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Semi-Invariants in Surgery

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Abstract. A semi-invariant in surgery is an invariant of a quadratic Poincaré complex which is defined in terms of a null-cobordism. We define five such gadgets: the semicharacteristic, the semitorsion, the cross semitorsion, the torsion semicharacteristic, and the cross torsion semicharacteristic. We describe applications to the evaluation of surgery obstructions, especially in the odd-dimensional case.

Key words. Quadratic Poincaré complex, cobordism, semi-invariant, surgery.

0. Introduction

Many invariants of odd-dimensional manifolds can be defined in terms of invariants of bounding even-dimensional manifolds. The linking form is determined by the intersection form, the semicharacteristic is determined by the Euler characteristic, the ρ-invariant is determined by the multisignature, the η-invariant is determined by the integral of the Hirzebruch L-genus, and the Rochlin invariant is determined by the signature, each with respect to the appropriate notion of bounding manifold. We call all such invariants *semi-invariants.* In this paper we deal with semi-invariants in surgery.

The surgery obstruction theory of Wall [14] was expressed in Ranicki [10] in terms of chain complexes with duality. The quadratic L-groups $L_n(A)$ of a ring with involution A are the cobordism groups of n-dimensional quadratic Poincaré complexes (C, ψ) over A. Here, C is an n-dimensional A-module chain complex and ψ is a quadratic structure on C which determines a quadratic Poincaré duality chain equivalence $(1 + T)\psi_0: C^{n-*} \to C$. The 'instant surgery obstruction' of [10] assigns to an n-dimensional normal map $(f, b): M^n \to X$ an n-dimensional quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi_1(X)]$ representing the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of [14]. The surgery obstruction is evaluated using appropriate invariants of the quadratic Poincaré complex.

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Surgery semi-invariants are invariants of odd-dimensional quadratic Poincaré complexes which are defined (more or less explicitly) in terms of bounding even-dimensional quadratic Poincaré pairs. Odd-dimensional surgery obstructions with finite fundamental group are largely determined by surgery semi-invariants.

We discuss five semi-invariants in surgery:

(i) semicharacteristic,
(ii) semitorsion,
(iii) cross semitorsion,
(iv) torsion semicharacteristic,
(v) cross torsion semicharacteristic.

In (ii) and (iii) we mean Whitehead torsion, and in (iv) and (v) we mean localization torsion. (We are still working on the meaning of the torsion semitorsion.)

Various special cases of the semi-invariants have appeared in the literature. The present paper consists of definitions, elementary properties and examples. We relate the semi-invariants to more traditional algebraic K-theory invariants involving the projective class and Whitehead torsion. The main innovation of direct computational significance is the algorithm of Section 5 for obtaining a situation in which the cross semi-invariants may be defined, generalizing a procedure of Pardon [8].

The *semicharacteristic* of a f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A with f.g. projective homology A-modules $H_*(C)$ is defined in Section 2 by

$$\chi_{1/2}(C, \psi) = \Sigma_{r=0}^{i} (-)^r [H_r(C)] \in \tilde{K}_0(A).$$

The semicharacteristic is the classic surgery semi-invariant. If A is semisimple, then every f.g. projective quadratic Poincaré complex over A has f.g. projective homology, $L_{2i+1}^p(A) = 0$ and the semicharacteristic defines isomorphisms

$$\chi_{1/2}\colon L_{2i+1}^h(A) \to \operatorname{coker}(L_{2i+2}^p(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_0(A))).$$

The *semitorsion* of a based f.g. free $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A with based f.g. free homology A-modules $H_*(C)$ is defined in Section 2 by

$$\begin{aligned}\tau_{1/2}(C, \psi) = \Sigma_{r=0}^{i} (-)^r \tau((1 + T)\psi_0 \colon H_{2i+1-r}(C)^* \to H_r(C))\\ + \tau(H_*(C) \to C) \in \tilde{K}_1(A),\end{aligned}$$

with $H_*(C) \to C$ any chain equivalence inducing the identity in homology. If A is a division ring, then every f.g. free quadratic Poincaré complex over A has f.g. free homology, $L_{2i+1}^h(A) = 0$ and the semitorsion defines isomorphisms

$$\tau_{1/2}\colon L_{2i+1}^s(A) \to \operatorname{coker}(L_{2i+2}^h(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_1(A)))$$

A $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) is *cross* if the Poincaré duality chain map $(1+T)\psi_0: C^{2i+1-*} \to C$ is an isomorphism. The *cross semi-torsion* of a based f.g. free cross $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A is defined in Section 3 by

$$\tau^{\dagger}_{1/2}(C,\psi) = \Sigma^{i}_{r=0} (-)^r \tau((1+T)\psi_0: C^{2i+1-r} \to C_r) \in \tilde{K}_1(A).$$

Let $S \subset A$ be a multiplicative subset in a ring with involution A, so that the localization $S^{-1}A$ inverting S is defined. Let $K_1(A, S)$ denote the Grothendieck group of stable isomorphism classes of S-torsion A-modules of homological dimension 1.

The *torsion semicharacteristic* of an $S^{-1}A$-acyclic f.g. projective $2i$-dimensional quadratic Poincaré complex (C, ψ) over A with S-torsion homology A-modules $H_*(C)$ of homological dimension 1 is defined in Section 6 by

$$\chi^{S}_{1/2}(C, \psi) = \Sigma^{i-1}_{r=0} (-)^r [H_r(C)] \in K_1(A, S).$$

Although, strictly speaking, the torsion semicharacteristic is an invariant of an even-dimensional complex (C, ψ), it is essentially an odd-dimensional invariant, since C is the resolution of an odd-dimensional chain complex of S-torsion A-modules of homological dimension 1.

The *cross torsion semicharacteristic* of a f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A with the chain equivalence $(1+T)\psi_0: C^{2i+1-*} \to C$ an $S^{-1}A$-isomorphism is defined in Section 7 by

$$\chi^{S\dagger}_{1/2}(C, \psi) = \Sigma^{i}_{r=0} (-)^r [\operatorname{coker}((1+T)\psi_0: C^{2i+1-r} \to C_r)] \in K_1(A, S).$$

Precursors of the surgery semi-invariants have been useful in the study of group actions on spheres (Madsen [5], Milgram [6] and Davis [2]), knot cobordism groups (Ranicki [11]), and group actions on homology spheres (Davis and Weinberger [4]). We hope our general treatment of the semi-invariants and the examples we bring together will be useful. Future possible geometric applications include propagation of group actions and the study of converses to Smith theory.

The cross invariants are one ingredient in a general program developed in Davis [3] for the evaluation of odd-dimensional surgery obstructions with finite fundamental group.

1. Algebraic Surgery Semi-Invariants

We shall assume that the reader is familiar with the basic definitions of the algebraic theory of surgery of [10, 11]. We start with a brief review of algebraic K-theory and the general theory (such as it is) of algebraic surgery semi-invariants.

We shall only be concerned with the reduced algebraic K-groups of a ring A

$$\tilde{K}_m(A) = \operatorname{coker}(K_m(\mathbb{Z}) \to K_m(A)) \quad (m = 0, 1).$$

An A-module chain complex C is *finite* if it is a finite-dimensional complex of based f.g. free A-modules. We assume that the reader is familiar with the (reduced) projective class of a finite-dimensional f.g. projective A-module chain complex C

$$[C] = \Sigma_{r=0}^{\infty}(-)^r[C_r] \in \tilde{K}_0(A).$$

and with the (reduced) torsion of a chain equivalence $f: C \to D$ of finite complexes

$$\tau(f: C \to D) \in \tilde{K}_1(A).$$

Given a ring A with involution $\bar{\ }: A \to A$ define the *dual* of a f.g. projective (left) A-module P by

$$P^* = \mathrm{Hom}_A(P, A), \quad A \times P^* \to P^*; \ (a, f) \to (x \to f(x)\bar{a}).$$

The dual of a morphism $f \in \mathrm{Hom}_A(P, Q)$ of f.g. projective A-modules is defined by

$$f^*: Q^* \to P^*; \quad g \to (h \to h(g(f(x)))).$$

The duality involutions on the reduced algebraic K-groups $\tilde{K}_m(A)$ $(m = 0, 1)$ by

$$*: \tilde{K}_0(A) \to \tilde{K}_0(A); \quad [P] \to [P]^* = [P^*],$$
$$*: \tilde{K}_1(A) \to \tilde{K}_1(A); \quad \tau(f: P \to Q) \to \tau(f)^* = \tau(f^*: Q^* \to P^*).$$

The n-duality involution $C \to C^{n-*}$ on chain complexes is such that

$$[C^{n-*}] = (-)^n[C]^* \in \tilde{K}_0(A),$$
$$\tau(f^*: D^{n-*} \to C^{n-*}) = (-)^n\tau(f: C \to D)^* \in \tilde{K}_1(A).$$

Given an n-dimensional quadratic Poincaré complex (C, ψ) define a K-theory invariant

$$\tau_0(C, \psi) = [C] \in \tilde{K}_0(A) \quad \text{if } C \text{ is f.g. projective},$$
$$\tau_1(C, \psi) = \tau((1 + T)\psi_0: C^{n-*} \to C) \in \tilde{K}_1(A) \text{ if } C \text{ is finite}.$$

Let $X \subseteq \tilde{K}_m(A)$ $(m = 0, 1)$ be a $*$-invariant subgroup. An n-dimensional quadratic Poincaré complex (C, ψ) is *X-Poincaré* if

for $m = 0$, C is f.g. projective with $\tau_0(C, \psi) \in X$,

for $m = 1$, C is finite with $\tau_1(C, \psi) \in X$.

$L_n^X(A)$ is the cobordism group of n-dimensional quadratic X-Poincaré complexes over A.

Given an $(n + 1)$-dimensional quadratic Poincaré pair $(D, C; \delta\psi, \psi)$ define a K-theory invariant $\tau_m(D, C; \delta\psi, \psi) \in \tilde{K}_m(A)$ such that

$$\tau_m(C, \psi) = \tau_m(D, C; \delta\psi, \psi) + (-)^n\tau_m(D, C; \delta\psi, \psi)^* \in \tilde{K}_m(A) \quad (m = 0, 1)$$

by

$$\tau_0(D, C; \delta\psi, \psi) = [D] \in \tilde{K}_0(A),$$
$$\tau_1(D, C; \delta\psi, \psi) = \tau((1 + T)(\delta\psi_0, \psi_0): (D/C)^{n+1-*} \to D) \in \tilde{K}_1(A)$$

with $m = 0$ if C, D are f.g. projective and $m = 1$ if C, D are finite.

Let $Y \subseteq X \subseteq \tilde{K}_m(A)$ $(m = 0, 1)$ be $*$-invariant subgroups, so that the Tate $\mathbb{Z}_2$-cohomology groups of X/Y are defined by

$$\hat{H}^n(\mathbb{Z}_2; X/Y) = \{g \in X/Y \mid g^* = (-)^n g\}/\{h + (-)^n h^* \mid h \in X/Y\}.$$

An n-dimensional quadratic Poincaré pair $(D, C; \delta\psi, \psi)$ is (X, Y)*-Poincaré* if

$$\tau_m(C, \psi) \in Y, \quad \tau_m(D, C; \delta\psi, \psi) \in X.$$

The relative quadratic L-groups $L_n^{X,Y}(A)$ are the cobordism group of n-dimensional quadratic (X, Y)-Poincaré pairs over A, such that there is defined an exact sequence

$$\cdots \to L_n^Y(A) \to L_n^X(A) \to L_n^{X,Y}(A) \to L_{n-1}^Y(A) \to \cdots.$$

The isomorphisms

$$L_n^{X,Y}(A) \to \hat{H}^n(\mathbb{Z}_2; X/Y); \quad (D, C; \delta\psi, \psi) \to \tau_m(D, C; \delta\psi, \psi)$$

establish an isomorphism between the relative L-theory sequence and the Rothenberg exact sequence

$$\cdots \to L_n^Y(A) \to L_n^X(A) \to \hat{H}^n(\mathbb{Z}_2; X/Y) \to L_{n-1}^Y(A) \to \cdots.$$

In particular, there are defined isomorphisms

$$\begin{aligned} &\ker(L_n^Y(A) \to L_n^X(A)) \to \operatorname{coker}(L_{n+1}^X(A) \to \hat{H}^{n+1}(\mathbb{Z}_2; X/Y)); \\ &(C, \psi) \to \tau_m(D, C; \delta\psi, \psi) \end{aligned} \tag{1.1}$$

sending the cobordism class of an X-null-cobordant n-dimensional quadratic Y-Poincaré complex (C, ψ) over A to the K-theory invariant of any $(n+1)$-dimensional quadratic X-Poincaré null-cobordism $(D, C; \delta\psi, \psi)$.

A finite n-dimensional quadratic Poincaré complex over A (C, ψ) is *simple* if

$$\tau_1(C, \psi) = 0 \in \tilde{K}_1(A).$$

The quadratic L-group $L_n^q(A)$ for $q = p$ (resp. h, s) is the cobordism group of n-dimensional quadratic Poincaré complexes (C, ψ) over A which are f.g. projective (resp., finite, simple)

$$L_*^p(A) = L_*^{\tilde{K}_0(A)}(A), \qquad L_*^s(A) = L_*^{\{0\} \subseteq \tilde{K}_1(A)}(A),$$
$$L_*^h(A) = L_*^{\{0\} \subseteq \tilde{K}_0(A)}(A) = L_*^{\tilde{K}_1(A)}(A).$$

The surgery semicharacteristic (Davis [2]) was interpreted in Milgram and Ranicki [7] from the point of view of the isomorphism (1.1). We repeat this interpretation here and in Section 2 below, along with the torsion analogue.

DEFINITION 1.2. (i) Let $X \subseteq \tilde{K}_0(A)$ be a $*$-invariant subgroup. The *universal semicharacteristic* is the isomorphism

$$\begin{aligned} &(U\chi)_{1/2} : \ker(L_n^X(A) \to L_n^p(A)) \to \operatorname{coker}(L_{n+1}^p(A) \to \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_0(A)/X)), \\ &(C, \psi) \to (U\chi)_{1/2}(C, \psi) = \tau_0(D, C; \delta\psi, \psi) \end{aligned}$$

defined for any f.g. projective n-dimensional quadratic X-Poincaré complex (C, ψ) over A which admits a f.g. projective null-cobordism $(D, C; \delta\psi, \psi)$.
(ii) Let $X \subseteq \tilde{K}_1(A)$ be a $*$-invariant subgroup. The *universal semitorsion* is the isomorphism

$$(U\tau)_{1/2}: \ker(L_n^X(A) \to L_n^h(A)) \to \operatorname{coker}(L_{n+1}^h(A) \to \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_1(A)/X)),$$
$$(C, \psi) \to (U\tau)_{1/2}(C, \psi) = \tau_1(D, C; \delta\psi, \psi)$$

defined for any finite n-dimensional quadratic X-Poincaré complex (C, ψ) over A which admits a finite null-cobordism $(D, C; \delta\psi, \psi)$. □

REMARK 1.3. Let P be a f.g. projective A-module such that

$$[P] = [P^*] \in \tilde{K}_0(A),$$

representing an element $[P] \in \hat{H}^{2*}(\mathbb{Z}_2; \tilde{K}_0(A))$. Then

$$\begin{aligned}[P] &= 0 \in \operatorname{coker}(L_{2i+2}^p(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_0(A))) \\ &= \ker(L_{2i+1}^h(A) \to L_{2i+1}^p(A))\end{aligned}$$

if and only if $P \oplus A^r$ admits a nonsingular $(-)^{i+1}$ quadratic form for some integer $r \geqslant 0$. □

2. The Semicharacteristic and the Semitorsion

We now specialize the algebraic surgery semi-invariants of Section 1 to odd-dimensional quadratic Poincaré complexes.

Given an A-module chain complex C and integers $i \leqslant j$, let $C[i,j]$ be the subquotient complex of C with

$$C[i,j]_r = C_r \quad \text{if } i \leqslant r \leqslant j, \ = 0 \text{ otherwise.}$$

We shall only be concerned with positive complexes, so that $C_r = 0$ for $r < 0$. Let $H_*(C)$ be the A-module chain complex defined by the homology A-modules and 0 differentials

$$H_*(C): \cdots \to H_{r+1}(C) \xrightarrow{0} H_r(C) \to \cdots \to H_0(C).$$

If $H_*(C)$ is a projective A-module complex there is defined a chain equivalence $i: H_*(C) \to C$ inducing the identity in homology, which is unique up to chain homotopy. This follows from the splitting of the exact sequences

$$0 \to \operatorname{im}(d: C_{r+1} \to C_r) \to \ker(d: C_r \to C_{r-1}) \to H_r(C) \to 0.$$

Choices of splitting maps $i: H_r(C) \to C_r$ $(r \geqslant 0)$ such that $di = 0$ define a chain equivalence $i: H_*(C) \to C$ inducing the identity in homology, and every such chain equivalence is of this type. If $i, i': H_r(C) \to C_r$ $(r \geqslant 0)$ are two sets of splitting maps there exist A-module morphisms $j: H_r(C) \to C_{r+1}$ $(r \geqslant 0)$ such that $i' - i = dj$, defining a chain homotopy $j: i \simeq i': H_*(C) \to C$.

An A-module chain complex is *H-projective* (resp., *H-finite*) if both C and $H_*(C)$ are f.g. projective (resp. finite) complexes.

The *semicharacteristic* of an H-projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A is defined by

$$\chi_{1/2}(C) = \Sigma_{r=0}^{i}(-)^r[H_r(C)] \in \tilde{K}_0(A).$$

(This is the standard definition of the surgery semicharacteristic, cf. Davis [2].) The following result verifies that this agrees with the universal semicharacteristic $(U\chi)_{1/2}(C, \psi)$ of Definition 1.2 (i)

PROPOSITION 2.1. (i) *An H-projective* (2i + 1)*-dimensional quadratic Poincaré complex* (C, ψ) *bounds an H-projective* (2i + 2)*-dimensional quadratic Poincaré pair* $(H_*(C)[i+1,\ 2i+1],\ C; 0, \psi)$.

(ii) *The semicharacteristic and the projective class are related by*

$$[C] = \chi_{1/2}(C) - \chi_{1/2}(C)^* \in \tilde{K}_0(A).$$

(iii) *Let* $X \subseteq \tilde{K}_0(A)$ *be a *-invariant subgroup. If* (C, ψ) *is an H-projective* (2i+1)*-dimensional quadratic X-Poincaré complex over A then*

$$\chi_{1/2}(C) = (C, \psi)$$
$$\in \operatorname{coker}(L_{2i+2}^p(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_0(A)/X)) = \ker(L_{2i+1}^X(A) \to L_{2i+1}^p(A)).$$

Proof. (i) The composite of the chain equivalence $C \to H_*(C)$ and the projection $H_*(C) \to H_*(C)[i+1, 2i+1]$ is a chain map $C \to H_*(C)[i+1,\ 2i+1]$ sending $\psi \in Q_{2i+1}(C)$ to $0 \in Q_{2i+1}(H_*(C)\,[i+1, 2i+1])$.

(ii) By the universal coefficient theorem and Poincaré duality we have that up to isomorphism

$$H_r(C) = H^{2i+1-r}(C) = H_{2i+1-r}(C)^*.$$

By this and by the homology invariance of projective class

$$\begin{aligned}[C] &= [H_*(C)] = \Sigma_{r=0}^{2i+1}(-)^r[H_r(C)] \\ &= \Sigma_{r=0}^{i}(-)^r[H_r(C)] + \Sigma_{r=i+1}^{2i+1}(-)^r[H_{2i+1-r}(C)^*] \\ &= \chi_{1/2}(C) - \chi_{1/2}(C)^* \in \tilde{K}_0(A).\end{aligned}$$

(iii) Immediate from (i) and Definition 1.2(i), since

$$\tau_0(H_*(C)[i+1, 2i+1], C; 0, \psi) = -\chi_{1/2}(C)^* \in \tilde{K}_0(A).$$ □

EXAMPLE 2.2. If A is semisimple every f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex over A is H-projective, $L_{2i+1}^p(A) = 0$ (by Ranicki [9]) and the semicharacteristic defines isomorphisms

$$\chi_{1/2}: L_{2i+1}^h(A) \to \operatorname{coker}(L_{2i+2}^p(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_0(A)));$$
$$(C, \psi) \to \chi_{1/2}(C).$$ □

The *semitorsion* of an H-finite $(2i + 1)$-dimensional quadratic Poincaré complex over A (C, ψ) is defined by

$$\begin{aligned}\tau_{1/2}(C,\psi)\\ &= \Sigma_{r=0}^{i}(-)^r \tau((1 + T)\psi_0 \colon H_{2i+1-r}(C)^* \to H_r(C))\\ &\quad + \tau(H_*(C) \to C) \in \tilde{K}_1(A),\end{aligned}$$

with $H_*(C) \to C$ any chain equivalence inducing the identity in homology. The following result verifies that this agrees with the universal semitorsion $(U\tau)_{1/2}(C,\psi)$ of Definition 1.2(ii).

PROPOSITION 2.3. (i) *An H-finite $(2i + 1)$-dimensional quadratic Poincaré complex (C,ψ) over A bounds an H-finite $(2i + 2)$-dimensional quadratic Poincaré pair $(H_*(C)[i + 1, 2i + 1], C; 0, \psi)$.*

(ii) *The semitorsion and the torsion are related by*

$$\tau(C,\psi) = \tau_{1/2}(C,\psi) - \tau_{1/2}(C,\psi)^* \in \tilde{K}_1(A).$$

(iii) *Let $X \subseteq \tilde{K}_1(A)$ be a $*$-invariant subgroup. If (C,ψ) is an H-finite $(2i + 1)$-dimensional quadratic X-Poincaré complex over A then*

$$\begin{aligned}\tau_{1/2}(C,\psi) &= (C,\psi) \in \operatorname{coker}(L_{2i+2}^h(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_1(A)/X))\\ &= \ker(L_{2i+1}^X(A) \to L_{2i+1}^h(A)).\end{aligned}$$

Proof. (i) The null-cobordism $(H_*(C)[i + 1, \; 2i + 1], \; C; 0, \psi)$ defined in Proposition 2.1(i) is H-finite.

(ii) By the sum formula for torsion applied to the homotopy commutative square of chain equivalences of finite A-module chain complexes

$$\begin{array}{ccc} C^{2i+1-*} & \xrightarrow{(1+T)\psi_0} & C \\ \big\downarrow & & \big\downarrow \\ H_*(C)^{2i+1-*} & \xrightarrow{((1+T)\psi_0)_*} & H_*(C) \end{array}$$

we have

$$\begin{aligned}\tau(C,\psi) - \tau(H_*(C) \to C) + \tau(H_*(C) \to C)^* &= \tau(H_*(C), \psi_*)\\ &= \Sigma_{r=0}^{2i+1}(-)^r \tau((1 + T)\psi_0 \, H_{2i+1-r}(C)^* \to H_r(C))\\ &= \Sigma_{r=0}^{i}(-)^r \tau((1 + T)\psi_0 \colon H_{2i+1-r}(C)^* \to H_r(C))\\ &\quad - \Sigma_{r=0}^{i}(-)^r \tau((1 + T)\psi_0 \colon H_{2i+1-r}(C)^* \to H_r(C))^* \in \tilde{K}_1(A).\end{aligned}$$

(iii) Apply Definition 1.2(ii) to the torsion of the H-finite null-cobordism $(H_*(C)[i+1,2i+1],\ C;0,\psi)$ of (C,ψ)

$$\begin{aligned}&\tau_1(H_*(C)[i+1,2i+1],\ C;0,\psi)\\&\quad=\tau_1(H_*(C)[i+1,2i+1],H_*(C);0,\psi)-\tau(C\to H_*(C))^*\\&\quad=-\tau_{1/2}(C,\psi)^*\in\tilde{K}_1(A).\end{aligned}$$ □

EXAMPLE 2.4. If A is a division ring every finite $(2i+1)$-dimensional quadratic Poincaré complex over A is H-finite, $L^h_{2i+1}(A)=0$ and the semitorsion defines isomorphisms

$$\begin{aligned}&\tau_{1/2}\colon L^s_{2i+1}(A)\to\operatorname{coker}(L^h_{2i+2}(A)\to\hat{H}^{2i+2}(\mathbb{Z}_2;\tilde{K}_1(A)));\\&(C,\psi)\to\tau_{1/2}(C,\psi).\end{aligned}$$ □

EXAMPLE 2.5. The 'Reidemeister torsion' invariant defined by Madsen [5], p. 205

$$\Delta_*(C)=\tau(C\to H_*(C))\in\tilde{K}_1(A)$$

for an H-finite $(2i+1)$-dimensional quadratic Poincaré complex over A (C,ψ) such that

$$\begin{aligned}&\tau((1+T)\psi_0)_*\colon H_{2i+1-r}(C)^*\to H_r(C)^*=0\in\tilde{K}_1(A)\quad(0\leqslant r\leqslant 2l+1),\\&\tau(C_r\to H_r(C))=0\in\tilde{K}_1(A)\quad(i\leqslant r\leqslant 2i+1)\end{aligned}$$

agrees with the semitorsion up to sign

$$\tau_{1/2}(C,\psi)=-\tau(C\to H_*(C))=-\Delta_*(C)\in\tilde{K}_1(A).$$

EXAMPLE 2.6. The 'first patching obstruction' of Milgram [6] is a semitorsion invariant. □

3. The Cross Semitorsion

A f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) over A is *cross* if the Poincaré duality chain equivalence

$$(1+T)\psi_0\colon C^{2i+1-*}\to C$$

is an isomorphism.

PROPOSITION 3.1. *A cross $(2i+1)$-dimensional quadratic Poincaré complex bounds.*

Proof. Define a null-cobordism $(C[i+1,2i+1],C;0,\psi)$, with $C\to C[i+1,2i+1]$ the projection. □

Define the *cross semitorsion* of a finite cross $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) over A

$$\tau^\dagger_{1/2}(C,\psi)=\Sigma^i_{r=0}(-)^r\tau((1+T)\psi_0\colon C^{2i+1-r}\to C_r)\in\tilde{K}_1(A).$$

PROPOSITION 3.2.(i) *The cross semitorsion is related to the torsion by*

$$\tau(C,\psi) = \tau^{\dagger}_{1/2}(C,\psi) - \tau^{\dagger}_{1/2}(C,\psi)^* \in \tilde{K}_1(A).$$

(ii) *Let* $X \subseteq \tilde{K}_1(A)$ *be a* $*$*-invariant subgroup. If* (C,ψ) *is a finite cross* $(2i+1)$*-dimensional quadratic* X*-Poincaré complex over* A *then*

$$\begin{aligned}\tau^{\dagger}_{1/2}(C,\psi) &= (C,\psi) \in \operatorname{coker}(L^h_{2i+2}(A) \to \hat{H}^{2i+2}(\mathbb{Z}_2;\tilde{K}_1(A)/X)) \\ &= \ker(L^X_{2i+1}(A) \to L^h_{2i+1}(A)).\end{aligned}$$

Proof. (i) The finite null-cobordism $(C[i+1,2i+1],C;0,\psi)$ has torsion

$$\tau_1(C[i+1,2i+1],C;0,\psi) = -\tau^{\dagger}_{1/2}(C,\psi)^* \in \tilde{K}_1(A),$$

so that

$$\begin{aligned}\tau(C,\psi) &= \tau_1(C[i+1,2i+1],C;0,\psi) - \tau_1(C[i+1,2i+1],C;0,\psi)^* \\ &= \tau^{\dagger}_{1/2}(C,\psi) - \tau^{\dagger}_{1/2}(C,\psi)^* \in \tilde{K}_1(A).\end{aligned}$$

(ii) Apply Definition 1.2(ii) to the null-cobordism of (i). □

EXAMPLE 3.3. Given a finite 2-group π and an element $x \in L^h_{2i+1}(\mathbb{Z}[\pi])$ Pardon [8, p. 143] described an algebraic procedure for representing x by a highly-connected finite $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi]$ such that $C_r = 0$ for $r \neq i, i+1$ and $H_i(C)$ is an odd torsion f.g. $\mathbb{Z}[\pi]$-module. Let (C',ψ') be the cobordant complex with

$$\begin{aligned}&C'_r = 0 \quad \text{for } r \neq i, i+1,\\ &d_{C'} = (1+T)\psi_0 : C'_{i+1} = C^{i+1} \to C'_i = C_i,\\ &(1+T)\psi'_0 = d_C : C'^{i+1} = C_{i+1} \to C'_i = C_i.\end{aligned}$$

The induced $(2i+1)$-dimensional quadratic Poincaré complex over $\mathbb{Z}_{(2)}[\pi]$

$$(C'',\psi'') = \mathbb{Z}_{(2)}[\pi] \otimes_{\mathbb{Z}[\pi]} (C',\psi')$$

is cross, and such that the invariants b_i of Pardon [8] are determined by the cross semitorsion

$$\tau^{\dagger}_{1/2}(C'',\psi'') \in \tilde{K}_1(\mathbb{Z}_{(2)}[\pi]).$$

See Section 5 below for an extension of this procedure to quadratic Poincaré complexes which are not highly-connected, and to other algebraic situations. Also, see Section 6 for the connection with the localization exact sequence.

4. Semilocal Rings

The *Jacobson radical* of a ring A is

$$\operatorname{rad}(A) = \bigcap_M \operatorname{Ann}(M)$$

where the intersection is over all simple left A-modules M, and the annihilator is defined by

$$\mathrm{Ann}(M) = \{r \in A \mid rM = 0\}.$$

$\mathrm{Rad}(A)$ is a two-sided ideal of A, and if I is a two-sided ideal contained in $\mathrm{rad}(A)$ there is defined an isomorphism of rings

$$A/\mathrm{rad}(A) \to (A/I)/\mathrm{rad}(A/I).$$

A matrix over A is invertible if and only if it is invertible over $A/\mathrm{rad}(A)$ (Bass [1], III, Section 2).

A ring A is *semilocal* if $A/\mathrm{rad}(A)$ is semisimple, and *local* if $A/\mathrm{rad}(A)$ is a division ring.

EXAMPLE 4.1. Given an integer $n > 1$ define the multiplicative subset

$$S_n = \{a \in \mathbb{Z} \mid (a, n) = 1\} \subset \mathbb{Z}.$$

The localization of $\mathbb{Z}$ at n

$$\mathbb{Z}_{(n)} = (S_n)^{-1}\mathbb{Z}$$

is semilocal for all n, and local if n is prime. Indeed, if $n = (p_1)^{e_1}(p_2)^{e_2}\cdots(p_r)^{e_r}$ is the prime factorization of n then

$$\mathrm{rad}(\mathbb{Z}_{(n)}) = p_1\mathbb{Z}_{(n)} \cap p_2\mathbb{Z}_{(n)} \cap \cdots \cap p_r\mathbb{Z}_{(n)},$$

and

$$\mathbb{Z}_{(n)}/\mathrm{rad}(\mathbb{Z}_{(n)}) = \mathbb{F}_{p_1} \times \mathbb{F}_{p_2} \times \cdots \times \mathbb{F}_{p_r}$$

by the Chinese Remainder Theorem. □

PROPOSITION 4.2. *Let R be a commutative semilocal ring and B an R-algebra which is finitely generated as an R-module. Then B is semilocal.*

Proof. Since $R \subset \mathrm{center}(B)$ and B is finitely generated over R, we have $\mathrm{rad}(R) \subset \mathrm{rad}(B)$ (see Bass [1], III, Section 2). Thus, $B/\mathrm{rad}(B)$ is a finite $(R/\mathrm{rad}(R))$-algebra and, hence, is Artinian. Since in addition $\mathrm{rad}(B/\mathrm{rad}(B)) = 0$, $B/\mathrm{rad}(B)$ is semisimple by Wedderburn theory. □

Proposition 4.2. shows that for a finite group π the rings $\mathbb{Z}_{(n)}[\pi]$ $(n > 1)$ are semilocal, which is of interest to computations of local surgery obstruction theory. Moreover, if π is a p-group for some prime p then $\mathbb{Z}_{(p)}[\pi]$ is actually a local ring (Bass [1], XI, 1.2).

5. How to Make a Complex Cross

Let $A \to B$ be a morphism of rings with involution. A f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A is *B-cross* if $B \otimes_A (C, \psi)$ is cross. We shall later be particularly concerned with the example $A = \mathbb{Z}[\pi] \to B = \mathbb{Z}_{(n)}[\pi]$ for a finite group π.

THEOREM 5.1. *Suppose that $A \to B$ is a morphism of rings with involution, that B is semilocal, and that the composite $A \to B \to F = B/\mathrm{rad}(B)$ is onto.*

Let (C, ψ) be a f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex over A such that $(\bar{C}, \bar{\psi}) = F \otimes_A (C, \psi)$ is a finite $(2i+1)$-dimensional quadratic Poincaré complex over F with $(\bar{C}, \bar{\psi}) = 0 \in L^h_{2i+1}(F)$. Then (C, ψ) is homotopy equivalent to a B-cross complex. □

The proof of Theorem 5.1 is somewhat involved, and is deferred until after the statement of Example 5.3 below. Remark (b) of Pardon [8], p. 143 is the highly-connected finite case of 5.1 with $A = \mathbb{Z}[\pi]$ for a finite 2-group π, $B = \mathbb{Z}_{(2)}[\pi]$, $F = \mathbb{F}_2$.

COROLLARY 5.2.(i) *For a local ring B $L^h_{2i+1}(B) = 0$ and the semitorsion defines isomorphisms*

$$\tau_{1/2}: L^s_{2i+1}(B) \to \mathrm{coker}(L^h_{2i+2}(B) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_1(B))).$$

(ii) *For a semilocal ring B the map $L^h_{2i+1}(B) \to L^h_{2i+1}(B/\mathrm{rad}(B))$ is injective.*

Proof. By taking $A = B$(ii) follows from 5.1. Part (i) follows from (ii), 1.2 and the fact that for any division ring F $\tilde{K}_0(F) = 0$ and, thus,

$$L^h_{2i+1}(F) = L^p_{2i+1}(F) = 0$$

by Ranicki [9]. □

EXAMPLE 5.3. Given a finite group π of order $|\pi|$ define a *-invariant subgroup

$$A = \mathrm{im}(\tilde{K}_1(\mathbb{Z}[\pi]) \to \tilde{K}_1(\mathbb{Z}_{(\pi)}[\pi])) \subseteq \tilde{K}_1(\mathbb{Z}_{(\pi)}[\pi]),$$

where $\mathbb{Z}_{(\pi)}$ is short for the localization $\mathbb{Z}_{(|\pi|)}$ of $\mathbb{Z}$ at $|\pi|$. If $\mathbb{Z}[\pi]$ is given the oriented involution $g \to g^{-1} (g \in \pi)$ then by Davis [3] the map $L^h_3(\mathbb{Z}[\pi]) \to L^A_3(\mathbb{Z}_{(\pi)}[\pi])$ is injective. Thus, if $n = 4j+3$ and $(f, b): M^n \to X$ is a degree one normal map to a finite orientable geometric Poincaré complex X with $\pi_1(X) = \pi$ the surgery obstruction $\sigma_*(f, b) \in L^h_n(\mathbb{Z}[\pi])$ is determined by a semicharacteristic and a cross torsion invariant. Indeed, consider the exact sequence

$$\cdots \to \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_1(\mathbb{Z}_{(\pi)}[\pi])/A) \to L^A_n(\mathbb{Z}_{(\pi)}[\pi]) \to L^h_n(\mathbb{Z}_{(\pi)}[\pi]) \to \cdots.$$

By 5.2(ii) the image of $\sigma_*(f, b)$ in $L^h_n(\mathbb{Z}_{(\pi)}[\pi])$ is detected by the surgery semicharacteristic in $\mathbb{Z}_{(\pi)}[\pi]/\mathrm{rad}(\mathbb{Z}_{(\pi)}[\pi])$, which can be easily computed as in Davis [2]. If this vanishes we may apply the algorithm of 5.1 (below) to the 'instant surgery obstruction' quadratic Poincaré complex $(C(f^!), \psi)$ representing $\sigma_*(f, b)$ given by Ranicki [10], to obtain a homotopy equivalent complex (C', ψ') such that

$\mathbb{Z}_{(n)} \otimes_{\mathbb{Z}[\pi]} (C',\psi')$ is cross. Then $\sigma_*(f,b)$ is determined by the cobordism class of this complex in $L^A_n(\mathbb{Z}_{(\pi)}[\pi])$, which by Proposition 3.2(ii) is determined by the cross semitorsion. See Davis and Weinberger [4] for geometric applications. □

Proof of 5.1. Each C_r is a f.g. free F-module (by hypothesis). Define

$$c_r = \dim_F \bar{C}_r \ (= 0 \text{ for } r < 0 \text{ and } > 2i + 1),$$

and let

$$\bar{B}_r = \operatorname{im}(\bar{d}: \bar{C}_{r+1} \to \bar{C}_r).$$

The proof is in two steps.

Step 1 Replace (C,ψ) by a homotopy equivalent complex for which

(i) $c_r = c_{2i+1-r}$ $(0 \leqslant r \leqslant 2i + 1)$,

(ii) there is a $g \in \operatorname{Hom}_F(\bar{B}^i, \bar{B}_i)$ such that $g + (-)^{i+1} g^*$: $\bar{B}^i \to \bar{B}_i$ is an isomorphism.

Proof. For $u, v \geqslant 0$ let $E = E(u, u + 1)^v$ be the contractible finite A-module chain complex defined by

$$d_E = 1: E_{u+1} = A^v \to E_u = A^v, \quad E_r = 0 \text{ for } r \neq u, u + 1.$$

Assume inductively that $c_r = c_{2i+1-r} (r \leqslant p < i)$ for some p. If $p < i - 1$ replace C by the chain equivalent complex

$$C' = C \oplus E(p + 1, p + 2)^{c_{2i-p}} \oplus E(2i - p - 1, 2i - p)^{c_{p+1}}$$

for which

$$c'_r = c'_{2i+1-r} \ (r \leqslant p + 1 < i).$$

By Poincaré duality $\chi(\bar{C}) = 0$, so that if $p = i - 1$

$$0 = \chi(\bar{C}) = \Sigma_{r=0}^{2i+1} (-)^r c_r = \Sigma_{r=0}^{i} (-)^r (c_r - c_{r-1}) = (-)^i (c_i - c_{i-1}),$$

and (i) holds.

Since $(\bar{C}, \bar{\psi}) = 0 \in L^h_{2i+1}(F)$ the surgery semicharacteristic

$$\chi_{1/2}(\bar{C}) = \Sigma_{r=0}^{i} (-)^r [H_r(\bar{C})] \in \operatorname{coker}(\tau_0: L^p_{2i+2}(F) \to \hat{H}^{2i+2}(\mathbb{Z}_2; \tilde{K}_0(F)))$$

is zero. The projective class of the chain complex

$$E: \bar{B}_i \to \bar{C}_i \to \bar{C}_{i-1} \to \cdots \to \bar{C}_0$$

is equal to the projective class of its homology

$$(-)^{i+1} [\bar{B}_i] + \Sigma_{r=0}^{i} (-)^r [\bar{C}_r] = \Sigma_{r=0}^{i} (-)^r [H_r(\bar{C})] \in K_0(F).$$

Each $\bar{C}_r$ is a f.g. free F-module (by hypothesis), so that $[\bar{C}_r] = 0 \in \tilde{K}_0(F)$ and

$$[\bar{B}_i] = (-)^{i+1} \chi_{1/2}(\bar{C}) = 0 \in \operatorname{coker}(\tau_0).$$

By Remark 1.3, $\bar{B}_i \oplus F^k$ satisfies (ii) for some $k \geqslant 0$. We then replace C by $C \oplus E(i,i+1)^k$.

It may now be assumed that (C,ψ) satisfies the conditions (i) and (ii) of Step 1. Before proceeding to Step 2 let us recall (from Ranicki [10, 11]) the precise definition of the quadratic structure $\psi \in Q_n(C)$.

Given a finite-dimensional A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on $\mathrm{Hom}_A(C^*, C)$ by the signed duality involution

$$T: \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p); \quad \psi \to T\psi = (-)^{pq}\psi^*$$

$$(\psi \in \mathrm{Hom}_A(C^p, C_q)).$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W: \cdots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \to$$

$$\cdots \to \mathbb{Z}[\mathbb{Z}_2\} \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2].$$

An *n-dimensional quadratic chain of C* $\{\psi_s\}$ is a chain

$$\{\psi_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C))_n,$$

as defined by a collection of A-module morphisms

$$\{\psi_s \in \mathrm{Hom}_A(C^{n-r-s}, C_r) | r \in \mathbb{Z}, s \geqq 0\}.$$

The *boundary* of $\{\psi_s\}$ is the $(n-1)$-dimensional quadratic chain

$$d\{\psi_s\} = \{(d\psi)_s\}$$

defined by

$$(d\psi)_s = d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}):$$
$$C^{n-r-s-1} \to C_r.$$

There are defined quadratic chains, boundaries, cycles ($d\{\psi_s\} = 0$), and in particular the *quadratic homology groups*

$$Q_n(C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C)).$$

A chain equivalence of A-module chain complexes $f: C \to D$ induces isomorphisms in the Q-groups

$$f_\%: Q_n(C) \to Q_n(D); \quad \{\psi_s\} \to \{f\psi_s f^*\}.$$

Returning to the $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) over A satisfying (i) and (ii) of Step 1, we now wish to obtain:

Step 2 Find a quadratic cycle representative $\{\psi_s\}$ of $\psi \in Q_{2i+1}(C)$ for which $(1+T)\psi_0: C^{2i+1-*} \to C$ is a chain B-isomorphism.

Proof. Every finite chain complex over a semisimple ring is chain isomorphic to a direct sum of a contractible complex and a complex with zero differentials. Explicitly, define F-module chain complexes C', D by

$$d_{\bar{C}'} = 0 \colon \bar{C}'_r = H_r(\bar{C}) \to \bar{C}'_{r-1} = H_{r-1}(\bar{C}),$$

$$d_{\bar{D}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \colon \bar{D}_r = \bar{B}_r \oplus \bar{B}_{r-1} \to \bar{D}_{r-1} = \bar{B}_{r-1} \oplus \bar{B}_{r-2}.$$

As F is a semisimple ring there exists a chain isomorphism

$$f \colon \bar{C} \to \bar{C}' \oplus \bar{D}$$

inducing

$$f_* = 1 \colon H_*(\bar{C}) \to H_*(\bar{C}' \oplus \bar{D}) = H_*(\bar{C}').$$

It follows from

$$c_r = c_{2i+1-r}, \qquad \bar{C}'_r \cong \bar{C}'^{2i+1-r} \quad (r \in \mathbb{Z}),$$

the condition (ii) above, and the fact that stable isomorphism implies isomorphism for modules over semisimple rings that

$$\bar{B}_z \oplus \bar{B}_{r-1} \cong \bar{B}^{2i+1-r} \oplus \bar{B}^{2i-r}.$$

By induction on r there exist F-module isomorphisms

$$h \colon \bar{B}^{2i-r} \to \bar{B}_r \quad (r \in \mathbb{Z}).$$

For $r = i$ we choose $h = g + (-)^{i+1} g^*$.

Define a quadratic cycle $\{\bar{\theta}_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_F(\bar{D}^*, \bar{D}))_{2i+1}$ by

$$\bar{\theta}_0 = \begin{pmatrix} 0 & h \\ (-)^r h & 0 \end{pmatrix} \colon$$

$$\bar{D}^{2i+1-r} = \bar{B}^{2i+1-r} \oplus \bar{B}^{2i-r} \to \bar{D}_r = \bar{B}_r \oplus \bar{B}_{r-1} \quad (r \geqslant i+1),$$

$$\bar{\theta}_1 = \begin{pmatrix} (-)^i g & 0 \\ 0 & 0 \end{pmatrix} \colon \bar{D}^i = \bar{B}^i \oplus \bar{B}^{i-1} \to \bar{D}_r = \bar{B}_r \oplus \bar{B}_{r-1},$$

$$\bar{\theta}_s = 0 \colon \bar{D}^{2i+1-r-s} \to \bar{D}_r \text{ otherwise}$$

for which $(1+T)\bar{\theta}_0 \colon \bar{D}^{2i+1-*} \to \bar{D}$ is an isomorphism.

The isomorphism

$$f_\% \colon Q_{2i+1}(\bar{C}) \to Q_{2i+1}(\bar{C}' \oplus \bar{D}) (= Q_{2i+1}(\bar{C}'))$$

sends $\bar{\psi} \in Q_{2i+1}(\bar{C})$ to the element $f_\%(\bar{\psi})$ with a representative quadratic cycle of the type $\{\bar{\psi}'_s \oplus \bar{\theta}_s\}$ for some quadratic cycle

$$\{\bar{\psi}'_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_F(\bar{C}', \bar{C}'))_{2i+1},$$

with $(1+T)\bar{\psi}'_0 \colon \bar{C}'^{2i+1-*} \to \bar{C}'$ a chain isomorphism. Thus for any quadratic cycle

$\{\psi_s\}$ representing $\psi \in Q_{2i+1}(C)$ there exists a quadratic chain

$$\{\bar{\rho}_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_F(\bar{C}^*, \bar{C}))_{2i+2}$$

such that the quadratic cycle

$$\begin{aligned}\{\bar{\psi}_s''\} &= \{\bar{\psi}_s\} + \bar{d}\{\bar{\rho}_s\} \\ &= \{f^{-1}(\bar{\psi}_s' \oplus \bar{\theta}_s')f^{*-1}\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_F(\bar{C}^*, \bar{C}))_{2i+1}\end{aligned}$$

has $(1+T)\bar{\psi}_0'': \bar{C}^{2i+1-*} \to \bar{C}$ a chain isomorphism. As $A \to F$ is onto it is possible to lift $\{\bar{\rho}_s\}$ onto a chain

$$\{\rho_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C))_{2i+2}.$$

The quadratic cycle

$$\{\psi_s''\} = \{\psi_s\} + d\{\rho_s\} \in (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C))_{2i+1}$$

represents $\psi \in Q_{2i+1}(C)$ and has $(1+T)\psi_0'': C^{2i+1-*} \to C$ a chain B-isomorphism, since a matrix over B is invertible if it is invertible over $B/\mathrm{rad}(B)$. □

6. Torsion Semi-Invariants

We are interested in the surgery invariants arising from the localization exact sequences in K- and L-theory. In Section 6 we deal with the 'torsion characteristic' and the 'torsion semicharacteristic', leaving the 'cross torsion semicharacteristic' to Section 7.

Let A be a ring with involution, and let $S \subset A$ be a multiplicative subset which is invariant under the involution. The localization $S^{-1}A$ of A inverting S is then a ring with involution and the inclusion defines a morphism of rings with involution

$$j: A \to S^{-1}A; \quad a \to \frac{a}{1}.$$

An A-module chain complex C is *S-acyclic* if $S^{-1}C = S^{-1}A \otimes_A C$ is an acyclic $S^{-1}A$-module chain complex.

An *(A, S)-module* M is an S-torsion A-module which admits a f.g. projective A-module resolution

$$0 \to P_1 \xrightarrow{d} P_0 \to M \to 0.$$

The *S-dual* of M is the (A, S)-module

$$\begin{aligned}&M\hat{} = \mathrm{Hom}_A(M, S^{-1}A/A), \\ &A \times M\hat{} \to M\hat{}; \quad (a, f) \to (x \to f(x)\cdot \bar{a}),\end{aligned}$$

with the dual f.g. projective A-module resolution

$$0 \to (P_0)^* \xrightarrow{d^*} (P_1)^* \to M\hat{} \to 0.$$

The localization exact sequence of Bass [1]

$$\cdots \to \tilde{K}_1(A) \xrightarrow{j_1} \tilde{K}_1(S^{-1}A) \xrightarrow{\partial} K_1(A,S)$$
$$\xrightarrow{\sigma} \tilde{K}_0(A) \xrightarrow{j_0} \tilde{K}_0(S^{-1}A) \to \cdots$$

involves the Grothendieck group $K_1(A,S)$ of stable isomorphism classes of (A,S)-modules, with

$$\partial\colon \tilde{K}_1(S^{-1}A) \to K_1(A,S);$$
$$\tau(S^{-1}d\colon S^{-1}P_1 \to S^{-1}P_0) \to [\mathrm{coker}(d\colon P_1 \to P_0)],$$
$$\sigma\colon K_1(A,S) \to \tilde{K}_0(A); \quad [M] \to [P_0] - [P_1].$$

The duality involution on $K_1(A,S)$ is defined by

$$*\colon K_1(A,S) \to K_1(A,S); \quad [M] \to [M\hat{\ }]$$

and is such that

$$j* = *j, \qquad \partial * = *\partial, \qquad \sigma * = -*\sigma.$$

An S-acyclic chain complex C is *(H,S)-torsion* if the homology A-modules $H_*(C)$ are (A,S)-torsion. In dealing with (H,S)-torsion complexes $H_*(C)[i,j]$ is to be interpreted as an S-acyclic f.g. projective complex D with a chain map $D \to C$ inducing isomorphisms

$$H_r(D) \to H_r(C) \quad \text{if } i \leqslant r \leqslant j,$$

and such that $H_r(D) = 0$ for $r \notin [i,j]$.

The *torsion characteristic* of a finite-dimensional S-acyclic A-module chain complex C is an invariant

$$\chi^S(C) \in K_1(A,S)$$

such that

(i) $\chi^S(C') = \chi^S(C) + \chi^S(C'') \in K_1(A,S)$ for a short exact sequence

$$0 \to C \to C' \to C'' \to 0,$$

(ii) if C is (H,S)-torsion

$$\chi^S(C) = \sum_i (-)^i [H_i(C)] \in K_1(A,S).$$

The torsion characteristic is such that

$$\sigma(\chi^S(C)) = [C] \in \tilde{K}_0(A),$$

and

$$\chi^S(C^{n-*}) = (-)^{n-1}\chi^S(C)^* \in K_1(A,S).$$

Localization exact sequences in L-theory have been developed by many authors, including Karoubi, Carlsson-Milgram, Pardon and Ranicki. We review this theory in the notation of Ranicki [11], Section 3.

Given a $*$-invariant subgroup $X \subseteq K_1(A, S)$ there is defined a localization exact sequence

$$\cdots \to L_n^{\sigma X}(A) \to L_n^{\partial^{-1}(X)}(S^{-1}A) \to L_n^X(A, S) \to L_{n-1}^{\sigma X}(A) \to \cdots$$

with $L_n^X(A, S)$ the cobordism group of S-acyclic $(n-1)$-dimensional quadratic Poincaré complexes (C, ψ) over A with

$$\chi^S(C) = (-)^n \chi^S(C)^* \in X \subseteq K_1(A, S).$$

Given $*$-invariant subgroups $Y \subseteq X \subseteq K_1(A, S)$ there is a Rothenberg type localization exact sequence

$$\cdots \to \hat{H}^{n+1}(\mathbb{Z}_2; X/Y) \to L_n^Y(A, S) \to L_n^X(A, S) \xrightarrow{\chi^S} \hat{H}^n(\mathbb{Z}_2; X/Y) \to \cdots,$$

By analogy with Definition 1.2:

PROPOSITION 6.1. *For any* (A, S) *and* $*$*-invariant subgroups* $Y \subseteq X \subseteq K_1(A, S)$ *there are defined isomorphisms*

$$\ker(L_n^Y(A, S) \to L_n^X(A, S)) \to \operatorname{coker}(L_{n+1}^X(A, S) \to \hat{H}^{n+1}(\mathbb{Z}_2; X/Y));$$
$$(C, \psi) \to \chi^S(D)$$

with (C, ψ) *an* S*-acyclic* $(n-1)$*-dimensional quadratic Poincaré complex over* A *such that*

$$\chi^S(C) \in Y \subseteq K_1(A, S),$$

and $(D, C; \delta\psi, \psi)$ *any* S*-acyclic* n*-dimensional quadratic Poincaré null-cobordism of* (C, ψ) *such that*

$$\chi^S(D) \in X \subseteq K_1(A, S),$$

in which case

$$\chi^S(C) = \chi^S(D) + (-)^n \chi^S(D)^* \in Y \subseteq X \subseteq K_1(A, S). \qquad \square$$

Define the torsion L-groups $L_*^q(A, S)$ for $q = p, h, s$ by

$$L_n^p(A, S) = L_n^X(A, S) \quad \text{where } X = \{0\} \subseteq K_1(A, S).$$
$$L_n^h(A, S) = L_n^X(A, S) \quad \text{where } X = \ker(\sigma) \subseteq K_1(A, S),$$
$$L_n^s(A, S) = L_n^X(A, S) \quad \text{where } X = \{0\} \subseteq K_1(A, S).$$

The pair (A, S) is *odd* if $\hat{H}^*(\mathbb{Z}_2; S^{-1}A/A) = 0$, or equivalently if $\hat{H}^*(\mathbb{Z}_2; A) \to \hat{H}^*(\mathbb{Z}_2; S^{-1}A)$ is an isomorphism. For odd (A, S) the symmetrization maps $1+T: Q_*(C) \to Q^*(C)$ are isomorphisms for S-acyclic finite-dimensional A-module chain complex C, and there is no difference between quadratic and symmetric structures on S-acyclic complexes. We refer to Ranicki [10, 11] for the symmetric version of the theory.

EXAMPLE 6.2. Any pair (A, S) with $\frac{1}{2} \in A$ is odd, with

$$\hat{H}^*(\mathbb{Z}_2; A) = \hat{H}^*(\mathbb{Z}_2; S^{-1}A) = \hat{H}^*(\mathbb{Z}_2; S^{-1}A/A) = 0. \qquad \square$$

EXAMPLE 6.3. Give the Laurent polynomial extension ring $\mathbb{Z}[t, t^{-1}]$ of $\mathbb{Z}$ the involution $\bar{t} = t^{-1}$, and let $P \subset \mathbb{Z}[t, t^{-1}]$ be the multiplicative subset of the polynomials $p(t) \in \mathbb{Z}[t, t^{-1}]$ such that $p(1) = 1$, such as the Alexander polynomials of knots. A finite-dimensional $\mathbb{Z}[t, t^{-1}]$-module chain complex C is P-acyclic if and only if $\mathbb{Z} \otimes_{\mathbb{Z}[t,t^{-1}]} C$ is acyclic. The pair $(\mathbb{Z}[t,t^{-1}], P)$ is odd, which may be verified as follows. Given an element

$$x = \frac{a(t)}{p(t)} \in P^{-1}\mathbb{Z}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}]$$

let $q(t) \in \mathbb{Z}[t, t^{-1}]$ be such that

$$p(t) = 1 + (1-t)q(t) \in \mathbb{Z}[t, t^{-1}].$$

For any $r \pmod 2$

$$x = y + (-)^r\bar{y} + (x - (-)^r\bar{x})z + a(t)p(t^{-1}) \in P^{-1}\mathbb{Z}[t, t^{-1}]$$

with y, z defined by

$$\begin{aligned} y = a(t)(tq(t) - q(t^{-1}) + q(t)q(t^{-1}) \\ - tq(t)q(t^{-1}))/p(t) \in P^{-1}\mathbb{Z}[t, t^{-1}], \end{aligned}$$

$$\begin{aligned} z = t^{-1}q(t^{-1}) - q(t) + q(t^{-1})q(t) \\ - t^{-1}q(t^{-1})q(t) \in \mathbb{Z}[t, t^{-1}] \subset P^{-1}\mathbb{Z}[t, t^{-1}]. \end{aligned}$$

Thus if

$$x - (-)^r\bar{x} \in \mathbb{Z}[t, t^{-1}] \subset P^{-1}\mathbb{Z}[t, t^{-1}],$$

representing an element $x \in \hat{H}^r(\mathbb{Z}_2; P^{-1}\mathbb{Z}[t,t^{-1}]/\mathbb{Z}[t,t^{-1}])$, then

$$x = y + (-)^r\bar{y} \in P^{-1}\mathbb{Z}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}]$$

and so

$$x = 0 \in \hat{H}^r(\mathbb{Z}_2; P^{-1}\mathbb{Z}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}]).$$

See Section 7.9 of Ranicki [11] for the identification

$$L^h_{n+3}(\mathbb{Z}[t, t^{-1}], P) = C_n \quad (n \geqslant 4)$$

of the torsion L-groups of $(\mathbb{Z}[t,t^{-1}],P)$ and the high-dimensional knot cobordism groups C_n. Since $\tilde{K}_0(\mathbb{Z}[t,t^{-1}])=0$

$$L^h_*(\mathbb{Z}[t,t^{-1}],P)=L^p_*(\mathbb{Z}[t,t^{-1}],P).$$

Given a knot $k\colon S^n\subset S^{n+2}$ with complement

$$X=\text{closure}(S^{n+2}-\text{neighbourhood of }k(S^n))$$

there is defined a $\mathbb{Z}$-homology equivalence $(n+2)$-dimensional normal map

$$(f,b)\colon(X,\partial X)\to(D^{n+3}\times S^1,S^n\times S^1)$$

which is the identity $\partial X\to S^n\times S^1$ on the boundary. The torsion characteristic of the knot cobordism class

$$\sigma_*(f,b)=[k]\in L^h_{n+3}(\mathbb{Z}[t,t^{-1}],P)=C_n$$

is the knot cobordism invariant defined by the Reidemeister torsion

$$\chi^P(\sigma_*(f,b))=\tau([k])\in\hat{H}^{n+3}(\mathbb{Z}_2;K_1(\mathbb{Z}[t,t^{-1}],P)).$$

Thd odd property of $(\mathbb{Z}[t,t^{-1}],P)$ gives a quick proof of Proposition 7.9.2 (ii) of [11] that for a P-acyclic $\mathbb{Z}$-module chain complex C the symmetrization chain maps

$$1+T\colon Q_*(C)\to Q^*(C)$$

are isomorphisms. In particular, this helps explain why only the symmetric structure of the Blanchfield form is relevant in the odd-dimensional knot cobordism group $L^h_{2i+2}(\mathbb{Z}[t,t^{-1}],P)=C_{2i-1}$. □

A $(2i+1)$-dimensional quadratic Poincaré complex over A which is H-projective bounds (Proposition 2.1(i)). There is a corresponding result for (H,S)-torsion $2i$-dimensional symmetric Poincaré complexes, but not in general for (H,S)-torsion quadratic Poincaré complexes.

PROPOSITION 6.4. *Given an (H,S)-torsion $2i$-dimensional quadratic Poincaré complex (C,ψ) over A there is defined an (H,S)-torsion $(2i+1)$-dimensional symmetric Poincaré null-cobordism $(H_*(C)[0,i-1],C;0,(1+T)\psi)$ of the symmetrization $(C,(1+T)\psi)$.*

Proof. The composite $C\to H_*(C)\to H_*(C)[0,i-1]$ sends $(1+T)\psi\in Q^{2i}(C)$ to $0\in Q^{2i}(H_*(C)[0,i-1])$. □

REMARK 6.5. The ith quadratic linking Wu class ([11], Section 3.3) of an (H,S)-torsion $2i$-dimensional quadratic Poincaré complex over $A(C,\psi)$

$$\begin{aligned}v^i_S(\psi)\colon H^i(C)\to\hat{H}^i(\mathbb{Z}_2;S^{-1}A/A)\\ =\{x\in S^{-1}A/A\,|\,\bar{x}=(-)^ix\}/\{y+(-)^i\bar{y}\,|\,y\in S^{-1}A/A\}\end{aligned}$$

is the obstruction to defining an (H,S)-torsion $(2i+1)$-dimensional quadratic Poincaré null-cobordism $(H_*(C)[i,2i-1],C;\delta\psi,\psi)$ of (C,ψ). See Section 2 of

Milgram and Ranicki [7] for an example realizing this obstruction, with Kervaire invariant

$$(C,\psi) = 1 \in L_3(\mathbb{Z}, \mathbb{Z} - \{0\}) = L_2(\mathbb{Z}) = \mathbb{Z}_2. \qquad \square$$

Define the *torsion semicharacteristic* of an (H,S)-torsion 2i-dimensional quadratic Poincaré complex (C,ψ) over A by

$$\chi^S_{1/2}(C) = \sum_{r=0}^{i-1} (-)^r [H_r(C)] \in K_1(A,S).$$

PROPOSITION 6.6. (i) *The torsion semicharacteristic and the torsion characteristic are related by*

$$\chi^S(C) = \chi^S_{1/2}(C) - \chi^S_{1/2}(C)^* \in K_1(A,S).$$

(ii) *If (A,S) is such that $\hat{H}^i(\mathbb{Z}_2; S^{-1}A/A) = 0$ then for any $*$-invariant subgroup $X \subseteq K_1(A,S)$ and any (H,S)-torsion $2i$-dimensional quadratic Poincaré complex over A (C,ψ) such that $\chi^S(C) \in X$*

$$\begin{aligned}(C,\psi) = \chi^S_{1/2}(C) &\in \ker(L^X_{2i+1}(A,S) \to L^p_{2i+1}(A,S)) \\ &= \operatorname{coker}(L^p_{2i+2}(A,S) \to \hat{H}^{2i+2}(\mathbb{Z}_2; K_1(A,S)/X)).\end{aligned}$$

Proof. (i) By the universal coefficient theorem and Poincaré duality we have that up to isomorphism

$$H_r(C) = H^{2i-r}(C) = H_{2i-1-r}(C)^\wedge,$$

so that by the homology invariance of the torsion characteristic

$$\begin{aligned}\chi^S(C) = \chi^S(H_*(C)) &= \sum_{r=0}^{2i-1} (-)^r [H_r(C)] \\ &= \sum_{r=0}^{i-1} (-)^r [H_r(C)] + \sum_{r=i}^{2i-1} (-)^r [H_{2i-1-r}(C)^\wedge] \\ &= \chi^S_{1/2}(C) - \chi^S_{1/2}(C)^* \in K_1(A,S).\end{aligned}$$

(ii) Immediate from (i), Proposition 6.1 and Remark 6.5. $\square$

EXAMPLE 6.7. Let $(f,b)\colon M^{2i} \to X^{2i}$ be a normal map of closed $2i$-dimensional manifolds with finite fundamental group $\pi_1(X) = \pi$, such that the kernels

$$K_*(M) = \ker(\tilde{f}_*\colon H_*(\tilde{M}) \to H_*(\tilde{X}))$$

are $(\mathbb{Z}[\pi], S)$-modules, with $S = \mathbb{Z} - \{0\} \subset \mathbb{Z}[\pi]$, and such that for odd i the Kervaire invariant is 0

$$\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \sigma_*(f,b) = 0 \in L_{2i}(\mathbb{Z}) = \mathbb{Z}_2.$$

Theorem 1.11 of Milgram and Ranicki [7] identifies

$$\begin{aligned}\sigma_*(f,b) &= \sigma(\chi^S_{1/2}(f)) \in \ker(L^h_{2i}(\mathbb{Z}[\pi]) \to L^p_{2i}(\mathbb{Z}[\pi])) \\ &= \operatorname{coker}(L^p_{2i+1}(\mathbb{Z}[\pi]) \to \hat{H}^{2i+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi]))),\end{aligned}$$

with

$$\chi^S_{1/2}(f) = \sum_{r=0}^{i-1} (-)^r [K_r(M)] \in K_1(\mathbb{Z}[\pi], S)$$

the torsion semicharacteristic of the S-acyclic $2i$-dimensional quadratic kernel $\sigma_*(f,b) = (C(f^!), \psi)$ over $\mathbb{Z}[\pi]$. □

The pair (A, S) is *simple* if every S-acyclic finite-dimensional A-module chain complex C is (H, S)-torsion, or equivalently if every f.g. S-torsion A-module is an (A, S)-module. ((A, S) is '0-dimensional' in the terminology of [11], p. 211).

EXAMPLE 6.8. For any Dedekind ring A the pair $(A, A - \{0\})$ is simple. □

EXAMPLE 6.9. If π is a finite group and p is a prime not dividing the order $|\pi|$ then the pair $(\mathbb{Z}[\pi], \{p\})$ is simple by Rim [12]. □

PROPOSITION 6.10. *For odd simple (A, S) and any $*$-invariant subgroup $X \subseteq K_1(A, S)$ the torsion semicharacteristic defines an isomorphism*

$$\begin{aligned}&\chi^S_{1/2} : L^X_{2i+1}(A, S) \to \operatorname{coker}(L^p_{2i+2}(A, S) \to \hat{H}^{2i+2}(\mathbb{Z}_2; K_1(A, S)/X)); \\ &(C, \psi) \to \chi^S_{1/2}(C).\end{aligned}$$

In particular, $L^p_{2i+1}(A, S) = 0$.

Proof. Immediate from Proposition 6.4 and Remark 6.5. □

EXAMPLE 6.11. Let A be any ring with involution which is of characteristic 0, so that $\mathbb{Z} \subseteq A$. Let $S \subset A$ be a multiplicative subset of odd integers, and let $T \subset A$ be the multiplicative subset of the integers coprime to each $s \in S$, so that $2 \in T$ and

$$\hat{H}^*(\mathbb{Z}_2; T^{-1}A) = 0.$$

The pair (A, S) is odd, since there is defined an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$S^{-1}A/A \to \mathbb{Q} \otimes A/T^{-1}A$$

inducing an isomorphism of the Tate $\mathbb{Z}_2$-cohomology groups

$$\hat{H}^*(\mathbb{Z}_2; S^{-1}A/A) \to \hat{H}^*(\mathbb{Z}_2; \mathbb{Q} \otimes A/T^{-1}A) = 0.$$ □

EXAMPLE 6.12. Given a finite group π let $S_{|\pi|} \subset \mathbb{Z}[\pi]$ be the multiplicative subset consisting of the integers coprime to $|\pi|$, so that

$$(S_{|\pi|})^{-1}\mathbb{Z}[\pi] = \mathbb{Z}_{(|\pi|)}[\pi].$$

If the order $|\pi|$ is even then the pair $(\mathbb{Z}[\pi], S_{|\pi|})$ is both odd and simple, and Proposition 6.10 applies. □

7. The Cross Torsion Semicharacteristic

In the first instance we define the *cross semicharacteristic* of a f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A

$$\chi^{\dagger}_{1/2}(C,\psi) = \sum_{r=0}^{i} (-)^r[C_r] \in \tilde{K}_0(A).$$

If (C,ψ) is H-projective and $d = 0: C_{i+1} \to C_i$ then

$$\chi^{\dagger}_{1/2}(C,\psi) = [C[0,i]] = [H_*(C)[0,i]] = \chi_{1/2}(C,\psi) \in \tilde{K}_0(A).$$

If (C, ψ) is cross then

$$[C] = \chi^{\dagger}_{1/2}(C,\psi) - \chi^{\dagger}_{1/2}(C,\psi)^* \in \tilde{K}_0(A),$$

and there is defined a null-cobordism $(C[i+1, 2i+1],\ C; 0, \psi)$ as in 3.1, with projective class

$$[C[i+1, 2i+1]] = -\chi^{\dagger}_{1/2}(C,\psi)^* \in \tilde{K}_0(A).$$

Let now (C, ψ) be a f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex over A which is $S^{-1}A$-cross, for some multiplicative subset $S \subset A$. (Perhaps (C, ψ) was constructed by 5.1.) The *cross torsion semicharacteristic* of (C, ψ) is defined by

$$\chi^{S\dagger}_{1/2}(C,\psi) = \sum_{r=0}^{i} (-)^r [\operatorname{coker}((1+T)\psi_0 : C^{2i+1-r} \to C_r)] \in K_1(A,S).$$

The image of $\chi^{S\dagger}_{1/2}(C,\psi)$ under $\sigma: K_1(A,S) \to \tilde{K}_0(A)$ is such that

$$\sigma(\chi^{\dagger}_{1/2}(C,\psi)) = \chi^{\dagger}_{1/2}(C,\psi) - \chi^{\dagger}_{1/2}(C,\psi)^* - [C] \in \tilde{K}_0(A).$$

PROPOSITION 7.1. (i) *Given an $S^{-1}A$-cross f.g. projective $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) over A there is defined a f.g. projective $(2i+2)$-dimensional quadratic $S^{-1}A$-Poincaré pair $(C[i+1, 2i+1], C; 0, \psi)$ such that the effect of surgery on (C,ψ) by this pair is a cobordant S-acyclic $(i-1)$-connected $(2i+1)$-dimensional quadratic Poincaré complex (C', ψ') over A with*

$$\chi^S(C') = \chi^{S\dagger}_{1/2}(C,\psi) \in K_1(A,S).$$

(ii) *If $X \subseteq K_1(A,S)$ is a $*$-invariant subgroup and (C,ψ) is as in* (i), *with*

$$\sum_{r=i+1}^{2i+1} (-)^r [C_r] \in \sigma(X) \subseteq \tilde{K}_0(A), \qquad \chi^{S\dagger}_{1/2}(C, \psi) \in X \subseteq K_1(A, S)$$

then $(C, \psi) \in L^{\sigma X}_{2i+1}(A)$ is the image of $(C', \psi') \in L^X_{2i+2}(A, S)$.

(iii) *If (C,ψ) is as in* (i) *with C a finite complex then*

$$\partial(\tau^{\dagger}_{1/2}(S^{-1}C, S^{-1}\psi)) = \chi^{S\dagger}_{1/2}(C,\psi) \in K_1(A,S).$$

Proof. (i) Up to chain equivalence C' is given by

$$C' : \cdots \to 0 \to C^{i+1} \xrightarrow{\begin{pmatrix}(1+T)\psi_0 \\ d^*\end{pmatrix}} C_i \oplus C^{i+2}$$

$$\xrightarrow{\begin{pmatrix} d & (1+T)\psi_0 \\ 0 & d^* \end{pmatrix}} C_{i-1} \oplus C^{i+3} \to \cdots$$

$$\to C_1 \oplus C^{2i+1} \xrightarrow{(d\,(1+T)\psi_0)} C_0$$

with $H_r(C') = 0 (0 \leqslant r \leqslant i-1)$ and such that $H_i(C')$ is an (A,S)-module. The (A,S)-modules defined by

$$M_r = \operatorname{coker}((1+T)\psi_0 : C^{2i+1-r} \to C_r) \quad (0 \leqslant r \leqslant i)$$

are such that there is defined an exact sequence

$$0 \to H_i(C') \to M_i \xrightarrow{d} M_{i-1} \to$$

$$\cdots \to M_1 \xrightarrow{d} M_0 \to 0.$$

The torsion characteristic of C' is thus

$$\chi^S(C') = \chi^S(H_*(C')) = (-)^i[H_i(C')]$$
$$= \sum_{r=0}^{i} (-)^r [M_r] = \chi^{S\dagger}_{1/2} C, \psi) \in K_1(A,S).$$

(ii) Immediate from (i).

(iii) Immediate from the definition of the map $\partial : \tilde{K}_1(S^{-1}A) \to K_1(A,S)$ in the K-theory localization sequence and the definitions of the semitorsion $\tau_{1/2}$ and the cross torsion semicharacteristic $\chi^{S\dagger}_{1/2}$

$$\tau^{\dagger}_{1/2} S^{-1}(C,\psi) = \sum_{r=0}^{i} (-)^r \tau(S^{-1}(1+T)\psi_0 : S^{-1}C^{2i+1-r} \to S^{-1}C_r) \in \tilde{K}_1(S^{-1}A),$$

$$\chi^{S\dagger}_{1/2}(C,\psi) = \sum_{r=0}^{i} (-)^r [\operatorname{coker}((1+T)\psi_0 : C^{2i+1-r} \to C_r)] \in K_1(A,S). \qquad \square$$

EXAMPLE 7.2 (Detection of $L^p_{2i+1}(\mathbb{Z}[\pi])$). Let π be a finite group, and give $\mathbb{Z}[\pi]$ the oriented involution $\bar{g} = g^{-1} (g \in \pi)$. As in 6.12 let $S = S_{|\pi|} \subset \mathbb{Z}[\pi]$ be the multiplicative subset of the integers coprime to $|\pi|$, so that

$$S^{-1}\mathbb{Z}[\pi] = \mathbb{Z}_{(\pi)}[\pi].$$

$(\mathbb{Z}[\pi], S)$ is simple by Rim [12]. Define the $*$-invariant subgroup

$$A = \ker(\partial: \tilde{K}_1(\mathbb{Z}_{(\pi)}[\pi]) \to K_1(\mathbb{Z}[\pi], S)).$$

By the result of Swan [13] the map $\tilde{K}_0(\mathbb{Z}[\pi]) \to \tilde{K}_0(\mathbb{Z}_{(\pi)}[\pi])$ is zero, so that there is defined a short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \to \tilde{K}_1(\mathbb{Z}_{(\pi)}[\pi])/A \to K_1(\mathbb{Z}[\pi], S)) \to \tilde{K}_0(\mathbb{Z}[\pi]) \to 0.$$

We write this as

$$0 \to G \to H \to K \to 0.$$

There is defined a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & & L^A_{2i+2}(\mathbb{Z}_{(\pi)}[\pi]) & \xrightarrow{\Gamma^s} & L^s_{2i+2}(\mathbb{Z}[\pi], S) & \xrightarrow{\partial^s} & L^h_{2i+1}(\mathbb{Z}[\pi]) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \\
\cdots \longrightarrow & & L^h_{2i+2}(\mathbb{Z}_{(\pi)}[\pi]) & \xrightarrow{\Gamma^p} & L^p_{2i+2}(\mathbb{Z}[\pi], S) & \xrightarrow{\partial^p} & L^p_{2i+1}(\mathbb{Z}[\pi]) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow \chi^S & & \downarrow \\
\cdots \longrightarrow & & \hat{H}^{2i+2}(\mathbb{Z}_2; G) & \longrightarrow & \hat{H}^{2i+2}(\mathbb{Z}_2; H) & \longrightarrow & \hat{H}^{2i+1}(\mathbb{Z}_2; K) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

In Davis [3] it is shown that the composite $\partial^p\alpha$ is 0, so that if (C, ψ) is a f.g. projective $(2i+1)$-dimensional S-acyclic quadratic Poincaré complex over $\mathbb{Z}[\pi]$ then its class in $L^p_{2i+1}(\mathbb{Z}[\pi])$ is determined by the torsion characteristic

$$\chi^S(C) = \sum_{r=0}^{2i+1} (-)^r[H_r(C)] \in \hat{H}^{2i+2}(\mathbb{Z}_2; H),$$

with $(C, \psi) = 0 \in L^p_{2i+1}(\mathbb{Z}[\pi])$ if and only if

$$\chi^S(C) = 0 \in \operatorname{coker}(\chi^S\Gamma^p: L^h_{2i+2}(\mathbb{Z}_{(\pi)}[\pi]) \to \hat{H}^{2i+2}(\mathbb{Z}_2; H)).$$

This was motivated by the result of Pardon [8] in the case of a finite 2-group π, $2i+1 \equiv 3 \pmod 4$ with C $(i-1)$-connected and finite.

The question is then given an arbitrary $(C, \psi) \in L^p_{2i+1}(\mathbb{Z}[\pi])$, what is the obstruction to finding a cobordant S-acyclic complex (C', ψ')? Consider the localization exact sequence

$$\cdots \to L^p_{n+1}(\mathbb{Z}[\pi], S) \to L^p_n(\mathbb{Z}[\pi]) \to L^h_n(\mathbb{Z}_{(\pi)}[\pi]) \to \cdots$$

in the above diagram. To detect an element $(C,\psi)\in L^p_{2i+1}(\mathbb{Z}[\pi])$ we first compute the semicharacteristic in $L^h_{2i+1}(\mathbb{Z}_{(\pi)}[\pi]/\text{rad})$. If this vanishes we apply the algorithm of 5.1 to find a homotopy equivalent $\mathbb{Z}_{(\pi)}[\pi]$-cross complex (C'',ψ''). Then 7.1 gives a cobordant S-acyclic complex (C',ψ') whose class in $L^p_{2i+1}(\mathbb{Z}[\pi])$ is determined by

$$\chi^S(C') = \chi^{S\dagger}_{1/2}(C'',\psi'')\in K_1(\mathbb{Z}[\pi],S). \qquad \square$$

EXAMPLE 7.3. (Detection of $L^h_{2i+1}(\mathbb{Z}[\pi])$). Let π, S be as in 7.2. In Davis [3] it is shown that if (C,ψ) is a finite $(2i+1)$-dimensional S-acyclic quadratic Poincaré complex over $\mathbb{Z}[\pi]$ then its class in $L^h_{2i+1}(\mathbb{Z}[\pi])$ is determined by its torsion characteristic

$$\chi^S(C) = \sum_{r=0}^{2i+1}(-)^r[H_r(C)]\in \hat{H}^{2i+2}(\mathbb{Z}_2;\ker(\sigma\colon K_1(\mathbb{Z}[\pi],S)\to \tilde{K}_0(\mathbb{Z}[\pi])))$$

if $2i+1\equiv 3(\text{mod }4)$ or if $2i+1\equiv 1(\text{mod }4)$ and $H_*(C)$ is split-symplectic. In particular, if $\chi^S(C)=0\in\hat{H}^{2i+2}(\mathbb{Z}_2;\ker(\sigma))$ then $(C,\psi)=0\in L^h_{2i+1}(\mathbb{Z}[\pi])$. (If $2i+1\equiv 3(\text{mod }4)$ the composite

$$L^s_{2i+2}(\mathbb{Z}[\pi],S)\to L^h_{2i+2}(\mathbb{Z}[\pi],S)\to L^h_{2i+1}(\mathbb{Z}[\pi])$$

is trivial.)

Consider the localization exact sequence

$$\cdots\to L^h_{n+1}(\mathbb{Z}[\pi],S)\to L^h_n(\mathbb{Z}[\pi])\to L^h_n(\mathbb{Z}_{(\pi)}[\pi])\to\cdots.$$

Given an element $(C,\psi)\in L^h_{2i+1}(\mathbb{Z}[\pi])$ the surgery semicharacteristic is the obstruction as above to finding a homotopy equivalent $\mathbb{Z}_{(\pi)}[\pi]$-cross (C'',ψ'') and a cobordant S-acyclic (C',ψ'). The torsion characteristic can then be computed by 7.1 (ii) and (iii). In the more difficult cases where the surgery obstruction is not determined by $\chi^S(C')$, one must examine invariants of the linking form

$$\lambda\colon H_i(C')\times H_i(C')\to \mathbb{Z}_{(\pi)}[\pi]/\mathbb{Z}[\pi].$$

(See Davis [3] for details.) $\square$

REMARK 7.4. The connection between Examples 5.3 and 7.3 is that the isomorphism

$$\partial^{-1}\colon \ker(\sigma\colon K_1(A,S)\to\tilde{K}_0(A))\to \text{coker}(j_1\colon \tilde{K}_1(A)\to\tilde{K}_1(S^{-1}A))$$

sends the torsion characteristic $\chi^S(C)\in\ker(\sigma)$ of a finite S-acyclic A-module chain complex C to the torsion $\tau(S^{-1}C)\in\text{coker}(j_1)$. (See Davis and Weinberger [4].) $\square$

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