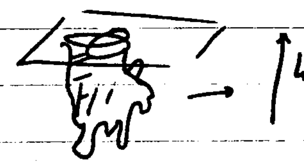
· K problem list here.

Reminder

$\mathcal{C} = \{$oriented knots in $\mathbb{R}^3, \#\} / \{$slice knots$\}$ is a group, the knot concordance group.

Not much is known — many open problems.

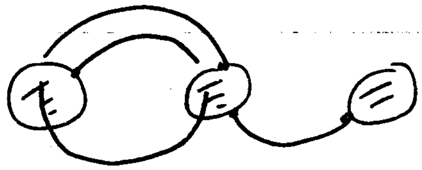
Can think of slice discs in Morse theory



A ribbon disc is a slice disc without local maxima.

When projected into $\mathbb{R}^3$, the disc is then a ribbon disc.

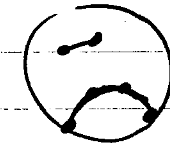
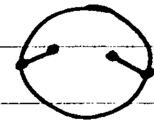
That is, to construct the knot, start with an unlink (local minima; can order so that these come first, then the saddles.) Now add bands connecting them until the result is a knot/disc.



arcs for saddle attaching.

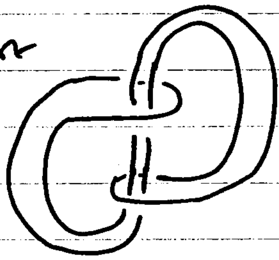
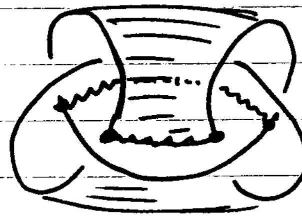
eg local ribbon singularity  are the only self-intersections

against clasp singularity 

Preimages in the actual disc look like  or & 

ribbon (one interior arc) (both arcs hit ∂)

Can locally resolve a ribbon singularity into 4-space to get back a slice disc. Not true for clasps.

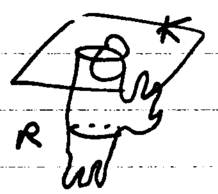
eg ribbon knot  or  ribbon!

Conjecture: slice $\Rightarrow$ ribbon [assumed not to be true?]

Fact the group $\dfrac{\{\text{knots}, \#\}}{\{\text{ribbon}\}}$ is $\cong e$!

Because: K is slice $\Leftrightarrow$ $\exists$ ribbon knot R s.t $K\#R$ is slice ⊛

[Have to realise that the groups are both constructed by semigroup quotient, ie. $K_1 \underset{\text{inc}}{\sim} K_2 \Leftrightarrow \exists$ slice knots S_1, S_2 s.t $K_1\#S_1 = K_2\#S_2$. Now using fact, add a ribbon to get version with ribbon] [ribbon # ribbon is ribbon]

To prove ⊛, just take  K R

split the saddles into ones making a disc below (with minima), annulus above.

Then, $K\#R$ is ribbon!

Look at K  R

new height function!

(needs a little work to make precise!)

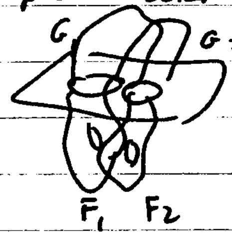
[**Rk** a general slice disc, pushed into $\mathbb{R}^3$,
can be altered so as to have only clasp singularities
~~But~~ - even triple points can be pushed off boundary
Actually, ribbon ~~knots~~ singularities do this too.

Rk Mike Freedman claims all pfs that start by reduction to clasp singularities are wrong!

—·—

Now want to show ℓ is non-trivial - can do this with additive knot invariants that vanish on ribbon knots.
(often more convenient to show)

Recall, linking # $lk(\ell_1, \ell_2)$ got from Seifert surface $F_1 \pitchfork \ell_2$ for example.
4-dim version: make each cpt bound F_i in B^4
& then take the intersection $F_1 \pitchfork F_2$.
Need only show this is indept of choice of surfaces, as then can compute using one Seifert surface & reproduce 3-dim defn.
Proof is just pick alternate G_1, G_2
& form



→ closed surfaces in $\mathbb{R}^4$ which have 0 intersection homologically (or: any surface bounds a 3-mfd.) (M = surface = arc

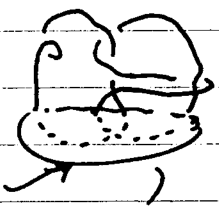
Thus, a link with non-zero linking # cannot be slice
(i.e. bound disjoint discs in 4-space)

Let F be a Seifert surface for K.
Then have the Seifert pairing on $H_1 T \cong \mathbb{Z}^{2g}$

$$S : H_1 T \times H_1 T \longrightarrow \mathbb{Z}$$

- may as well view knots as boundaries of discs-with-bands, & then just compute the linking #s & framings of the bands.

(push off first curve upwards, still have crossings down here



)

From S we can get the Alexander poly $A(K) = \det(S - tS^T) \in \mathbb{Z}[t^{\pm}]$
Must show indept of basis chosen and the choice of surface.

The signature $\sigma(K) = \sigma$ of symmetric form $S + S^T$.
again indept of choice of S.

[Rk: independence of choice of S is again from Morse theory
the elementary changes are handle additions
which effect $S \longmapsto \left(\begin{array}{c|c} S & \vdots \\ \hline 0 \cdots & \begin{matrix} 0 & \\ & 0 \end{matrix} \end{array}\right)$]

Other rk: can normalise $A(K) \in \mathbb{Z}[t^{\pm}]$ precisely by
① $A(K)(t) = A(K)(t^{-1})$ (true anyway up to powers of
② $A(K)(1) = +1$.
($S - S^T$ = intersection form J = ± 1)

Conway polynomial $C_K(z)$ is $z = t^{1/2} + t^{-1/2}$ version of this

Thm If K is slice then $\exists$ a half-rank direct summand L in $H_1 F$ s.t $S|_L = 0$

Coroll (a) K slice $\Rightarrow \sigma(K) = 0$
(b) K slice $\Rightarrow$ $AP(K)$ is of form $f(t).f(t^{-1})$ (up to

Pf of coroll. (b) $S = \left(\begin{array}{c|c} 0 & A \\ \hline B & * \end{array}\right)$ so $\det(S - tS^T)$ is ~~$\det *$~~

$\pm \det(A - tB^T).\det(B - tA^T) = f.\bar{f}$ up to units.

(a) For signature have $\sigma\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$; the matrix
$S-S^T$ is non-singular, $\therefore$ $S+S^T$ is over $\mathbb{Z}_2$, $\therefore$
$S+S^T$ is over $\mathbb{Z}$ (in fact det = odd) $\therefore$ over $\mathbb{Q}$ have ± 1
eigenvalues.

- One way to complete is to split off $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ then by induction.
- or: $\begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix}^T \begin{pmatrix} 0 & A \\ A^T & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & A^{-1T}BA^{-1} \end{pmatrix}$ & now row-col. $\to \begin{pmatrix} 0 & \\ 1 & \end{pmatrix}$
provided can divide by 2
- or: I claim $\begin{pmatrix} +1 & & & \\ & \ddots & & \\ & & +1 & \\ & & & -1 \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$ $\left.\right\}^r$ $\left.\right\}^s$ has at most $\min(r,s)$ lin indep square-0

Rk $\exists$ also a 4-mfd signature definition.

Ex Trefoil: $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ so $S+S^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, det +ve $\therefore$ same

$\therefore \sigma \neq 0$

Fig-8: $S = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ $S+S^T = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ det -ve $\therefore$ opp sign

$\therefore \sigma = 0$

agrees with fact that σ is additive & fig 8 is amphichiral
$\therefore$ $2\sigma(K) = \sigma(K \# \bar{K}) = 0$.

But $A(\text{fig-8}) = \det\begin{pmatrix} 1-t & -t \\ 1 & -1+t \end{pmatrix} = -(1-t)^2 + t$

$= -t^2 + 3t - 1 \mathrel{\hat{=}} -t + 3 - t^{-1}$

but this isn't $f \cdot \bar{f}$. (~~as~~ $A(-1)$ is not a square, in fact)

$\therefore$ { fig-8 is a 2-torsion elt in $\mathcal{C}$.
i.e. { trefoil is an ∞-order elt in $\mathcal{C}$.

Some twist knots

trefoil K_{-1}

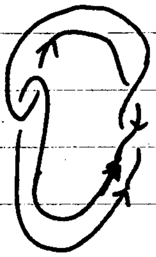
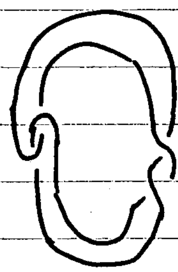
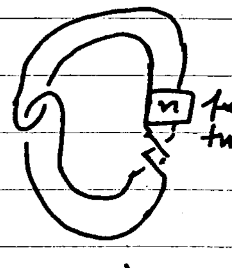


figure-8 K_1



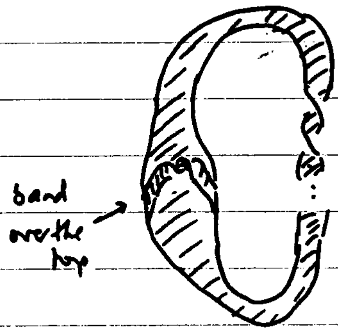
general $K_n =$



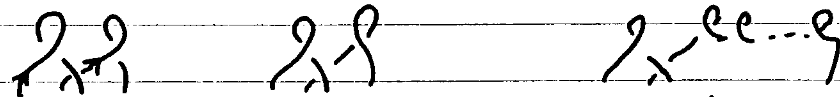
(K_0 = unknot)

(Hopf link bounds  or, ~~twisted~~ ~~torus~~ twisted annul

Seifert surfaces: (essentially got by resolving the obvious disc with clasp singula



Thus, as a banded surface,



K_{-1} $\quad$ K_{+1} $\quad$ K_n

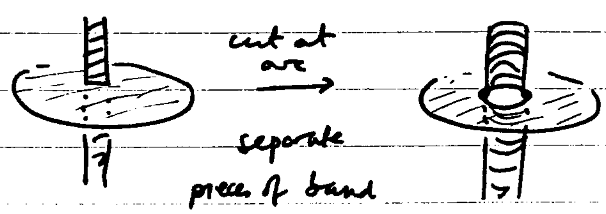
$S = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$ $\quad$ $S = \begin{pmatrix}-1 & 1\\ 0 & +1\end{pmatrix}$ $\quad$ $S = \begin{pmatrix}-1 & 1\\ 0 & n\end{pmatrix}$

the upper right '1', for example, is linking between the left band & the pushed-off (up) right band.

we constructed $\sigma : \mathcal{C} \to \mathbb{Z}$ the signature,
by using the theorem that any slice knot K with a Seifert surface F has a Lagrangian $L \subseteq H_1 F$ (a half rank direct-summand for which $S|_L = 0$)

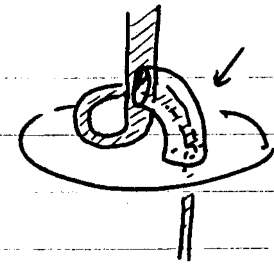
Proof of this theorem.
Warmup exercise is to prove for ribbon knots that σ is well-de to define σ
(Enough to show actually that σ vanishes on ribbon knots for some Seifert surface, because we know $\mathcal{C}$ = knots/slice $\overset{?}{=}$ knots/ribbon, and σ is independent of choice of surfac

ribbon singularities can be resolved:



the result is a manifold!
(Unknot not really important; shearing was!)

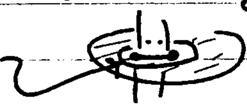
Alternative, easier picture:



add a ~~band~~ tube, feed the band through it. Knot is actually the same!

isotopy $\cong$ (undo kinks)

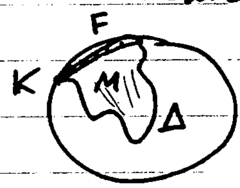
Desingularising the whole ribbon like this gives a disc + one tube for each ribbon singularity, and with obvious Lagrangian: circles surrounding the ribbon cuts



the unlink of circles as basis

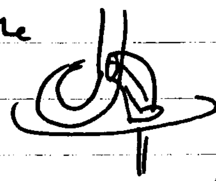
<u>Actual pf of thm 1</u> (which is stronger)

<u>Step 1</u>: $\exists$ oriented submfd $M^3 \subseteq D^4$ with boundary $F \cup_K \Delta$ (SS, slice)



Just use same arg as for construction of SS itself: H^1 always representable.

For the ribbon case, it's actually a solid handlebody; push in the bands in the



picture to B^4, then fill in the tube with a 2-handle. ([illegible])

Adding collar, get h-body.

<u>Or</u>: sweep the Morse picture through in B^4 using the obvious discs & fingers:

now at the ribbon singularity: extending the tube

touching is a 1-handle attachment; then can carry on isotoping the band through.

<u>Step 2</u>. Claim $\ker\left[H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})\right]$ is a Lagrangian subspace in $H_1(\partial M)$.

This completes the proof of the theorem as follows:

if $a, b \in \ker\, H_1 \partial M \to H_1 M$ $\quad\quad$ ($F \cup \text{disc} = \partial M$)

$$H_1\partial M = H_1 F$$

then they bound 2-cycles A, B in M whose intersection in B^4 is $\emptyset$ empty when B is pushed off M in the normal direction $\perp M^3 \subseteq B^4$. (so as to compute $S(a,b) = lk(a, b^+) = A \cap B^+$.)

To move from $\mathbb{Q}$ to $\mathbb{Z}$ coeff statement is no problem; although the $\mathbb{Z}$-kernel of $H_1F \to H_1M$ might not be a direct summand, use instead

$$L = \{ a \in H_1F : \exists\ n \in \mathbb{Z}-\{0\} \text{ s.t. } n\cdot a \in \ker(H_1F \to H_1M) \}$$

Its rank is the same as that of the kernel, & it's a direct summand because H_1F/L is torsion-free.

so $0 \to L \to H_1F \to H_1F/L \to 0$ splits.

Note also that $S(na, b) = 0 \Rightarrow S(a,b) = 0$ because these are integers, so $S|_L = 0$ as required.

Pf of step 2: use les over $\mathbb{Q}$ of $(M, \partial M)$.

Kernel k

$$0 \to H_3 N,\partial N \to H_2 \partial N \to H_2 N \to H_2(N,\partial) \to H_1 \partial \xrightarrow{\ H_1 N\ } H_1(N,\partial) \to H_0 \partial \to H_0 N \to 0$$

$\mathbb{Q}$-dims: $\quad a \quad b \quad c \quad d \quad e \quad d \quad c \quad b \quad a$

Now use Lefschetz duality & Kronecker duality.

Then, use the two halves of exact sequence:

$$k - d + c - b + a = 0 \quad ; \quad k - e + d - c + b - a = 0$$

$$\Rightarrow \underline{2k = e}.$$

Now can use the twist knots K_n to show $\mathcal{C}$ is not finitely-generated.

Define twisted signatures $\sigma_\omega : \mathcal{C} \to \mathbb{Z}$.

Pick $\omega \in S^1$ ~~a root of unity~~ ~~but~~ (actually, any $\omega \in \mathbb{C}$ will do) but [illegible]

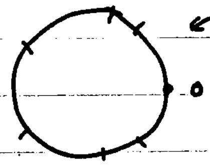
$$\sigma_\omega(K) = \sigma\left((1-\omega) S + (1-\bar{\omega}) S^T \right)$$

A hermitian form so has a signature.

The form is $= (1-\omega)(S - \bar{\omega} S^T)$

so <u>det</u> $= (1-\omega)^{2g} \cdot A_K(\bar{\omega})$ in terms of A. Polynomial

The finitely-many zeroes of the AP mean f.-many places at which the form becomes degenerate, & $\therefore$ the signature jumps around the unit circle.



← conjugate pairs of AP roots.

<u>Thm</u> $\sigma_\omega : \mathcal{C} \to \mathbb{Z}$ is a hom

(ie σ_ω vanishes on slice knots).

<u>Pf</u> K slice $\Rightarrow (1-\omega) S + (1-\bar{\omega}) S^T = \left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array} \right)$

and so the proof is OK as before provided $A_K(\omega) \neq 0$ (degenerate case doesn't work).

So general def is to define σ_ω = average of the two limits, extending the def to the bad points.

<u>Next time</u>. Will start to prove

<u>Thm</u> K_n is slice $\Leftrightarrow n = 0, 2$. (Casson-Gordon)

<u>Pf</u> $\sigma(K_n) = 0 \Leftrightarrow n > 0$ kills negative ones.

For $n \geq 0$ C-G invts are defined & vanish exactly for $n = 0, 2$

($\sigma_\omega(K_n) = 0$ for $n > 0$ but show for $n < 0$ that the K_n are independent in $\mathcal{C}$)

— · —

<u>Thm</u> (a) K_n slice $\Leftrightarrow n = 0, 2$

(b) $\{K_n\}$ are independent in $\mathcal{C}$ for $n < 0$ or $(4n+1) = l^2$ (n

(c) If $n > 0$ and $4n+1$ is not a square, then $\{K_n\}$ are $\mathbb{Z}_2$-independent in $\mathcal{C}$

<u>Coroll</u> : <u>All</u> K_n are <u>distinct</u> in $\mathcal{C}$ except for unknot $K_0 = K_2$.

(Scholium?)
(Coroll of <u>pf</u>)

<u>Conj</u> : All K_n are <u>$\mathbb{Z}$-independent</u> (other than $K_0 = K_2 = 0$) in $\mathcal{C}$

Calculation

$\sigma_\omega : \mathcal{C} \to \mathbb{Z}$ defined by $\sigma\big((1-\omega)S + (1-\bar\omega)S^T\big)$

i.e. $\sigma\big((1-\omega)(S-\bar\omega S^T)\big)$ $\qquad S = \begin{pmatrix}-1 & 1\\ 0 & n\end{pmatrix}$

Compute $\sigma(K_n) : S^1 \to \mathbb{Z}$, these will jump at roots of Alexander poly $A_K(\bar\omega) = \det(S-\bar\omega S^T)$

$$A_{K_n}(t) = A_n(t) = \det\left[\begin{pmatrix}-1 & 1\\ 0 & n\end{pmatrix} - t\begin{pmatrix}-1 & 0\\ 1 & n\end{pmatrix}\right]$$

$$= \det\begin{pmatrix}t-1 & 1\\ -t & n(1-t)\end{pmatrix} = -n(1-t)^2 + t$$

$$= -nt^2 + (2n+1)t - n$$

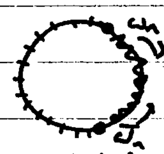
divide by $-n$: $t^2 - \frac{2n+1}{n} + 1$; roots $\frac{2n+1}{2n} \pm \frac{\sqrt{\frac{4n^2+4n+1}{4n^2} - 4}}{2}$

i.e. $\frac{(2n+1) \pm \sqrt{4n+1}}{2n}$

So A_n has real roots $\Leftrightarrow n>0$

$\Rightarrow$ σ is constant round circle at $\sigma = 0$ (

For $n<0$, roots are on unit circle since $\omega_n \cdot \bar\omega_n = 1$ (const coeffs) which are all distinct from one another.

$\sigma = -2$  $\sigma = 0$

computed at -1 as $\sigma\begin{pmatrix}-2 & 1\\ 1 & 2n\end{pmatrix}$; det is +ve for

but negative defin

[Andrew: # changes in sign of principal minors of a square matrix = its signature.]

Thus, have proved half of (b).

Now take $n>0$. The AP is irreducible $\Leftrightarrow$ $4n+1$ is a square. (let's think with rational coeffs ; irreducibility the same over $\mathbb{Z}$ & $\mathbb{Q}$). So assume not a square.

For any symmetric, irreducible polynomial $p \in \mathbb{Q}[t^{\pm 1}]$

Define a hom $h_p : \mathcal{C} \to \mathbb{Z}_2$

$K \mapsto$ exponent of p in A_K, mod 2.

(factorised over PID $\mathbb{Q}[t^{\pm 1}]$)

This is additive under # because Al multiplies.

Need to show it vanishes on slice knots.

But for these, $A_K = f\cdot\bar{f}$ so the exponent of $p=\bar{p}$ is even in it. □

For case (c) have $A_n = -nt + (2n+1) - nt^{-1}$

distinct irreducible symmetric polys $\therefore K_n$ are $\mathbb{Z}_2$-independent using the appropriate h_p functions.

The hard case is to do with $4n+1 =$ square case, ie (b). Done below then ⇒ Schobin (invariants of the K_n's are different)

The conjecture that all $\{K_n\}_{(\neq 0,2)}$ are $\mathbb{Z}$-independent in $\mathcal{C}$ is partially known by Livingston – Naik.

— · —

Suppose $4n+1 = \ell^2$ is ~~[illegible]~~

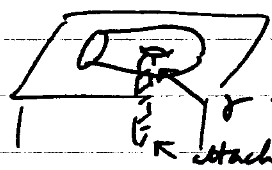
then $4n = (\ell-1)(\ell+1)$, $\ell = 2m+1$ so $\underline{n = m(m+1) \quad m>0}$

For $m=1$ it's K_2, the Stevedore's knot which is actually slice: Claim is proved by finding that $\gamma = -s+\ell$ is a vector with square 0:

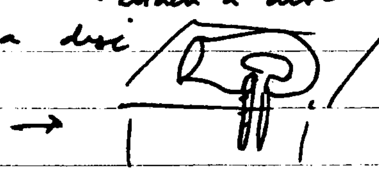
$$(-1,1)\begin{pmatrix}-1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = 0$$

this curve on the SS is the unknot visibly is a 0-framed unknot.

K_2 is therefore ribbon:



now surger the surface to a disc



to do the surgery & get two discs need the 0-framing on the unknotted curve.

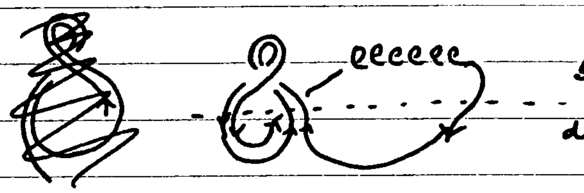
Lemma If $n = m(m+1)$ then the curve $\gamma = (-m)s + l$ has self-linking zero.

Pf $(-m, 1)\begin{pmatrix} -1 & 1 \\ 0 & m(m+1) \end{pmatrix}\begin{pmatrix} -m \\ 1 \end{pmatrix} = 0$

Def K is called <u>algebraically slice</u> if $\exists$ a Lagrangian in H_1
(a half rank direct summand L in H_1F with $S|_L = 0$)

Rk (a) K slice $\Rightarrow$ K alg slice
(b) The $K_{m(m+1)}$ are alg slice.

$m=2$ $(n=6)$. $\gamma = -2s + l$:



This is the <u>trefoil</u> in fact, so is not slice by existing thm

Similarly for $K_{m(m+1)}$ get $\gamma = (m, m+1)$ torus knot; slice $\Leftrightarrow$ $m=1$
(Actually $\exists$ two possibly 'γ's with 0 self-linking, but each is an $(m, m+1)$ torus knot

Remaining part of thm (a) follows from

Citations
(a) = Casson-Gordon
(b) = Trotter $n<0$
Jiang $4n+1$ pr
COT remaining

<u>Thm</u> (COT)
If K is a genus-1 knot with $A_K \neq 1$
then K slice $\Rightarrow$ $\exists\ \gamma \subseteq F$ s.t $S(\gamma, \gamma) = 0$ & γ is a generator of H_1F s.t $\int_{\omega \in S^1} \sigma_\omega(\gamma) = 0$.

(it's not known whether any slice knot actually has a slice γ-curve, but this result is a substitute).

Still need to prove

Thm 2 The twist knots K_n, $4n+1 = l^2$, are $\mathbb{Z}$-independent in $\mathcal{C}$ (but are algebraically slice).

Thm? If a genus-1 knot is slice and $\Delta_K \neq 1$ then $\exists$ a curve $\gamma \subseteq F$ with $S(\gamma,\gamma) = 0$ and $\int_{\omega \in S^1} \sigma_\omega(\gamma) = 0$

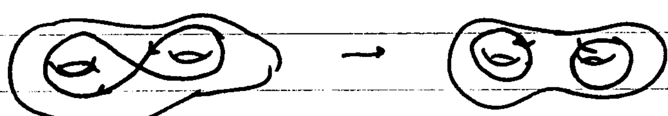
Survey of homology, intersection forms, linking forms in low dimensions

Homology

Fact 1. X any space, $i = 0,1,2,3$, then $H_i(X)$ is isomorphic to $\Omega_i(X)$ oriented bordism. $\{f: M^i \to X\}$ closed mfd / $\{$ boundary of $F: W^{i+1} \to X\}$
($\Omega_i(M,f) \mapsto f_*[M]$)

Group given by $\amalg$, inverse by reversing orientation
Proof is that first 'essential' singularity in oriented mfd is cone on $\mathbb{CP}^2$. (with $\mathbb{Z}_2$, cone on $\mathbb{RP}^2$!)

Fact 2 X^n a mfd of dim $n \le 4$ then homology classes $H_i(X)$ are represented by (embedded) submanifolds (except for $i=n$ because only $\pm 1 \times$ generator works!)

eg. 

increases # components

Same for $i=2$, $n=4$. by connect-sum using annulus SS for Hopf (increases genus)

NB in 3-mfds, immersed genus = embedded genus.

Intersection pairings

X^n a closed oriented mfd. Then PD gives

$$0 \to \mathrm{Ext}^1(H_{p-1}X, \mathbb{Z}) \quad H_{n-p}(X) \to H^p(X) \twoheadrightarrow \mathrm{Hom}(H_p X; \mathbb{Z}) \to 0$$

canonical $\cong$ $\mathrm{Hom}(\mathrm{Tor}\, H_{p-1}X, \mathbb{Q}/\mathbb{Z})$ (non-can $\cong$ $\mathrm{Tor}\, H_{p-1}$)

slant product with $[M]$ ($\leftarrow$ cap product with $[M]$)

So rewrite as

$$\cancel{0 \to \mathrm{Ext}\, H_{n-p} X \to H_{n-p} X \to}$$

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}^1(H_{p-1}X;\mathbb{Z}) & \to & H^p X & \to & \mathrm{Hom}(H_pX/\mathrm{Tor},\mathbb{Z}) & \to & 0 \\ & & (b) \uparrow \cong & & \uparrow \cap [M] & & \uparrow \cong (a) & & \\ 0 & \to & \mathrm{Tor}\, H_{n-p}X & \to & H_{n-p}X & \to & H_{n-p}X/\mathrm{Tor} & \to & 0 \end{array}$$

Thus (a) get intersection form $I_X : \dfrac{H_{n-p}X}{\mathrm{Tor}} \times \dfrac{H_pX}{\mathrm{Tor}} \to \mathbb{Z}$

(b) linking pairing $lk_X : \mathrm{Tor}\, H_{n-p}X \times \mathrm{Tor}\, H_{p-1}X \to \mathbb{Q}/\mathbb{Z}$

which are both non-singular.

Geometrically: these pairings are given by counting intersections (with sign). Linking # by taking multiple of one class, getting bounded guy, intersecting & dividing.

$$lk_X(A_{n-p}, B_{p-1}) = \# \frac{A_{n-p} \cap \beta_p}{m} \in \mathbb{Q}/\mathbb{Z}$$

where $\partial\beta_p = m.B_{p-1}$ $(m \neq 0)$ (This is independent of m.)

These are particularly interesting "in middle dimensions".

i.e. (a) $p = n-p$ (b) $n-p = p-1$.

n even $= 2p$ $\quad n$ odd. $= 2p-1$ $\quad (p = \lceil \tfrac{n}{2} \rceil)$

Then symmetry is that $I_X(A_p, B_p) = (-1)^p I_X(B, A)$

$lk_X(a_p, b_p) = (-1)^{p-1} lk_X(a, b)$

Thus, $n=4$: symmetric intersection form

$n=3$: symmetric linking form

For mfds with boundary (compact oriented) $\quad [M] \in H_n(M, \partial M)$

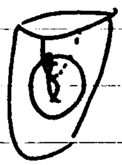
Let's put relative in cohomology in upper ses. above.

Thus $I_X : \dfrac{H_{n-p}(X)}{\mathrm{Tor}} \times \dfrac{H_p(X,\partial X)}{\mathrm{Tor}} \to \mathbb{Z}$

$$lk_X : \mathrm{Tor}\, H_{n-p}(X) \times \mathrm{Tor}_{p-1}(X, \partial X) \to \mathbb{Q}/\mathbb{Z}$$

Cannot deal with symmetry here. But can at least use $H_p(X) \to H_p(X, \partial X)$ to get a nicer pairing

(Rk: relative homology classes are representable by proper submfds of $(M,\partial M)$).



$$\begin{array}{c} H_1(M,\partial M) \times H_1 M \to \mathbb{Z}. \\ \uparrow \\ H_1 M \qquad\qquad \text{ok.} \end{array}$$

But of course relative classes don't intersect!

$H_1(D^2,\partial D^2) = 0$!

eg $(W^4, \partial W = M)$ cpt oriented 4-mfd.

$$\begin{tikzcd} H_2 W \arrow[r] \arrow[dr, "I_W"'] & H_2(W,M) \arrow[r, "(**)"] & H_1 M \arrow[r] & H_1 W \\ & H^2 W \arrow[u, phantom, "\cong"] \arrow[d, "(*)"] & & \\ & \mathrm{Hom}(H_2, \mathbb{Z}) & & \end{tikzcd}$$

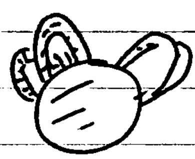
Can calculate $(H_1 M, lk_M)$ from $(H_2 W, I_W)$.

Assume $H_1 W = 0$: then get ~~iso~~ isoms $(*)$ and surjection $(**)$

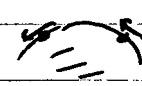
Then have $H_2 W \xrightarrow{I_W} (H_2 W)^* \to H_1 M \to 0$

Concrete example: $W = B^4 \cup$ (framed link L) 2-handles.

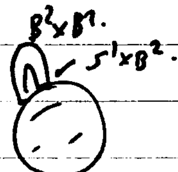
think of 2-mfds,



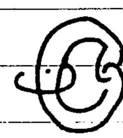
1-handles attach along framed link



now in 4-dim:



links describe core



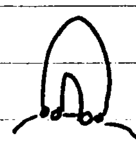
then thicken to $S^1 \times B^2$, framings $\mathbb{Z}$.

It measures self-linking of the boundaries of

two cores

just like

against



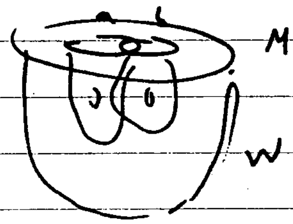
Thus get $n \times n$ linking matrix.

NB $W \simeq \cdot \cup n$ 2-cells so $H_2 \cong \mathbb{Z}^n$. $\pi_1 W = 0$ $H_1 W = 0$

have $\mathbb{Z}^n \xrightarrow[\text{lk matrix}]{A} \mathbb{Z}^n \to H_1 M \to 0$

(H_2 represented by cores of handles $\cup$ cones on link cpts, or Seifert surfaces. Use cores $\cap$ S.S to compute linking # as intersection)

The linking form on M is determined by I_W.
For a torsion element in M (let's assume I_W has finite cokernel, in fact) can always lift to a class in $H_2(W, M) \twoheadrightarrow H_1 M$



start with $a, b \in H_1 M$:
have relative classes $A, B \in H_2(W, M)$.
Now take $m.B$ s.t it comes from an element of $H_2 W$ and take intersection - then divide by m.

$$\mathbb{Z}^n \xrightarrow{I} \mathbb{Z}^n \to \text{coker} \to 0$$

get pairing $(x, y) \in$ coker $\mapsto \mathbb{Z}(\tilde{x}, I^{-1}\tilde{y})$ ← inverse over rationals ← intersection of $\tilde{x}$ with $I^{-1}\tilde{y}$

lifts into $\mathbb{Z}^n$

Recall: framed link $L \subseteq S^3 \to W^4_L$, $M^3_L = \partial W^4_L$.
M^3_L = surgery on S^3.

<u>Thm</u> Any closed oriented 3-mfd is of the form M_L.
(Bing, Rochlin, Lickorish, Wallace ...?)

<u>Pf</u> To prove $\Omega_3^{\#} = 0$ à la Rochlin (probably)
use Pontryagin - Thom construction, $\Omega_3 = \pi_3(MSO)$
MSO = the Thom spectrum. Then, have Hurewicz isomorphism to $H_3(MSO)$ because $\Omega_{\leq 2}$ are zero by classification of 1- & 2-manifolds. Then, Thom isomorphism says

$H_3(MSO) = H_3(BSO) = 0$.

Actually can then continue to get $\Omega_4 \cong \mathbb{Z}$.

Then, step 2 is to ~~cancel~~ surger ~~away the~~ the 4-mfd on circles to kill π_1. (The circles are embedded, neighborhoods are trivial bundles ...).

Finally cancel handles to get only 0-h $\cup$ 2-hs.

<u>Addendum</u>: can assume all framings are even.

This comes from $\Omega_3^{spin} = 0$, some handleslides and Wu formula.

eg. $L = \bigcirc^{+1}$; $W_L = \mathbb{CP}^2 - \text{ball}$, is not spin.

<u>Corollary</u> Any knot $S^2 \overset{K}{\subseteq} S^4$ is slice; extends over a 3-ball in B^5.

(Kervaire) in higher dims.

<u>Proof</u>

Pick a Seifert surface $F^3 \subseteq S^4$, $\partial F^3 = K$.

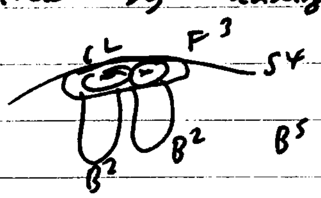
(Use the correct construction via a map $S^4 - K \to S^1$ representing generator of H^1 of complement)

(Alexander duality $\Rightarrow H^1(S^4 - S^2) \cong \mathbb{Z}$)

By the theorem F is obtained as surgery on B^3 (cap it off with B^3, use thm, take off at the end)

Conversely can surger F to B^3 along some $L \subseteq F$.

Now it's no problem; need only to surger along these by adding 2-disks in B^5:



each circle of L in S^4 bounds a disc in B^5. They're <u>disjoint</u>.

Now need to discuss framings: thicken each $B^2 \subseteq B^5$ up to a $B^2 \times B^2$; need <u>even</u> framing on link to do it right.

Recall that normal bundle of disc B^2 in B^5 is trivial, but then can always find in trivial $\mathbb{R}^3$-bundle on S^2 a 2-plane sub-bundle with any <u>even</u> Euler class.

Thus, surgeries possible ambiently $\Leftrightarrow$ even framings.

(Alt. proof via cohomotopy:
cobordism of framed 3-mfds inside S^4
is classified by $[S^4, S^1] = 0$. Think this works.)

Back to $\mathcal{C}_1 = \dfrac{\{S^1 \subseteq S^3\}}{\partial\{B^2 \subseteq B^4\}}$

$\Rightarrow$ def of $\mathcal{C}_n = \dfrac{\{S^n \subseteq S^{n+2}\}}{\partial\{B^{n+1} \subseteq B^{n+3}\}}$ higher-dim cobordism groups.

(rk: codim -2 the only interesting case, at least in PL case. Schönflies for codim 1, see Rourke & Sanderson for PL).
(rk: the difficulty corresponds to π_1 being non-trivial)
(The proof of unknottedness in codim - 3, say, starts with observing that the complement has same π_1 & homology as unknot complement ...)

Thm For $n \geq 4$, the groups $\mathcal{C}_n$ are 4-periodic in n:

(smooth or PL)

(a) $\mathcal{C}_{even} = 0$ $\qquad$ $\mathcal{C}_0 = 0$ $\qquad$ $\mathcal{C}_1$ unknown

(b) $\mathcal{C}_{4k+1} \cong AC^-$ $\qquad$ Plus $\qquad$ $\mathcal{C}_2 = 0$

(c) $\mathcal{C}_{4k-1} \cong AC^+$ $\qquad$ $0 \to \mathcal{C}_3 \to AC^+ \to \mathbb{Z}_2 \to 0$

$\boxed{\text{signature}(S+S^T)/8 \bmod 2 \text{ Rochlin.}}$

Recall defn of algebraic conc groups:

$$AC_n^{\pm} = \dfrac{\left\{ \begin{array}{l} \text{Seifert forms } S: \mathbb{Z}^n \to (\mathbb{Z}^n)^* \\ \text{with } S \pm S^T \text{ an isomorphism} \end{array} \right\}}{\left\{ \begin{array}{l} \text{metabolic forms: those with} \\ \text{a Lagrangian summand.} \end{array} \right\}}$$

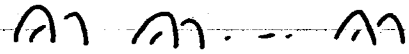
it has maps $\twoheadrightarrow \overset{\infty}{\oplus} \mathbb{Z} \overset{\infty}{\oplus} \mathbb{Z}_2$
via twisted signatures and factorisations of Alexander poly.

Rk AC^- was original gp; AC^+ is the appropriate thing in the other dimensions

Rk the map (Seifert form) : $\mathcal{C}_n \to AC^{\pm}$ is clear.

Rk $AC^{\pm}$ in fact are $\cong \overset{\infty}{\oplus} \mathbb{Z} \overset{\infty}{\oplus} \mathbb{Z}_2 \overset{\infty}{\oplus} \mathbb{Z}_4$ some additional stuff.

Lemma $C_1 \to AC^-$ is onto.

Proof Use the picture of a knot with banded Seifert surface. Just insert twists of bands & linking #s appropriately.  denotes unknot.

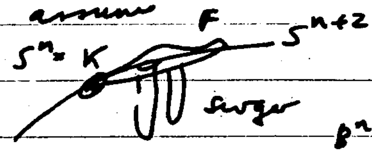
So given any S s.t. $S - S^T = J$ (symplectic) can do it.

As J is the only skew isomorphism, up to isomorphism, have done..

(In higher dimensions, it's plumbing)

Sketch pf of the thm of $C_{n\geq 4}$

Pick a Seifert surface for the knot. By ambient surgeries below the middle dimension, can assume F is alterable to a slice disc $B^{n+1} \subseteq B^{n+3}$; at least in the even case (a) □



If instead $S^{2k-1} \subseteq S^{2k+1}$ then may assume
$\cap\!\!|$ F^{2k} $\subseteq$

F^{2k} is a connected sum $\#(S^k \times S^k) - \mathring{B}^{2k}$ by again surgering below middle dimension. (quite hard)

∴ after this concordance have analogue of $\# S^1 \times S^1 -$ d

which occurs in the $1 \subseteq 3$ dim case.

Now S is again the linking form on the middle homology of this thing inside S^{2k+1}. Arrange as before to get the onto case $\to AC^\pm$ or a appropriate ⊛

Final step: if S has a Lagrangian in $H_k F^{2k}$ then original knot was slice: true when $k > 1 \to$ ⊕

⊛ [Stabilise onto $\# S^k \times S^k$ to make the symmetric bilinear forms all isomorphic to standard one. (add an odd (± 1) to make it diagonalise...)] NOT RIGHT!

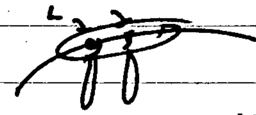
⊕ → To do this, represent the Lagrangian by a link $L = \amalg S^k \subseteq F^{2k} \subseteq S^{2k+1}$

For $k \geq 3$ use Whitney trick to make them disjoint.

For $k=2$ need stable Whitney trick.

Final lemma:

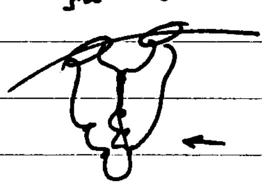
If $L: \amalg S^k \to S^{2k+1}$ has trivial linking numbers ~~then~~ and $k > 1$ then L is slice (in fact trivial!)

(Then we're done as usual by surgery  .

The classical dimension is the case where the dimension didn't drop & make progress.) (eg $k=2$: started trying to slice a knot $S^3 \subseteq S^5$: at this stage have to do $S^2 \subseteq S^5$.

Proof of lemma

$S^k \quad S^k$



S^{2k+1}

B^{2k+2}

$\leftarrow$ immersed discs, possibly intersect & self-intersect.

Pair them up with Whitney discs; ambient dimension is ≥ 6 so we're OK.

Erratum to high dimensional concordance theorem

Did not carefully specify the ~~category~~ we were working in (Diff, PL or topological - locally flat)

The theorems last time applied to PL version.

$$\text{For } n \geq 4, \quad e_n^{PL} = e_n^{TOP} = \begin{cases} 0 & n \text{ even} \\ AC^+ & n = 4k-1 \\ AC^- & n = 4k+1 \end{cases}$$

whereas in the smooth case there is a problem with realizing symmetric forms using plumbing: can get exotic spheres knotted.

So same result e_n^{DIFF} for $n \neq 4k \pm 1$

BUT

$$0 \to e^{DIFF}_{4k-1} \to Ae^+ \to \mathbb{Z}_{n_k} \to 0$$

$$\| $$

$$\frac{\left\{ \begin{array}{l} S : \mathbb{Z}^n \to (\mathbb{Z}^n)^* \\ S + S^T \text{ isom} \end{array} \right\}}{\text{metabolics}}$$

(have Lagrangians)

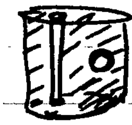
$\swarrow$ an even form, so $8 \mid \sigma$

$$S \longmapsto \frac{\sigma(S+S^T)}{8}$$

where n_k = order of Kervaire – Milnor exotic sphere Σ_{4k-1} in the group of exotic spheres Θ_{4k-1}

Recall $\Theta_{4k-1} = \dfrac{\{\text{mfds homeo to } S^{4k-1}\}_{\#}}{\{\text{those diffeo to } S^{4k-1}\}}$ is a finite group

(Proof that $\Sigma \# \bar{\Sigma}$ diffeo to S^{4k-1} is h-cobordism

$\Sigma \times I$ − a tube and a ball 

Andrew:

In the $4k+1$ case there is a similar Arf invariant problem:

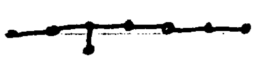
$$0 \to \Theta^{DIFF}_{4k+1} \to \mathcal{A}C^{-} \to \{0 \text{ or } \mathbb{Z}_2\} \to 0$$

? Kervaire invariant problem: unknown!

The problem in the previous proof was trying to realise an even unimodular form stabilised by hyperbolics, geometrically.

<u>Thm</u> (Serre)(1962) $(S+S^T) \oplus$ hyperbolic $\cong \bigoplus_{i=1}^{k} E_8 \oplus$ hyperbolic

with $8k = \sigma(S+S^T)$

$E_8 =$  '2' on diagonal, adjacency gives a '−1'.

To realise this as an intersection form of a Seifert surface $W^{4k} \subseteq S^{4k+1}$ do plumbing

$$W_L^{4k} = B^{4k} \cup_L (2k\text{-handles}) \simeq \bigvee S^{2k}$$

L = link of $(2k-1)$-spheres in S^{4k-1} (framed) $(\pi_{2k-1}(SO(2k)) = \mathbb{Z}$

Always $\pi_1(W_L) = 1$ (except)

$$H_* = \mathbb{Z}^{\# L} \text{ in middle dimension.}$$

intersection form $H_2 \to H_2^*$ is linking matrix.

The boundary has π_1 trivial also, homology sphere if linking form is non-singular (use the l.e.s. + Lefschetz duality

$\therefore$ for $k > 1$ by h-cobordism, $\partial W_L \underset{\text{homeo}}{\cong} S^{4k-1}$

but it is by defn Σ_{4k-1} Kervaire – Milnor sphere $\partial(E_8)$

(<u>indep</u> of choice of link: classified up to isotopy by linking form)

Also this construction W^{4k} always embeds in S^{4k+1},
is a Seifert surface & knot pair. So this shows
~~$e^{DIFF}_{4k-1} \twoheadrightarrow$~~ $e^{PL \text{ or } TOP}_{4k-1} \twoheadrightarrow AC^+$ but not in smooth case
because $\Sigma_{4k-1} \underset{\text{diffeo}}{\not\cong} S^{4k-1}$

(Only need to do the E_8-case by Serre's thm)

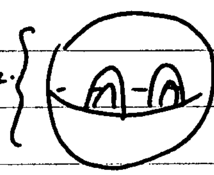
Rk $n_2 = 28$. Σ^7 generates Θ_7. (Kervaire-Milnor)

Will prove ⊛ that any $W^{4k}_L \subseteq S^{4k+1}$. Even in $k=1$; a corollary is that

Cor Any closed oriented 3-mfd embeds in S^5.
(Rk: Whitney's best case would give $2n$, 6 in this case)
(Andrew: "the great theorem of 1965!")

⊛ Proof consider $W_L \times I$; the $B^{4k} \cup$ (2k-handles) (0-handle) is still of this form, $B^{4k+1} \cup$ (2k-handles)
but they attach along a trivial link because of the dimension shift. $\amalg S^{2k-1} \subseteq S^{4k}$ (bound discs, ∃ self-intersections but no problem when moving the knot across!)
~~Therefore $W_L \times I$~~
Then (use only that the link is now slice) can attach ambiently!

S^{4k}



B^{4k+1} – thickened sl

B^{4k+1}

The only omission here is framing; inside the trivial framed $4k+1$ dimensional handles, can find all $4k$-plane bundles ~~here~~ with any even Euler classes.

(or: framings stabilised $\pi_{2k-1}(SO(2k)) \longrightarrow \pi_{2k-1}(SO(2k+1))$

$e(TS^{2k}) = 2$ (Euler class) $\quad \downarrow e \quad \mathbb{Z}$

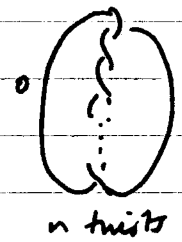
$\pi_{2k}(S^{2k}) \longrightarrow \pi_{2k-1}(SO(2k)) \longrightarrow \pi_{2k-1}(SO(2k+1)) \longrightarrow 0$

$1 \mapsto TS^{k}$.)

∃ more complicated choices than we need.
only plumb to TS^{2k} & these are kernel of stabilisation

Thm If a 3-manifold $M \cong M_L$ (L a framed link) for a 0-framed link L which is the union of two slice links, then M^3 embeds in S^4.

$\underline{\underline{\text{eg}}}$ $L =$



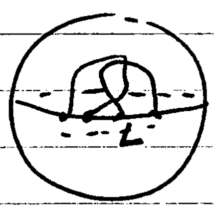
$H_1(M_L) = \operatorname{coker}\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \mathbb{Z}_n \oplus \mathbb{Z}_n$

actually $L(n,1) \# L(n,1)$

but a single lens space does not embed in S^4.

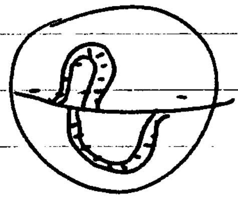
Proof

$S^4 =$



If the whole link were slice, it would be easy: $W_L \subseteq S^4$ as before. 0-framing necessary to be able to thicken the slice discs to "negative 2-handles" inside B^4.

Now if the individual components are slice, just subtract!



$W_L \not\subseteq S^4$ but the 3-manifold does.

(W_L with one handle surgered to a 1-handle does embed eg slice knot $\cup$ O ($\leftarrow$ unknot).)

Rk Homology bordism group of $\mathbb{Z}HS$ / homology balls can be obstructed by Furuta(?) – Stern.

Any $\mathbb{Z}HS$ embeds topologically in S^4 (Freedman)

– an interesting open question.

Next must consider 4-mfds with 1-handles $W^4 = 0 \cup 1 \cup 2\text{-h}.$

Because $\pi_1(B^4 - \text{slice disc}) \cong \mathbb{Z}$.

But such guys can still be drawn with link pictures.

Thm (Laudenbach)

A closed oriented 4-mfd N^4 with a handle decomp is determined by its 2-skeleton. (Any diffeo of $\# S^1 \times S^2$ extends over $\# \natural S^1 \times B^3$

Rk His proof is by classification of the MCG of $\# S^1 \times S^2$.
A particular element is the Gluck twist

$$S^1 \times S^2 \to S^1 \times S^2$$
$$(x, v) \mapsto (x, x.v) \qquad x \in SO(2) \subseteq SO(3) \text{ axis}$$

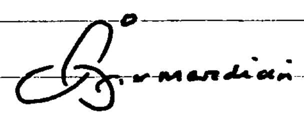
Rk can cut $S^2 \times B^2$ & reglue in a 4-mfd using Gluck, which doesn't extend this way $\begin{matrix} S^1 \times S^2 \\ \vdots \\ B^2 \times S^2 \end{matrix}$
so S^4 surgered like this might be an exotic smooth structure on S^4.

Cameron Gordon showed that certain 2-knots for which Gluck twist $\cong S^4$ are actually different 2-knots $S^2 \subseteq S^4$

Back to classical knots

$$\begin{matrix} S^1 & \subseteq & S^3 \\ \cap & & \cap \\ D^2 & \hookrightarrow & D^4 \end{matrix} \quad \text{slice knot}$$

Lemma 1 If K is slice then M^3_K = 0-surgery on K bounds a 4-manifold W^4 with



(i) $H_1 W \xleftarrow[i_*]{\cong} H_1 M = \mathbb{Z}$

(ii) $H_2 W = 0$

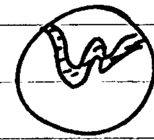
(iii) $\pi_1 W$ is normally generated by the meridian m.

Thm (Freedman)
The converse holds in the topological category (locally-f

Rk. (~~Proof of~~ Lemma 2: if (i), (ii) are satisfied then K is slice in some homology 4-ball — that is, it is homologically slice.)

Rk there are no known obstructions able to tell the difference between homologically slice & actually slice!

Proof of lemma 1

Define $W^4 = B^4 -$ thickened slice disc. 

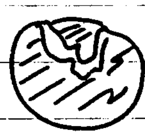
Then $\partial = M^3_K$. (0-framing because push-off of knot along the core doesn't link)

Then (i)(ii) follows from Alexander duality. [or MV for $X \cup_{S^1 \times S^2} h = B^4$]

(iii) comes by ~~that~~ the fact that adding relation (merid = 1) kills the group, which is obviously true geometrically.

If B^4 was a homology ball instead then (i) & (ii) still true. But the fundamental group statement ~~would~~ could fail.

Proof that (i)(ii) $\Rightarrow$ slice in a homology 4-ball (lemma 2)

Start with the  $= W^4$, now attach the 2-handle to W^4 along meridian. This is a homology ball, (~~...~~ ($\partial = M^3$ in particular) and K is slice.

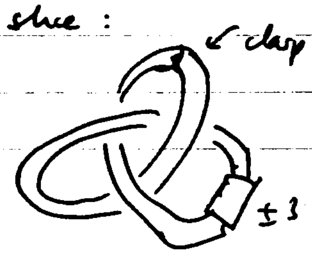
(the knot is the cocore of the 2-handle.)

Proof of the thm. Start as above, make smooth homology 4-ball B as above. If we have (iii) then $\pi_1 B = \phi$ hence homotopy 4-ball. But then by Poincaré conjecture it is homeo to B^4. (This is a "strange" homeomorphism, after which the slice disc looks strange but it's locally flat because the homeo carries its regular nbhd with it)

(i.e. it is a locally-flat topological ^not smooth^ embedding $B^2 \subseteq B^4$.)

Open problem: smooth case of this: homotopy 4-ball $\overset{\text{diffeo}}{\cong} B^4$?

Rk $\exists$ examples of knots which are topologically locally-flat slice but not smoothly slice:



Whitehead double of 0-framed trefoil

[Proof: construction $\Rightarrow$
$\not\Leftarrow$ gauge theory]

What 4-manifolds does M_K (0-surgery on K) bound? (i.e. want obstruction to M_K^3 bounding such 4-mfd given)

(a) $M_K = \partial(W_K = B^4 \cup_K \text{2-handle})$

Trouble with this is it's simply-connected but has $H_2 \neq 0$

Want to get one with $\pi_1 \cong \mathbb{Z}$ $\Rightarrow H_1 \cong \mathbb{Z}$

Lemma 3 (a) $\exists$ always a W^4 with $\partial = M_K$ and $H_1 M \xrightarrow{i_*} H_1 W \cong \mathbb{Z}$.
(and $\pi_1 \cong \mathbb{Z}$)

(– so the $H_2 W =$ condition is essentially the obstruction to slicen

(b) Any W^4 satisfying (ii) $(H_2 = 0)$ with $\pi_1 W \cong \mathbb{Z}$ has twisted signatures $\sigma(I_{\tilde{W}} \otimes \mathbb{C}_\omega) = \sigma_\omega(K)$ original definition.

$\uparrow$ intersection form on universal cover

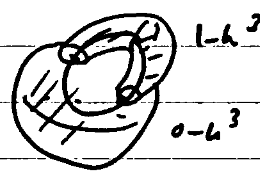
$\omega \in S^1$ gives character of $\mathbb{Z}$.

– If condition $H_2 W = 0$ then so will $I_{\tilde{W}}$ and so these signatures will therefore be zero.

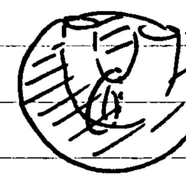
Drawing 4-mfds with 0-1-2 handles (oriented!)

Already know the 'no-1-handles' case.

Actual 3 dim picture:



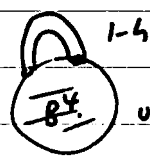
$\cong$



subtract a tube

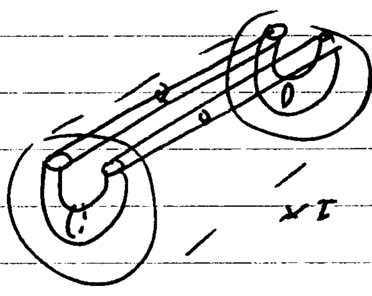
generators of π_1 illustrated.

Now take this picture $\times I$:



B^4 – (a 2-handle) really is above picture $\times I$!

D

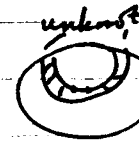


boundary 2 tori for the hole.

$\checkmark$ subtracting (arc $\times I$) $\times$ original B^3 from $B^3 \times I$; on the boundary see two end holes, & two annuli

By convention, draw this as

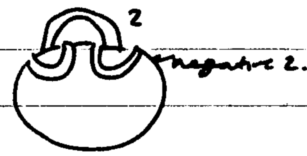
meaning



unknot

remove (std) 2-handle from B^4.

Every dotted circle comes with a canonical disc, 
it should be pushed down into B^4 & then removed.

Then, draw an oriented 4-mfd by unlink of 's (draw "fla
& then 2-h framed link.
(attach by general position → 

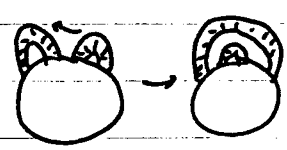
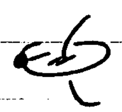
⇝ 2-h attaching maps disjoint)
They can link the 1-handle discs.

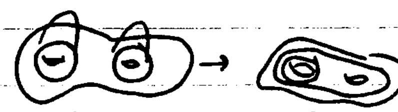
Can read off π_1 very easily. Generators = meridians of 1-handles, relations = attaching curves

Eg on 

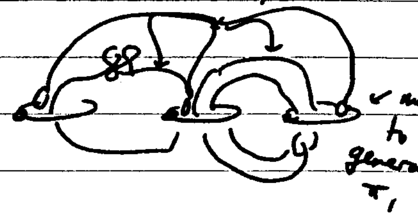
: Gompf found this mfd inside K3 surface
– if it bounded a W^4 as above, could give ~~i~~ & contradict Donaldson ($\nexists$ $E_8+E_8+\ldots$ forms smoothly
Actually K slice ⇒ ±1-surgery bounds a homology ball
& it is there things he actually found mods
(ie homology sphere is the splitting surface).
Eg Freedman – Taylor ⇒ any ⊕ intersection form can be split as a "connect-sum" along homology spheres.

———.———.———.

Thm The diffeo type of W_L is unchanged under the following moves on L:
(0) Isotopy of L
(1) Handleslide of 1 on 1, 2 on 1, 2 on 2.
(= isotopy of attaching maps of handles over each other)
(2) Cancellation of a 1-2 pair;
geometric intersection #1 of disc of 1-h & 2-h curve  

(push off other things first!) 
arbitrary paths of discs

Obs on computing $\pi_1(W_L)$ & $H_* W_L$ by cellular chain complex.
Also can do computations of homology of universal cover;
$\mathbb{Z}\pi$ replaces every $\mathbb{Z}$ in chain complex
Then read intersection matrix with $\mathbb{Z}\pi_1$-coefficients
(Need to connect all handles to a basept via an arc)
Then count class in π_1 for any intersection pt; go along 2-handle, back to basept along 1-handle arc.



(NB Really are utilising the spanning disc of the 1-handle O in which the intersection points live.)

Rk $\pi_2(W_L) = \pi_2(\tilde{W}_L) = H_2(\tilde{W}_L)$ can be computed as kernel of ∂_2 map over $\mathbb{Z}\pi$.

Rk when moving 1-handle arc and Abe, cannot do a band-sum through the discs.

Rk If $L = L_1 \sqcup L_2$ then $W_L = W_{L_1} \natural W_{L_2}$
$\partial W_L = \partial W_{L_1} \# \partial W_{L_2}$

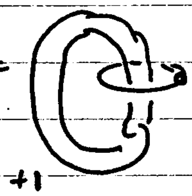
Coroll Adding a ± 1-framed unknot doesn't change the boundary of W_L, as $W_{L \sqcup O^{\pm 1}} = W_L \# \overline{\mathbb{CP}^2}$

Kirby's thm Two manifolds $M^3_{L_1} \cong M^3_{L_2}$ $\Leftrightarrow$ L_1 can be obtained from L_2 by a (finite) sequence of handleslides & blowing up / down.

Rk For any smooth manifold, handle decompositions exist (Morse function & gradient-like vector field).
They are unique up to handleslides (isotopy) & cancellation.
(Cerf) (generic g.l. v.f is required to make transverse ascending & descending mfds.)

This isn't good for $W_L\,(0,1,2)$ because 3 & 4-handles may appear.
Casson shows ∃ examples where 3-handle appearance is required.

<u>But</u> (I claim) ∃ a calculus for <u>closed</u> 4-manifolds.

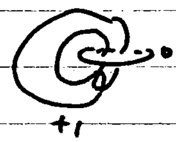
<u>example</u> $W =$ 

$\pi_1 = \mathbb{Z} = \langle m \rangle$
$\pi_2 = \mathbb{Z}[\pi_1 W] = \mathbb{Z}[m^{\pm 1}]$
$(W \simeq S^1 \vee S^2)$

$H_2 = \mathbb{Z}$ with $\epsilon : \pi_2 \to H_2$ being $m \mapsto 1$ (augmentation)

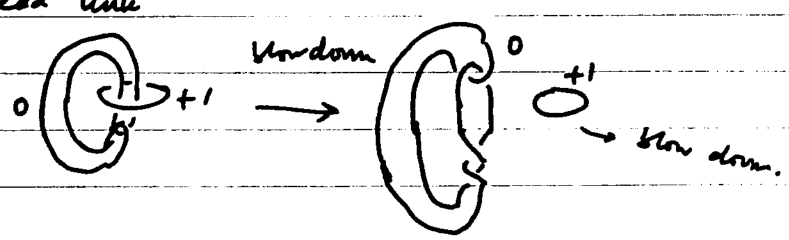
I_W = Intersection form is $(+1)$
Can also compute $I_{\tilde{W}}$ in the cover.
The 2-sphere in W can be pictured as core of 2-handle
Can render it as a 1-self-intersection guy in B^4.
& then compute $I_{\tilde{W}} = (m + m^{-1} - 1) \in \mathbb{Z}[m^{\pm 1}]$
($\hat{=}$ AP of trefoil)

<u>NB</u> Can't use core ∪ Seifert surface because this isn't a sphere, and in fact <u>doesn't</u> lift to the universal cover because of its π_1.

<u>Further</u>: to compute boundary, 

← do handle-trading to replace "−ve" by "+ve" handle

symmetry of Whitehead link



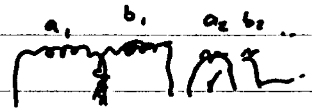
So <u>0-surgery on trefoil</u>.

<u>Rk</u> This is explicit example of $\pi_1 = \mathbb{Z}$ meridian-gen W^4 bounded by 0-surgery on the knot. Can get AP + all twisted σ's from W^4.

Theorem Let K be a knot, M_K the surgery on K.
(a) Then there exists a 4-mfd W s.t. $\partial W = M_K$, $\pi_1 W \cong \mathbb{Z}$, $\boxed{\sigma =}$ (generated by m)
(b) Any 4-mfd W as in (a) also satisfies ① $\pi_2(W)$ is a free $\mathbb{Z}[\mathbb{Z}]$-module of rank $= \operatorname{rank}_{\mathbb{Z}} H_2(W)$ $\left(= H_2(\tilde{W})\right)$
② Intersection form λ on W has
$\det(\lambda) =$ Alexander poly of K
$\sigma_z(\lambda) = \sigma_z(K) \quad \forall z \in S^1$.

(Remember: K slice $\Rightarrow$ K homologically slice $\Leftrightarrow$ $M_K = \partial$ of a 4-mfd N with $H_1(N) \cong H_1 M_K$, $H_2(N) = 0$).
(Thus, for knots with $\Delta \neq 1$, a 4-mfd with $\pi_1 \cong \mathbb{Z}$ won't do to slice the knot)

Proof (a): pick a Seifert surface for K, draw in band form, and then add handles:
in terms of linking #s of bands (Seifert form) see

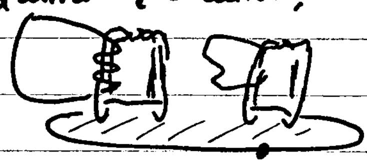


(bands can go anywhere)

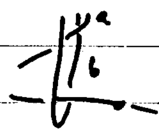
Seifert matrix $\begin{pmatrix} A & C \\ 1+C^T & B \end{pmatrix}$ '1' from symplectic basis

$A = A^T \quad B = B^T \quad S - S^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

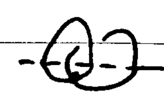
Now make W_0^4: 'a' curves & 'b' curves, 0-framed but put correct twists on



At the bottom see



reflecting the original intersection.



So: W_0^4 = 0-handle $\cup$ 1-handle $\cup$ $2g \times$ 2-handle
π_1 obviously $\cong \mathbb{Z}$ and homotopy equiv to $S^1 \vee \bigvee^{2g} S^2$.
(gen by meridian of 1-handle.)
Thus $\pi_2 \; (= H_2(\tilde{W}_0))$ is $\bigoplus^{2g} \mathbb{Z}\pi_1(W_0)$ $\quad$ ($\& \; H_2 W_0 = \mathbb{Z}^{2g}$)

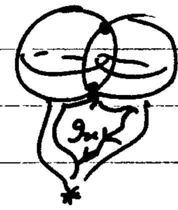
Now compute the intersection form λ_W on $H_2\tilde{W} = \pi_2$.

Given $f_1, f_2 : S^2 \hookrightarrow W_0$; take

$$(f_1, f_2) \longmapsto \sum_{x \in f_1 \cap f_2} \epsilon_x \cdot g_x$$

ϵ_x = sign of dpt
g_x = double pt loop at x.

(NB: $\exists$ a basept somewhere, as we are using $\underline{\pi_2}$ here: picture



NB Also, have to have spheres here, or at least condition on π_1 of surface if we were using non-spherical classes in H_2.

The form $\lambda(f_1, f_2)$ is hermitian :

$$\lambda(f_1, f_2) = \overline{\lambda(f_2, f_1)} \quad \text{where} \quad \bar{g} = g^{-1} \text{ on } \mathbb{Z}\pi_1.$$

(signs are unchanged but loops g_x reverse)

~~Now $H_2(W_0, M_K; \mathbb{Z}[\mathbb{Z}])$~~

(universal cover)

$$\begin{array}{ccccccc} \text{Now } H_2(W_0; \mathbb{Z}[\mathbb{Z}]) & \to & H_2(W_0, \partial W_0; \mathbb{Z}[\mathbb{Z}]) & \to & H_1(M_K; \mathbb{Z}[\mathbb{Z}]) & \to & H_1(W_0; \mathbb{Z}[\mathbb{Z}]) \\ \| & & \|2 & & \| & & \| \\ H_2\tilde{W}_0 = \pi_2 W_0 & \xrightarrow{\lambda} & H^2(W_0; \mathbb{Z}[\mathbb{Z}]) & & H_1(\bar{M}_K) & & 0 \\ & & \|2 & & \text{inf cyclic cov} & & \\ & & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\pi_2 W_0, \mathbb{Z}[\mathbb{Z}]) & & & & \end{array}$$

Hence $H_1(\bar{M}_K) = \operatorname{coker} \lambda$; $\therefore$ $\det \lambda = AP$ of $M_K = AP$ of K

(char poly of $H_1\bar{M}_K$ as $\mathbb{Z}[\mathbb{Z}]$-module)

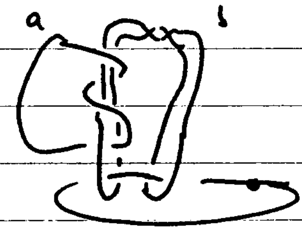
(well-defined up to units ; determinant got by taking basis, & changes alter by units)

(didn't use any special properties of W_0 here really)

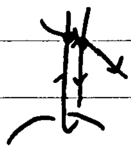
Now compute

$$\lambda_{W_0} = \begin{pmatrix} A & 1 + (t-1)C \\ 1 + (t^{-1}-1)C^T & (t-1)(t^{-1}-1)B \end{pmatrix} \qquad 2g \times 2g \text{ over } \mathbb{Z}[t^{\pm 1}]$$

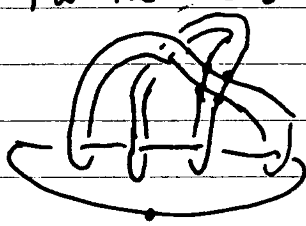
The spheres all come as (core of 2-h) $\cup$ (disc which is a null-homotopy of the attaching curve in $S^1 \times B^3$)

eg: 

In homotopying an a away from a b, pick up pairs of intersection pts except right at the end (one extra one.)

Pairs: The two differ by one revolution through the 1-handle  hence the "$t-1$"
(& then use hermitian to fill in the other corner!)

For the b-b pairs, see stuff like



get 4 intersection points, when followed get
$$(t-1)(t^{-1}-1)\,B.$$

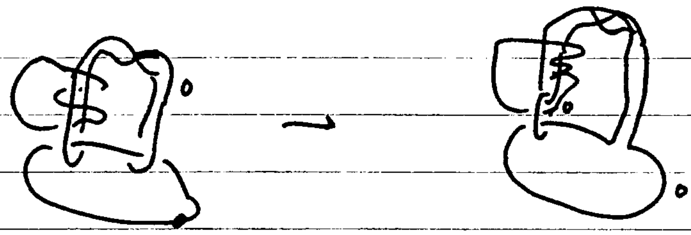
NB twists in the 'b' bands were added to make diagonal entries right; actually those twists were

 i.e. 

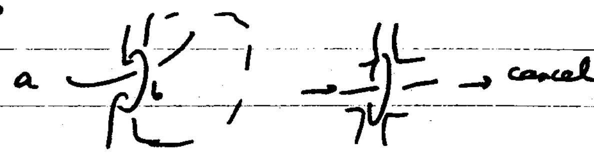
same arg; computing λ - self-intersection of a sphere always "four times bigger" than one might expect 
(Difference between λ & μ)

EX After a base change involving $(1-t)^{-1}$, λ_{W_0} agrees with
$$(1-t)S + (1-t^{-1})S^T$$
whose signatures at $t=z$ are by def $\sigma_z(K)$.
Hence $\sigma_z(\lambda)$ (= by def σ of λ with $t=z$). $=\sigma_z(K)$
Rk $t=1$ cannot invert $(1-t)$ but then $\begin{pmatrix} A & 1 \\ 1 & 0 \end{pmatrix}$ & $\sigma=0$; agrees.

To see $\partial W_0 = M_K$ (Rk: remove S. surface, pushed into B^4 & then attach round handles to turn this into 0-surgery).
- shrink the 'b' curves to unknots, twisting up the ⊂⊃ guys.
Now surger it to a 2-handle, slide on the "a" curves & circumcise away:



→ slide each end of the big curve:



so have formed ∂(bands) where the a-curves were!

Finally to prove (c) for an arbitrary W^4 with $\partial = M_K$, π_1 use the theorem:

Thm (Kreck) If W_1, W_2 are 4-mflds with the same bdry

$$\begin{array}{ccc} \partial W_1 \subseteq W_1 & \xrightarrow{u_1} & \\ \wr\wr & & K(\pi,1) \\ \partial W_2 \subseteq W_2 & \xrightarrow{u_2} & \end{array}$$

u_i are isos on π_1 commuting

then $W_1 \# \pm \mathbb{CP}^2$'s $\cong W_2 \# \pm \mathbb{CP}^2$'s $\Leftrightarrow u_*[W_1 \cup_\partial W_2] = 0$ in $H_4(\pi)$

= Stable diffeomorphism classification of 4-mfds (with ∂).

In our case, $H_4(\mathbb{Z}) = 0$ so all other constructions W_0^4 are stably diffeo to our constructed one.
Then, H_2 is stably equivalent → in fact actually free;
det & σ not affected by $\#\overline{\mathbb{CP}}^2$ which preserves original

Rk: above thm stated with $S^2 \tilde{\times} S^2$ is precisely "$\ell\sigma = 0$" on RH.
But for $\# S^2 \times S^2$ need also $w_2(W_1) = w_2(W_2)$ in $H^2(\pi; \mathbb{Z}_2) \cup \{\infty\}$
(if universal cover of $\tilde{W}_i$ is spin, can pull back w_2 to $H^*\pi$
but otherwise "∞").

$$0 \to H^2(\pi; \mathbb{Z}_2) \to H^2(W; \mathbb{Z}_2) \to H^2(\tilde{W}; \mathbb{Z}_2) \to \cdots$$
$$\omega? \qquad w_2(W) \longrightarrow w_2(\tilde{W})$$

& then some spin bordism stuff.

L^2 – homology

Atiyah's original defn of L^2 Betti numbers.
Suppose M is a (compact) Riemannian manifold,
and $\bar{M} \xrightarrow{\Gamma} M$ a regular cover, assume infinite group
so $\bar{M}$ is non-compact.
Idea: measure the space of smooth L^2-integrable p-forms
on $\bar{M}$ (do any exist?)
It will be 0 or inf-dim, so want a measurement
of dimension which is 0 $\Leftrightarrow$ no such groups.
Thus, the analytic L^2- Betti number is

$$b_p^{(2)}(M) = \lim_{t\to\infty} \int_{\text{a fund domain}} tr_{\mathbb{C}} \left(e^{-t\Delta_p}(x,x) \right) dx$$

Δ_p = Laplacian on p-forms $\Omega^p(\bar{M})^{\Gamma}$ (lift the metric)
$e^{-t\Delta_p}(x,x)$ = heat kernel
$tr_{\mathbb{C}}$ means usual matrix trace of the section of $End(\Lambda^p T)$
– i.e. fibrewise trace.
NB the group Γ ~~acts~~ of the covering is brought in by
choice of fundamental domain, i.e. it isn't just
an invariant of $\bar{M}$.

(Idea : 0 for circle & tori. Not zero for surfaces & $\mathbb{H}^2$ cover

Alternative approach
If $\tilde{X} \to X$ is the universal ~~regular~~ cover, X a finite CW complex
(assume for convenience). If M is a $\pi_1 X$ - module
then can define $H_p(X; M) = H_p(C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1 X]} M)$
For example, $\pi_1 X \twoheadrightarrow \Gamma$ lets us take $H_p(X; \mathbb{Z}[\Gamma])$
$= H_p(C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi_1 X} \mathbb{Z}\Gamma) = H_p(\bar{X})$ $\quad \bar{X}$ the cover corresponding
to Γ. $H_p(\bar{X})$ is a $\mathbb{Z}\Gamma$- module, but algebraically
it is not at all understood. (might not even be
finitely-generated ! non-Noetherian ring means submodules of f.g
things not necessarily f-g ...

Idea embed $\mathbb{Z}[\Gamma] \hookrightarrow \mathcal{N}\Gamma$ von Neumann algebra of Γ; a ring with much simpler representation theory

Then

def $H_p^{(2)}(\bar{X}) = H_p(X; \mathcal{N}\Gamma)$ is L^2-homology

$b_p^{(2)}(\bar{X}) = \dim_{\mathcal{N}\Gamma} H_p(X; \mathcal{N}\Gamma) \in [0, \infty)$

["Handbook of geometry" Wolfgang Lück]

Examples

(a) $|\Gamma| < \infty \Rightarrow \mathcal{N}\Gamma = \mathbb{C}[\Gamma]$ is semisimple, algebra is ok

(b) $\Gamma = \mathbb{Z}$ then $\mathcal{N}\Gamma = L^\infty(S^1; \mathbb{C})$

$\mathbb{C}[\Gamma]$ sits inside it as Laurent polynomials.

Def Take $\ell^2(\Gamma)$ = square-summable series $\{ \Sigma a_g g : a_g \in \mathbb{C}, \Sigma |a_g|^2 < \infty \}$

No longer an algebra, but is a Hilbert space on which Γ acts on the left & right; a $\mathbb{C}[\Gamma]$-bimodule.

Then $\mathcal{N}\Gamma = \mathcal{B}(\ell^2\Gamma)^{\Gamma_{\text{right}}}$ = bounded operators on $\ell^2\Gamma$ commuting with the right-mult.

$\mathbb{C}\Gamma \hookrightarrow \mathcal{N}\Gamma$ because it obviously does.

NB by von Neumann's double commutant theorem, $\mathcal{N}\Gamma$ is the pointwise closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(\ell^2\Gamma)$.

($T_n v$ converges $\forall v \Leftrightarrow T_n$ converges

(weak or strong ~~topology~~ but not operator norm.)

Trace $\operatorname{tr}: \mathcal{N}\Gamma \to \mathbb{C}$ $\qquad e \in \ell^2\Gamma$ is the "identity" element.

$a \mapsto \langle a(e), e \rangle$

so that for $a \in \mathbb{C}\Gamma$, $\operatorname{tr}(a) = a_e$ = coeff of e.

(not usual augmentation)

It satisfies $\operatorname{tr}(ab) = \operatorname{tr}(ba)$

proof: Enough to show true for $a \in \mathcal{N}\Gamma$, $b \in \Gamma$ because it's linear & cts. But obviously

$\langle ag(e), e \rangle = \langle (ae)g, e \rangle = \langle ae, eg^{-1} \rangle = \langle ae, g^{-1}e \rangle = \langle g(ae),$

using bimodule property & that L, R g-actions are isometries.

tr_Γ is faithful, i.e. $tr(a^*a)=0 \Leftrightarrow a=0$.

This is because $\langle a^*a.e, e\rangle = \langle ae, ae\rangle = 0$

$\Rightarrow ae=0$, but then $ag=0 \ \forall g\in\Gamma$ by rt mult

& then by continuity, $a.\overline{\mathbb{C}\Gamma}=0$ so $a=0$.

Def Let P be a f-g projective $\mathcal{N}\Gamma$-module, i.e. $P=\mathrm{im}(p)$ for some $p\in M_n(\mathcal{N}\Gamma)$ with $p^2=p$ (orth: $p=p^*$)

Then, $\dim_{\mathcal{N}\Gamma} P = tr_\Gamma(p)$. (= matrix trace of '$tr_\Gamma$'s, really)

$\in\mathbb{R}$ because $p=p^*$.

(and $tr(a^*)=\overline{tr(a)}$)

In fact a positive real number, $tr(p^*p)=\langle pe,pe\rangle>0$

[Rk: can define L^2-Betti no's $\otimes\ \ell^2\Gamma$ but need a complex where one uses 'closure' of im(differential) to factor out by. This is being handled by $\mathcal{N}\Gamma$]

Thm (Farber, Lück) (?)
$\mathcal{N}\Gamma$ is a semi-hereditary ring: any f-g submodule of a projective module is projective.

Cor Given $\overline{X}\xrightarrow{\Gamma}X$, there are f g proj modules P_i, Q_i s.t

$$0\to P_i\to Q_i\to H_i^{(2)}(\overline{X})\to 0$$

$\mathcal{N}\Gamma$-module

(" homological dimension 1" (finitely-presented))

Thus can define (for $H_i^{(2)}(X)$, which isn't nec. projective)

$$b_i^{(2)}(\overline{X}) = \dim_{\mathcal{N}\Gamma} Q_i - \dim_{\mathcal{N}\Gamma} P_i$$

Rk Actually, can extend the dimension to all modules over $\mathcal{N}\Gamma$: Wolfgang Lück.

Rk about $\Gamma = \mathbb{Z}$: $\ell^2(\mathbb{Z}) \cong$ (Fourier expansion) $L^2(S^1;\mathbb{C})$

but taking closure in bounded operators gets to L^∞ = all bounded functions. Trace becomes $\int_{S^1}$

because in $\langle f.e, e\rangle$, 'e' is the constant fn. '1'. and inner product is integral.

—.

Given $h^* = h \in M_n(N\Gamma)$, want to define the L^2-signature. $\sigma^{(2)}_\Gamma(h) \in \mathbb{R}$. [use both as an endo & as a]

Theorem (functional calculus).

There are f.g. projective $N\Gamma$-modules P_+, P_-, P_0 s.t after a base change, $h = \begin{pmatrix} 0 & & \\ & +1 & \\ & & -1 \end{pmatrix}$

($h \mapsto aha^*$; think of as a hermitian form here)

Then $\sigma^{(2)}_\Gamma(h) = \dim_{N\Gamma} P_+ - \dim_{N\Gamma} P_-$; $|\sigma| \leq n$.

(Rk $\dim_{N\Gamma}(N\Gamma) = 1$; for $p = p^2 = p^*$ acting on one $N\Gamma$, $0 \leq tr(p) \leq 1$. Just use $(1-p)^* = (1-p)$ as well, so $tr\, p \geq 0$, $tr(1-p) \geq 0$.

Proof (that $\exists$ such a decomposition)

Think of h as a hermitian operator in $B((\ell^2\Gamma)^n)$

Then spec(h) is a compact subset of $\mathbb{R}$.

(Bounded $\Rightarrow$ cpt ; hermitian $\Rightarrow$ real)

(spec = all λ s.t. $h - \lambda . id$ not invertible)

Then given any bounded function f: Spec$(h) \to \mathbb{R}$, there is a herm. operator $f(h)$ with "spec$(f(h)) = f($Spec

The proof of this is that for polynomial f, $f(h)$ is defined uniquely. In general, f = ptwise limit of polynomials f_i: define $f(h)$ = ptwise limit of f_i(

by defn lives in $M_n(N\Gamma)$

Rk $L^\infty(\text{spec } h, \mathbb{R}) \longrightarrow M_n(N\Gamma)$

$f \mapsto f(h)$

is a C^*-algebra map.

Now define f_+, f_-, f_0 by 'Heaviside' types,

$1 = f_+ + f_- + f_0$ & the function δ_0.

all projections. $\rightarrow$ ~~$h = f_+(h) + f_-(h) + f_0(h)$ orthog projns.~~

($\underline{\text{NB}}$ for C^* algs need continuous functions f.)

Recall $H_i^{(2)}(\bar{X}) = H_i(X; N\Gamma) = H_i(\bar{X}) \otimes_{\mathbb{Z}\Gamma} N\Gamma)$ $\bar{X} \xrightarrow{\Gamma\text{-cover}} X$

where $\mathbb{Z}\Gamma \subseteq \mathbb{C}\Gamma \subseteq N\Gamma = B(\ell^2\Gamma)^{\Gamma \text{right}}$

a completion of $\mathbb{C}\Gamma$ under pointwise convergence of operators.

Can actually think of a chain complex

$C_i(\bar{X}) \otimes_{\mathbb{Z}\Gamma} N\Gamma = (N\Gamma)^{\# i\text{-cells}}$

with boundary maps being matrices $d_i \in M_{m,n}(N\Gamma)$

$\underline{\text{Properties}}$ (how to compute $H_*^{(2)}$ easily)

Lück's survey

① Homotopy invariance:

if
$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\simeq]{f} & Y \end{array}$$
is a hty equivalence then $H_i^{(2)}(X) \cong H_i^{(2)}(Y)$

NB. (This would be surprising using the analytic defn of L^2 Betti numbers.)

Recall that $H_i(\bar{X})$ is a not quite projective module:

it can be given by $0 \to P_i \to Q_i \to H_i^{(2)}(\bar{X}) \to 0$

where $H_i^{(2)}(\bar{X})$ = f.g. proj $\oplus$ torsion part

measured by dimension $\in [0,\infty)$ — measured by Novikov-Shubin invariants.

② Euler-Poincaré formula: $\chi(X) = \sum (-1)^i b_i(X) = \sum (-1)^i b_i^{(2)}(\bar{X})$

(for a finite ~~cw~~ complex)

③ Multiplicativity under finite covers (!!)

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\Gamma_0} & X' \\ {\scriptstyle\Gamma}\downarrow & \swarrow {\scriptstyle n} & \\ X & & \end{array}$$

Γ_0 index n in Γ. $\Rightarrow b_i^{(2)}(X')_{\Gamma_0} = n \cdot b_i^{(2)}(X)_\Gamma$

(ie. $X = \bar{X}/\Gamma$ $X' = \bar{X}/\Gamma_0$ $\Gamma_0 \leq \Gamma$ index n

so $X' \xrightarrow{n} X$

& Proof of ③ is that $\Sigma(-1)^i b_i = \Sigma(-1)^i c_i = \Sigma(-1)^i \dim_{N\Gamma}(C_i \otimes_{\mathbb{Z}\Gamma} N\Gamma)$
$= \Sigma(-1)^i b_i^{(2)}(\bar{X})$

just comes from additivity of $\dim_{\mathbb{Q}}$ under s. exact sequence & similarly for the von Neumann $\dim_{N\Gamma}$.

Proof of ⑤: Algebraic statement necessary is that $\Gamma_0 \leq \Gamma$ induces $N\Gamma_0 \to N\Gamma$; if M is an $N\Gamma$-module then can restrict to $N\Gamma_0$ with dimension given by: $\dim_{N\Gamma_0}(\mathrm{res}_{N\Gamma_0} M) = n \cdot \dim_{N\Gamma} M$.

Atiyah conjecture
Γ torsion-free $\Rightarrow b_i^{(2)}(X)_\Gamma \in \mathbb{N}$
- known for elementary amenable groups, free groups, class is closed under certain operations. (not hyperbolic yet

④ Poincaré duality: if X is a closed oriented manifold of dim n, then $b_i^{(2)}(X)_\Gamma = b_{n-i}^{(2)}(X)_\Gamma$
Pf is that the usual chain equivalence between homology & cohomology complexes, tensored with $\otimes N\Gamma$, remains a ch.e. ($\exists$ subtlety: need L^2-cohomology but it is very close to homology)

⑧ Künneth formula: $b_n^{(2)}(X_1 \times X_2)_{\Gamma_1 \times \Gamma_2} = \sum_{p+q=n} b_p^{(2)}(X_1)_{\Gamma_1} b_q^{(2)}(X_2)_{\Gamma_2}$

⑥ If X connected, then $b_0^{(2)}(X)_\Gamma = \begin{cases} 1/|\Gamma| & |\Gamma| \text{ finite} \\ 0 & \text{otherwise} \end{cases}$

Look at $C_1(\bar{X}) \to C_0(\bar{X}) \to H_0(\bar{X}) \to 0$
$\quad\;\;\; (\mathbb{Z}\Gamma)^n \to \mathbb{Z}\Gamma$
(use hty invariance)

e_i

1-cells have $e_i \mapsto (g_i - 1)$ $\quad [e_i] = g_i \in \pi_1 X$

Then obvious for finite group case; $H_0(\bar{X}) = \mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module which has dimension $1/|\Gamma|$.

Infinite case: get $(N\Gamma)^n \xrightarrow{d} N\Gamma \to H_0(\bar{X})$
claim: dim image = 1

NB also a limiting property for residually-finite groups: can express $b_i^{(2)}$ as weighted limit of usual ~~...~~ Betti numbers.

Ex • S^1 with group $\mathbb{Z}$: $b_0^{(2)} = b_1^{(2)} = 0$ from connectedness, infinite group and χ formula. (or duality) (or self-covering

• This does the torus too

• For a surface Σ_g, $b_0 = b_2 = 0$, $b_1 = 2g-2$
(& universal cover)

• For free group F_n ($= \vee S^1$) have $b_0 = 0$, $b_1 =$ ~~...~~ $n-1$
(thinking of $b_i^{(2)}$(group) via $K(\pi,1)$).

• Knot complement: ? need to find out! guess all zero.
L^2-torsion = A. Polynomial, I think.

Also for 3-mfld guess all Betti #s = 0 & then the
(!) L^2-torsion is hyp volume. — use Mayer-Vietoris
or Gromov norm & Heegaard splitting.

~~Defn~~ Let A be a $\mathbb{C}$-algebra with involution $*$.
Pick $a \in A$, have $\mathrm{Spec}(a)$. (complete normed with $\|aa^*\| = \|a\|^2$)

A is a C^*-algebra $\Leftrightarrow$ $\mathrm{Spec}(a)$ is compact
check! and there is a $*$-ring homomorphism
$C^0(\mathrm{Spec}(a);\mathbb{C}) \to A$
taking $\mathrm{id}_{\mathrm{Spec}(a)} \mapsto a$
const $1 \mapsto 1$

[funcs to S^1 $\mapsto$ unit... ; poly $\mapsto$ poly(... ; real-val $\mapsto$ herm...]

Basically, 'a' generates an abelian subalgebra which means it equals $C^0(\mathrm{Spec}(a))$ by Gelfand-Naimark or whatever.

A is a von Neumann algebra $\Leftrightarrow$? same as above with bounded $L^\infty(\mathrm{Spec}(a);\mathbb{C}) \to A$
$\Downarrow$
(A is a C^* alg too) - - - - ($C^0(\mathrm{Spec}(a)) \subseteq L^\infty$ because $\mathrm{Spec}(a)$ is compact.)

This is (I think) an intrinsic characterisation of a VN algebra — no Hilbert space in sight!

the Hilbert space on which $\mathcal{N}\Gamma$ acts

Thm If $h^* = h \in M_n(\mathcal{N}\Gamma) = A$ then $\exists$ a decomp of the Hilbert space
$\mathcal{H} = (\ell^2\Gamma)^n = P_0 \oplus P_+ \oplus P_-$ orthogonal
s.t. $h = \begin{pmatrix} 0 & & \\ & +1 & \\ & & -1 \end{pmatrix}$

Pf: use the functions f_+ etc to make
$f_\pm(h)$, $f_0(h)$. ($\mathrm{Spec}(h) \subseteq \mathbb{R}$ as hermitian)
$P_+ = \overline{\mathrm{im}\, f_+(h)}$ ... ; the projections all commute
(this decomp came from commutative algebra of fns)
~~...~~ with $\underset{(\text{not } h)}{\mathrm{id}_A} = f_+(h) + f_-(h) + f_0(h)$.

Rk Positive operator $\overset{\text{def}}{=}$ herm. operator with Spec ≥ 0.
Such a thing is always of the form b^*b for some b,
just take the square root! (using functional calc) (b will be hermitian)

So $a^*ha = \begin{pmatrix} 0 & & \\ & +1 & \\ & & -1 \end{pmatrix}$. $a = b^*. 1. b$.

Correction Functional calculus works for normal operators
$aa^* = a^*a$.
Then the correct statement of it is:
$\forall$ given a, $\exists$ a measure μ_a on $\mathrm{Spec}(a) \subseteq \mathbb{C}$
with a $*$-hom $L^\infty(\mathrm{Spec}(a), \mu_a) \longrightarrow A$.

eg $\mu_a(\{\lambda\}) = \dim_A(E_\lambda)$ if $\exists\, A \xrightarrow{tr} \mathbb{R}$ (type II)
λ-eigenspace

Example $\Gamma = \mathbb{Z}$ $C^0(S^1) = C^*(\mathbb{Z})$ (Q: if $E_\lambda \neq 0$, does $\dim_A \neq$
probably yes for II, certainly for $\mathcal{N}\Gamma$)

$$\begin{array}{ccc}
\ell^1\mathbb{Z} & & C^0(S^1) \\
\cap & & \cap \\
 & & L^\infty S^1 = \mathcal{N}\mathbb{Z} \\
 & \xleftarrow{\text{Fourier}} & \cap \\
\ell^2\mathbb{Z} & \cong & L^2 S^1 \\
\cap & & \cap \\
\ell^\infty\mathbb{Z} & & L^1 S^1
\end{array}$$

norm-closure of polynomials

Pointwise closure of poly fns (by multiplication)

The FT doesn't actually take L^∞ to ℓ^1 or anything so easy.

(alg) $(p=p^*=p^2)$

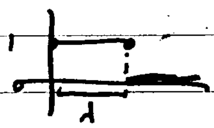
Note the projections in $\mathcal{N}\mathbb{Z}$ correspond 1-1 with measurable sets in S^1.

so $p=p^*$ $\Rightarrow$ $p\in L^\infty(S^1,\mathbb{R})$

$p^2=p$ $\Rightarrow$ p has values 0 or 1 almost everywhere.

so view as a characteristic function

Consequence Any positive real number occurs as a dimension of a proj module over $\mathcal{N}\Gamma$: just use the function p

$p=$  to define $p\cdot\mathcal{N}\Gamma$ with $\dim_{\mathcal{N}\Gamma} = p$. $(0\leq p$

Now add copies of $\mathcal{N}\Gamma$ to make larger numb

Ex $K_0(\mathcal{N}\mathbb{Z}) \cong L^\infty S^1$ $(=\mathcal{N}\mathbb{Z})$

$\exists$ tr : $M_n(\mathcal{N}\mathbb{Z}) \longrightarrow \mathcal{N}\mathbb{Z}$ $\left(\xrightarrow{\int} \mathbb{R}\right.$ our standard trace but don't need to do this

(a 'universal' trace on an abelian algebra)

(universal signature)

For $h=h^*\in M_n(\mathcal{N}\mathbb{Z})$ can define $\sigma^{(2)}_{univ}(h)\in\mathcal{N}\mathbb{Z}$ by diagonalising h as before and looking at the difference between P_+, P_- in $K_0(\mathcal{N}\mathbb{Z})=\mathcal{N}\mathbb{Z}$

Lemma Given $h^*=h$ in $M_n(\mathcal{N}\mathbb{Z})$ then for almost all $z\in S^1$ we have

$$\sigma^{(2)}_{univ}(h)(z) = \underbrace{\sigma_{ordinary}(h(z))}_{\equiv\,\sigma_1} \in \mathbb{R}$$

(need to choose a "representative" of the fn in L^∞ to make this make sense, really!)

(i.e really a twisted signature

$\mathrm{sig}\, h(z) = (z-1)S + (\bar z-1)S^T$

Remark: If $h\in M_n(\mathbb{C}[z])$ then $\sigma_1(h(z))$ is a step function with jumps at zeros of $\det h$ and values $\in\mathbb{Z}$ (it's an ordinary signature). Knowing this a.e. determines it completely if we average the values at the jumps.

Proof: "Functional calculus commutes with substitution".

Then $\sigma^{(2)}_{univ}(h)(z) = tr\,(p_+(h)(z) - p_-(h)(z))$

Now $p_+(h) = \lim_i p_i(h)$ where p_i are matrices of polynomials

(concrete expression of functional calculus: pointwise limit.)

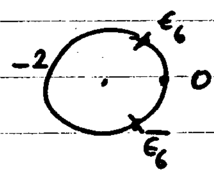
Now substituting the z is right: $p_+(h)(z) \underset{a.e.}{=} \lim_i (p_i(h)(z))$

Now # pos eigenvalues of $h(z) = tr\,(p_+(h(z)))$ just doing functional calculus ~~off~~ on $M_n\mathbb{C}$! $= tr\,(p_+(h)(z))$.

(Rk: Spec $h(z)$ is discrete subset: just the eigenvalues with multiplicities

Coroll $\sigma^{(2)}_{\mathbb{Z}}(h) = \int_{S^1} \sigma^{(2)}_{univ}(h) = \int_{S^1} \sigma_1(h(z))\, dz \in$

(this only works because $\mathbb{Z}$ is abelian: otherwise no universal trace

Ex $\sigma_{\mathbb{Z}}^{(2)}(\text{trefoil}) = \int_{S^1}$  $= -\frac{4}{3} \neq 0$

($\det h(z) = (z-\epsilon_6)(z-\bar{\epsilon}_6) = z^2 - z + 1$ well-def up to powers of z.
alt: $(z+z^{-1}) - 1$

Ex can work out for all the twist knots K_n.
or for torus knots.

Def Let X^{4k} be a compact oriented mfd, $\rho: \pi_1 X \to \Gamma$ a representation. Then define
$$\sigma_\Gamma^{(2)}(X,\rho) = \sigma_\Gamma^{(2)}(\lambda_X^{(2)}) \in \mathbb{R}$$
where $\lambda_X^{(2)}$ is a hermitian form ~~on the~~ of X defined as follows via an intersection form:

Rk to define σ for herm h_P form on a proj module P, not free
use $P \oplus Q = (N\Gamma)^n$
& take $\sigma(h_P \oplus 0_Q)$ as the def

$$\lambda_X^{(2)} : H_{2k}(X;\mathcal{N}\Gamma) \to H_{2k}(X,\partial X;\mathcal{N}\Gamma) \overset{PD}{\cong} H^{2k}(X;\mathcal{N}\Gamma) \to \mathrm{Hom}_{\mathcal{N}\Gamma}(H_{2k}(X;\mathcal{N}\Gamma),\mathcal{N}\Gamma$$

But H_{2k} might not be projective $\mathcal{N}\Gamma$-module!

(this map is not nec. iso — ∃ a spectral sequence instead of UCT. But its defn is the usual one.)

Lück. Given M a f-gen module over $\mathcal{N}\Gamma$, define the torsion of M $T(M) = \{m \in M : f(m) = 0 \ \forall$ linear $f : M \to \mathcal{N}\Gamma$

(the usual defn fails for non-comm rings.

Now $\lambda_X^{(2)}$ is viewed as a pairing on $\dfrac{H_{2k}(X;\mathcal{N}\Gamma)}{T(-)}$

Thm (Lück) $M/\text{torsion}$ is projective. (like f.g. ab gps!)

(ie any M is of the form $M/T(M) \oplus T(M)$.)

L^2-signature thm (Atiyah)

If Y^{4k} is closed oriented mfd then for any rep ρ, $\sigma_\Gamma^{(2)}(Y,\rho) = \sigma_1(Y) \quad \forall \rho$.

(Actually this is true for any elliptic operator!)

(eg lift a twisted Dirac operator to the Γ-cover, compute the index via $\dim_{\mathcal{N}\Gamma}$.)

(Actually the above σ thm is true when Y is a Poincaré cplx

Lemma (additivity) If $\pi_1(X_i^{4k}) \xrightarrow{\rho_i} \Gamma$ are given s.t. $\partial X_1 \underset{\phi}{\cong} \partial X_2$, $\pi_1 \partial X_1 \overset{\phi_*}{\cong} \pi_1 \partial X_2$, $\rho_1 \searrow \Gamma \swarrow \rho_2$

then $\sigma_\Gamma^{(2)}(X_1,\rho_1) + \sigma_\Gamma^{(2)}(X_2,\rho_2) = \sigma_\Gamma^{(2)}(X_1 \cup_\phi X_2, \rho_1 \cup \rho_2)$.

Proof exactly the same as usual — decompose the module as proj ⊕ torsion, ignore the torsion, blah blah...

Cor

The reduced L^2-signature $\tilde{\sigma}^{(2)}_\Gamma(X,\rho) = \sigma^{(2)}_\Gamma(X,\rho) - \sigma_1(X)$ is an invariant of the boundary $(\partial X, \rho|_{\partial X})$

Rk Given (M^{4k-1}, g, ρ), ~~Riem metric~~ can ~~also~~ define $\left\{ \begin{array}{l} g = \text{metric} \\ \rho = \text{rep} \end{array} \right.$

$$\tilde{\eta}^{(2)}_\Gamma(M, g, \rho) = \eta^{(2)}_\Gamma(M, g, \rho) - \eta(M, g)$$

which is independent of the metric g ! ⊛ (~~η-invts not signature then~~)

(i.e. $= \int L\text{-form} - \sigma(\ )$)

(Thm of Gromov & Cheeger)

By Atiyah - Patodi - Singer, $\tilde{\eta}_\Gamma(M,\rho) = \tilde{\sigma}^{(2)}_\Gamma(X,\rho)$ if $M = \partial X$.

Thus reduced L^2-signature is defined when $\nexists$ a null-bordism.

⊛ The η-invt is got by eigenvalues of σ operator - making a ζ-function.

Def Given a knot $K \subseteq S^3$, and a rep $\rho : \pi_1(S^3_K) \to \Gamma$, (0-surgery)
define $\tilde{\sigma}^{(2)}_\Gamma(K,\rho) = \tilde{\eta}^{(2)}_\Gamma(S^3_K, \rho)$

Q For what Γ & ρ is this a concordance invariant?

Ex $\Gamma = \mathbb{Z}$, ρ = abelianisation $\pi_1 \to H_1$, get infinite cyclic cover. Then

$$\tilde{\sigma}^{(2)}_{\mathbb{Z}}(K,\rho) = \sigma^{(2)}_{\mathbb{Z}}(W_K, \rho)$$

the nice 4-mfd with $\pi_1 = \mathbb{Z}$ bounded by S^3_K.

and this is $\tilde{\sigma}^{(2)}_{\mathbb{Z}}(\lambda^{(2)}_W)$
which is $\int_{S^1} \sigma_z(K)\, dz$

Casson–Gordon invariants, revisited

Recall the reduced L^2-signature $\sigma_\Gamma^{(2)}(K,\rho) \in \mathbb{R}$

given $\rho : \pi_1(S^3_K) \to \Gamma$

eg $\Gamma = \mathbb{Z}$, $\rho =$ abelianisation, have $\tilde{\sigma}^{(2)}_{\mathbb{Z}}(K,\rho) = \int_{S^1} \sigma_z(k)\, dz$

really comes from 1-dim reps of $\mathbb{Z}$

What we really want are choices of Γ, ρ such that these are concordance invariants.

Define $\Gamma = \underbrace{\frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]}}_{= A} \rtimes \mathbb{Z}$ where $\mathbb{Z} = \langle m \rangle$

conjugation acting by t (mult!)

A metabelian group ; A is actually inf-dim'l $\mathbb{Q}$-vector space.

Analogous to $\mathbb{Q} = ff(\mathbb{Z})$, $\mathbb{Q}/\mathbb{Z}$

There are no finite-dim reps of this gp except those we factoring through $\mathbb{Z}$, or finite gp. So have to use $\ell^2\Gamma$ to make an interesting representation.

Now define a rep variety $Rep^*(\pi, \Gamma) = \{\rho : \pi \to \Gamma : \rho(\text{meridian}) = m\}$

Note that they all factor through $\pi/\pi^{(2)}$ ← second comm gp because $\Gamma^{(2)}$ is trivial.

(in fact, $\Gamma^{(1)} = [\Gamma,\Gamma] = A$.)

Further, we know $\pi/\pi^{(2)} = \frac{\pi^{(1)}}{\pi^{(2)}} \rtimes \pi/\pi^{(1)}$ "meridian".

(i.e the map $0 \to \frac{\pi^{(1)}}{\pi^{(2)}} \to \frac{\pi}{\pi^{(2)}} \to \pi/\pi^{(1)} = \mathbb{Z} \to 0$ always splits)

So, $\rho \in Rep^*$ is given by $\bar{\rho} \rtimes id : \frac{\pi^{(1)}}{\pi^{(2)}} \rtimes \mathbb{Z} \to A \rtimes \mathbb{Z}$.

where $\bar{\rho}$ must be a hom of ab gps $\frac{\pi^{(1)}}{\pi^{(2)}} \to A$ commuting with the action of m.

Def $A_K = \frac{\pi^{(1)}}{\pi^{(2)}} \otimes \mathbb{Q}$ ~ the rational Alexander module of K.

Thus, $Rep^*(\pi,\Gamma) = \mathrm{Hom}_{\mathbb{Q}[m^{\pm 1}]}(A_K, A) =$ a finite-dim'l $\mathbb{Q}$-vector space.

(Rk: can work with integers here, but it's easier not to!)

The f-dimensionality comes because A_K is a f-gen torsion module over $\mathbb{Q}[t^{\pm 1}]$ ((*) will prove shortly)

Thm [COT] / possibly "nnp" but doesn't actually matter whether connected co

If K is slice then there is a subvector space $R \subseteq Rep^*(\pi,$ of half dimension s.t. $\sigma^{(2)}_{\Gamma}(K,\rho) = 0 \quad \forall \rho \in R$.

In fact, R comes from a Lagrangian of the Blanchfield form

(*) $M = S^3_K$ o-surgery, $\bar{M}$ its inf. cyclic cover. $\pi = \pi_1 M$

Then $\pi_1(\bar{M}) = \pi^{(1)}$; $H_1(\bar{M}) = \pi^{(1)}/\pi^{(2)} \Rightarrow A_K$ which is by definition $\pi^{(1)}/\pi^{(2)} \otimes \mathbb{Q}$ is just $H_1(\bar{M};\mathbb{Q})$.

To prove it's f-g torsion over $\mathbb{Q}[t^{\pm 1}]$: use Gysin sequence with $\mathbb{Q}$ coefficients (as always from now on, probably!)

$$H_3\bar{M} \to H_3 M \xrightarrow{?} H_2(\bar{M}) \xrightarrow{1-t} H_2(\bar{M}) \to H_2 M \xrightarrow{?} H_1\bar{M} \xrightarrow{1-t} H_1\bar{M} \to H_1 M \to H_0\bar{M} \xrightarrow{1-t} H_0\bar{M}$$

chains are f-g ; $\mathbb{Q}Q^{\pm 1}$ Noetherian $\Rightarrow$ f-g. $\quad \mathbb{Q} \xrightarrow{0} \mathbb{Q}$

(Rk: think of the fibration $\bar{X} \to X \to S^1$ & take its Gysin sequence; the point is, $\bar{X} \to X$ defined by $X \to S^1$, and $\bar{X}$ maps to the homotopy fibre as a htpy equ ~~Then~~

So actually just used

$$H_1\bar{M} \xrightarrow{1-t} H_1\bar{M} \to H_1 M \to H_0\bar{M} \xrightarrow{1-t} H_0\bar{M}$$
$$\mathbb{Q} \cong \mathbb{Q} \xrightarrow{0} \mathbb{Q}$$

so $H_1\bar{M} \xrightarrow{1-t} H_1\bar{M} \to 0$

$1-t$ can't be onto if there's a free part over $\mathbb{Q}[t^{\pm 1}]$ (it's not invertible) $\therefore$ $H_1\bar{M}$ is torsion.

And it's f-g because chains are f-g free, $\mathbb{Q}[t^{\pm 1}]$ is PI $\therefore$ Noetherian, and so subquotient is f-g.

Rk Same argument applies to any finite CW cplx with $H_1(X) = \mathbb{Q}$;
$\Rightarrow H_1$ (inf cyclic cover; $\mathbb{Q}$) is a f-g torsion $\mathbb{Q}[t^{\pm 1}]$-module.

$m = t$!

Blanchfield form $\quad A_K = H_1(\bar{M};\mathbb{Q}) \cong H_1(M;\ \mathbb{Q}[t^{\pm1}])$ by defn

now apply Poincaré duality with $\quad \wr\wr$

twisted coefficients $\quad H^2(M;\mathbb{Q}[t^{\pm1}])$

From the coefficient sequence

$$0 \to \mathbb{Q}[t^{\pm1}] \to \mathbb{Q}(t) \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm1}]} \to 0$$

get the Bockstein

$$? \to H^1\left(M;\ \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm1}]}\right) \xrightarrow{\cong} H^2(M;\ \mathbb{Q}[t^{\pm1}]) \to ?$$

but $\mathbb{Q}(t)$ is flat over $\mathbb{Q}[t^{\pm1}]$ (it's quotient field)

so both ? groups are zero; they are $H^*(M;\mathbb{Q}(t))$

got by $H^*(M;\mathbb{Q}[t^{\pm1}]) \otimes \mathbb{Q}(t)$ (flatness)

torsion $\qquad = 0 ..$

Thus $\quad A_K = H_1(\bar{M};\mathbb{Q}) \cong H^2(--) \cong H^1(--) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}[t^{\pm1}]}(H^1(\bar{M}); \ldots$

$\qquad \wr\wr \; A_K^{\wedge}$

the last comes from spectral sequence UCT

for twisted coefficients.

$A_K^{\wedge}$ is Pontryagin dual (do hom into quotient field thing

eg $\mathbb{Z}_n^{\wedge} = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z})$

So $\quad Bl : A_K \times A_K \to A \quad$ is a hermitian unimodular form

$=$ linking form on $H_1(\bar{M};\mathbb{Q})$.

Geometrically: do the usual linking form construction,

multiplying elt of H_1 till zero, bounding it with a surface, intersecting

and then dividing by the chosen multiple.

(This shows hermitian, by the usual argument)

The theorem constructs Lagrangian $R \leq \mathrm{Rep}^*(\pi,\Gamma) = \mathrm{Hom}_{\mathbb{Q}[m^{\pm1}]}(A_K, A)$

$= A_K^{\wedge} \cong A_K$
Blanchfield.

Proof of the theorem (that $\exists$ R, a Lagrangian)

Let $W = B^4 -$ slice disc, so that $\partial W = S^3_K$ and $H_1 W \cong H_1 M$, $H_2 W = 0$. (homological slice in this sense will be enough to prove this; recall earlier notes about this; so a knot which is slice in a rational homology ball will also do this).

Look at

$$H_3(\tilde W,\tilde M) \to H_2(\tilde M) \to H_2(\tilde W) \hookrightarrow H_2(\tilde W,\tilde M) \to H_1(\tilde M) \to H_1(\tilde W) \twoheadrightarrow H_1(\tilde W,\tilde M) \xrightarrow{0} H_0(\tilde M) \xrightarrow{\cong} H_0(\tilde W)$$

$0 \cong 0$

by Gysin sequence before; uses $H_2 W = 0$ here

$$0 \to H_2\tilde W \to H_2(\tilde W,\tilde M) \to H_1(\tilde M) \xrightarrow{i_*} H_1(\tilde W) \to H_1(\tilde W,\tilde M) \to 0$$

torsion modules

$L = \ker i_*$; define $L = R$ as the image of this map

Claim ① $\dim_{\mathbb{Q}} L = \frac{1}{2}\dim_{\mathbb{Q}} H_1\tilde M$

② If $\rho: \pi \to \Gamma$ comes from L ($\sim R$) then it extends to $\tilde\rho: \pi_1 W \to \Gamma$.

The point here is that we can now calculate the $L^{(2)}$ signature of M,ρ via the 4-mfd with its extended ρ:

$$\tilde\sigma^{(2)}_\Gamma(K,\rho) = \sigma^{(2)}_\Gamma(W,\tilde\rho) - \underset{\substack{\parallel\\0}}{\sigma_1(W)}$$

Final step: can show $b_2^{(2)}(W,\tilde\rho) = 0 \implies \sigma^{(2)}_\Gamma(W,\tilde\rho) = 0$

To see ~~Lagrangian~~ dimension count:

$$\begin{array}{ccccccccccc}
0 \to & H_2\tilde W & \to & H_2(\tilde W,\tilde M) & \to & H_1\tilde M & \to & H_1\tilde W & \to & H_1(\tilde W,\tilde M) & \to 0\\
 & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \to & H_1(\tilde W,\tilde M)^\wedge & \to & H_1(\tilde W)^\wedge & \to & H_1(\tilde M)^\wedge & \to & H_2(\tilde W,\tilde M)^\wedge & \to & H_2(\tilde W)^\wedge & \to 0
\end{array}$$

$\downarrow$ Bl all isom

(NB For W, have $H_i(\tilde W) = H_i(W;\mathbb{Q}(t^{\pm1})) \overset{PD}{\cong} H^{4-i}(W,M;\mathbb{Q}(t^{\pm1}))$

& $\uparrow\cong$ Bockstein

again, no higher Ext gps, a bit more tricky. (univ coeff th.) $\mathrm{Hom}(H_{3-i}(W,M), \frac{\mathbb{Q}(t)}{\mathbb{Q}(t^{\pm1})}) \cong H^{3-i}(W,M;\frac{\mathbb{Q}(t)}{\mathbb{Q}(t^{\pm1})})$)

again the Bockstein's an isom because of the fact that all these are torsion modules, & have $H_*(-;\mathbb{Q}(t)) = 0$.

So have $H_i\tilde W \xrightarrow{\cong} H_{3-i}(\tilde W,\tilde M)^\wedge$ Blanchfield

So ① reduces that $\dim_{\mathbb{Q}} N^\wedge = \dim N$ (check for cyclic modules!)
see that the dims on top & bottom sequence agree, reversed. So same as the usual Lagrangian argument.

The extension part ② follows because $x \in L$ inside $H_1 \bar{M}$ maps to function $Bl(x, -)$ & comes from elt ~~...~~ $H_1 \hat{W}^\wedge$ ~~...~~
can use this to extend ρ:

$$\frac{\pi_1 W}{\pi_1 W^{(2)}} = H_1(\bar{W}) \rtimes \mathbb{Z} \longleftarrow H_1 \bar{M} \rtimes \mathbb{Z}$$

use $y \rtimes id$: $H_1(\bar{W}) \rtimes \mathbb{Z} \to \Gamma = A \rtimes \mathbb{Z}$, $\quad H_1\bar{M} \rtimes \mathbb{Z} \xrightarrow{\rho} \Gamma = A \rtimes \mathbb{Z}$

Lagrangian also because $Bl_M(x, x') = Bl_W(y, i_*(x')) = 0$

|| (NB) Basic idea: inf cyclic cov of M^3 behaves like a surface Σ^2; "slice W^4" cov to $\bar{W}$ behaves like N^3, $\partial N = \Sigma$.
i.e. TRUE for fibred knots; others behave homologically the same.

Lemma $\boxed{*}$

Let Γ be a poly-torsion-free-abelian group (PTFA)
(finite extensions by t.f. abelian gps, iterated) (solvable)
If $H_1(W^4; \mathbb{Q}) \cong H_1(\partial W; \mathbb{Q}) \cong \mathbb{Q}$ and $\rho: \pi_1 W \to \Gamma$
then $\sigma^{(2)}_\Gamma(W, \rho) = b_2 W$.

Rk $\pi = \pi_1(S^3 - K)$ a knot gp
then $\pi/\pi^{(n)}$ is PTFA.
Strebel showed t-free (clearly solvable)

~~Using the ZEDT.~~

Rk: also, if Γ is PTFA
then $\mathbb{Z}\Gamma$ has an Ore skew-field
of fractions $\mathcal{K}\Gamma$
~~- $\emptyset$ zero-divisors,~~ and all pairs as^{-1}
can be reversed c'd. $\mathbb{Z}\Gamma \subseteq \mathcal{K}\Gamma$ ...

An example sequence of PTFA gps, "universal" wrt maps of knot complements to solvable groups:

$$\Gamma_0 = \mathbb{Z} \longleftarrow \Gamma_1 = \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z} = \frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]} \rtimes \Gamma_0 \longleftarrow \Gamma_2 = \frac{\mathcal{K}\Gamma_1}{\mathbb{Q}[\Gamma_1]} \rtimes \Gamma_1 ?$$ NO not quite

$\langle \mu \rangle$ generator acts by mult by t.

torsion-free abelian

"secondary Blanchfield pairing"

(metabelian)

Used the fact that $\mathbb{Q}\Gamma_0$ was a PID.
Not true for $\mathbb{Q}\Gamma_1$ so have to alter it a bit.

$$\mathbb{Q}\Gamma_1 = \mathbb{Q}\left[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]}\right]_\alpha [m^{\pm 1}] \subseteq \mathcal{K}\left(\mathbb{Q}\left[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]}\right]\right)_\alpha [m^{\pm 1}]$$

"twisted poly ring" a PID! (non-comm.)

So Γ_2 is really given as $\frac{\mathcal{K}\Gamma_1}{\mathcal{K}[\Gamma_1^{(1)}][m^{\pm 1}]} \rtimes \Gamma_1$

$\mathcal{K}\left[\mathbb{Q}\left[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}\Gamma_0}\right]\right]$ as above. $\equiv$ commutator subgroup of Γ_1.

Don't want to do this really, ought to be able to get by without it..

$$\ldots \quad \Gamma_{n+1} = \frac{\mathcal{K}\Gamma_n}{\mathcal{K}[\Gamma_n^{(1)}][m^{\pm 1}]} \rtimes \Gamma_n$$

"universal rationally solvable groups" based on $\mathbb{Z}$
(corr to H_1 of knot complement; could try with links & $\mathbb{Z}^n$)
There are lots of maps of knot groups into these.

<u>Main thm</u> (COT)
If K is slice then it is (h-) solvable for all $h \in \frac{1}{2}\mathbb{N}$

Def: (0)- solvable $\Leftrightarrow$ $\mathrm{Arf}(K) = 0$
$\Rightarrow$ $\exists$ a well-def obstruction to being slice
$B_0 \in L^0(\mathcal{K}\Gamma_0) / L^0(\mathbb{Z}\Gamma_0)$

$\left(\xrightarrow{\sigma(z) = \int_{S^1}} \mathbb{R} \right.$

$\left. \xrightarrow[z \neq S^1]{\tilde{\sigma}_2} \mathbb{Z} \right)$

the usual obstruction for being algebraically slice; the Blanchfield form. This gives the usual twisted signatures etc.
B_0 is essentially form $(1-m)S + (1-m^{-1})S^T$, hermitian matrix over gp ring, invertible over the quotient field.
L-theory = non-deg symm forms over the field / lagrangians
quotienting by $L^0(\mathbb{Z}\Gamma_0)$ corresponds to lack of signature defn at $z = 0$; thus, reduced signature essentially.

$\frac{1}{2}$-solvable $\Leftrightarrow$ ~~$B_0=0$~~ K algebraically slice, $\Rightarrow B_0 = 0$

(~~Rk~~: the actual def of (h-) solvable is in terms of gropes. ~~solvable~~ (knot bound a $\overset{\text{symm}}{\text{grope}}$ of height $h+2$ in B^4. can define $\frac{1}{2}$ integer case ...)

1-solvable $\Rightarrow$ $\exists$ a Lagrangian L_0 inside B_0 (this is actually $\Leftarrow$)
s.t. $\forall\ l_0 \in L_0$, $\exists$ a well-def obstruction
$B_1(l_0) \in L^0(\mathbb{Q}\Gamma_1)/L^0(\mathbb{Z}\Gamma_1) \xrightarrow{\tilde\sigma_{\Gamma_1}^{(2)}} \mathbb{R}$

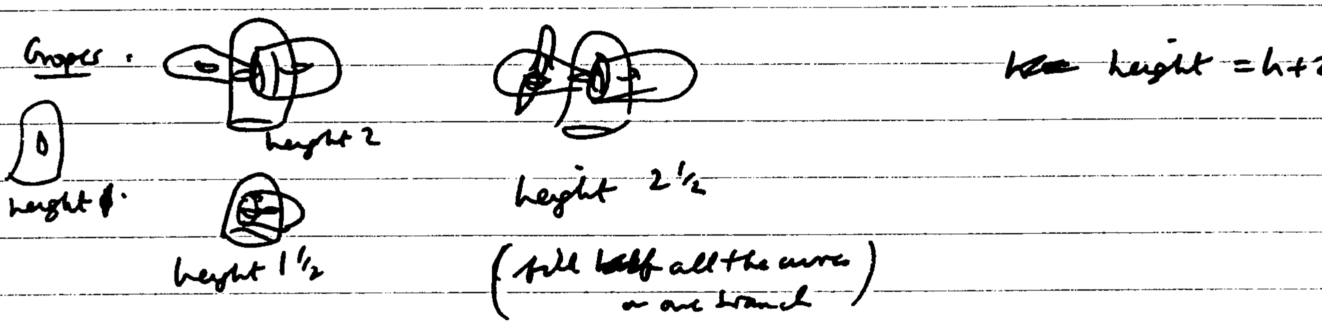
i.e. there is a rep extending to a 4-mfd which defines an intersection form. This doesn't depend on choice of extension.
There's a unique $\tilde\sigma^{(2)}$ function

$1\frac{1}{2}$-solvable $\Rightarrow$ all Casson-Gordon invariants vanish.
($\Leftarrow$ if we defined C-G "correctly")
$\Rightarrow$ $\exists$ $L_0 \in B_0$ s.t. $B_1(l_0) = 0$ $\forall\ l_0 \in L_0$
$\Rightarrow \exists\ L_1(l_0) \in B_1(l_0)$.

In general:
($n \in \mathbb{N}$) n-solvable $\Rightarrow$ $\exists\ L_0$ s.t. $\forall\ l_0 \in L_0$, $\exists\ L_1(l_0) \subseteq B_1(l_0)$
s.t. $\forall\ l_1 \in L_1$, $\exists\ L_2(l_0, l_1) \subseteq B_2(l_0, l_1)$ s.t. $\forall\ l_2 \in$
... and $\exists$ a well-def obstruction $B_n(l_0, \ldots l_{n-1})$
in $L^0(\mathbb{Q}\Gamma_n)/L^0(\mathbb{Z}\Gamma_n) \xrightarrow{\sigma_{\Gamma_n}^{(2)}} \mathbb{R}$ can measure the obstruction.

$(n+\frac{1}{2})$-solvable $\Rightarrow$ n-solvable, and $B_n = 0$.

Gropes:



Thm $\exists$ knots ~~K~~ which are 2-solvable but not $2\frac{1}{2}$-solvable
{ in fact infinitely-many } an inf-gen subgp. ($\therefore$ not slice)

Proof of lemma $\boxed{*}$

Will actually prove that $\dim_{\mathcal{K}\Gamma}(H_2(W;\mathcal{K}\Gamma)) = b_2(W)$ in homological alg sense, not L^2. Why?

Recall Γ is PTFA: can do $\mathbb{Q}\Gamma \subseteq \mathcal{N}\Gamma$ completion.

Any vN alg satisfies Ore condition, w.r.t. all non-zero-divisors, so has an embedding $\mathcal{N}\Gamma \subseteq \mathcal{U}\Gamma$; not a skew field, but any f-pres module is projective, & quite nice.

(eg if $\Gamma = \mathbb{Z}$; $\mathcal{N}\Gamma = L^\infty(S^1)$ (bounded ops) ; $\mathcal{U}\Gamma =$ all measurable functions! (= "unbounded" operators on ...) (eg $1/x$ for $x \neq 0$)

Then can do a ~~rational~~ division closure of $\mathbb{Q}\Gamma \subseteq \mathcal{N}\Gamma$

$$\begin{array}{ccc} \mathbb{Q}\Gamma & \subseteq & \mathcal{N}\Gamma \\ \cap & & \cap \\ \mathcal{K}\Gamma & \subseteq & \mathcal{U}\Gamma \end{array}$$

by closing in $\mathcal{U}\Gamma$. (This closure actually is the Ore localization if that exists?)

Now Ore localization is flat so that

① $\dim_\Gamma H_2(W;\mathcal{N}\Gamma) = \dim_\Gamma(H_2(W;\mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma)$

(can extend dim fn but not trace!) ie $\mathrm{tr}: \mathcal{N}\Gamma \to \mathbb{C}$ doesn't extend to $\mathcal{U}\Gamma$.

eg $\int_{L^\infty}$ doesn't extend to $\int_{\text{meas}}$

But: $\overset{\dim}{\mathrm{tr}_\Gamma}: K_0(\mathcal{N}\Gamma) \to \mathbb{R}$ does extend to $K_0(\mathcal{U}\Gamma) \to \mathbb{R}$ because any projection $p: (\mathcal{U}\Gamma)^n \twoheadrightarrow Q$ satisfies $p^2 = p = p^*$ which makes it bounded because of normal algebra $\|p\| <$ so OK.

② By flatness $\dim_\Gamma(H_2(W;\mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma) = \dim_\Gamma H_2(W;\mathcal{U}\Gamma)$

③ Also $H_2(W;\mathcal{K}\Gamma) \otimes_{\mathcal{K}\Gamma} \mathcal{U}\Gamma = H_2(W;\mathcal{U}\Gamma)$

(Rk = Proof of Atiyah conjecture for PTFA: ~~If Atiyah conj true for Γ then~~ $\mathcal{K}\Gamma =$ Ore localization, dimension is an integer over a skew field (it's ...) and this shows L^2 Betti #s are integers)

Thus $H_2(W;\mathcal{K}\Gamma)$ free (over a skew field)

$\Rightarrow$ so is $H_2(W;\mathcal{U}\Gamma)$ $\therefore$ so is $\dim_\Gamma(H_2(W;\mathcal{U}\Gamma)) \in \mathbb{Z}$

twisted Betti #

$b_2^\Gamma(W)$ $\equiv$ by def

Proof of the lemma. $\dim_{\mathcal{K}\Gamma}\left(H_2(W;\mathcal{K}\Gamma)\right) = b_2 W$

Claim: $b_0^\Gamma = b_1^\Gamma = b_3^\Gamma = 0$ for W.

Then $b_2^\Gamma = b_2$ by $\chi^\Gamma = \chi$!

To see this part, consider $C_*^\Gamma W$ & $C_*^{\mathbb{Z}} W$, overall ranks agree.

Whereas $\chi = b_2$ follows from $b_3 = 0$ from exact sequence

$H^0\partial W \xrightarrow{\cong} H^0 W \to H^1(W,\partial W) \to H^1 W \xrightarrow{\cong} H^1 \partial W$.

Proof of claim: $H_0(W;\mathcal{K}\Gamma) = \dfrac{\mathcal{K}\Gamma}{\{(1-g)k : g\in\Gamma,\ k\in\mathcal{K}\Gamma\}}$ co-invariants of Γ-action on $\mathcal{K}\Gamma$.

as usual for connected complex.

For $g \neq 1$, $1-g$ is invertible so this is zero.

Now $b_1^\Gamma = 0 \Rightarrow b_3^\Gamma = 0$ by the same exact sequence argument for twisted cohom.

Rk $H^i(X;\mathcal{K}\Gamma) = \mathrm{Hom}_{\mathcal{K}\Gamma}(H_i(X;\mathcal{K}\Gamma);\mathcal{K}\Gamma)$ by UCT for skew

(in general, spectral sequence needed, but zero for).

$\therefore\ b^i = b_i$.

All that remains is $b_1^\Gamma(W) = 0$: and ρ_* is $\neq 0$ on $H_1(\ ;\mathbb{Q})$

Lemma. Let X be a connected finite CW complex, ρ a rep : $\pi_1 X \to \Gamma$ to a PTFA group. If $b_1^{\mathbb{Q}}(X) = 1$ then $b_1^{\mathcal{K}\Gamma}(X) = 0$. (Rk: PTFA $\Rightarrow$ $\mathcal{K}\Gamma$ exists, $\mathbb{Z}\Gamma \subseteq \mathcal{K}\Gamma$.)

Pf

Let $S^1 \to X$ induce an iso on $H_1(X;\mathbb{Q})$.

Pull back $\hat{X} \xrightarrow{\Gamma} X$ to $\hat{S}^1 \to S^1$. Then look at chain compl

$C_*(\hat{X}, \hat{S}^1;\mathbb{Q})$ is a free $\mathbb{Q}\Gamma$-chain complex s.t. $H_i(C_*\otimes_{\mathbb{Q}\Gamma}\mathbb{Q}) =$

for $i=0,1$. (it's $H_*(X,S^1)$).

Algebraic claim: in this setting, it follows that

$H_i(C_*\otimes_{\mathbb{Q}\Gamma}\mathcal{K}\Gamma) = 0$ for $i = 0,1$. (would be true for $i = 0 \ldots$

From this, the lemma follows because $C_*(\hat{S}^1)$ is the complex $\mathbb{Q}\Gamma \xrightarrow{z-1} \mathbb{Q}\Gamma$ where $\pi_1 S^1 \to \pi_1 X \to \Gamma$, $1 \mapsto z$.

Now on $\otimes\mathcal{K}\Gamma$, see $z-1$ is invertible because it isn't zero in H_1 (by assumption). Thus $H_*(\hat{S}^1;\mathcal{K}\Gamma) = 0 \Rightarrow H_*(X;\mathcal{K}\Gamma) = 0$ for

Proof of claim: use the following theorem

Thm (Strebel) If Γ is PTFA and $f : F \to F'$ is a map of free $\mathbb{Q}\Gamma$-modules s.t. $f \otimes \mathbb{Q}$ is injective, then ~~$\mathbb{Z}\otimes_{\mathbb{Z}}$~~ f is injective.

example: $\Gamma = \mathbb{Z}$, $\mathbb{Q}[t^{\pm 1}] \xrightarrow{\times f} \mathbb{Q}[t^{\pm 1}]$; if $f(1) \neq 0$ in $\mathbb{Q}$ so (i.e. $f \otimes \mathbb{Q} \neq 0$) then mult by f is injective)

but $\mathbb{Q}[t^{\pm 1}]^n \xrightarrow{f} \mathbb{Q}[t^{\pm 1}]$; just need to show $\det \neq 0$

& this is because $\det(1) \neq 0$

But the general case is a non-commutative Γ ; must iterate this kind of argument.

Consequence: if rank F = rank F' then $f \otimes \mathbb{Q}$ is $\cong$ $\Rightarrow$ $f \otimes \mathcal{K}\Gamma$ is $\cong$.

This is because $\mathcal{K}\Gamma$ is flat over $\mathbb{Q}\Gamma$, so $\otimes$ preserves injectivity but now is between vector spaces of equal dim.

This is same as saying that $\mathbb{Q}\Gamma \hookrightarrow \mathcal{K}\Gamma$

$\mathbb{Q}\Gamma$ embeds in its Cohn localisation $\mathrm{Cohn}_\varepsilon(\mathbb{Q}\Gamma)$

(which is universal w.r.t. the above property of $\mathcal{K}\Gamma$.)

Proof of claim (cont'd): have a free chain cplx over $\mathbb{Q}\Gamma$ s.t. $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q})$

for $i = 0 \ldots n$.

(n=1 in our case)

$$\begin{tikzcd} \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] \arrow[l, bend right, "h_1"'] & C_0 \arrow[r] \arrow[d, "\varepsilon"] \arrow[l, bend right, "h_0"'] & 0 \\ \cdots \arrow[r] & C_2 \otimes_{\mathbb{Q}\Gamma} \mathbb{Q} \arrow[r] & C_1 \otimes_{\mathbb{Q}\Gamma} \mathbb{Q} \arrow[r] \arrow[l, bend left] & C_0 \otimes_{\mathbb{Q}\Gamma} \mathbb{Q} \arrow[r] \arrow[l, bend left] & \cdots \end{tikzcd}$$

pick some "partial chain homotopies"

as far as possible to split the ∂ maps.

(i.e. as far as acyclic cplx) (NB, there are chain homoto[pies] increasing w/ dim by one!

Lift these to h_i maps on the upper chain complex (by freeness)

Now $f_0 = \partial \circ h_0 : C_0 \circlearrowleft$ satisfies $f_0 \otimes \mathbb{Q} = id$ $\Rightarrow$

$f_0 \otimes \mathcal{K}\Gamma$ is an isom (by Strebel).

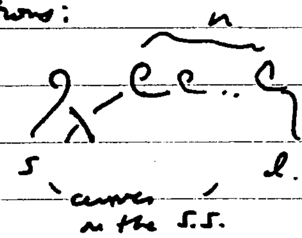
Let $f_1 = \partial h_1 + h_0 \partial \ : C_1 \circlearrowleft$, again $f_1 \otimes \mathbb{Q} = id$ $\Rightarrow$ $f_1 \otimes \mathcal{K}\Gamma = $ iso[m]

etc. Then this shows the 'h_i' form partial chain htpies

zero to isomorphisms ; i.e the f_*'s on homology are iso & vanis[h]

$\therefore$ $H_i = 0$ for $i = 0 \ldots n$.

Let's now give example application / calculations:

Thm : Recall the twist knots K_n = 

so Seifert matrix $\begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$

Recall alg slice $\Leftrightarrow$ $4n+1$ is a square

$\Leftrightarrow$ $n = m(m+1) \geq 0$

and K_0, K_2 actually are slice.

The curves $\gamma_1 = -m \cdot s + l$, $\gamma_2 = (m+1)s + l$ are the only two curves on F s.t. $lk(\gamma_i, \gamma_i^+) = 0$. [$\exists$ exactly 2 i… because of quadr… equation (make precise…

eg $n=6$, $m=2$; both curves are trefoils

see p 223 Kauffman "on knots". for any m, both knots are torus knots $(m, m+1)$. Thus $m = 0, 1$ cases are unknots, & hence K_0, K_2 are slice. The other cases aren't unknots : can we use this to show that the knots aren't actually slice (though they are alg slice).

Thm The knots K_n, $n = m(m+1)$, $m \geq 2$, are $\mathbb{Z}$- independent in the group $\mathcal{C}$.

Thm (COT/CG). Let K be algebraically slice genus - 1 knot with non-trivial Alexander polynomial. If K is slice then $\sigma_{\mathbb{Z}}^{(2)}(\gamma_i) = 0$ for one of the two curves γ_i on a SS which have $lk(\gamma, \gamma^+) = 0$.

$\therefore$ the higher order invariant of K = the lower order ($\int$ twisted) for signatures

This thm $\Rightarrow$ the K_n thm by just checking that $\sigma_{\mathbb{Z}}^{(2)}(T_{m,m+1}) \neq 0$. (shows not slice) & a bit more work for independence.

QED

Proof of COT/CG

Let F be a genus-1 SS for K.

(i) The curves $\{\gamma_i\}$ in H_1F are Lagrangians (in fact the Lagrangians of the Seifert form. (a signature-0 ~~sym.~~ herm. bil form. on $\mathbb{Z}^2$).

They don't generate H_1 though.

($\underline{Q}$ what does the determinant = H_1(a 3-mfd) with induced linking form mean?)

Recall $H_1F \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow[\lambda = (1-t)S + (1-t^{-1})S^T]{} H_1F^* \otimes \mathbb{Z}[t^{\pm 1}] \twoheadrightarrow$ A-module of K $\to 0$ $= H_1(S^3_K; \mathbb{Z}[t^{\pm 1}])$

is a hermitian restriction of the Blanchfield form.

(i.e. $Bl(a,b) = \lambda^{-1}(\bar{a})(\bar{b})$ where λ is invertible over $\mathbb{Q}(t)$ & we take result modulo $\mathbb{Q}[t^{\pm 1}]$)

So Lagrangians in S give Lagrangians of Blanchfield form that is $\Leftrightarrow$, in fact; $\{\gamma_1, \gamma_2\} \leftrightarrow \{l_0, l_1\} \in \{L_0, L_1\}$

Define for each $i = 0, 1$:

$$\rho(l_i)\ \pi_1(S^3_K) = \pi \twoheadrightarrow \pi/\pi^{(2)} = A_K \rtimes \mathbb{Z} \xrightarrow[Bl(l_i, -) \rtimes id_{\mathbb{Z}}]{} \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z} = \Gamma$$

(Remember $A_K \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$, $0 \neq l_i \in L_i \otimes \mathbb{Q} \cong \mathbb{Q}$.)

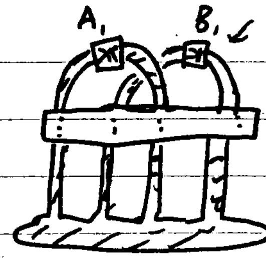
Final lemma:

$$\tilde{\sigma}^{(2)}_{\Gamma}(K, \rho(l_i)) = \sigma^{(2)}_{\mathbb{Z}}(\gamma_i)$$

independent of the choice of $l_i \in L_i$ ("scale")

In this setting, the main thm now says that K slice $\Rightarrow \exists$ a Lagrangian $L \subseteq Bl_0$, s.t $\forall l \in L$, $\sigma_{\Gamma}^{(2)}(K, \rho(l)) =$

There are here just two Lagrangians. Hence we conclude that K slice $\Rightarrow \sigma_{\mathbb{Z}}^{(2)}(\gamma_i) = 0$ for one of the γ's, which can be checked not to be true for torus knots or knot 1), m



Last lemma

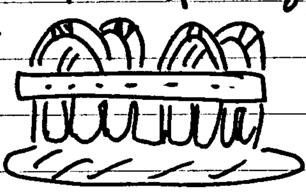
If K has a genus-1 S.S. F and $k \subseteq F$ has self-linking number 0, and is primitive in H_1 ($\Rightarrow$ then we can think of it as one of the bands of the surface) Then

$$\sigma^{(2)}_{\Gamma}(K, \rho_K) = \tilde{\sigma}^{(2)}_{\mathbb{Z}}(k)$$

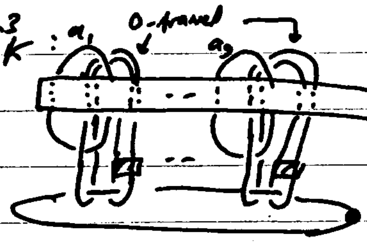
where
$$\begin{array}{ccc} \rho_K : \pi_1(S^3_K) & \longrightarrow & \Gamma \\ \big\downarrow \text{mod } \pi^{(2)} & & \| \\ A_K \rtimes \mathbb{Z} & \longrightarrow & \dfrac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z} \\ & Bl(k,-) \rtimes id_{\mathbb{Z}}. & \end{array}$$

[Eg If K is one of the alg. slice' twist knots $K_{m(m+1)}$, m... then k is an $(m, m+1)$ torus knot, for which $\tilde{\sigma}^{(2)}_{\mathbb{Z}}(k) \neq 0$, $\Rightarrow K$ is not slice

Recall picture of W_0, with boundary S^3_K:

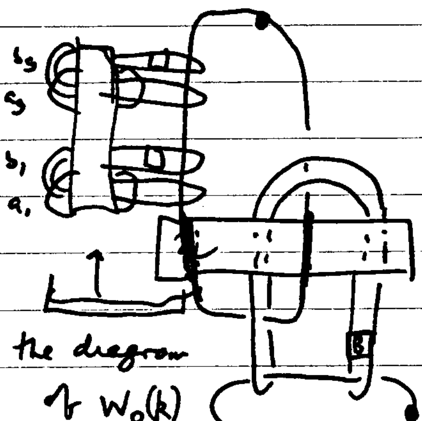


$\rightsquigarrow$



W_0

but also define a new manifold W with $\partial = S^3_K$:



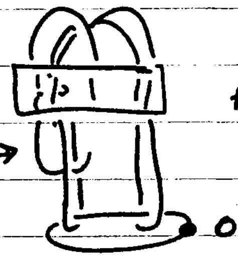
} corresponds to original K

This is built by taking $W_0(K)$ and $W_0(k)$ then splicing the 'a_1' curve of $W_0(K)$ to the base 1-handle of W...

← (not a good picture)

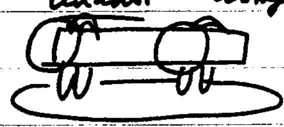
better one: —

Actually, idea is that $V_0(K)$ has



this curve is 'K' on the surface.

Write this k as an unknot using the surgery for the 'k' Seifert surface, i.e. this thing



unknot (surgered in k)

So now sphere in: the point is that the 'S' curve of $W_0(K)$ can wind all around this.

<u>Proof.</u> $S_K = \begin{pmatrix} 0 & l+1 \\ l & B \end{pmatrix}$ $\quad l = lk(k, \text{dual band } \gamma)$, $B =$ twisting of dual band γ.

So that AP of K is $\pm ((l+1) - tl)(l - t(l+1))$.

Now compute $\pi_1 W$: generators M, m meridians of K, k.

($\langle M \rangle$ will be the $\mathbb{Z}$ in $\Gamma = \frac{\mathbb{Q}(t)}{\Phi[t]} \rtimes \mathbb{Z}$ now).

$\pi_1 W = \langle M, m : m \cdot [M, \gamma] = 1$, where $\gamma = m^l$ really $\rangle$

(Computed because of the B curve in W; links M twice in opp directions, m once + the band-linking m^l's)

Hence $m \in [\pi_1 W, \pi_1 W]$ (can actually see the genus-1 surface which it bounds: tubed disc bounded by the B-curve, punctured at m.)

So $\pi_1 W = \langle$ (normal closure of m) $\rtimes_{\mathbb{Z}} \langle M \rangle$

Now work ~~with~~ modulo second commutators, i.e. all conjugates of m commute. So

$$\frac{\pi_1 W}{(\pi_1 W)^{(2)}} \cong \frac{\mathbb{Z}[t^{\pm 1}]}{lt - (l+1)} \rtimes \langle M \rangle$$

using notation $s = m$ additively.

so that $ls = m^l$, $ts = m^M$

(i.e. we rewrite the conj action as a mult. action.)

M acts by mult by t.

So there's a commu diagram:

$$\begin{array}{ccc} \pi_1(S^3_K) & \longrightarrow & \Gamma \\ \downarrow & \nearrow \rho & \uparrow \\ \pi_1(W) & \twoheadrightarrow & \dfrac{\mathbb{Z}[t]}{lt-(l+1)} \rtimes \mathbb{Z} \end{array}$$

note that $m \in \pi_1 W / (\pi_1 W)^{(2)}$ maps to a non-zero elt in Γ.

Thus $\hat{\sigma}^{(2)}_\Gamma(K, \rho_K) = \tilde{\sigma}^{(2)}_\Gamma(W^4, \rho)$

We will now calculate the RHS via the computation of $H_2(W; \mathcal{N}\Gamma)$. Let $\Gamma_0 = \mathrm{im}(\rho) \leq \Gamma$, have $\langle m \rangle'' \cong \mathbb{Z} \leq \Gamma_0$

[By algebraic lemma, $\Gamma_1 \leq \Gamma_2 \Rightarrow$ get commuting diag

$$\begin{array}{ccc} L^0(\mathcal{N}\Gamma_1) & \longrightarrow & L^0(\mathcal{N}\Gamma_2) \\ & \tilde{\sigma}^{(2)}_{\Gamma_1} \searrow \quad \swarrow \tilde{\sigma}^{(2)}_{\Gamma_2} & \\ & \mathbb{R} & \end{array}$$

so OK to use Γ_0.]

Claim Then: $H_2(W; \mathcal{N}\Gamma_0) \cong \underbrace{H_2(W_0(k); \mathcal{N}(\langle m^{\pm 1}\rangle))}_{\text{this has desired } \tilde{\sigma}^{(2)} \text{ for A-4 lemma.}} \otimes_{\mathbb{Z}[m^{\pm 1}]} \mathcal{N}\Gamma_0$.

Why? $\Gamma_0 = \pi_1 W / (\pi_1 W)^{(2)}$

Compute the second homology geometrically; the 2-handles from $W_0(k)$ give a basis upstairs, because they obviously lift. Intersections computed using $m \in \mathbb{Z}$ become ones computed using $m \in \Gamma_0$, hence the tensoring up as hermitian module. Now use the algebraic lemma again; the map $L^0(\mathcal{N}\Gamma_1) \to L^0(\mathcal{N}\Gamma_2)$ is exactly tensoring up (induction).

So, ultimately $\tilde{\sigma}^{(2)}_\Gamma(K, \rho_K) = \tilde{\sigma}^{(2)}_\Gamma(W) \underset{\text{(alg lemma)}}{=} \tilde{\sigma}^{(2)}_{\Gamma_0}(W) \underset{\text{(alg lemma)}}{=} \tilde{\sigma}^{2}_{\mathbb{Z}}(W_0(k)) \underset{\text{(defn)}}{=} \tilde{\sigma}^{(2)}_{\mathbb{Z}}(k)$

this is why we never needed to bother about surjectivity of $\pi_1 \xrightarrow{\rho} \Gamma$ in our choices: using the image is enough

Note on alg. lemma: it is rather remarkable that signature on a small gp extends to a bigger one.
Not usually true for invariants of L-groups.

But it works because
$$\begin{array}{ccc} \mathcal{N}\Gamma_1 & \longrightarrow & \mathcal{N}\Gamma_2 \\ & {\scriptstyle tr}\searrow \quad \swarrow{\scriptstyle tr} & \\ & \mathbb{C} & \end{array}$$
commutes, really.

<u>NB</u> $\mathcal{N}$ is <u>not</u> a functor! $\mathbb{Z} \to 1$ doesn't give
$$\mathcal{N}(\mathbb{Z}) = L^\infty(S^1) \to \mathbb{C} = \mathcal{N}(1).$$
It works for the C^* algebra $C^0(S^1)$ by evaluation at 1. But cannot evaluate $L^\infty(S^1)$ anywhere.
However it is functorial for injections: if $\Gamma_1 \le \Gamma_2$ then $\mathcal{N}\Gamma_1 \hookrightarrow \mathcal{N}\Gamma_2$.
Recall $\mathbb{C}\Gamma_1 \subseteq \mathcal{N}\Gamma_1 = \mathcal{B}(\ell^2\Gamma_1)^{\Gamma_1\text{-rt-inv}}$
But the H. space completion $\ell^2\Gamma_1 \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2 = \ell^2\Gamma_2$
Then send $\mathcal{N}\Gamma_1 \longrightarrow \mathcal{N}\Gamma_2$
$$a \longmapsto \text{closure of } a \otimes_{\Gamma_1} id_{\Gamma_2}.$$
in fact gives an injection.
<u>trace</u>$_\Gamma$ is just matrix element of $1_\Gamma \equiv e_\Gamma$ $\langle e_\Gamma a, e_\Gamma \rangle_{\ell^2\Gamma}$
so that $\langle e_1 a, e_1 \rangle_{\ell^2\Gamma_1} = \langle e_1 a, e_1 \rangle_{\ell^2\Gamma_1} \cdot \langle e_2, e_2 \rangle_{\mathbb{C}\Gamma_2}$
$$= \langle (e_1 \otimes e_2)(a \otimes id), (e_1 \otimes e_2) \rangle_{\ell^2\Gamma_1 \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2}$$
$$(\text{complete}) \quad = tr_{\Gamma_2}(a \otimes id).$$

<u>Qn</u> what happens to image of $\sigma^{(2)}_{\Gamma_1}$ as we look at
$$\begin{array}{ccccc} L^0(\mathcal{N}\Gamma_1) & \longrightarrow & L^0(\mathcal{N}\Gamma_2) & \longrightarrow & ? \\ & \searrow \quad \swarrow & & & \\ & \mathbb{R} & & & \end{array}$$
(continuous? discrete?)