

Proposition 8, together with [4], furnishes the

COROLLARY (Linearization Theorem). — Every M -regular difference equation of type $\text{Diff}(\mathbb{R}^n)$, \mathbb{R}^n over B is isomorphic to a linear one.

Using [2] we obtain.

COROLLARY. — Let H be a separable Hilbert space. Then every M -regular difference equation of type $GL(H)$, H over B is isomorphic to the one given by $f(b+m) = f(b)$.

Proposition 8 implies that the equivariant K -theory based on M -regular linear difference equations is isomorphic with $K(\quad) \cdot M \setminus$. Yet the equivariant K -theory based on all linear difference equations has some interest. In general, they will be different from each other.

Finally it should be remarked that these ideas could be used in studying difference approximations of completely integrable systems of total differential equations.

LITERATURE

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EQUIVARIANT STABLE HOMOTOPY THEORY

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Equivariant maps between spheres.

Let G be a finite group.

A finite-dimensional real vector-space V on which G acts linearly will be called a G -module. Its one-point compactification S^V is a sphere with G -action, in which we shall regard ∞ as a base-point. Our object is to describe the homotopy-classes of equivariant maps between such spheres.

For each G -module V there is a concept of *suspension*: if X is a G -space with base-point x_0 we define the suspension $S^V X$ as

$$S^V \wedge X = (S^V \times X) / ((\infty \times X) \cup (S^V \times x_0)).$$

If X and Y are G -spaces with base-points then $[X; Y]_G$ denotes the set of homotopy-classes of base-point-preserving G -maps $X \rightarrow Y$. There is a suspension-map $[X; Y]_G \rightarrow [S^V X; S^V Y]_G$ for any G -module V . One can order the isomorphism-classes of G -modules by

$$V \leq V' \iff V \text{ is isomorphic to a submodule of } V';$$

then one defines the set of stable equivariant maps

$$\{X; Y\}_G = \lim_{\substack{\uparrow \\ V}} [S^V X; S^V Y],$$

which is an abelian group. (Strictly speaking the limit is taken over the category of G -modules and embeddings).

PROPOSITION 1. — $[S^{V \oplus W}; S^W]_G$ is independent of W if W is sufficiently large, and can be identified with the set of cobordism-classes of compact V -framed G -manifolds.

The terminology in the proposition is explained by

DEFINITION 1. — If V is a G -module, a compact G -manifold M is called V -framed if there is given a stable G -isomorphism φ_M of its tangent bundle T_M with $M \times V$, i.e. if there is given a G -module W and an isomorphism of G -vector-bundles $T_M \oplus (M \times W) \cong M \times (V \oplus W)$. Such a manifold is said to bound if there is a G -manifold N with boundary M and a stable isomorphism of T_N with $N \times (V \oplus W)$ which induces φ_M .

The proof of Proposition 1 depends essentially on the concept of "consistent transversality" introduced by Wasserman [2]. Details can be found in [1].

Proposition 1 makes clear in particular what plays the role of the *degree* of a map in the equivariant theory. Recall that one defines for any group G its *Burnside ring* $A(G)$ as the Grothendieck group of the category of finite G -sets, i.e. $A(G)$ is the free abelian group on the set of conjugacy-classes of subgroups of finite index in G . Then we have

COROLLARY. — For large W , $[S^W; S^W]_G \cong A(G)$ as rings, where the multiplication in $[S^W; S^W]_G$ is composition of maps, and that in $A(G)$ corresponds to forming the product of G -sets.

Thus the equivariant homotopy class of a map $S^W \rightarrow S^W$ is determined by the degrees of its restrictions to the fixed-point subsets of the subgroups H of G ; and the diagram

$$\begin{array}{ccc} [S^W; S^W]_G & \longrightarrow & A(G) \\ \downarrow & & \downarrow \epsilon_H \\ [(S^W)^H; (S^W)^H] & \xrightarrow{\text{degree}} & \mathbb{Z} \end{array}$$

commutes, where ϵ_H assigns to a G -set S the cardinal of S^H .

PROPOSITION 2. — If $V = \mathbb{R}^n$ with trivial G -action then

$$[S^{V \oplus W}; S^W]_G \cong \bigoplus_H \pi_n^S(BW_H),$$

where the sum is taken over the conjugacy classes of subgroups H of G , π_n^S denotes stable homotopy, and $W_H = N_H/H$, where N_H is the normalizer of H in G .

Proof. — If M is a V -framed G -manifold then the isotropy-group must be constant on each component of M , for if g is an element of the isotropy group at x then g acts trivially on the tangent-space to M at x , and so leaves fixed all the geodesics through x . But if M has all its isotropy-groups conjugate to H one can write it as $(G/H) \times_{W_H} M^H$, where M^H , the H -invariant part of M , is a free W_H space.

Thus a general V -framed manifold can be written $\bigsqcup_H (G/H) \times_{W_H} M_H$, where M_H is a V -framed free W_H -manifold. The cobordism-classes of such M_H can be identified with $\pi_n^S(BW_H)$, and Proposition 2 follows.

Equivariant stable cohomology theory

For any pair $Y \subset X$ of compact G -spaces and any virtual G -module α (i.e. any $\alpha \in RO(G)$) let us define

$$\omega_G^\alpha(X, Y) = \lim_{\substack{\longrightarrow \\ V}} [S^V(X/Y); S^{V+\alpha}] = \{X/Y; S^\alpha\}.$$

This is a generalized cohomology theory in the sense that it satisfies obvious homotopy, exactness, and excision axioms (for any pair (X, Y) there is a boundary homomorphism $\omega_G^\alpha(Y) \rightarrow \omega_G^{\alpha+1}(X, Y)$). It has the additional stability property that $\tilde{\omega}_G^\alpha(X) \cong \tilde{\omega}_G^{\alpha+V}(S^V X)$ for any X and V . Furthermore it is universal among cohomology theories with those four properties.

On free G -spaces and trivial G -spaces one can express ω_G^α in terms of ordinary stable homotopy, at least when $\alpha \in \mathbb{Z} \subset RO(G)$, as follows.

PROPOSITION 3. — If X is a free compact G -space, then

$$\omega_G^n(X) \cong \omega^n(X/G) = \pi_S^n(X/G).$$

PROPOSITION 4. — If G acts trivially on X then

$$\omega_G^n(X) \cong \bigoplus_H \{X; S^n BW_H^+\},$$

where BW_H^+ is the union of BW_H with a disjoint base-point.

As ordinary stable homotopy coincides with homology when tensored with the rationals, and as classifying-spaces for finite groups have trivial rational homology, one deduces from Proposition 4 that $\omega_G^n(X) \otimes \mathbb{Q} \cong A(G) \otimes H^n(X; \mathbb{Q})$ when G acts trivially on X . More generally one has

PROPOSITION 5. — For any compact G -space X , and any $\alpha \in RO(G)$,

$$\omega_G^\alpha(X) \otimes \mathbb{Q} \cong \bigoplus_H H^{\alpha_H}(X^H; \mathbb{Q})^{W^H}$$

where, if $\alpha = V - W \in RO(G)$, $\alpha_H = \dim V^H - \dim W^H$.

It is easy to see that $\{S^V; S^W\}_G$ is a finitely generated abelian group, so Proposition 5 implies the

COROLLARY. — $\{S^V; S^W\}_G$ is finite unless $\dim V^H = \dim W^H$ for some subgroup H of G .

The equivariant J-homomorphism ⁽¹⁾.

The relationship between equivariant stable cohomology as defined here and equivariant K -theory is precisely analogous to that in the classical case. There is a J -homomorphism

$$J : KO_G^{-1}(X) \rightarrow \omega_G^0(X)$$

(from the additive group KO_G^{-1} to the multiplicative group of ω_G^0) defined by the usual Hopf construction. Its image can be determined in the following way.

The Adams operations ψ^k act on $KO_G(X)$, and hence on the profinite completion $KO_G(X)^\wedge$. They define an action of \mathbb{Z} on $KO_G(X)^\wedge$ which is continuous when \mathbb{Z} is given the profinite topology, and so the action extends to an action of the profinite completion $\hat{\mathbb{Z}}$. The group of units $\hat{\mathbb{Z}}^*$ of this ring is the product of the subgroup (± 1) with a topologically cyclic group Γ . Let α be a generator of Γ . Then $\psi^\alpha : KO_G(X)^\wedge \rightarrow KO_G(X)^\wedge$ extends to a transformation of multiplicative cohomology theories, and so one can define a new multiplicative cohomology theory J_G^* with a multiplicative transformation $J_G^* \rightarrow KO_G^{*\wedge}$ fitting into an exact triangle

$$\cdots \rightarrow J_G^* \rightarrow KO_G^{*\wedge} \xrightarrow{\psi^\alpha - 1} KO_G^{*\wedge} \rightarrow \cdots$$

(1) The proofs of the results in this section depend on the work of Sullivan on the Adams conjecture.

Thus there is a short exact sequence

$$0 \rightarrow \text{coker}(\psi^a - 1) \rightarrow J_G^* \rightarrow \ker(\psi^a - 1) \rightarrow 0. \quad \dots (\dagger)$$

In terms of the theory J_G^* one can describe the J -homomorphism as follows. The Hurewicz homomorphism $\omega_G^* \rightarrow KO_G^{*\wedge}$ factorizes through J_G^* , giving a multiplicative transformation $h: \omega_G^* \rightarrow J_G^*$. In view of the exact sequence (\dagger) one sees that this assigns to an element of stable cohomotopy its d - and e -invariants in the sense of Adams. The J -homomorphism $J: KO_G^{-1}(X) \rightarrow \omega_G^0(X)$ factorizes through $\tilde{J}_G^0(X)$ to give an exponential map $J: \tilde{J}_G^0(X) \rightarrow \omega_G^0(X)$. If G is a p -group the composite $hJ: \tilde{J}_G^0(X) \rightarrow J_G^0(X)$ is an isomorphism between the additive group $\tilde{J}_G^0(X)$ and the multiplicative group $1 + \tilde{J}_G^0(X)$. Then $\tilde{J}_G^0(X)$ is a direct summand in the multiplicative group $1 + \omega_G^0(X)$.

The definition of an equivariant cohomology theory.

In conclusion I shall mention two facts which tend to support the use of all real representations for suspending in equivariant stable homotopy theory, and the indexing of equivariant cohomology theories by $RO(G)$.

The first is the generalization of the construction of Eilenberg-MacLane spaces as the infinite symmetric products of spheres.

PROPOSITION 6. — If A is a topological abelian group with G -action, and V is a G -module, there is a G -space $B^V A$ and a G -homotopy-equivalence

$$A \rightarrow \text{Map}(S^V; B^V A);$$

and if $A = \mathbb{Z}$ with trivial G -action one can take $B^V A = F(S^V)$, the free abelian group on S^V .

The second is the generalization of a theorem of Barratt and Quillen. If S is a finite G -set let us write Σ_S for the group of G -automorphisms of S . One can form an associative monoid $\Gamma_G = \bigsqcup_S B\Sigma_S$, the sum being over all finite G -sets S .

The monoid can also be written $\prod_H \bigsqcup_{n \in \mathbb{N}} B(\Sigma_n \wr W_H)$, where H runs through the conjugacy-classes of subgroups G , and $\Sigma_n \wr W_H$ denotes the semi-direct product

$$\Sigma_n \wr (W_H \times \dots \times W_H).$$

PROPOSITION 7. — The classifying-space for Γ_G is homotopy-equivalent to that of $\lim_{\vec{V}} \text{Map}_G(S^V; S^V)$, where V runs through all G -modules.

The theorem of Barratt and Quillen tells one that

$$B(\bigsqcup_n B(\Sigma_n \wr W_H)) \simeq B(\Omega^\infty S^\infty(BW_H^+)),$$

so one deduces.

$$\text{COROLLARY. — } \lim_{\vec{V}} \text{Map}_G(S^V; S^V) \simeq \prod_H \Omega^\infty S^\infty(BW_H^+).$$

This is of course just a restatement of Proposition 4, but it provides a completely different proof of it.

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