ON THE HOMOLOGY INVARIANTS OF KNOTS

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1. Let k be an oriented knot in 3-dimensional euclidean space R^3 and V a closed tubular neighbourhood of k. The boundary of V is a torus T, and $W = R^3 - V + T$ is the closed complement of V. An oriented Jordan curve (i.e. a homeomorph of a circle) on T which bounds on V (on $R^3 - V + T$) but not on T is called a meridian

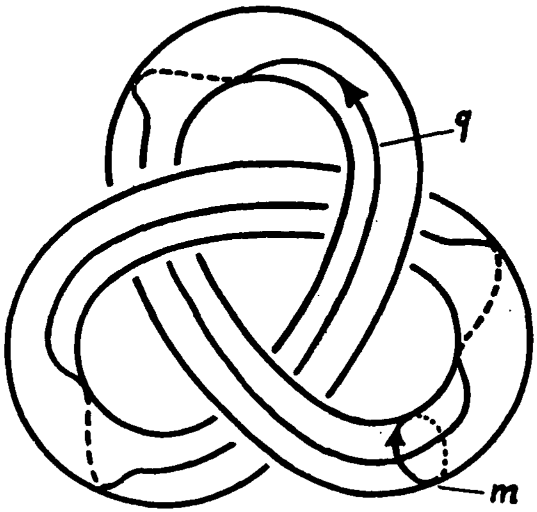

FIG. 1.

(longitudinal circuit). If m_1 and m_2 are any two meridians, one has $m_1 \sim \pm m_2$ on T; likewise $q_1 \sim \pm q_2$ on T for any pair q_1, q_2 of longitudinal circuits.

By a topological mapping ϕ one can carry V into a tubular neighbourhood V^* of an unknotted curve k^* in such a way that the longitudinal circuit of V is carried into a longitudinal circuit q^* of V^*. The 3-space in which V^* lies will be designated by R^{3*}. Fig. 1 and Fig. 2 show the situation in the case when k is a trefoil knot.

Let l be an arbitrary knot in the interior of V. Then l, as a 1-cycle in V, is homologous in V to some multiple of k, say

$$l \sim nk \quad \text{on } V. \tag{1}$$

By a suitable orientation of l we can arrange that $n \geqslant 0$. Fig. 3 shows an example in which k is a trefoil knot and $n = 0$.

The knot l is carried by the topological mapping ϕ into a knot l^* in the interior of V^*. The purpose of this paper is to prove the two following theorems:

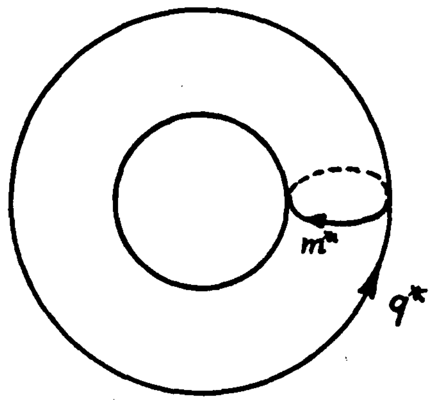

FIG. 2.

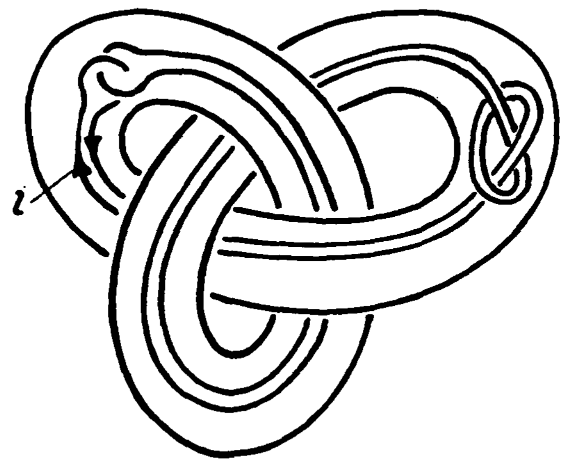

FIG. 3.

THEOREM I. *For $n = 0$ the homology invariants of l and l^* are the same.*

In other words: if M_g and M_g^* are the g-sheeted cyclic covering manifolds† of R^3 with the branch lines l and l^* respectively, the homology groups and linking invariants of M_g and M_g^* are equal for

† Seifert-Threlfall, *Lehrbuch der Topologie* (Leipzig 1934) §§ 58 and 77.

$g = 2, 3, \ldots$, and the homology groups of M_∞ and M_∞^*, considered as groups with operators,† are isomorphic.

THEOREM II. *Between the L-polynomials $\Delta_l(x)$, $\Delta_{l^*}(x)$, $\Delta_k(x)$ of the knots l, l^*, and k the following equation holds*

$$\Delta_l(x) = \Delta_{l^*}(x)\Delta_k(x^n). \tag{2}$$

In the case $n = 0$ formula (2) reduces to

$$\Delta_l(x) = \Delta_{l^*}(x). \tag{3}$$

For we then have

$$\Delta_k(x^n) = \Delta_k(x^0) = \Delta_k(1).$$

But it is known that $\Delta_k(1) = 1$ for any knot k. Besides (3) is a consequence of Theorem I.

In the special case when the knot l lies on the boundary T of V, formula (2) expresses a theorem due to Burau.‡ A theorem due to Alexander§ to the effect that the L-polynomial of a composite knot is the product of the L-polynomials of the factors is another special case of Theorem II (here $n = 1$).

Theorem I illustrates the limits of the homology invariants of a knot in so far as the properties of knot k do not appear in the homology invariants of l. A special case of this fact is the theorem of Whitehead's‖ on the L-polynomial of a 'doubled knot'.

2. Proof of Theorem I.

The g-sheeted cyclic covering manifold M_g is the union of the complexes V_g and W_g corresponding to the decomposition of R^3 into V and W. V_g is the g-sheeted cyclic covering manifold of V with branch line l. W_g decomposes into g homeomorphs $W', W'', \ldots, W^{(g)}$ of W, since every closed curve of W is homologous to zero in $R^3 - l$, n being equal to zero. The intersection of V_g and $W^{(\gamma)}$ is a torus T_γ ($\gamma = 1, 2, \ldots, g$). Let q_γ be the covering of the longitudinal circuit q lying on T_γ, and let $a_1, a_2, \ldots, a_t$ be a set of generators of the homology

† H. Seifert, 'Über das Geschlecht von Knoten', *Math. Annalen*, 110 (1934), 571–92.

‡ W. Burau, 'Kennzeichnung der Schlauchknoten', *Abh. math. Semin. Hamburg. Univ.* 9 (1932), 125.

§ J. W. Alexander, 'Topological invariants of knots and links', *Trans. American Math. Soc.* 30 (1928), 275–306.

‖ J. H. C. Whitehead, *J. of London Math. Soc.* 12 (1937), 63.

group of dimension 1 of V_g. Then the homology group of V_g is defined by a set of m relations

$$\sum_{\tau=1}^{t} \rho_{\mu\tau} a_\tau \sim 0 \quad \text{on } V_g \quad (\mu = 1, 2,..., m). \tag{4}$$

Let

$$q_\gamma \sim \sum_{\tau=1}^{t} \sigma_{\gamma\tau} a_\tau \quad \text{on } V_g \quad (\gamma = 1, 2,..., g). \tag{5}$$

Then the homology group of M_g is defined by the relations (4) and

$$\sum_{\tau=1}^{t} \sigma_{\gamma\tau} a_\tau \sim 0 \quad \text{on } M_g \quad (\gamma = 1, 2,..., g). \tag{6}$$

On the other hand let us consider the g-sheeted cyclic covering manifold $M_g^* = V_g^* + W_g^*$ of R^{3*} with branch line l^*. Corresponding to the mapping ϕ of V on V^* (cf. § 1) there exists a homeomorphic mapping ϕ_g of V_g on V_g^* which carries the torus T_γ into the torus T_γ^*, the longitudinal circuit q_γ into the longitudinal circuit q_γ^* and the set of generators $a_1, a_2,..., a_t$ of V_g into the set of generators $a_1^*, a_2^*,..., a_t^*$ of V_g^*. Then we have the relations

$$\sum_{\tau=1}^{t} \rho_{\mu\tau} a_\tau^* \sim 0 \quad \text{on } V_g^* \quad (\mu = 1, 2,..., m) \tag{4*}$$

and

$$q_\gamma^* \sim \sum_{\tau=1}^{t} \sigma_{\gamma\tau} a_\tau^* \quad \text{on } V_g^* \quad (\gamma = 1, 2,..., g), \tag{5*}$$

since ϕ_g is a homeomorphic mapping. It follows that the homology groups of M_g and M_g^* are isomorphic.

In order to determine the linking invariants of M_g, we consider (besides $a_1, a_2,..., a_t$) t 1-dimensional chains $a_1', a_2',..., a_t'$ on V_g such that $a_\tau' \sim a_\tau$ on V_g and a_τ and a_λ' do not intersect for $\tau, \lambda = 1, 2,..., t$. Because of formulae (4) and (5), there are 2-chains $A_1, A_2,..., A_m$ and $B_1, B_2,..., B_g$ on V_g such that

$$\text{boundary } A_\mu = \sum_{\tau=1}^{t} \rho_{\mu\tau} a_\tau \quad (\mu = 1, 2,..., m)$$

and

$$\text{boundary } B_\gamma = \sum_{\tau=1}^{t} \sigma_{\gamma\tau} a_\tau - q_\gamma \quad (\gamma = 1, 2,..., g).$$

Then the linking invariants of M_g are determined by the $t(t+g)$ intersection numbers†

$$S(A_\tau, a_\sigma'), \qquad S(B_\gamma, a_\sigma')$$
$$(\tau, \sigma = 1, 2,..., t;\ \gamma = 1, 2,..., g).$$

If we define $a_\tau'^*$, A_τ^*, B_γ^* as the images of a_τ', A_τ, B_γ under the mapping ϕ_g, it follows that

$$S(A_\tau, a_\sigma') = S(A_\tau^*, a_\sigma'^*), \qquad S(B_\gamma, a_\sigma') = S(B_\gamma^*, a_\sigma'^*),$$

† H. Seifert, 'Die Verschlingungsinvarianten der zyklischen Knotenüberlagerungen', *Abh. math. Semin. Hamburg. Univ.* 11 (1935), 84–101.

provided that the orientation of V_g is carried into the orientation of V_g^* under the mapping ϕ_g. Therefore the linking invariants of M_g and M_g^* are the same.

The assertion that the 1-dimensional homology groups of M_∞ and M_∞^* are operator isomorphic follows from the fact that they are obtained from the operator isomorphic groups of V_∞ and V_∞^* by adding the relations $q_\gamma \sim 0$ and $q_\gamma^* \sim 0$ $(-\infty < \gamma < +\infty)$.

3. For the proof of formula (2) we make use of the following facts (cf. Seifert):† For any knot c there can always be found an orientable surface F without singularities whose boundary is c. By cutting R^3 along F we obtain a bounded 3-dimensional manifold $\bar{M}$ whose boundary consists of the two exposed faces of the cut, i.e. of F and a homeomorphic copy xF of F. Let h be the genus of F, let $a_1, a_2,..., a_{2h}$ be a (1-dimensional) homology basis of F and let $xa_1, xa_2,..., xa_{2h}$ be the corresponding basis of xF. Then there are homologies of the form

$$a_i - \sum_{j=1}^{2h} \gamma_{ij}(a_j - xa_j) \sim 0 \quad \text{in } \bar{M} \quad (i = 1, 2,..., 2h). \tag{7}$$

All homologies between $a_1,..., a_{2h}$, $xa_1,..., xa_{2h}$ that exist in $\bar{M}$ are consequences of (7).

The matrix $\Gamma = (\gamma_{ij})$, from which all homology invariants of c can be derived, may be called a *homology matrix* of c. The matrix Γ is uniquely determined up to the choice of the spanning surface F and its homology basis $a_1, a_2,..., a_{2h}$. The L-polynomial $\Delta_c(x)$ of c is the coefficient determinant of the system (7)

$$\Delta_c(x) = |\mathrm{E} - \Gamma + x\Gamma|, \tag{8}$$

where E is the unit matrix of order $2h$.

4. We may assume $n > 0$, since for $n = 0$ Theorem II is a consequence of Theorem I. We begin by constructing an oriented non-singular surface F_{l^*} bounded by l^*. To this end we choose on the boundary T^* of V^* a set of n non-intersecting longitudinal circuits $q_1^*, q_2^*,..., q_n^*$ and orient them so that they all become homologous to k^* in V^*. Since we have the homology $l^* \sim \sum_{\nu=1}^{n} q_\nu^*$ in V^*, it follows that there exists in V^* an oriented non-singular surface $F_{l^*}^0$ with boundary $l^* - \sum_{\nu=1}^{n} q_\nu^*$. From $F_{l^*}^0$ we obtain the desired surface F_{l^*} by

† H. Seifert, 'Über das Geschlecht von Knoten', *Math. Annalen*, 110 (1934), 571–92.

adjoining n non-intersecting 2-cells $F_{q_1^*}, F_{q_2^*},\dots, F_{q_n^*}$ which lie in $W^* = R^{3*}-V^*+T^*$ and have the boundaries $q_1^*, q_2^*,\dots, q_n^*$ respectively.

Next we construct a surface F_l bounded by l. The homeomorphic mapping ϕ^{-1} of V^* upon V carries the surface $F_{l^*}^0$ into a surface F_l^0 whose boundary consists of l and n longitudinal circuits $q_1, q_2,\dots, q_n$ of V, images of $q_1^*, q_2^*,\dots, q_n^*$ respectively. Since $q_1 \sim 0$ in W, there exists an oriented non-singular surface F_{q_1} in W with boundary q_1. By an isotopic deformation F_{q_1} can be carried into a 'parallel' surface

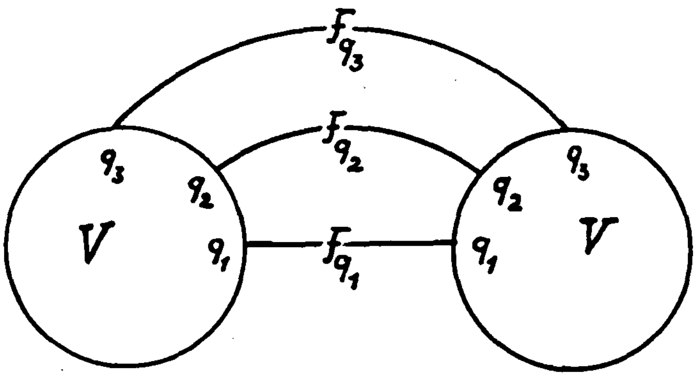

FIG. 4.

F_{q_2} in W with boundary q_2 such that F_{q_1} and F_{q_2} do not intersect. By a second deformation F_{q_2} can be carried into a surface F_{q_3} in W with boundary q_3 such that F_{q_3} intersects neither F_{q_1} nor F_{q_2}, and so on. $F_l^0, F_{q_1}, F_{q_2},\dots, F_{q_n}$ form together an orientable non-singular surface F_l bounded by l. Fig. 4 shows the situation in a schematic cross-section; F_l^0 is omitted and $n = 3$.

The genus of F_l is obviously

$$h_{l^*}+nh_k,$$

where h_{l^*}, h_k denote the genera of F_{l^*} and F_{q_1} respectively.

5. I shall now construct a homology basis of dimension 1 on F_l. Let

$$a_1^{(1)}, a_2^{(1)},\dots, a_{2h_k}^{(1)}$$

be a homology basis on F_{q_1} and

$$a_1^{(\nu)}, a_2^{(\nu)},\dots, a_{2h_k}^{(\nu)} \quad (\nu = 1, 2,\dots, n) \tag{9}$$

the basis on F_{q_ν} which corresponds to it with respect to the abovementioned deformation of F_{q_1} into F_{q_ν}. On F_{l^*} we select a homology basis

$$b_1^*, b_2^*,\dots, b_{2h_{l^*}}^*. \tag{10}$$

One can assume that these chains all lie on F^0_{l*}, since the 2-cells $F^*_{q_1}, F^*_{q_2},\ldots, F^*_{q_n}$ can obviously be avoided. By the homeomorphic mapping ϕ^{-1} of V^* on V the chains (10) are carried into the chains

$$b_1, b_2,\ldots, b_{2h_{l*}} \tag{11}$$

on F^0_l. The chains (9) and (11) constitute the desired homology basis of F_l.

6. In order to obtain the L-polynomial of the knot l we cut R^3 along the surface F_l according to the general rule of § 3. I shall use the following notation. By the cutting process the complexes R^3, V, W, T go into $\bar{R}^3$, $\bar{V}$, $\bar{W}$, $\bar{T}$. The two exposed faces of the cut are designated by F_l and xF_l. F_l consists of the $n+1$ surfaces $F_{q_1}, F_{q_2},\ldots, F_{q_n}$, F^0_l, and similarly xF_l is the union of the surfaces $xF_{q_1}, xF_{q_2},\ldots, xF_{q_n}$, xF^0_l. The homology basis $a^{(1)}_1,\ldots, a^{(1)}_{2h_k}, a^{(2)}_1,\ldots, a^{(n)}_{2h_k}, b_1, b_2,\ldots, b_{2h_{l*}}$ of F_l corresponds to the homology basis $xa^{(1)}_1,\ldots, xa^{(1)}_{2h_k}, xa^{(2)}_1,\ldots, xa^{(n)}_{2h_k}$, $xb_1, xb_2,\ldots, xb_{2h_{l*}}$ of xF_l. The notation used in R^{3*} differs only in the addition of a superscript star.

7. In $\bar{R}^{3*}$ we have relations of the form (cf. § 3)

$$b^*_i - \sum_{j=1}^{2h_{l*}} \gamma^{(l*)}_{ij}(b^*_j - xb^*_j) \sim 0 \text{ in } \bar{R}^{3*} \quad (i = 1, 2,\ldots, 2h_{l*}). \tag{12}$$

The homology matrix of the knot l^* is

$$\Gamma_{l*} = (\gamma^{(l*)}_{ij}). \tag{13}$$

The left side of (12), being a chain in $\bar{V}^*$ and homologous to zero in $\bar{R}^{3*} = \bar{V}^* + \bar{W}^*$, must be homologous to a chain on $\bar{V}^* \cap \bar{W}^* = \bar{T}^*$. Thus it is homologous to a linear combination of the chains $q^*_1, q^*_2,\ldots, q^*_n$:

$$b^*_i - \sum_{j=1}^{2h_{l*}} \gamma^{(l*)}_{ij}(b^*_j - xb^*_j) \sim \sum_{j=1}^{n} \alpha_j q^*_j \quad \text{on } \bar{V}^*. \tag{14}$$

By the homeomorphic mapping ϕ^{-1} of V^* on V the homology (14) corresponds to the following homology on $\bar{V}$:

$$b_i - \sum_{j=1}^{2h_{l*}} \gamma^{(l*)}_{ij}(b_j - xb_j) \sim \sum_{j=1}^{n} \alpha_j q_j \quad \text{on } \bar{V}. \tag{15}$$

If we consider this homology in $\bar{R}^3 = \bar{V} + \bar{W}$, it simplifies to

$$b_i - \sum_{j=1}^{2h_{l*}} \gamma^{(l*)}_{ij}(b_j - xb_j) \sim 0 \quad \text{in } \bar{R}^3 \quad (i = 1, 2,\ldots, 2h_{l*}), \tag{16}$$

since $q_j \sim 0$ in $\bar{W}$.

We still need the homologies belonging to the $a_i^{(\nu)}$. They have the following general form (cf. § 3):

$$a_i^{(\nu)} - \sum_{j=1}^{2h_k} \sum_{\mu=1}^{n} \gamma_{ij}^{\nu\mu}(a_j^{(\mu)} - xa_j^{(\mu)}) \sim \sum_{j=1}^{2h_k} \eta_{ij}^{(\nu)}(b_j - xb_j) \quad \text{in } \bar{R}^3. \tag{17}$$

The left side of (17) is a chain in $\overline{W}$, the right side a chain in $\overline{V}$. Therefore there is a certain chain on $\overline{V} \cap \overline{W} = \overline{T}$ to which either side is homologous (in $\overline{V}$ or $\overline{W}$ respectively). The most general such chain is a linear combination of $q_1, q_2, \ldots, q_n$, but these are ~ 0 in $\overline{W}$. So it follows that

$$a_i^{(\nu)} - \sum_{j=1}^{2h_k} \sum_{\mu=1}^{n} \gamma_{ij}^{\nu\mu}(a_j^{(\mu)} - xa_j^{(\mu)}) \sim 0 \quad \text{in } \overline{W}$$

$$(i = 1, 2, \ldots, 2h_k;\ \nu = 1, 2, \ldots, n). \tag{18}$$

In order to determine the matrices

$$\Gamma^{\nu\mu} = (\gamma_{ij}^{\nu\mu})$$

we identify in $\overline{W}$ the surfaces F_{q_2} and $xF_{q_2}, \ldots, F_{q_n}$ and xF_{q_n}. Hereby $\overline{W}$ goes into a complex $\overline{W}_{q_1}$, which may be described as the complex W cut along F_{q_1}. The chains $a_i^{(\mu)}$ and $xa_i^{(\mu)}$ are thereby identified ($\mu = 2, 3, \ldots, n$), so that (18) reduces to

$$a_i^{(1)} - \sum_{j=1}^{2h_k} \gamma_{ij}^{11}(a_j^{(1)} - xa_j^{(1)}) \sim 0 \quad \text{in } \overline{W}_{q_1} \quad (i = 1, 2, \ldots, 2h_k). \tag{19}$$

But this is exactly the system of relations (7) formed for the knot q_1 and the surface F_{q_1}. So we see from (19) that Γ^{11} is just the homology matrix of q_1 or, what is the same thing, of k (k and q_1 are equivalent knots, since k can be deformed in V into q_1). If we designate the homology matrix of k by Γ_k, we have the result $\Gamma^{11} = \Gamma_k$, and in the same way one proves

$$\Gamma^{\nu\nu} = \Gamma_k \quad (\nu = 1, 2, \ldots, n). \tag{20}$$

Now we note that the homologies

$$xa_i^{(\nu)} \sim a_i^{(\nu+1)} \quad \text{in } \overline{W} \quad (i = 1, 2, \ldots, 2h_k;\ \nu = 1, 2, \ldots, n-1) \tag{21}$$

hold, provided that the q_ν have been enumerated in the right way. But we know that (16) and (18) are a complete system of homologies in $\bar{R}^3$ between the chains b_i, $a_j^{(\nu)}$, xb_i, $xa_j^{(\nu)}$. So (21) must be a consequence of (16) and (18) and therefore, because of the special form (16) and (18), of (18) alone.

If we write the variables $a_i^{(\nu)}$ and $xa_i^{(\nu)}$ in the order

$$a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)};\ xa_i^{(1)}, xa_i^{(2)}, \ldots, xa_i^{(n)},$$

the coefficient matrices of (18) and (21) are

$$\left.\begin{array}{cccc|cccc} E-\Gamma_k & -\Gamma^{12} & . & -\Gamma^{1n} & \Gamma_k & \Gamma^{12} & . & \Gamma^{1n} \\ -\Gamma^{21} & E-\Gamma_k & . & -\Gamma^{2n} & \Gamma^{21} & \Gamma_k & . & \Gamma^{2n} \\ . & . & . & . & . & . & . & . \\ -\Gamma^{n1} & -\Gamma^{n2} & . & E-\Gamma_k & \Gamma^{n1} & \Gamma^{n2} & . & \Gamma_k \end{array}\right. \tag{22}$$

and

$$\left.\begin{array}{ccccc|ccccc} 0 & -E & 0 & . & 0 & E & 0 & . & 0 & 0 \\ 0 & 0 & -E & . & 0 & 0 & E & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & -E & 0 & 0 & . & E & 0. \end{array}\right. \tag{23}$$

In both matrices we add the right half to the left and obtain

$$\left.\begin{array}{cccc|cccc} E & 0 & . & 0 & \Gamma_k & \Gamma^{12} & . & \Gamma^{1n} \\ 0 & E & . & 0 & \Gamma^{21} & \Gamma_k & . & \Gamma^{2n} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & E & \Gamma^{n1} & \Gamma^{n2} & . & \Gamma_k \end{array}\right. \tag{22'}$$

and

$$\left.\begin{array}{cccccc|ccccc} E & -E & 0 & . & 0 & 0 & E & 0 & . & 0 & 0 \\ 0 & E & -E & . & 0 & 0 & 0 & E & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & E & -E & 0 & 0 & . & E & 0. \end{array}\right. \tag{23'}$$

The rows of (23′) can be linear combinations of the rows of (22′) only if

$$\Gamma^{\nu\mu} = \begin{cases} \Gamma_k - E & (\nu > \mu), \\ \Gamma_k & (\nu < \mu). \end{cases}$$

Hence we find for the homology matrix Γ_l of l (see (16) and (18))

$$\Gamma_l = \left[\begin{array}{ccc|c} \Gamma^{11} & . & \Gamma^{1n} & 0 \\ . & . & . & . \\ \Gamma^{n1} & . & \Gamma^{nn} & 0 \\ \hline 0 & . & 0 & \Gamma_{l^*} \end{array}\right] = \left[\begin{array}{cccc|c} \Gamma_k & \Gamma_k & . & \Gamma_k & 0 \\ \Gamma_k - E & \Gamma_k & . & \Gamma_k & 0 \\ . & . & . & . & . \\ \Gamma_k - E & \Gamma_k - E & . & \Gamma_k & 0 \\ \hline 0 & 0 & . & 0 & \Gamma_{l^*} \end{array}\right],$$

and for the L-polynomial $\Delta_l(x)$ of l

$$\Delta_l(x) = |\mathrm{E}-\Gamma_l+x\Gamma_l|$$
$$= |\mathrm{E}-\Gamma_{l^*}+x\Gamma_{l^*}|.\begin{vmatrix} \mathrm{E}-\Gamma_k+x\Gamma_k & . & -\Gamma_k+x\Gamma_k \\ . & . & . \\ \mathrm{E}-\Gamma_k+x(\Gamma_k-\mathrm{E}) & . & \mathrm{E}-\Gamma_k+x\Gamma_k \end{vmatrix}. \tag{24}$$

The first factor is the L-polynomial $\Delta_{l^*}(x)$ of l^*; the second factor has in the diagonal $\mathrm{E}-\Gamma_k+x\Gamma_k$, above the diagonal $-\Gamma_k+x\Gamma_k$, and below the diagonal $\mathrm{E}-\Gamma_k+x(\Gamma_k-\mathrm{E})$. This determinant can be computed as follows. Subtract successively the $(n-1)$th row from the nth, the $(n-2)$th from the $(n-1)$th,..., the first row from the second. There results

$$\begin{vmatrix} \mathrm{E}-\Gamma_k+x\Gamma_k & -\Gamma_k+x\Gamma_k & . & -\Gamma_k+x\Gamma_k & -\Gamma_k+x\Gamma_k \\ -x\mathrm{E} & \mathrm{E} & . & 0 & 0 \\ 0 & -x\mathrm{E} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & -x\mathrm{E} & \mathrm{E} \end{vmatrix}.$$

Next add x times the nth column to the $(n-1)$th, x times the $(n-1)$th column to the $(n-2)$th, etc. This gives

$$\begin{vmatrix} \mathrm{E}-\Gamma_k+x^n\Gamma_k & -\Gamma_k+x^{n-1}\Gamma_k & . & -\Gamma_k+x\Gamma_k \\ 0 & \mathrm{E} & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \mathrm{E} \end{vmatrix}$$
$$= |\mathrm{E}-\Gamma_k+x^n\Gamma_k| = \Delta_k(x^n).$$

Thus (24) becomes

$$\Delta_l(x) = \Delta_{l^*}(x)\Delta_k(x^n).$$

This completes the proof.