Wall's surgery obstruction groups for $G \times Z$

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Introduction

An obstruction theory for the general surgery problem has recently been defined by C. T. C. Wall in [31] and [32]. Let $(X, \partial X)$ be a manifold pair with the dimension of X at least five. Let $G = \pi_1 X$ and let $w: G \longrightarrow Z_2$ be the first Stiefel-Whitney class of X. Let v be a vector bundle over X of the same fibre-homotopy type as the normal bundle of X. Let

$$\varphi: (M, \partial M) \longrightarrow (X, \partial X)$$

be a map of degree one of manifold pairs whose restriction to the boundary is a simple homotopy equivalence of boundaries. Let F be a stable framing of $\tau M \oplus \varphi^* v$, τM the tangent bundle of M. Then Wall defines an invariant $\theta(M, \varphi, F)$, depending only on the cobordism class of this triple. This invariant lies in an abelian group $L_n^s(G, w)$, or just $L_n(G, w)$, that depends functorially on (G, w); here $n = \dim X$. It vanishes on a "surgery problem" (M, φ, F) if and only if (M, φ, F) is cobordant to (N, ψ, E) with ψ a simple homotopy equivalence. If we omit the adjective "simple" in this description, the appropriate obstruction lies in an abelian group $L_n^h(G, w)$ that also depends functorially on (G, w). Surgery was previously studied in the simply-connected case by Kervaire and Milnor [10], Novikov [20] and W. Browder. Interesting results in the non-simply connected case were first obtained by W. Browder in [1].

The first five sections of the present paper are devoted to proving a "Künneth formula" for Wall's surgery obstruction groups. Let Z denote the integers. Let K^{n-2} be a closed smooth of piecewise linear (PL) manifold, $n \geqq 7$. Let G be the fundamental group of K, and let $w: G \longrightarrow Z_2$ be the first Stiefel-Whitney class of K. Let w_1 be the composite of w with the natural projection of $G \times Z$ onto G.

THEOREM 5.1. *There is a split exact sequence*

$$0 \longrightarrow L_n^s(G, w) \xrightarrow{j_*} L_n^s(G \times Z, w_1) \xrightarrow{\alpha(K)} L_{n-1}^h(G, w) \longrightarrow 0 .$$

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The map j_* is induced by inclusion, and a splitting of $\alpha(K)$ corresponds geometrically to taking cartesian products with a circle.

It is well known (see [10]) that

$$L_n(e) = \begin{cases} 0 & n \equiv 1 \pmod 2 \\ Z & n \equiv 0 \pmod 4 \\ Z_2 & n \equiv 2 \pmod 4 . \end{cases}$$

Hence Theorem 5.1 leads to the computation of the groups $L_n(G, 0) = L_n(G)$ for G a free abelian group. In view of Sullivan's reformulation of the theory of Browder and Novikov (see [23], [28], and [32]), knowledge of the groups $L_n(G)$ can be used to attack the problem of classifying manifolds with fundamental group G. In this paper we concentrate, in §6 and §7, on smooth closed five-manifolds with fundamental groups Z, $Z \oplus Z$, and $Z \oplus Z_2$. For example, we obtain the following.

THEOREM. (See Corollary 6.8.) *Let M be a smooth, closed, orientable five-manifold with fundamental group Z. Then the number of closed smooth manifolds of the same homotopy type as M is finite and at most equals the number of elements of $H^2(M; Z_2)$.*

COROLLARY. *If M has the homotopy type of $S^1 \times S^4$, then M is diffeomorphic to $S^1 \times S^4$.*

COROLLARY 6.9. (See also [25].) *Any h-cobordism of $S^1 \times S^3$ with itself is a product.*

COROLLARY 6.10. (See also [25].) *Let $\varphi: S^3 \to S^5$ be a smooth embedding. Then φ is (ambient) isotopic to the standard inclusion if and only if $S^5 - \varphi S^3$ has the homotopy type of a circle.*

Corollary 6.10 is similar to some results of Levine [15] for higher dimensions.

Using more of the work of Sullivan, we obtatin a Hauptvermutung for 5-manifolds with fundamental group Z.

THEOREM 6.11. *Let $h: K \to M$ be a topological homeomorphism of smooth, closed, orientable 5-manifolds with fundamental group Z. Then h is homotopic to a diffeomorphism.*

Other applications of Theorem 5.1 *via* Browder-Novikov theory include for example the classification of PL manifolds of the same homotopy type as the torus T^n, $n \geqq 5$. Some of the applications will be the subject of a future paper with W-C. Hsiang. They have been announced in [7]. (See also [34].) Other authors [1], [12], and [14] have used the applications of Theorem 5.1 to help study the annulus conjecture, the triangulations of topological manifolds and the Hauptvermutung.

We prove Theorem 5.1 by a primarily geometric analysis of the surgery obstruction groups. We confine ourselves to the differentiable situation; as usual everything can be translated to the PL category. The main idea is to split a surgery problem with fundamental group $Z \times G$ into two surgery problems with fundamental group G by means of the splitting theorem of [5, 6] or the fibering theorem of [3, 4]. This idea was used by W. Browder in [1]. We do this in §2 and §3, after a preliminary section outlining the elements of Wall's surgery obstruction theory that we need.

This approach quickly forces the introduction of the groups $L_n^h(G, w)$. Suppose that $h: M^m \to K \times S^1$, $m \geqq 6$, is a simple homotopy equivalence, and assume for simplicity that K and M are closed. Let $z \in S^1$. Then the splitting theorem of [5, 6] tells us that h is homotopic to k, where $k^{-1}(K \times z)$ is a smooth submanifold and where the restriction of k is a homotopy equivalence of this submanifold with $K \times z$. However, we have no hope to obtain a simple homotopy equivalence in this way, in general. In fact, if $f: P \to K$ is a homotopy equivalence, $f \times \mathrm{id}_{S^1}$ is always simple [13].

In § 4 we compare the groups $L_n^s(G, w)$ and $L_n^h(G, w)$. Let $\mathrm{Wh}(G)$ be the Whitehead group of G. Let * be the natural involution defined using w. (See [18].) Let $A_n(G, w)$ be the quotient of the subgroup of all τ in $\mathrm{Wh}(G)$ satisfying $\tau = (-1)^n\tau^*$ by the subgroup of all elements of the form $\tau + (-1)^n\tau^*$.

PROPOSITION 4.1. *There is a natural exact sequence*

$$\cdots \longrightarrow L_n^s(G, w) \longrightarrow L_n^h(G, w) \longrightarrow A_n(G, w) \longrightarrow L_{n-1}^s(G, w) \longrightarrow \cdots .$$

Finally, in § 5, we interpret the work of §§ 2–4 to obtain information about the surgery obstruction groups for $Z \times G$. Although these obstruction groups have purely algebraic definitions, a purely algebraic derivation of a result like 5.1 seems to be quite difficult.

A word about style. In the derivation of Theorem 5.1, we pay more detailed attention to the geometric aspects of the argument that to the algebraic ones. The reader who is familiar with [32, § 5 and § 6] will be able to fill in the missing details easily. I hope that the reader who is unfamiliar with [32] will still be able to follow the general line of argument.

A preliminary form of Theorem 5.1 was first announced in [24]. A result along the lines of Theorem 5.1 has been found independently by C. T. C. Wall, using a different line of argument.

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1. Wall's surgery obstruction groups

Let C be the category whose objects are pairs (G, w), G a group and w a homomorphism of G into Z_2, and whose morphisms are group homomorphisms such that the obvious diagram commutes. Every finite Poincaré complex X (see [31] or [32]) determines functorially an object $(\pi_1 X, wX)$ of C; let $w(X)(b) = +1$ if b preserves orientation and -1 if b reverses orientation. There is a sequence of functors L_n, satisfying $L_n = L_{n+4}$ for all n, which plays the role of the range of a surgery obstruction for $n \geqq 5$; each L_n is a functor from C to the category of abelian groups. More precisely, let X^n be a connected, compact, smooth, and oriented manifold; and let v be its stable normal bundle. (Actually one needs only a finite Poincaré complex with a suitable vector bundle; see [32].) Here an oriented manifold is a manifold together with an orientation element in $H_n(X, \partial X; Z^t)$, the n^{th} relative homology groups with twisted integer coefficients. Let $B_n(X, v)$ be the cobordism classes of triples (M^n, φ, F), M a compact smooth oriented manifold; $\varphi\colon (M, \partial M) \to (X, \partial X)$ a map which pulls back the twisted integer coefficients, carries the orientation class of M to that of X, and induces a simple homotopy equivalence of ∂M with ∂X (see [18] for a definition of simple homotopy equivalence); and F a stable framing of $\tau M \oplus \varphi^* v$. A cobordism (W, Φ, G) of (M, φ, F) and (M_1, φ_1, F_1) is a representative of an element of $B_{n+1}(X \times I; v \times I)$ such that $\partial W = M \cup \partial_0 W \cup M_1$, with $M \cap \partial_0 W = \partial M$ and $M_1 \cap \partial_0 W = \partial M_1$ and with $\partial_0 W$ an s-cobordism of ∂M with ∂M_1; and such that $\Phi(x) = (\varphi(x), 0)$ if $x \in M$, $\Phi(x) = (\varphi_1(x), 1)$ if $x \in M_1$, $G \mid M = F$, $G \mid M_1 = F_1$, and $\Phi(\partial_0 W) \subseteq \partial X \times I$. For $n \geqq 5$ there is a map

$$\theta\colon B_n(X, v) \longrightarrow L_n(\pi_1 X, wX)$$

such that $\theta[M, \varphi, F] = 0$ if and only if this class contains (N, ψ, H) with ψ a simple homotopy equivalence. (Note that if X is just a finite Poincaré complex and v a vector bundle over X, the Thom class of v must be stably spherical if $B_n(X, v) \neq \emptyset$.)

The groups $L_n(\pi_1 X, wX)$ are not too large in the sense that each of their elements is the obstruction of some surgery problem with boundary. In fact, we have the following result of Wall [31, p. 274] and [32].

THEOREM 1.1. *Let X^{m-1}, m greater than six, be a smooth manifold. Let v be the stable normal bundle of X. Let η be a given element of $L_m(\pi_1 X, wX)$.*

Let φ_1 be a simple homotopy equivalence of M^{m-1} and X which induces a simple homotopy equivalence of the (possibly empty) boundaries. Let F_1 be a stable framing of $\tau M \oplus \varphi_1^* v$. Then there is a map of manifold triads,

$$\varphi: (W, \partial_- W, \partial_+ W) \longrightarrow (X \times I, X \times 0 \cup \partial X \times I, X \times 1)\ ,$$

and a stable framing F of $\tau W \oplus \varphi^*(v \times I)$ such that

(1) *$\partial W = M \times 0 \cup \partial M \times I$ and $\varphi(x, t) = (\varphi_1(x), t)$ for x in ∂M or $t = 0$;*

(2) *φ induces a simple homotopy equivalence of $\partial_+ W$ with $X \times I$;*

(3) *F extends F_1 $(\tau W \oplus \varphi^*(v \times I) \mid M = \tau M \oplus \varphi_1^* v \oplus \theta^1)$; and*

(4) *$\theta[W, \varphi, F] = \eta$.*

The next result is a consequence of [32, Ths. 3.1 and 3.2]. (See also [31, Lem. 7.3].)

THEOREM 1.2. *Let $\varphi: (W^n, \partial W) \rightarrow (X^n, \partial X)$, n greater than six and X connected, be a map of compact smooth manifolds which pulls back the twisted integer coefficients and carries the orientation class of W onto an orientation class of X. Let v be the stable normal bundle of X, and let F be a stable framing of $\tau W \oplus \varphi^* v$. Assume that $\partial X = \partial_1 X \cup \cdots \cup \partial_k X$, where each $\partial_i X$ is connected and where $\partial(\partial_i X)$ is the union of the submanifolds $\partial_i X \cap \partial_j X$, $i \neq j$. Let $\partial W = \partial_1 W \cup \cdots \cup \partial_k W$ be a similar decomposition of ∂W into submanifolds, except that $\partial_1 W, \cdots, \partial_k W$ need not be connected. Suppose that $\varphi(\partial_i W)$ is contained in $\partial_i X$ and that φ induces simple homotopy equivalences of $\partial(\partial_i W)$ with $\partial(\partial_i X)$. Let $\varphi_i = \varphi \mid \partial_i W: \partial_i W \rightarrow \partial_i X$. Let F_i be the restriction of F to $\partial_i W$. Let g_i be the map of $L_{n-1}(\pi_1(\partial_i X), w(\partial_i X))$ into $L_{n-1}(\pi_1 X, wX)$ induced by inclusion. Then in $L_{n-1}(\pi_1 X, wX)$,*

$$0 = g_1(\theta[\partial_1 W, \varphi_1, F_1]) + \cdots + g_k(\theta[\partial_k W, \varphi_k, F_k])\ .$$

The next two propositions also follow from [31] and [32].

PROPOSITION 1.3. *Let (M, φ, F) represent an element of $B_n(X, v)$. Then if $-M$ denotes the manifold M with the orientation class reversed, $\theta(-M, \varphi, F) = -\theta(M, \varphi, F)$.*

Note that this proposition follows from Theorem 1.2 by considering $M \times I$.

PROPOSITION 1.4. *Suppose that $X^n = Y \times I$ and let (W, φ, F) and (W_1, φ_1, F_1) represent elements of $B_n(X, v)$. Assume that $\partial W = M \cup \partial_0 W \cup M_1$ where $\partial_0 W$ meets M in ∂M, meets M_1 in ∂M_1, and is an s-cobordism of ∂M with ∂M_1; assume also that $\varphi(M)$, $\varphi(\partial_0 W)$, and $\varphi(M_1)$ are contained in $Y \times 0$, $\partial Y \times I$, and $Y \times 1$, respectively, and that φ induces simple homotopy equivalences of M, $\partial_0 W$, and M_1 with $Y \times 0$, $\partial Y \times I$, and $Y \times 1$, respectively. Let $\partial W_1 = M_1 \cup \partial_0 W_1 \cup M_2$ be a similar decomposition of ∂W_1. Suppose that $\varphi \mid M_1 = i \circ \varphi_1 \mid M_1$, where $i(y, t) = (y, 1 + t)$, and suppose that the restrictions* *of F and F_1 to M_1 agree. Let $W_2 = W \cup_{M_1} W_1$, and define φ_2: $W_2 \to Y \times [0, 2]$ by setting $\varphi_2(x)$ to be $\varphi(x)$ for x in W and $i \circ \varphi_1(x)$ for x in W_1. Let $F_2 = F \cup_{M_1} F_1$. Then in $L_n(\pi_1 X, wX)$,*

$$\theta(W_2, \varphi_2, F_2) = \theta(W, \varphi, F) + \theta(W_1, \varphi_1, F_1) .$$

Also, suppose that $f: X^n \to Y$ is a simple homotopy equivalence and $v = f^*u$. Let (M, φ, F) represent an element of $B_n(X, v)$. Then

$$L_n(f_*)\theta(M, \varphi, F) = \theta(M, f \circ \varphi, F) .$$

One can also define a surgery obstruction theory for the problem of modifying a map to get a homotopy equivalence. The entire discussion above carries over to this case, with "simple homotopy" replaced by just "homotopy" and "s-cobordism" by "h-cobordism". We denote the corresponding bordism set by $B^h_n(X, v)$ and the obstruction group by $L^h_n(\pi_1 X, wX)$, but we continue to write θ for the map which assigns to each cobordism class its surgery obstruction. This theory is given by Wall for $n = 2k$ in [31]; for $n = 2k + 1$ one can construct this theory along the lines of [32, § 6]. By $L_n(G, w)$ or $L^s_n(G, w)$ we always mean the obstruction groups for the surgery problem to obtain a simple homotopy equivalence. "Theorem 1. ih", $i = 1, 2, 3, 4$ means the analogue of Theorem 1.i for the theory involving just homotopy equivalences.

We conclude this section with an indication of some of the main points in the algebraic definition of the obstruction groups $L_n(G, w)$. Let $\Lambda = Z(G)$ be the integral group-ring of G. Λ is a ring with involution $(\sum \alpha_g g)^- = \sum w(g)\alpha_g g^{-1}$. A special $(-1)^k$-hermitian form over Λ is a triple (H, λ, μ), where H is a stably free (right) Λ-module with a preferred class of s-bases, and where λ: $H \times H \to \Lambda$ and μ: $H \to \Lambda/I$, $I = \{x - (-1)^k\bar{x} \mid x \in \Lambda\}$, satisfy the following.

(i) If $x \in H$, $y \to \lambda(x, y)$ is a Λ-homomorphism;

(ii) $\lambda(x, y) = (-1)^k(\overline{\lambda(y, x)})$;

(iii) $\lambda(x, x) \equiv \mu(x) + (-1)^k\overline{\mu(x)} \pmod I$;

(iv) $\mu(x + y) - \mu(x) - \mu(y) \equiv \lambda(x, y) \pmod I$;

(v) $\mu(x\alpha) = \bar{\alpha}\mu(x)\alpha$ (α in Λ, x in H); and

(vi) the adjoint $A\lambda$: $H \to \mathrm{Hom}_\Lambda(H; \Lambda)$ is a simple isomorphism of stably free s-based Λ-modules. (Note that by using the conjugation one can convert the natural left module structure on $\mathrm{Hom}_\Lambda(H; \Lambda)$ to a right module structure.) The group $L_{2k}(G, w)$ is defined to be the reduced Grothendieck group of special $(-1)^k$-hermitian forms under direct sum. The zero element in $L_{2k}(G, w)$ is represented by (H, λ, μ), where H is free and has a preferred base $\{e_1, \cdots, e_r, f_1, \cdots, f_r\}$ such that $\lambda(e_i, e_j) = \lambda(f_i, f_j) = \mu(e_i) = \mu(f_j) = 0, 1 \leqq i,$

$j \leqq r$, and $\lambda(e_i, f_j) = \delta_{ij}$. Such a form is called a standard kernel and the above base is called the standard base. The group $L^h_{2k}(G, w)$ is defined similarly, except that H is not given a preferred base and the adjoint map $A\lambda$ is only required to be an isomorphism of Λ-modules.

To define $L_{2k+1}(G, w)$, let (K_r, λ_r, μ_r) be a $(-1)^k$-hermitian standard kernel with base $e^r_1, \cdots, e^r_r, f^r_1, \cdots, f^r_r$ as in the definition of standard kernel above. Let $SU^k_r(\Lambda)$ denote the group of automorphisms of this standard kernel. There is a natural inclusion of $SU_r(\Lambda)$ in $SU_{r+1}(\Lambda)$ (we drop the k for notational ease); let $SU(\Lambda)$ denote the limit. Let $RU(\Lambda)$ be the subgroup generated by $[SU(\Lambda); SU(\Lambda)]$ and the element σ which has a representative in $SU_1(\Lambda)$ with matrix $\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$ with respect to e^1_1, f^1_1. Then $L_{2k+1}(G, w) = SU(\Lambda)/RU(\Lambda)$. $RU(\Lambda)$ is also generated by σ and a subgroup $TU(\Lambda) = \lim_{r\to\infty} TU_r(\Lambda)$, where $TU_r(\Lambda) \subseteq SU_r(\Lambda)$ consists of those automorphisms leaving the subspace generated by $e^r_1, \cdots, e^r_r$ invariant. To define $L^h_{2k+1}(G, w)$, one forgets about the preferred class of base and instead gets a group $U_r(\Lambda)$ of automorphisms which are not necessarily simple with respect to the base $\{e^r_1, \cdots, e^r_r, f^r_1, \cdots, f^r_r\}$. Then as above, $L^h_{2k+1}(G, w)$ is a suitable quotient of $U(\Lambda)$ by a subgroup, which we denote by $RU^h(\Lambda)$, generated by σ and $TU^h(\Lambda)$.

2. A surgery invariant

Throughout this section let g be a differentiable fibration of the compact connected smooth manifold X^n over S^1. Let L be the fiber $g^{-1}(z)$. Assume that $n \geqq 6$. If ∂X is non-empty, assume that $n \geqq 7$ and that $g \mid \partial X$ is also a differentiable fibration. Assume also that g induces an epimorphism of fundamental groups with kernel G and that, if ∂X is non-empty, the restrictions of g to each component of ∂X also induce epimorphisms of fundamental groups.

Let v be a vector bundle over X with spherical Thom class (e.g., a normal bundle). Let (M, φ, F) represent an element of $B_n(X, v)$. After a homotopy of φ as a map of the pair $(M, \partial M)$ to the pair $(X, \partial X)$, we can assume z, the basepoint of S^1, is a regular value of $g \circ \varphi$ and of $g \circ \varphi \mid \partial M$. Let $N = \varphi^{-1}L$. Then we may also assume that $\varphi \mid \partial N \colon \partial N \to \partial L$ is a (not necessarily simple) homotopy equivalence; this follows from [6, Th. 2.2], since $\varphi \mid \partial M \colon \partial M \to \partial X$ is a simple homotopy equivalence (see [5, Cor. 2] for a statement of the result we are using here in the absolute case). Let $\psi \colon N \to L$ be the restriction of φ. Then $F \mid N$ is a stable framing of $\psi^*(v \mid L) \oplus \tau N$. Define

$$\alpha_g(M, \varphi, F) = \theta(N, \psi, F \mid N) \in L^h_{n-1}(G, w \mid L) .$$

PROPOSITION 2.1. *$\alpha_g(M, \varphi, F)$ is well-defined and depends only upon the cobordism class of (M, φ, F). If φ is a simple homotopy equivalence, then*

$\alpha_g(M, \varphi, F) = 0$.

PROOF. The orientation class of N is determined by that of M and the standard orientation of S^1. We must have that ψ carries this class onto an orientation class of L. Let X_L and M_N be the manifolds obtained by splitting X and M along L and N respectively. (See [2]). The map φ naturally induces a map $\varphi_L: M_N \to X_L$; it suffices to check that φ_L carries an orientation class to an orientation class. For then the same is true of the restriction of φ_L to the boundary of M_N. But ∂M_N is the union of $\partial M_{\partial N}$ with two copies of N, and ∂X_L has a similar decomposition and φ_L respects these decompositions; so it follows easily using the Mayer-Vietoris sequence that ψ has the desired property.

So let x be in $X - L$. By transversality and using the fact that $X - L$ is connected and the homotopy extension property, one can find a homotopy φ_t of φ such that $\varphi_t = \varphi$ on N, all t in $[0, 1]$, $\varphi_t^{-1}L = N$ all t in $[0, 1]$, and $\varphi_1^{-1}x = D$, a finite set of points of $M - N$ which meet each component of $M - N$. Let $p_1, \cdots, p_k$ be the points of D. Let U_i be a neighborhood of p_i in $M - N$ for each i, such that U_i meets U_j only when $i = j$. Then

$$H_n(M, M - D) = \bigoplus_{i=1}^k H_n(U_i, U_i - p_i) = H_n(M_N, M_N - D) .$$

In the orientable case, we have the following commutative diagram (all coefficients Z).

$$\begin{array}{ccc}
H_n(M, \partial M) & \xrightarrow{(\varphi_1)_*} & H_n(X, \partial X) \\
\downarrow j & & \downarrow \\
H_n(M, M - D) & \xrightarrow{(\varphi_1)_*} & H_n(X, X - x) \\
\downarrow \cong & & \downarrow \cong \\
H_n(M_N, M_N - D) & \xrightarrow{((\varphi_1)_L)_*} & H_n(X_L, X_L - x) \\
\uparrow k & & \uparrow \cong \\
H_n(M_N, \partial M_N) & \xrightarrow{(\varphi_L)_*} & H_n(X_L, \partial X_L)
\end{array}$$

The maps k and j are monic, and

$$j[M] = [U_1] + \cdots + [U_k] = k[M_N] ,$$

where $[U_i]$ is the image of the orientation class $[M]$ in $H_n(M, M - p_i) = H_n(U_i, U_i - p_i)$. The bottom square commutes because φ_L and $(\varphi_1)_L$ are homotopic as maps of pairs. That φ_L carries the orientation class onto an orientation class is now clear. In the non-orientable case, one has to write in twisted integer coefficients in the above diagram and to check that ψ pulls back the twisted coefficients. This is easy.

Continuing with (M, φ, F) as above, suppose now that W^{n+1} is a compact smooth manifold with $\partial W = M \cup \partial_0 W \cup M_1$, where there is a diffeomorphism f of $\partial M \times I$ to $\partial_0 W$ such that $M \cap \partial_0 W = \partial M = f(\partial M \times 0)$ and $M_1 \cap \partial_0 W = \partial M_1 = f(\partial M \times 1)$. For simplicity we identify $\partial_0 W$ with $\partial M \times I$. Let Φ be a map of W to $X \times I$ with the following properties:

(1) $\Phi \mid M = \varphi$;

(2) $\Phi(M_1)$ is contained in $X \times 1$;

(3) $\Phi(\partial_0 W)$ is contained in $\partial X \times I$; and

(4) if $\varphi_1: M_1 \to X \times 1$ is the restriction of Φ, then φ_1 is transverse to $L \times 1$, the restriction of φ_1 to ∂M_1 is transverse to $\partial L \times 1$, and if $N_1 = \varphi_1^{-1}(L \times 1)$, then the restriction of φ_1 induces a homotopy equivalence of ∂N_1 with $\partial L \times 1$.

Suppose also that we are given a stable framing G of $\Phi^*(v \times I) \oplus \tau W$ which extends F, and let $F_1 = G \mid M_1$. Let $\psi_1: N_1 \to L \times 1$ be the restriction of φ_1. Then we want to show that

$$\theta(N, \psi, F \mid N) = \theta(N_1, \psi_1, F_1 \mid N_1)$$

in $L^h_{n-1}(G, wL)$. This will show that $\alpha_g(M, \varphi, F)$ is independent of the choices of the various homotopies in the definition (take $W = M \times I$) and also that it depends only upon the class of (M, φ, F) in $B_n(X, v)$.

After a homotopy of Φ relative M and M_1, we can also assume that Φ is transverse to $L \times I$ and that $\Phi \mid \partial W$ is transverse to $\partial(L \times I)$. Let $P = \Phi^{-1}(L \times I)$ and let $\bar{\varphi}: P \to L \times I$ be the restriction of Φ. $G \mid P$ is a stable framing of $\tau P \oplus \bar{\varphi}^*((v \mid L) \times I)$. Let $N_0 = \partial P \cap (\partial M \times I)$; then $\partial P = N \cup N_1 \cup N_0$. Let $L_1, \cdots, L_k$ be the components of ∂L, and let $P_i = (\bar{\varphi} \mid N_0)^{-1}(L_i \times I)$. Let $\bar{\varphi}_i: P_i \to L_i \times I$ be the restriction of $\bar{\varphi}$. Let $\eta_i = \theta(P_i, \bar{\varphi}_i, G \mid P_i)$. Let h_i be the inclusion induced map of $L^h_{n-1}(\pi_1 L_i, wL_i)$ into $L^h_{n-1}(G, wL) = L^h_{n-1}(\pi_1(L \times I), w(L \times I))$. Then by Theorem 1.2 h

$$\theta(N, \psi, F \mid N) - \theta(N_1, \psi_1, F_1 \mid N_1) + h_1\eta_1 + \cdots + h_k\eta_k = 0 .$$

The minus sign is due to the fact that the orientation of N_1 is usually taken to be the negative of its orientation as a part of ∂P, while that of N is taken to be equal to its orientation as a part of ∂P, or *vice-versa*. (One could certainly be more precise, if tedious, than we are here concerning the orientations.) Hence to prove the equation we want, it will suffice to show that $\eta_1 = \cdots = \eta_k = 0$. But this follows immediately from [6, Th. 2.2], applied component by component to $\Phi \mid \partial M \times I: \partial M \times I \to \partial X \times I$; i.e., since this map is a simple homotopy equivalence, we can assume after a homotopy of Φ that $\bar{\varphi}_i$ is a homotopy equivalence. (See also [5, Cor. 2] for a statement of [6, Th. 2.2] in the absolute case.)

It remains to show that if φ is a simple homotopy equivalence, then

$\alpha_g(M, \varphi, F) = 0$. But it follows immediately from [6, Th. 2.2](see [5, Cor. 2]) that in the definition of $\alpha_g(M, \varphi, F)$, one may take $\psi = \varphi \mid N \colon N \to L$ to be a (not necessarily simple) homotopy equivalence. This completes the proof of Proposition 2.1.

Later we will use Proposition 2.1 to help define a split epimorphism of $L_n(G \times Z)$ onto $L^h_{n-1}(G)$. For the moment however, we state only the following consequence.

COROLLARY 2.2. *Let X^n be a compact connected smooth manifold. Assume that n is greater than five and greater than six if ∂X is not empty. Let v be a vector bundle over X. Let (M, φ, F) represent an element of $B_n(X, v)$. Let H be a framing of η, the trivial line bundle over S^1. Then $F \times H$ is a stable framing of $\tau(M \times S^1) \oplus (\varphi \times \mathrm{id}_{S^1})^*(v \times S^1)$, and the following are equivalent.*

(1) *(M, φ, F) is cobordant to (N, ψ, G) with ψ a homotopy equivalence; and*

(2) *$(M \times S^1, \varphi \times S^1, F \times H)$ is cobordant to $(P, \bar\varphi, E)$, with $\bar\varphi$ a simple homotopy equivalence.*

PROOF. The statement about $F \times H$ follows from the fact that $\tau(M \times S^1) \oplus (\varphi \times S^1)^*(v \times S^1) = (\tau M \oplus \varphi^* v) \times \eta$. That (1) implies (2) is straightforward. (See [13]). Assume (2). Let g be the natural projection of $X \times S^1$ onto S^1. Then by Proposition 2.1, $\alpha_g(M \times S^1, \varphi \times S^1, F \times H) = 0$. But by definition, this also equals $\theta(M, \varphi, F)$.

3. Another surgery invariant

In this section we define a surgery invariant β_g on triples on which α_g vanishes. Most of the ideas have already appeared in the last section, but the details are about twice as tedious.

If μ is an embedding of $S^i \times D^{n-i}$ into the interior of N^n, let $w(N, \mu)$ denote the elementary cobordism obtained by surgery using μ. For example, we can take $w(N, \mu) = N \times I \cup D^{i+1} \times D^{n-i}$, with $(\mu(x), 1)$ identified with x. We usually identify N with $\partial_- w(N, \mu)$. Also, we view D^k as contained in D^{k+1} by the standard embedding on the first k coordinates. The next lemma tells us how to extend a cobordism of a submanifold in some cases. The proof is trivial and so is omitted.

LEMMA 3.1. *Let (M, φ, F) represent an element of $B_n(X, v)$, X a compact smooth connected manifold, and v a normal bundle of X. Let g be a differentiable map of X onto S^1 with regular value z. If ∂X is not empty, assume that z is also a regular value of $g \mid \partial X$ and in the image of $g \mid \partial X$. Let $L =$*

$g^{-1}z$, and assume that φ is transverse to L, and that $\varphi \mid \partial M$ is transverse to ∂L. Let $N = \varphi^{-1}L$. Let $\mu\colon S^i \times D^{n-i-1} \to \operatorname{Int} N$ be an embedding. Let $p\colon w(N, \mu) \to L$ be an extension of $(\varphi \mid N) \circ \pi_1\colon N \times I \to L$, and let G be a stable framing of $p^*v \oplus \tau(w(N, \mu))$ which extends $(F \mid N) \times I$. Then there is an embedding $\bar{\mu}$ of $S^i \times D^{n-i}$ into $\operatorname{Int} M$ extending μ and such that $\bar{\mu}^{-1}N = S^i \times D^{n-i-1}$; an extension $\Phi\colon w(M, \bar{\mu}) \to X$ of φ and p such that $\Phi(\partial M \times I)$ is contained in ∂X and such that $\Phi, \Phi \mid \partial_+ w(M, \bar{\mu}), \Phi \mid \partial M \times I$, and $\Phi \mid \partial M \times \partial I$ are transverse to $L, L, \partial L$ and ∂L respectively; and a stable framing K of $\Phi^*v \oplus \tau(w(M, \bar{\mu}))$ extending $F \times I$ and G.

Remark. Suppose that Φ is a map of $(W^{n+1}, \partial W)$ into $(X^n, \partial X)$. Assume that $\partial W = M \cup \partial_0 W \cup M_1$, with the usual properties. Identify $\partial_0 W$ with $\partial M \times I$. Then there is a smooth function f from W to I, with non-degenerate critical points, such that $M = f^{-1}(0)$, $M_1 = f^{-1}(1)$, and $f(x, t) = t$ if x is in ∂M and t in I. We have a map $(\Phi, f)\colon W \to X \times I$. Let v be a vector bundle over X. Then since $v \times I$ is the pullback of v under the natural projection of $X \times I$ onto X, $\tau W \oplus \Phi^*v = \tau W \oplus (\Phi, f)^*(v \times I)$. If Φ induces a simple homotopy equivalence of ∂M with ∂X, then (Φ, f) induces a simple homotopy equivalence of $\partial M \times I$ with $\partial X \times I$. If Φ or some restriction of Φ is transverse to a submanifold K of X, then (Φ, f) or the corresponding restriction of (Φ, f) is transverse to $K \times I$.

LEMMA 3.2. *Let f be a map of $(Z^n, \partial Z^n)$ into $(X^n, \partial X)$. Let g be a differentiable fibration of the connected manifold X over S^1, with connected fibre L. Assume that $g \mid \partial X$ is also a differentiable fibration. Suppose that f and $f \mid \partial Z$ are transverse to L and ∂L respectively, and let $N = f^{-1}L$. Then if $f\colon Z \to X$ and $f \mid N\colon N \to L$ are homotopy equivalences, so is $f_L\colon Z_N \to X_L$, the map determined by f.*

(Recall, for example, that X_L is the manifold obtained by splitting X along L.)

The proof of Lemma 3.2 is fairly straightforward and is left to the reader.

Now we return to the situation of § 2. Assume the hypotheses of the first paragraph of that section. Again let v be a vector bundle over X, and let (M, φ, F) represent an element of $B_n(X, v)$. Again we can assume after a homotopy of φ as a map of the pair $(M, \partial M)$ to the pair $(X, \partial X)$ that z, the basepoint of S^1, is a regular value of $g \circ \varphi$ and of $g \circ \varphi \mid \partial M$. By [6, Th. 2.2] and the homotopy extension property, we can again assume after another homotopy that if $N = \varphi^{-1}L$ (recall $L = g^{-1}z$), then φ induces a homotopy equivalence of ∂N with ∂L. It follows from Lemma 3.2 (applied componentwise) that φ also induces a homotopy equivalence of $(\partial M)_{\partial N}$ with $(\partial X)_{\partial L}$.

Now suppose that $\alpha_g(M, \varphi, F) = 0$. Let $\psi = \varphi \mid N: N \to L$. By definition $\theta(N, \psi, F \mid N) = 0$. Hence $(N, \psi, F \mid N)$ is cobordant to $(\bar{N}, \bar{\psi}, G)$, with $\bar{\psi}$ a homotopy equivalence. Hence by Lemma 3.1, (M, φ, F) is cobordant to $(\bar{M}, \bar{\varphi}, K)$ such that $\bar{\varphi}$ is transverse to L and its restriction to $\partial\bar{M}$ to ∂L, and such that $N = (\bar{\varphi})^{-1}L, \bar{\varphi} \mid \bar{N} = \bar{\psi}$, and $K \mid \bar{N} = G$. The map $\bar{\varphi}$ induces $\bar{\varphi}_L: \bar{M}_{\bar{N}} \to X_L$. The restriction of $\bar{\varphi}_L$ induces a homotopy equivalence of $(\partial\bar{M})_{\partial\bar{N}}$ with $(\partial X)_{\partial L}$. We also have, for example, that $\partial(X_L)$ is the union of $(\partial X)_{\partial L}$ with two disjoint copies of L which meet $(\partial X)_{\partial L}$ in the two disjoint copies of ∂L which make up the boundary of $(\partial X)_{\partial L}$. Since $\bar{\psi}$ is a homotopy equivalence which induces a homotopy equivalence of boundaries, it follows, e.g., by using the Mayer-Vietoris sequence and several applications of the Van-Kampen theorem, that $\bar{\varphi}_L$ also induces a homotopy equivalence of boundaries. Let v_L be the pull-back of v under the quotient map of X_L onto X. Then $\tau(\bar{M}_{\bar{N}}) \oplus \bar{\varphi}_L^* v_L$ is the pull-back of $\tau M \oplus \bar{\varphi}^* v$ by the quotient map of $\bar{M}_{\bar{N}}$ onto $\bar{M}$; hence K pulls back to a stable framing K_L of $\tau(\bar{M}_{\bar{N}}) \oplus \bar{\varphi}_L^* v_L$ *via* the quotient map. So $(\bar{M}_{\bar{N}}, \bar{\varphi}, K_L)$ represents an element of $B_n^h(X_L, v_L)$. We define

$$\beta_g(M, \varphi, F) = \theta(\bar{M}_{\bar{N}}, \bar{\varphi}_L, K_L) ,$$

an element of $L_n^h(G, wX_L)$. (Since g is a fibration, the quotient map induces an isomorphism of $\pi_1 X_L$ with G.)

PROPOSITION 3.3. *β_g is well-defined on triples (M, φ, F) such that $\alpha_g(M, \varphi, F) = 0$ and depends only upon the cobordism class of such a triple. If φ is a simple homotopy equivalence, $\beta_g(M, \varphi, F) = 0$.*

PROOF. The proof that $\bar{\varphi}_L$ carries an orientation class to an orientation class is similar to the first part of the proof of Proposition 2.1 and so is omitted.

Let $\Phi: W^{n+1} \to X \times I$. Suppose that $\partial W = M \cup \partial_0 W \cup M_1$, where $\partial_0 W$ is diffeomorphic to $\partial M \times I$, meets M in ∂M and M_1 in ∂M_1, and is a cobordism of ∂M and ∂M_1. Let (M, φ, F) be as in the definition of β_g. Assume that $\Phi \mid M = (\varphi, 0)$ and that $\Phi(\partial_0 W)$ is contained in $\partial X \times I$. (Hence Φ induces a simple homotopy equivalence of $\partial_0 W$ with $\partial X \times I$.) Assume that $\Phi^{-1}(X \times 0) = M$ and $\Phi^{-1}(X \times 1) = M_1$. Let $\varphi_1: M_1 \to X \times 1$ be the restriction of Φ. Assume that φ_1 and its restriction to ∂M_1 are transverse to $L \times 1$ and $\partial L \times 1$ respectively. Let $N_1 = \varphi_1^{-1}(L \times 1)$. Assume that φ_1 induces homotopy equivalences of N_1 with $L \times 1$ and of ∂N_1 with $\partial L \times 1$. Let G be a stable framing of $\Phi^*(v \times I) \oplus \tau W$, with $G \mid M = F$. Then we want to show that

$$\theta(M_N, \varphi_L, F_L) = \theta\big((M_1)_{N_1}, (\varphi_1)_L, (G \mid M_1)_L\big) .$$

This will show that $\beta_g(M, \varphi, F)$ is independent of the initial homotopies in the definition, of the choice of the cobordism in the definition, and of the cobordism class of (M, φ, F).

Let $P = \Phi^{-1}(L \times I)$, and let W_P be the manifold obtained by splitting W along P. Let

$$\Phi_L \colon W_P \longrightarrow (X \times I)_{L\times I} = X_L \times I$$

be the map induced by Φ. Let G_L be the stable framing of $\tau(W_P) \oplus \Phi_L^*(v_L \times I)$ which is obtained from G by pulling back *via* the quotient map of W_P onto W. Now $\partial W_P = P' \cup P'' \cup (\partial W)_{\partial P}$, where the quotient map carries P' and P'' diffeomorphically onto P. Moreover, $(\partial W)_{\partial P} = M_N \cup (M_1)_{N_1} \cup (\partial_0 W)_{\bar{P}}$, where $\bar{P} = P \cap \partial_0 W$. Since Φ induces a simple homotopy equivalence of $\partial_0 W$ with $\partial X \times I$, we can apply [6, Th. 2.2] and the homotopy extension property; this enables us to assume, after a homotopy of Φ relative M and M_1, that the restriction of Φ is a homotopy equivalence of $\bar{P}$ with ∂L.

We also have, similarly,

$$\partial X_L \times I = (L' \times I) \cup (L'' \times I) \cup (X_L \times 0) \cup (X_L \times 1) \cup (\partial X_{\partial L} \times I) .$$

Φ_L carries $(\partial_0 W)_{\bar{P}}$ to $\partial X_{\partial L} \times I$, M_N to $X_L \times 0$, $(M_1)_{N_1}$ to $X_L \times 1$, and, after a possible change of notation, P' to $L' \times I$ and P'' to $L'' \times I$. Φ_L induces homotopy equivalences of all the corresponding intersections of these portions of the boundaries. Hence we are in a position to apply Theorem 1.2h. We want to show that this gives the desired result. By Lemma 3.2, the restriction of Φ_L induces a homotopy equivalence of $(\partial_0 W)_{\bar{P}}$ with $\partial X_{\partial L} \times I$. Hence θ vanishes on the triples obtained by restricting Φ_L to maps of the various components of $(\partial_0 W)_{\bar{P}}$ into the corresponding components of $\partial X_{\partial L} \times I$. Let $\eta = \theta(P', \Phi_L \mid P', G_L \mid P')$ and let μ be defined similarly, but with P' replaced by P''. Let

$$i' \colon L_n^h\big(\pi_1(L' \times I), w(L' \times I)\big) \longrightarrow L_n^h\big(\pi_1(X_L \times I), w(X_L \times I)\big)$$

be induced by inclusion. Let i'' be defined similarly, but with L' replaced by L''. Then in $L_n^h(G, wX_L) = L_n^h\big(\pi_1(X_L \times I), w(X_L \times I)\big)$, we have that

$$\theta(M_N, \varphi_L, F_L) - \theta\big((M_1)_{N_1}, (\varphi_1)_L, (G \mid M_1)_L\big) + i'\eta + i''\mu = 0 .$$

The minus sign is due to the fact that the orientation of $(M_1)_{N_1}$ is usually taken to be the negative of its orientation as a part of ∂W_P, while the orientation of M_N is taken to agree with its orientation as part of ∂W_P, or *vice-versa*. However, in the definition of μ and η we take the orientations to agree with those of the various manifolds considered as parts of boundaries. (We could certainly be more precise in keeping track of the orientations.)

Hence it suffices to show that $i'\eta + i''\mu = 0$. Let $q \colon X_L \to X$ and $p \colon W_P \to W$ be the quotient maps. The following diagram commutes.

$$\begin{array}{ccc} \pi_1 L' & & \\ \Big\downarrow{\scriptstyle (q \mid L'')_*^{-1} \circ (q \mid L')_*} & \searrow^{i'} & \\ & & \pi_1 X_L \\ & \nearrow_{i''} & \\ \pi_1 L'' & & \end{array}$$

Here we use i' and i'' to denote inclusion induced maps, again. This diagram commutes because $q_*: \pi_1 X_L \to \pi_1 X$ is a monomorphism and because

$$q \circ i'' \circ (q \mid L'')^{-1} \circ (q \mid L') = q \circ i' .$$

Hence it suffices to show (see § 1) that, in $L_n^h\big(\pi_1(L \times I), w(L \times I)\big)$,

$$\theta\big(P'', q \circ (\Phi_L \mid P''), G_L \mid P''\big) + \theta\big(P', q \circ (\Phi_L \mid P'), G_L \mid P'\big) = 0 .$$

However, this follows from the facts that $(p \mid L'')^{-1} \circ (p \mid L')$ is a diffeomorphism of these triples that reverses orientation and that θ is a cobordism (and hence a diffeomorphism) invariant. This completes the proof that β_g is a well-defined cobordism invariant of triples (M, φ, F) on which α_g vanishes.

To complete the proof of Proposition 3.3, suppose that we are given (M, φ, F) such that φ is a simple homotopy equivalence of M with X. We can assume that φ and $\varphi \mid \partial M$ are transverse to L and to ∂L, as usual. By [6, Th. 2.2] and the homotopy extension theorem, we can assume that if $N = \varphi^{-1}L$, then φ restricts to homotopy equivalence of N with L and of ∂N with ∂L. Then, by Lemma 3.2, $\varphi_L: M_N \to X_L$ is a homotopy equivalence. Hence $\beta_g(M, \varphi, F) = \theta(M_N, \varphi_L, F_L) = 0$. This completes the proof of this proposition.

4. An exact sequence

Let G be a finitely presented group and let $w: G \to \{+1, -1\}$ be a homomorphism. Let $\Lambda = Z(G)$ be the integral group ring of G. Then Λ has an involution given by $(\sum \alpha_g g)^- = \sum w(g)\alpha_g g^{-1}$. Using this involution, we define an operation * on matrices: $\big((a_{ij})\big)^* = \big((\bar{a}_{ji})\big)$. This operation induces an automorphism (conjugation), also called *, of $\mathrm{Wh}(G)$ (see [18]). Let

$$A_n(G, w) = \{\sigma \in \mathrm{Wh}(G) \mid \sigma = (-1)^n \sigma^*\}/\{\tau + (-1)^n \tau^* \mid \tau \in \mathrm{Wh}(G)\}.$$

PROPOSITION 4.1. *There is a natural exact sequence*

$$\cdots \longrightarrow L_n^s(G, w) \longrightarrow L_n^h(G, w) \longrightarrow A_n(G, w) \longrightarrow L_{n-1}^s(G, w) \longrightarrow \cdots .$$

This sequence is due to Rothenberg (unpublished), who derived it by geometric methods. We will give algebraic definitions of the maps and will indicate the main points of a proof of exactness that uses a combination of algebra and geometry (mostly algebra). Naturality will be obvious. We assume a familiarity with [31, § 3] and [32, §§ 5, 6]; most of the omitted details are quite straightforward, assuming this knowledge.

We assume for simplicity that $w = 0$ and we write the proposed sequence as follows, with $2k - 2 \geqq 5$:

$$\longrightarrow L^s_{2k}(G) \xrightarrow{a} L^h_{2k}(G) \xrightarrow{b} A_{2k}(G)$$
$$\xrightarrow{c} L^s_{2k-1}(G) \xrightarrow{d} L^h_{2k-1}(G) \xrightarrow{e} A_{2k-1}(G) \xrightarrow{f} L^s_{2k-2}(G) \longrightarrow$$

The map a is induced by the "forgetful functor" which forgets the preferred class of basis. That is, if by $u(H)$ we denote the underlying Λ-module of an s-based Λ-module, then

$$a[H, \lambda, \mu] = [u(H), \lambda, u] .$$

(As usual, [ ] denotes the equivalence class of whatever appears inside.)

To define b, suppose (H, λ, μ) represents an element of $L^h_{2k}(G)$, with H free. Choose a basis for H and let M be the matrix of λ with respect to this basis. Let $b[H, \lambda, \mu] = [M] \in A_{2k}(G)$. Since $M = \pm M^*$, the class of M in $\mathrm{Wh}(G)$ is self-conjugate. If M' is the matrix of λ with respect to a different basis, then $M' = BMB^*$ for some suitable matrix B, and so in $A_{2k}(G)$, $[M] = [M']$. Also, the matrix $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$ represents zero in $\mathrm{Wh}(G)$. It now follows easily that b is really a well-defined homomorphism.

It is obvious that $b \circ a = 0$. If $b[H, \lambda, \mu] = 0$, we can suppose after sufficient stabilization that if M is the matrix of λ with respect to some basis, then there is an invertible matrix B of the same size as M, such that, in $\mathrm{Wh}(G)$, $[M] = [B] + [B^*]$. But then B determines a change of basis such that if M' is the matrix of λ with respect to the new basis, M' represents zero in $\mathrm{Wh}(G)$. Hence, with respect to this new basis, $A\lambda \colon H \longrightarrow \mathrm{Hom}_\Lambda(H; \Lambda)$ is simple and so $[H, \lambda, \mu] \in \text{Image } a$.

To define c, let $\tau \in \mathrm{Wh}(G)$ be a self-conjugate element. Let M be a matrix, of size r say, representing τ. Then we want to define $c[\tau]$ to be the class of the following matrix in $L^s_{2k-1}(G)$:

$$\begin{pmatrix} M & 0 \\ 0 & (M^*)^{-1} \end{pmatrix}$$

This matrix (and all subsequent matrices in this section) is written with respect to the standard base $e^r_1, \cdots, e^r_r, f^r_1, \cdots, f^r_r$ of the standard kernel of dimension $2r$. This really defines a simple automorphism of this kernel because $[M] - [M^*] = 0$ in $\mathrm{Wh}(G)$. If M represents zero in $\mathrm{Wh}(G)$, then the above matrix represents an element of $TU(\Lambda) \subseteq RU(\Lambda)$ (see [32]) and so it is easy to see that we at least get a homomorphism

$$\bar{c} \colon \{\tau \in \mathrm{Wh}(G) \mid \tau = \tau^*\} \longrightarrow L^s_{2k-1}(G) .$$

To show that c is well-defined, it suffices to show that if $\tau = \xi + \xi^*$, $\bar{c}(\tau) = 0$. But in this case we can take for M a matrix of the form

$$\begin{pmatrix} P & 0 \\ 0 & P^* \end{pmatrix}$$

and so $\bar{c}(\tau)$ has the representative

$$\left(\begin{array}{cc|cc} P & 0 & & \\ 0 & \pm P^* & \multicolumn{2}{c}{0} \\ \hline & & (P^*)^{-1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & \pm P^{-1} \end{array}\right).$$

Multiplying on both sides by

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & (-1)^k I & 0 & 0 \end{pmatrix}$$

which represents an element of $RU(\Lambda)$ (see [32]), we get

$$\begin{pmatrix} P & 0 & 0 & 0 \\ 0 & \pm P^{-1} & 0 & 0 \\ 0 & 0 & (P^*)^{-1} & 0 \\ 0 & 0 & 0 & \pm P^* \end{pmatrix}$$

which represents an element of $TU(\Lambda) \subseteq RU(\Lambda)$.

The proof that Ker c = Image b is the one place in the proof where we use geometry. Hence we postpone it until the end.

The map $d: L^s_{2k-1}(G) \rightarrow L^h_{2k-1}(G)$ is defined to be the homomorphism induced by the inclusion of $SU(\Lambda)$ in $U(\Lambda)$. It is clear that $d \circ c = 0$; in fact $d \circ c(x)$, $x \in A_{2k}(G)$, has a representative in $TU^h(\Lambda)$. Suppose $d(y) = 0$. Then we can find a representative matrix for y which is a product of $(2r \times 2r)$-matrices of the form

$$C = \begin{pmatrix} A & B \\ 0 & (A^*)^{-1} \end{pmatrix},$$

A non-singular and representative of a self-conjugate element of Wh(G), and the $(2r \times 2r)$-stabilization of $\sigma = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$, σ_1 say. The matrix C above represents an element in the image of c, since it differs from

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

by an element of $TU_{2r}(\Lambda)$. Also, the element $\sigma_1 \in SU_{2r}(\Lambda)$ represents zero in $L^s_{2k-1}(G)$; and so $y \in \text{Image } c$.

To define $e: L^h_{2k-1}(G) \to A_{2k-1}(G)$, let $\alpha \in U_r(\Lambda)$ represent an element of $L^h_{2k-1}(G)$. We want to define $e[\alpha]$ to be the torsion of α with respect to any base of the underlying Λ-module of the standard kernel of dimension $2r$. It is elementary to check that this at least gives a map of $L^h_{2k-1}(G)$ into $\text{Wh}(G)$ which is well-defined. That the image lies in the subgroup $\{\tau \mid \tau + \tau^* = 0\}$ follows by considering a matrix for α with respect to the standard base of the standard kernel, and so by composing with the quotient map, we get e.

It is obvious that $e \circ d = 0$. Suppose $e[\alpha] = 0$, α in $U_r(\Lambda)$. Then, after some stabilization if necessary, we can find $\beta \in U_r(\Lambda)$, with matrix

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

with respect to the standard basis, such that $\beta\alpha$ has zero torsion. Hence $\beta\alpha \in SU_r(\Lambda)$. But β represents zero in $L^h_{2k-1}(G)$, and so $[\beta\alpha] = [\alpha]$. Hence $[\alpha] \in \text{Image } d$.

To define the map $f: A_{2k-1}(G) \to L^s_{2k-2}(G)$, let $\tau \in \text{Wh}(G)$ be such that $\tau = -\tau^*$. Let M be a $2r$ by $2r$ matrix representing τ, and let (K_r, λ_r, μ_r) be the standard kernel of dimension $2r$. By applying M to the standard basis of K_r, we get a new based module K' with the same underlying Λ-module as K_r. Since $\tau + \tau^* = 0$, $A\lambda: K' \to \text{Hom}_\Lambda (K', \Lambda)$ is still simple, and so (K', λ_r, μ_r) is a special $(-1)^{k-1}$-hermitian form. We define $\bar{f}(\tau) = [K', \lambda_r, \mu_r]$. It is easy to see that this defines a homomorphism

$$\bar{f}: \{\tau \in \text{Wh}(G) \mid \tau + \tau^* = 0\} \longrightarrow L^s_{2k-2}(G) .$$

If $\tau = \xi - \xi^*$, then we can choose for τ a representative M of the form

$$M = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} .$$

But then the isomorphism $K_r \to K'$ of s-based Λ-modules determined by M preserves λ_r and μ_r, and so (K_r, λ_r, μ_r) is isomorphic to (K', λ_r, μ_r), i.e., $\bar{f}(\tau) = 0$. Hence $\bar{f}$ induces a map $f: A_{2k-1}(G) \to L^s_{2k-2}(G)$.

That $f \circ e = 0$ is clear. In fact, given $\alpha \in U_r(\Lambda)$, α provides an isomorphism of the representative of $f \circ e[\alpha]$ constructed in the definition above and a standard kernel of the appropriate dimension. On the other hand, suppose $f(x) = 0$, x in $A_{2k-1}(G)$. Choose a representative matrix M, and let (K', λ_r, μ_r) be constructed from (K_r, λ_r, μ_r) as in the definition of f above. Then, after stabilization of M if necessary (this corresponds to adding standard kernels), we may assume that there is an isomorphism $\alpha'\colon (K_r, \lambda_r, \mu_r) \to (K', \lambda_r, \mu_r)$. Since K_r and K' have the same underlying Λ-module, α' clearly determines an element $\alpha \in U_r(\Lambda)$. It is clear that $e[\alpha] = x$.

It is obvious that $a \circ f = 0$. If $a[H, \lambda, \mu] = 0$, then we can suppose that $(u(H), \lambda, u) \cong (u(K_r), \lambda_r, \mu_r)$, at least after stabilization. This isomorphism induces a Λ-module isomorphism of the based Λ-modules H and K_r; let $\tau \in \mathrm{Wh}(G)$ be its torsion. Using the fact that $A\lambda$ and $A\lambda_r$ are simple with respect to the appropriate bases, an easy calculation shows that $\tau + \tau^* = 0$. It is clear that if $x = +[\tau]$ in $A_{2k-1}(G)$, $f(x) = [H, \lambda, \mu]$. (Actually one should take $[-\tau]$, but since $A_{2k-1}(G)$ has exponent two, this makes no difference.)

Now we turn to the question of exactness of the sequence

$$L^h_{2k}(G) \xrightarrow{\ b\ } A_{2k}(G) \xrightarrow{\ c\ } L^s_{2k-1}(G).$$

The algebra at this point seems to be more formidable than for the rest of the sequence, and so we resort to geometry.[1] (The rest of the proof could also be given using geometry.) First we interpret the maps b and c geometrically. Let (H, λ, μ), H free over Λ, represent an element of $L^h_{2k}(G)$, and let X^{2k-2} be a closed orientable manifold with $\pi_1 X = G$. By the construction in [31, § 7] or [32, § 5], we can find (W, φ, F) representing an element $B^h_{2k}(X \times D^2, \nu)$, $\nu =$ normal bundle of $X \times D^2$, such that $K_k(W, \Lambda) = H$, $K_i(W, \Lambda) = 0$ if $i \neq k$, and such that λ and μ are the intersection and self-intersection forms of W over Λ, as defined in [31]. In particular, (see [32]), we can take $\partial W = (X \times D^1_-) \cup \partial_+ W$, with $\varphi(\partial_+ W) \subseteq X \times D^1_+$ and with $\varphi \mid X \times D^1_- =$ identity. (As usual $\partial D^2 = D^1_+ \cup D^1_-$.) In this construction, W is obtained from $(\partial_- W) \times I = (X \times D^1_-) \times I$ by adding some handles of index k, and these handles determine a Λ-basis of $K_k(W, \partial_- W, \Lambda) = K_k(W, \Lambda) = H$. Let M be the matrix of λ with respect to this base. Recall also that $\varphi_1 = \varphi \mid \partial_+ W : \partial_+ W \to X \times D^1_+$ is a homotopy equivalence.

LEMMA 4.2. *φ_1 has torsion* $\pm[M] \in \mathrm{Wh}(G)$.

PROOF. If X_1 is a space, let $\widetilde{X}_1$ denote its universal covering space. Let $M(\varphi)$ and $M(\varphi_1)$ denote mapping cylinders of φ and φ_1 respectively. In computing $\tau(\varphi_1)$ we usually consider the chain complex $C_*(\widetilde{M(\varphi_1)}, \widetilde{\partial_+ W})$, but it is easy to see that it suffices to consider $C_*(\widetilde{M(\varphi)}, \widetilde{\partial_+ W})$; $\tau(\varphi_1)$ is the torsion of this latter complex. (Here, of course, we chose suitable triangulations of the spaces involved and assume the maps are cellular.)

Consider the exact sequence

$$0 \longrightarrow C_*(\widetilde{W}, \widetilde{\partial_+ W}) \longrightarrow C_*(\widetilde{M(\varphi)}, \widetilde{\partial_+ W}) \longrightarrow C_*(\widetilde{M(\varphi)}, \widetilde{W}) \longrightarrow 0 .$$

[1] R. Sharpe has found an algebraic proof based on Wall's decomposition of elements of $RU(\Lambda)$.

This is an exact sequence of based chain complexes. The homology sequence of this short exact sequence is just the (two-term) sequence $\mathcal{H}$:

$$\longrightarrow 0 \longrightarrow H_{k+1}(\widetilde{M(\varphi)}, \widetilde{W}) = K_k(W, \Lambda) \xrightarrow{\partial} H_k(\widetilde{W}, \widetilde{\partial_+ W}) = K_k(\widetilde{W}, \widetilde{\partial_+ W}, \Lambda) \longrightarrow 0 \longrightarrow .$$

$K_k(W, \Lambda) = H$ already has a preferred basis and we give $K_k(\widetilde{W}, \widetilde{\partial W}, \Lambda)$ the basis determined by the "dual handles" to the handles attached to $(\partial_- W) \times I$ to construct W. Then it follows from [31, Th. 2.5], and the definition of $\tau(\mathcal{H})$, the torsion of $\mathcal{H}$, that $\tau(\mathcal{H}) = \pm[M]$.

We have the formula (see [18])

$$\tau(C_*(\widetilde{M(\varphi)}, \widetilde{\partial_+ W})) - \tau(C_*(\widetilde{W}, \partial_+ \widetilde{W})) - \tau(C_*(\widetilde{M(\varphi)}, \widetilde{W})) = \tau(\mathcal{H}) .$$

The last two torsions on the left should be understood as defined using the bases of the homology groups $H_k(\widetilde{W}, \partial_+ \widetilde{W})$ and $H_{k+1}(\widetilde{M(\varphi)}, \widetilde{W})$ just defined (see [18]). It is not hard to check that these two torsions actually vanish. Hence $\tau(\varphi_1) = \pm[M]$.

Now suppose (W', φ', F') represents an element $B^h_{2k}(X \times D^2, \nu)$, has the property that $W' = \partial_+ W' \cup X \times D^1_-$, $\varphi'(\partial_+ W') \subseteq X \times D^1_+$, and $\varphi' \mid X \times D^1_- =$ identity. Suppose also that $\theta(W', \varphi', F') = [H, \lambda, u] = \theta(W, \varphi, F)$, (W, φ, F) as above. Then, using the standard computation for the torsion of a composite of two maps, the duality formula for the torsions of an h-cobordism (see [18]) and the last part of Th. 5.8 of [32], suitably modified for the theory of surgery obstructions to obtain homotopy equivalences, one can prove the following.

LEMMA 4.3. *$\exists \tau \in \mathrm{Wh}(G)$ such that $\tau(\varphi_1) - \tau(\varphi_1') = \tau + \tau^*$, where $\varphi_1' = \varphi' \mid \partial_+ W' \colon \partial_+ W' \to X \times D^1_+$.*

Lemma 4.2 and Lemma 4.3 give a complete interpretation of the map $b\colon L^h_{2k}(G) \to A_{2k}(G)$.

Now we interpret the map $c\colon A_{2k}(G) \to L^s_{2k-1}(G)$ geometrically. Suppose $\varphi\colon (N^{2k-1}, \partial N) \to (Y, \partial Y)$ is a homotopy equivalence of connected $(2k-1)$-manifolds, such that $\varphi \mid \partial N\colon \partial N \to \partial Y$ is a simple homotopy equivalence. Let F be such that (N, φ, F) represents an element of $B^s_{2k-1}(Y, \nu(Y))$. Let $\tau = \tau(\varphi)$, and assume $\tau = \tau^*$. (This actually follows; see [6, Prop. 1.26].)

LEMMA 4.4. *$\theta(N, \varphi, F)$ has a representative in $SU_r(\Lambda)$, r sufficiently large, with matrix of the form*

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

with respect to the standard basis $\{e_1^r, \cdots, e_r^r, f_1^r, \cdots, f_r^r\}$, *where* $[A] = \pm\tau$ *in* $\mathrm{Wh}(G)$.

Remarks. (1) When we pass to $A_{2k}(G)$, the sign does not matter.

(2) Suppose $Y = X \times I$, and suppose $\tau = \tau^*$, $\tau \in \mathrm{Wh}(G)$. Then there is an h-cobordism N of $X \times 0$ and a map φ: $(N, \partial_- N, \partial_+ N) \to (X \times I, X \times 0, X \times 1)$, a homotopy equivalence with $\varphi \mid \partial_- N =$ identity, such that $\tau(\varphi) = \tau$. We actually have $\tau(\varphi) = \varphi_*(-\tau(N, X))$, and $\tau(N, X)$ is arbitrary by Theorem 11.1 of [18]. Let $\varphi_1 = \varphi \mid \partial_+ N \colon \partial_+ N \to X \times 1$. Then using the duality formula and the formula for the torsion of a composite (see [18]),

$$\tau(\varphi_1) = \tau(\varphi) + \varphi_*\big(\tau(N, \partial_+ N)\big) = \tau(\varphi) + \varphi_*\big(\tau(N, X)\big)^* = \tau - \tau^* = 0\ .$$

In particular, we can find F such that (N, φ, F) represents an element of $B^s_{2k-1}\big(X \times I, \nu(X \times I)\big)$. This remark, together with Lemma 4.4, gives a geometric interpretation of c.

PROOF OF 4.4. As in Lemma 4.2, the proof involves chasing Wall's definition of surgery obstructions and using the sum formula, [18, Th. 3.2]. So let $U \subseteq \operatorname{Int} N$ be the union of disjointly embedded copies of $S^{k-1} \times D^k$ representing a set of generators of $K_{k-1}(N, \Lambda)$. The diagram (1) of [32, § 6] reduces to the following diagram of based Λ-modules, where all coefficients are in Λ and $N_0 = cl(N - U)$:

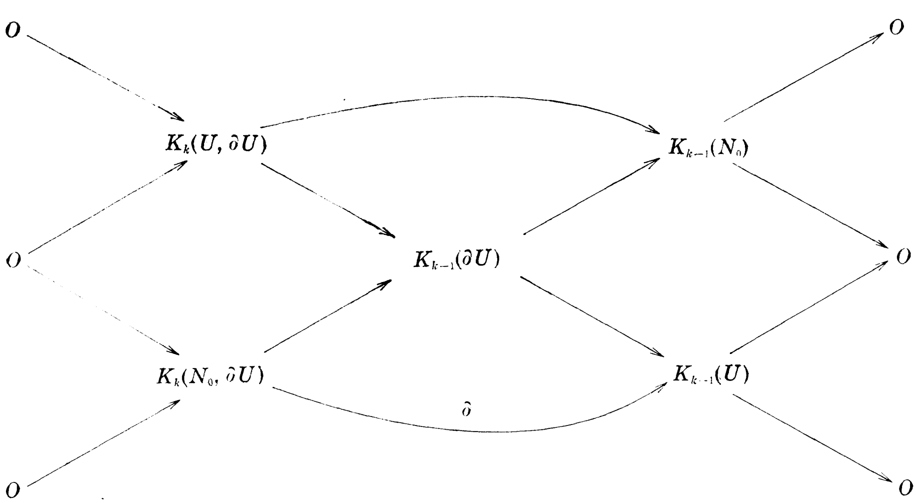


FIG. 1

Let r be the number of embedded copies of $S^{k-1} \times D^k$, and let $\alpha \in SU_r(\Lambda)$ be a representative of the element in $L^s_{2k-1}(G)$ determined as in [32, § 6] by this diagram. Choose bases for $K_k(N_0, \partial U)$ and $K_{k-1}(U)$ in the preferred class of s-bases. This may require some stabilization (i.e., adding some trivially embedded $S^{k-1} \times D^{k}$'s) for $K_k(N_0, \partial U)$; the preferred base of $K_{k-1}(U)$ is the obvious one. Let A_1 be the matrix of ∂ with respect to these bases. Then, with respect to the standard base $\{e_1^r, \cdots, e_r^r, f_1^r, \cdots, f_r^r\}$ we can take

$$\alpha = \begin{pmatrix} 0 & A_1 \\ \pm(A_1^*)^{-1} & 0 \end{pmatrix}. \qquad \text{(The sign depends on } k.\text{)}$$

Since

$$\begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

is in $RU_r(\Lambda)$ (the sign again depends upon k), α is equivalent to

$$\begin{pmatrix} A_1 & 0 \\ 0 & (A_1^*)^{-1} \end{pmatrix}.$$

Hence it suffices to show that we can take $A = A_1$, i.e., that the torsion of ∂: $K_k(M, U) \to K_{k-1}(U)$ is $\pm\tau$.

To see that $\tau(\partial) = \pm\tau$, recall that in Wall's procedure of [32, § 6], one first assumes that $\varphi^{-1}(D) = U$, D a disk in Int Y. After taking mapping cylinders, we can assume that φ: $(N, U) \to (Y, D)$ is an inclusion (D is no longer a disk, but is still contractible). Choosing appropriate triangulations, the preferred class of bases of $K_k(N, U, \Lambda) = H_{k+1}(Y, N \cup D, \Lambda)$ is the one which makes the torsion of $C_*(\widetilde{Y}, \widetilde{N \cup D})$ vanish. The obvious base of $H_k(C_*(\widetilde{D}, \widetilde{U})) = K_{k-1}(U, \Lambda)$ is the one making the torsion of $C_*(\widetilde{D}, \widetilde{U}) = C_*(D, U) \otimes_Z \Lambda$ vanish when computed with respect to this base. (Here $\widetilde{Z}$ denotes the inverse image of $Z \subseteq Y$ in the universal cover $\widetilde{Y}$ of Y under the projection map $\widetilde{Y} \to Y$.)

We have the following exact sequence:

$$0 \longrightarrow C_*(\widetilde{D}, \widetilde{U}) = C_*(\widetilde{N \cup D}, \widetilde{N}) \longrightarrow C_*(\widetilde{Y}, \widetilde{N}) \longrightarrow C_*(\widetilde{Y}, \widetilde{N \cup D}) \longrightarrow 0 .$$

The homology sequence of this exact sequence is just $\mathcal{H}$:

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K_k(N, U, \Lambda) \xrightarrow{\partial} K_{k-1}(U, \Lambda) \longrightarrow 0 \longrightarrow \cdots .$$

So by [18, Th. 3.2],

$$\tau = \tau(C_*(\widetilde{Y}, \widetilde{N})) = \tau(\mathcal{H}) = \pm\tau(\partial) = \pm[A_1] .$$

This completes the proof of Lemma 4.4.

Now we are ready to complete the proof of Proposition 4.1. We state what remains to be proved as a separate lemma.

LEMMA 4.5. *The sequence* $L_{2k}^h(G) \xrightarrow{b} A_{2k}(G) \xrightarrow{c} L_{2k-1}^s(G)$ *is exact.*

PROOF. Let $x \in L_{2k}^h(G)$. Then $x = \theta(W, \varphi, F)$, with (W, φ, F) as in the paragraph preceding Lemma 4.2; this follows from Theorem 1.1h. By Lemma 4.2 and Lemma 4.3, $b(x)$ is the class of the torsion of $\varphi \mid \partial_+ W \colon \partial_+ W \to X \times D'_+ = X \times I$. By Lemma 4.5, $cb(x) = \theta(\partial_+ W, \varphi \mid \partial_+ W, F \mid \partial_+ W)$. But this last term vanishes, because (W, φ, F) is a cobordism of $(\partial_+ W, \varphi \mid \partial_+ W, F \mid \partial_+ W)$ with $(\partial_- W, \varphi \mid \partial_- W, F \mid \partial_- W)$ and $\varphi \mid \partial_- W$ is a simple homotopy equivalence. Thus $c \circ b = 0$.

Suppose

$$\bar{\varphi} \colon (N, \partial_- N, \partial_+ N) \longrightarrow (X \times I, X \times 0, X \times 1)$$

is a homotopy equivalence with torsion τ, where $[\tau]$ in $A_{2k}(G)$ is any given element. We can assume $\bar{\varphi} \mid \partial_- N \colon \partial_- N \to X \times 0$ is the identity. Say $c([\tau]) = 0$. Then $\exists (W', \varphi', F')$, a cobordism of $(N, \bar{\varphi}, \bar{F})$, $\bar{F}$ some suitable framing, such that if $P = \partial_+ W'$,

$$\varphi' \mid P \colon (P, \partial_- P, \partial_+ P) \longrightarrow (X \times I \times 1, X \times 0 \times 1, X \times 1 \times 1)$$

is a simple homotopy equivalence. After making suitable identifications, it is not hard to find (W, φ, F), $\varphi \colon W \to X \times D^2$, such that $\varphi \mid \partial_- W \colon \partial_- W \to X \times D_1^-$ is the identity, such that $\varphi(\partial_+ W) \subseteq X \times D'_+ = X \times I$ and such that $\varphi \mid \partial_+ W = \bar{\varphi}$ and $F \mid \partial_+ W = \bar{F}$. By Lemma 4.2, $b(\theta(W, \varphi, F)) = [\tau]$ in $A_{2k}(G)$. This concludes the proof of Proposition 4.1.

One can give geometric interpretations for other maps in Rothenberg's exact sequence. For example, let $\varphi \colon (N, \partial N) \to (Y, \partial Y)$ be a homotopy equivalence of smooth $(2k-2)$-manifolds, with $\varphi \mid \partial N \colon \partial N \to \partial Y$ a simple homotopy equivalence. Let $G = \pi_1 Y$. Let $\tau = \tau(\varphi) \in \mathrm{Wh}(G)$ be the torsion of φ, with $\tau = -\tau^*$. (That this is always the case follows from [6, Prop. 1.26].) Let F be a stable framing of $\tau N \oplus \varphi^* \nu(Y)$. Then

$$\theta(N, \varphi, F) = e[\tau] \in L^s_{2k-2}(G) .$$

The only tricky point here is the obstruction to modifying φ is represented by the zero module with an s-basis; to realize this basis one has to stabilize by performing surgery on some trivially embedded $(k-1)$-spheres. Combining this with Lemma 4.4, we get the following result, which we state for later use.

PROPOSITION 4.6. *Let $\varphi \colon (N, \partial N) \to (Y, \partial Y)$ be a homotopy equivalence of smooth n-manifolds, such that $\varphi \mid \partial N \colon \partial N \to \partial Y$ is a simple homotopy equivalence. Let $\tau = \tau(\varphi) \in \mathrm{Wh}(G)$ be the torsion of φ. Let F be a stable framing of $\tau N \oplus \varphi^*(\nu(Y))$. Let γ denote the map $A_{n+1}(G, w) \to L^s_n(G, w)$ of Rothenberg's exact sequence, with $w = w(Y)$ and $G = \pi_1 Y$. Then*

$$\gamma[\tau] = \theta(N, \varphi, F) .$$

5. Computation of $L_n(Z \times G)$, G finitely presented

Let $X^n = Y \times I$, Y a connected closed manifold, and let v be a vector bundle over X. Then by $\tilde{B}^s_n(X, v)$ we denote those classes of $B^s_n(X, v)$ with representatives (M, φ, F) such that ∂M has Y as a connected component and $\varphi(y) = (y, 0)$ for all y in Y. Let $\tilde{B}^h_n(X, v) \subseteq B^h_n(X, v)$ be similarly defined.

THEOREM 5.1. *Let $n \geqq 7$. Let K^{n-2} be a closed, connected, smooth manifold with fundamental group G. Let $w: G \to Z_2$ be the orientation map of K. Let w_1 be the composite of w and the natural projection of $G \times Z$ onto G. Let v be the stable normal bundle of $K \times I$. Let $l_*: L^s_n(G, w) \to L^s_n(G \times Z, w_1)$ be the map induced by inclusion. Then there is a split exact sequence.*

$$0 \longrightarrow L^s_n(G, w) \xrightarrow{l_*} L^s_n(G \times Z, w_1) \xrightarrow{\alpha(K)} L^h_{n-1}(G, w) \longrightarrow 0 .$$

The map $\alpha(K)$ has a splitting $j(K)$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{B}^h_{n-1}(K \times I, v) & \xrightarrow{\times S^1} & \tilde{B}^s_n(K \times I \times S^1, v \times S^1) \\ \downarrow \theta & & \downarrow \theta \\ L^h_{n-1}(G, w) & \xrightarrow{j(K)} & L^s_n(G \times Z, w_1) \end{array}$$

Remarks. (1) The map $\times S^1$ above is defined by $[M, \varphi, F] \times S^1 = [M \times S^1, \varphi \times S^1, F \times H]$, where H is the standard framing of τS^1. Note that taking the product with S^1 always kills the torsion (see [13]).

(2) Given G and $w: G \to Z_2$, we can find K as in Theorem 5.1 with $(\pi_1 K, wK) = (G, w)$ if and only if G is finitely presented. This is well-known; given G one chooses a presentation, takes a connected sum of trivial and non-trivial $(n-3)$-sphere bundles over S^1 to realize the free group on the generators of G with suitable values under w, and then attaches 2-handles along embedded circles determined by the relations.

(3) Let e be the trivial group. Then it is well-known (see [10] or [31]) that

$$L_n(e) = \begin{cases} Z & n \equiv 0 \pmod 4 \\ 0 & n \equiv 1 \pmod 2 \\ Z_2 & n \equiv 2 \pmod 4 . \end{cases}$$

Also, Wh $(Z^k) = 0$, by [27].

Hence, if Z^k is the free abelian group on k generators, Theorem 5.1 allows us to compute the groups $L_n(Z^k) = L_n(Z^k, w)$, w trivial. For example

$$L_n(Z) = \begin{cases} Z & n \equiv 0 \mod 4 \\ Z & n \equiv 1 \mod 4 \\ Z_2 & n \equiv 2 \mod 4 \\ Z_2 & n \equiv 3 \mod 4 \end{cases}$$

$$L_n(Z \oplus Z) = \begin{cases} Z \oplus Z_2 & n \equiv 0 \mod 4 \\ Z \oplus Z & n \equiv 1 \mod 4 \\ Z \oplus Z_2 & n \equiv 2 \mod 4 \\ Z_2 \oplus Z_2 & n \equiv 3 \mod 4 . \end{cases}$$

PROOF OF 5.1. The first step in the proof is to define the map $\alpha(K)$. By Theorem 1.1 every element of $L_n^s(G \times Z, w_1)$ is of the form $\theta(W, \varphi, F)$, where (W, φ, F) represents an element of $B_n^s(K \times I \times S^1, v \times S^1)$ and where ∂W is the disjoint union of $K \times S^1$ and $\partial_+ W$, and $\varphi(x, y) = (x, 0, y)$ for x in K and y in S^1. Let g be the natural projection of $K \times I \times S^1$ onto S^1. Then we define

$$\alpha(K)\big(\theta(W, \varphi, F)\big) = \alpha_g(W, \varphi, F) .$$

It actually follows from [32, Th. 5.8 and 6.5] that $\alpha(K)$ is well-defined. However, this will also follow once we show that a relation $\alpha(K)$ is additive, since by Proposition 2.1 $\alpha_g(W, \varphi, F) = 0$ if $\theta(W, \varphi, F) = 0$.

To prove that $\alpha(K)$ is additive, let (W, φ, F) be as in the last paragraph. Then by [6, Th. 2.2] applied to the restriction of φ to $\partial_+ W$, the homotopy extension property, and the usual transversality theorems, we can assume after a homotopy relative $\partial_- W = K \times S^1$ of φ as a map of the pair $(W, \partial_+ W)$ to the pair $(K \times I \times S^1, K \times 1 \times S^1)$ that φ is transverse to $K \times I \times z$, that $\varphi | \partial_+ W$ is transverse to $K \times 1 \times z$, and that if $N = \varphi^{-1}(K \times I \times z)$, the restriction of φ induces a homotopy equivalence of ∂N with $K \times \partial I \times z$. Let $L = K \times z$, z the basepoint of S^1. Then from now on, we identify $L \times I$ and $K \times I \times z$ by the obvious map.

Let (W', φ', F') also represent an element of $B_n^s(K \times I \times S^1, v \times S^1)$. Assume that $\partial_- W' = \partial_+ W$ and that the composites of $\varphi | \partial_+ W$ and $\varphi' | \partial_- W'$ with the natural projection of $K \times I \times S^1$ onto $K \times S^1$ are equal. Also suppose that $F' | \partial_- W' = F | \partial_+ W$. We can take (W', φ', F'), subject to these requirements, so that $\theta(W', \varphi', F')$ is any given element of $L_n^s(G \times Z, w_1)$, by Theorem 1.1. As in the last paragraph, we can also assume that φ' is transverse to $L \times I$, that $\varphi' | \partial_+ W'$ is transverse to $L \times 1$, and that if N' is the inverse image of $L \times I$ under φ', then φ' restricts to a homotopy equivalence of $\partial N'$ with $L \times \partial I$.

Let W'' be the union of W and W' with $\partial_+ W$ and $\partial_- W'$ identified by the identity map. Define $\varphi'': W'' \to K \times I \times S^1$ as follows; let $h(x, t, y) = (x, t/2, y)$

and let $k(x, t, y) = (x, 1/2(t+1), y)$ if x is in K, t in I, and y in S^1; then set $\varphi''(u) = h\varphi(u)$ if u is in W and set $\varphi''(u) = k\varphi'(u)$ if u is in W'. Let $F'' = F \cup F'$, a stable framing of $(\varphi'')^*(v \times S^1) \oplus \tau W''$. Then by Proposition 1.4,

$$\theta(W'', \varphi'', F'') = \theta(W, \varphi, F) + \theta(W', \varphi', F') .$$

Let $h = \varphi' \mid \partial_- W'$. We can take φ' to be a composite of a map $\varphi'_1: W' \to (\partial_- W') \times I$ and $h \times 1$, where $\varphi'_1 \mid \partial_- W' = \text{id}$. Let $\psi'_1: N' \to (\partial_- N') \times I$ be the restriction of φ'_1. Let (W_1, φ_1, F_1) represent an element of $B^s_n(K \times I \times S^1, v \times S^1)$ with $\varphi_1 \mid \partial_- W_1 = \text{id}$ and $\theta(W_1, \varphi_1, F_1) = \theta(W', \varphi', F')$. Let $X = (K \times [-1, 0] \times S^1) \cup W \cup (\partial_- W' \times I)$, with identifications along common boundaries. Let $W_2 = W_1 \cup W \cup (\partial_- W' \times I)$ similarly. Let $k(x, t) = (x, -t)$ and let $\varphi_2: W_2 \to X_2$ be $k\varphi$, the identity, and φ'_1 on the respective summands. Let F_2 be the union of appropriate framings. Then, as in 1.4, $\theta(W_2, \varphi_2, F_2) = 0$. We can assume, if necessary after prior modification, that $\pi_1 N \to \pi_1 W$ is identified by φ with the inclusion of G in $G \times Z$. Then using [6, Th. 2.2] it follows that $\alpha_g(W_1, \varphi_1, F_1) = ((h \mid \partial_+ N) \times 1)_* \theta(N', \psi'_1, F' \mid N')$, and so $\alpha(K)(\theta(W_1, \varphi_1, F_1)) = \alpha_g(W', \varphi', F')$.

Hence to show that $\alpha(K)$ is additive, we must show that

$$\alpha_g(W'', \varphi'', F'') = \alpha_g(W, \varphi, F) + \alpha_g(W', \varphi', F') .$$

To see this, let N'' be the inverse image of $L \times I$ under φ''. Let $\psi: N \to L \times I$, $\psi': N' \to L \times I$, and $\psi'': N'' \to L \times I$ be restrictions of φ, φ', and φ'' respectively. Then N'' is the union of N and N' with $\partial_+ N$ and $\partial_- N'$ identified by the identity, and $\psi''(x) = h\psi(x)$ for x in N and equals $k\psi'(x)$ for x in N'. Clearly $F'' \mid N'' = (F \mid N) \cup (F' \mid N')$. Hence by Proposition 1.4

$$\theta(N'', \psi'', F'' \mid N'') = \theta(N, \psi, F \mid N) + \theta(N', \psi', F' \mid N')$$

in $L^h_{n-1}(G, w)$. But by definition of α_g, this is just what we have to prove. Hence $\alpha(K)$ is a well-defined homomorphism.

Every element of $L^h_{n-1}(G, w)$ is of the form $\theta(N, \psi, F)$, where (N, ψ, F) represents an element of $B^h_n(K \times I, v)$ with $\partial_- N = K$ and $\psi(x) = (x, 0)$ if x is in K. We define $i(K)$ by

$$i(K)(\theta(N, \psi, F)) = \theta([N, \psi, F] \times S^1) .$$

(See *Remark* (1) above.) Using 1.4, it is easy to check that as a relation $i(K)$ is additive, and it is trivial that if $\theta(N, \psi, F) = 0$, then θ also vanishes on the product of this triple with S^1. Hence $i(K)$ is a well-defined homomorphism. It is trivial from the definitions that $\alpha(K) \circ i(K) = \text{identity}$ and that the diagram in the statement of Theorem 5.1 commutes.

Remark. Given any tangential simple homotopy equivalence $h: P^{n-2} \to K$, a procedure analogous to the above can be used to define a split epimorphism $\alpha(K, h)$ of $L_n^s(G \times Z, w_1)$ onto $L_{n-1}^h(G, w)$ and a splitting $i(K, h)$ of this epimorphism. In particular $\alpha(K) = \alpha(K, \mathrm{id})$ and $i(K) = i(K, \mathrm{id})$. We do not know in general whether or not $\alpha(K, h)$ and $i(K, h)$ are independent of h. However, these maps are independent of the cobordism class of the tangential simple homotopy equivalence h. (See also *Remark* 2, at the end of this section.)

The next step of the proof is to use the results of § 3 to define a map

$$\beta(K): \ker \alpha(K) \longrightarrow L_n^h(G, w) = L_n^h\big(\pi_1(K \times I \times I), w(K \times I \times I)\big) .$$

Let

$$\beta(K)\big(\theta(W, \varphi, F)\big) = \beta_g(W, \varphi, F) ,$$

where (W, φ, F) represents an element of $B_n(K \times I \times S^1, v \times S^1)$ such that $\partial_- W = K \times S^1$, $\varphi(x, y) = (x, 0, y)$ for x in K and y in S^1, and $\alpha_g(W, \varphi, F) = 0$. Again, it actually follows from [32, Thms. 5.8 and 6.5] that $\beta(K)$ is well-defined. However, this also follows once we prove that as a relation, $\beta(K)$ is additive, since by Proposition 3.2, β_g vanishes on (W, φ, F) if θ does.

The proof that $\beta(K)$ is additive is similar to the proof that $\alpha(K)$ is additive, only slightly harder. Let (W, φ, F) be as in the last paragraph. As in the discussion of $\alpha(K)$, we can assume that φ and $\varphi \mid \partial_+ W$ are transverse to $L \times I$ and $L \times \partial I$, respectively, and that the restriction of φ induces a homotopy equivalence of $\partial\big(\varphi^{-1}(L \times I)\big)$ with $L \times \partial I$. Let (W', φ', F') be such that $\alpha_g(W', \varphi', F') = 0$, $\partial_- W' = \partial_+ W \times 2$, $\varphi(x)$ and $\varphi'(x, 2)$ have the same first and third components (their second components differ by unity) for all x in $\partial_+ W$, and $F(x) = F'(x, 2)$ for all x in $\partial_+ W$. Once again, we can also assume that φ' and $\varphi' \mid \partial_+ W'$ are transverse to $L \times I$ and $L \times 1$ respectively, and that the restriction of φ' is a homotopy equivalence of $\partial\big(\varphi'^{-1}(L \times I)\big)$ with $L \times \partial I$. By Theorem 1.1 we can insist, in addition to the above conditions, that $\theta(W, \varphi, F)$ and $\theta(W', \varphi', F')$ be any two given elements of the kernel of $\alpha(K)$. Let $W'' = W \cup \big(\partial_+ W \times [1, 2]\big) \cup W'$, where $\partial_+ W$ is identified with $\partial_+ W \times 1$ by the obvious map and $\partial_- W'$ with $\partial_+ W \times 2$ by the identity. Define $h_i(x, t, y) = (x, (i-1)/3 + t/3, y)$ for $i = 1, 2, 3$ and for x in K and y in S^1. Now define $\varphi'': W'' \to K \times I \times S^1$ by letting $\varphi''(y) = h_1\varphi(y)$ if y is in W, $\varphi''(u, t) = h_2(\varphi u, t - 1)$ if u is in $\partial_+ W$ and t in $[1, 2]$, and $\varphi''(y) = h_3\big(\varphi'(y)\big)$ if y is in W'. Let $F'' = F \cup \big(F \mid \partial_+ W \times [1, 2]\big) \cup F'$, a stable framing of $\varphi''^*(v \times S^1) \oplus \tau W''$. Then

$$\theta(W'', \varphi'', F'') = \theta(W, \varphi, F) + \theta(W', \varphi', F') .$$

Similarly to the proof for α_g, we want to show that

$$\beta_g(W'', \varphi'', F'') = \beta_g(W, \varphi, F) + \beta_g(W', \varphi', F') .$$

Note that $\alpha_g(W'', \varphi'', F'') = 0$ because $\alpha(K)$ is a homomorphism.

If (W, φ, F) is cobordant to (Y, η, G), there is a diffeomorphism of ∂Y with ∂W. For simplicity we consider $\partial W = \partial Y$. By Lemma 3.1 we can find (Y, η, G) such that $\varphi \mid \partial W: \partial W \to K \times \partial I \times S^1$ and $\eta \mid \partial Y: \partial Y \to K \times \partial I \times S^1$ are homotopic, $F \mid \partial W = G \mid \partial Y$, η and $\eta \mid \partial Y$ are transverse to $L \times I$ and $L \times \partial I$ respectively, and η restricts to homotopy equivalences of ∂N with $L \times \partial I$ and of N with $L \times I$, $N = \eta^{-1}(L \times I)$. By definition,

$$\beta_g(W, \varphi, F) = \theta(Y_N, \eta_L, G_L) .$$

Let (Y', η', G') be cobordant to (W', φ', F') and have properties analogous to those of (W, φ, F) (i.e., add primes to the beginning of this paragraph). Now let $H: \partial_+ Y \times [1, 2] \to K \times I \times S^1$ be such that $H(y, 1)$ and $\eta(y)$ have the same first and third coordinates, $H(y, 1)$ has zero as second coordinate, $H(y, 2)$ and $\eta'(y, 2)$ have the same first and third coordinates, and $H(y, 2)$ has second coordinate one. (Recall that $\partial_+ W = \partial_- W' \times 2$.) H is a simple homotopy equivalence. Hence by [6, Th. 2.2](see [5, Cor. 2] also) we can suppose in addition that H is transverse to $L \times I$ and that if $M = H^{-1}(L \times I)$, H restricts to a homotopy equivalence of M with $L \times I$. By Lemma 3.2, it follows that H_L is also a homotopy equivalence.

Let $Y'' = Y \cup (\partial_+ Y \times [1, 2]) \cup Y'$, with $\partial_+ Y$ identified with $\partial_+ Y \times 1$ in the obvious way and $\partial_- Y'$ identified with $\partial_+ Y \times 2$ by the identity. Define $\eta'': Y'' \to K \times I \times S^1$ by letting $\eta''(y) = h_1\eta(y)$ if y is in Y, $\eta''(y) = h_2H(y)$ if y is in $\partial_+ Y \times [1,2]$, and $\eta''(y) = h_3\eta'(y)$ if y is in Y'. Let $G'' = G \cup (G \mid \partial_+ Y \times I) \cup G'$. Let $N'' = \eta''^{-1}(L \times I)$. Then the restriction of η'' induces a homotopy equivalence of N'' with $L \times I$, and by definition

$$\beta_g(W'', \varphi'', F'') = \theta(Y''_{N''}, \varphi''_L, G''_L) .$$

But we have that $Y''_{N''} = Y_N \cup (\partial_+ Y \times I)_M \cup Y'_{N'}$, with $\partial_+ Y_N$ identified with $\partial_-[(\partial_+ Y \times I)_M] = \partial_+ Y_N \times 0$ by the obvious map, and with $\partial_- Y'_{N'} (= (\partial_- Y')_{\partial_- N'})$ identified to $\partial_+[(\partial_+ Y \times I)_M]$ similarly. Also, $(K \times I \times S^1)_{L \times I} = K \times I \times I$, and so $\eta''_L: Y''_{N''} \to K \times I \times I$. If x is in K and s and t in I, and if $i = 1, 2, 3$, let $k_i(x, t, s) = (x, (i - 1)/3 + t/3, s)$. Then $\eta''_L(y) = k_1\eta_L(y)$ if y is in Y_N, $\eta''_L(y) = k_2H_L(y)$ if y is in $(\partial_+ Y \times I)_M$, and $\eta''_L(y) = k_3\eta'_L(y)$ if y is in $Y'_{N'}$. Also, $G''_L = G_L \cup (G \mid \partial_+ Y \times I)_L \cup G'_L$. Hence by Proposition 1.4h

$$\beta_g(W'', \varphi'', F'') = \theta(Y_N, \eta_L, G_L) + \theta((\partial_+ Y \times I)_M, H_L, E_L) + \theta(Y'_{N'}, \eta'_L, G'_L) ,$$

where $E = G \mid \partial_+ Y \times I$. The middle term of the right side vanishes because H_L is a homotopy equivalence. This completes the proof that $\beta(K)$ is additive. Hence $\beta(K)$ is a well-defined homomorphism.

Now suppose that we are given an element ξ of $L^s_n(G, w)$. Let (Q, ψ, E)

represent an element of $\tilde{B}_n^s(K \times I \times I, v \times I)$ with $\theta(Q, \psi, E) = x$. Then we can suppose that $\partial Q = (K \times 0 \times I) \cup (K \times I \times \partial I) \cup \partial_+ Q$ and $\psi \mid (\partial Q - \partial_+ Q) = \mathrm{id}$. We can also suppose that if $e: K \times I \times 0 \to K \times I \times 1$ is $e(x, t, 0) = (x, t, 1)$. Then $e^*(E \mid K \times I \times 1) = E \mid K \times I \times 0$. Let P be obtained from Q by identifying $e(x, t, 0)$ with $(x, t, 0)$, let $\varphi: Q \to K \times I \times S^1$ be induced by ψ, and let F be determined by E; i.e., set

$$(P_L, \varphi_L, F_L) = (Q, \psi, E) .$$

LEMMA 5.2. *Let* $l_*: L_n^s(G, w) \to L_n^s(G \times Z, w_1)$ *be the map induced by inclusion. Then* $l_*(\xi) = \theta(P, \varphi, F)$, ξ *and* (P, φ, F) *as above.*

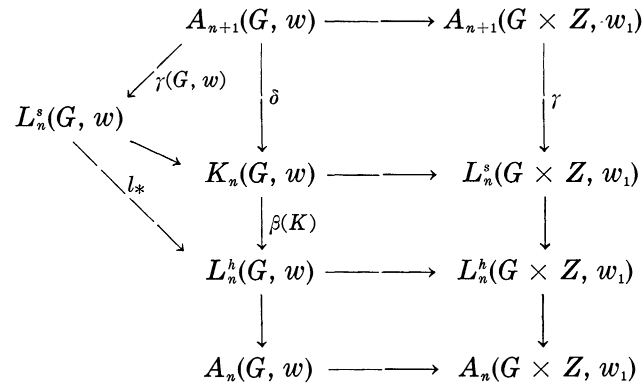
The proof of this lemma is straightforward using the definitions in [32, § 5 and § 6] of $\theta(P, \varphi, F)$. Also, there is a similar interpretation of the inclusion induced map $L_n^h(G, w) \to L_n^h(G \times Z, w_1)$.

Let $K_n(G, w)$ be the kernel of $\alpha(K): L_n^s(G \times Z, w_1) \to L_{n-1}^h(G, w)$. It is clear from Lemma 5.2 and the definition of $\alpha(K)$ that $\alpha(K) \circ l_* = 0$. Hence we can view l_* as a map of $L_n^s(G, w)$ into $K_n(G, w)$; l_* is obviously a monomorphism. To complete the proof we must show that l_* is an epimorphism.

To see that l_* is onto $K_n(G, w)$, let δ be the composite

$$A_{n+1}(G, w) \xrightarrow{\gamma(G, w)} L_n^s(G, w) \xrightarrow{l_*} K_n(G, w) ,$$

where $\gamma(G, w)$ is the appropriate map in Rothenberg's exact sequence (Proposition 4.1). Using Lemma 5.2 the analogue of Lemma 5.2 for the groups L_n^h, and the naturality of Rothenberg's exact sequence 4.1, it is easy to see that we have the following commutative diagram.



Now let $x \in K_n(G, w)$. Then by Proposition 4.1 the image of x in $A_n(G \times Z, w_1)$ is zero. The lowest horizontal map in the diagram is a monomorphism. Hence by Proposition 4.1 again $\beta(K)(x)$ is the image of some ξ in $L_n^s(G, w)$. Hence $\beta(K)(x - l_*\xi) = 0$. So we may as well assume to begin with that $\beta(K)(x) = 0$ as well as $\alpha(K)(x) = 0$.

Now we state another elementary lemma.

LEMMA 5.3. *Let (M, φ, F) represent an element of $B^s_n(K \times I \times S^1, v \times S^1)$ with φ transverse to L and $\varphi \mid \partial M$ to ∂L. Let $N = \varphi^{-1}L$. If (P, ψ, G) is obtained from (M_N, φ_L, F_L) by an elementary cobordism, then $(P, \psi, G) = \big((M_1)_{N_1}, (\varphi_1)_L, (F_1)_L\big)$, where (M_1, φ_1, F_1) is the result of surgery* (i.e., *the top end of an elementary cobordism of triples constructed*) *using an embedding $\mu: S^i \times D^{n-i} \to M - N$.*

It follows from this lemma and from Lemma 3.1 that $x = \theta(M, \varphi, F)$; where (M, φ, F) represents an element of $\tilde{B}^s_n(K \times I \times S^1, v \times S^1)$; where φ is transverse to L and $\varphi \mid \partial M$ to ∂L; and where if $\varphi^{-1}L = N$, $\varphi \mid \partial N: \partial N \to \partial L$, $\varphi \mid N: N \to L$, $\varphi_{\partial L}$: $(\partial M)_{\partial N} \to \partial(K \times I \times S^1)_{\partial L}$, and $\varphi_L: M_L \to K \times I \times I$ are all homotopy equivalences. It follows easily that φ itself is a homotopy equivalence. But then, according to [6, Lem. 2.1], $\tau(\varphi) \in \mathrm{Wh}\,(G \times Z)$ is in the image of $\mathrm{Wh}(l)$: $\mathrm{Wh}\,(G) \to \mathrm{Wh}\,(G \times Z)$, $l: G \to G \times Z$ the inclusion. (In fact, $\tau(\varphi)$ is the image under $\mathrm{Wh}\,(l)$ of the difference $\tau(\varphi_L) - \tau(\varphi \mid N)$. See [5, Th. 4] for a statement of the result we are using in the absolute case.) On the other hand, $\tau(\varphi) = (-1)^{n+1}\tau(\varphi)^*$ and by Proposition 4.6, if z is the element of $A_{n+1}(G \times Z, w_1)$ represented by $\tau(\varphi)$, then $\gamma(z) = x$. So there is μ in $A_{n+1}(G, w)$ whose image is z under the map induced by l; and since the above diagram commutes, $\delta(\mu) = x$. Hence $x = \delta(\mu) = l_* \circ \gamma(G, w)(\mu) = l_*(t)$ some $t \in L^s_n(G, w)$. This proves that $l_*\big(L^s_n(G, w)\big) = K_n(G, w)$ and so completes the proof of Theorem 5.1.

COROLLARY 5.4. *Let $B_{n+1}(G, w)$ be the cokernel of the inclusion induced map $L^h_{n+1}(G, w) \to L^h_{n+1}(G \times Z, w_1)$, and let $C_{n+1}(G, w)$ be the cokernel of the inclusion induced map $A_{n+1}(G, w) \to A_{n+1}(G \times Z, w_1)$. Let L^{n-1} be a closed smooth manifold with $(\pi_1 L, wL) = (G, w)$ and let K^{n-2} be as in Theorem* 5.1 (*with* $n \geqq 7$). *Then there is a commutative diagram with exact rows and columns.*

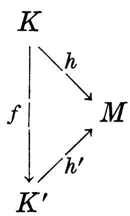
$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^s_{n+1}(G, w) & \longrightarrow & L^s_{n+1}(G \times Z, w_1) & \xrightarrow{\alpha(L)} & L^h_n(G, w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^h_{n+1}(G, w) & \longrightarrow & L^h_{n+1}(G \times Z, w_1) & \longrightarrow & B_{n+1}(G, w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1}(G, w) & \longrightarrow & A_{n+1}(G \times Z, w_1) & \longrightarrow & C_{n+1}(G, w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^s_n(G, w) & \longrightarrow & L^s_n(G \times Z, w_1) & \xrightarrow{\alpha(K)} & L^h_{n-1}(G, w) & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Remarks. (1) Let $C(G, \text{id})$ be as in [3] or [24] and suppose it vanishes. Then it follows from [5] that $C_{n+1}(G, w) = 0$, and so from Corollary 5.4 we recover the result announced in [24].

(2) In general the map $\alpha(K)$ depends on the choice of K. In fact, given G, this map really depends even on the way in which we identify $\pi_1 K$ with G. One could conjecture that this is the worst that can happen; i.e., $\alpha(K)$ and $\alpha(K')$ differ by an automorphism of $L^h_{n-1}(G, w)$ induced by an automorphism of G with itself. We know of no way to prove this, however.

6. Five-manifolds with fundamental group Z

Let M be a closed, connected, smooth manifold. Then by a homotopy smoothing of M we mean a simple homotopy equivalence $h: K \to M$ of closed smooth manifolds. Two homotopy smoothings h and h' are said to be concordant if and only if there is a diffeomorphism $f: K \to K'$ such that the following diagram commutes up to homotopy.



This definition is due to Sullivan. We denote the concordance classes of homotopy smoothings of M by $hS(M)$. Our aim in this section is to determine $hS(M)$ for M a 5-manifold with fundamental group Z.

Let F_k be the space of basepoint preserving homotopy equivalences of S^k. We can view F_k as contained in F_{k+1} by suspension. We can also consider the proper homotopy equivalences of R^k with itself as contained in F_k; in particular $O(k) \subseteq F_k$. If F is the direct limit of the F_k, then the infinite orthogonal group O is contained in F. F is an H-space under the Whitney sum construction, and this also induces an H-space structure on F/O. We have a fibration $F \to F/O$. In fact, there is an exact sequence

$$O \longrightarrow F \longrightarrow F/O \longrightarrow BO \longrightarrow BF ;$$

i.e., if X is a complex, the following sequence of abelian groups is exact.

$$[X, O] \longrightarrow [X, F] \longrightarrow [X, F/O] \longrightarrow [X, BO] \longrightarrow [X, BF] .$$

Here [,] means homotopy classes of maps. If $X = S^n$, the first map is just the J-homomorphism in dimension n; in fact $\pi_n F = \pi_{n+k} S^k$, k large. In particular, F/O is connected, $\pi_1(F/O) = 0$, $\pi_2 F \to \pi_2(F/O) = Z_2$ is an isomorphism,

$\pi_3(F/O) = \pi_5(F/O) = 0$, and $\pi_4(F/O) \to \pi_4(BO)$ is a monomorphism (it sends a generator to 24 times a generator).

A map of M into F/O can be viewed as an equivalence class of fibre-homotopy trivializations of stable vector bundles over M; two such trivializations are equivalent if they differ stably a stable vector bundle equivalence of the given vector bundles. This essentially amounts to viewing F/O as the fibre $BO \to BF$. The natural map $[M, F/O] \to [M, BO]$ can be interpreted as assigning to each class trivializations the classifying map of the class of stable vector bundles being trivialized.

Let $h: K \to M$ be a simple homotopy equivalence of closed smooth manifolds. They as in [28, § II], we can define $\eta(h) \in [M, F/O]$, the "characteristic F/O-bundle of h". Namely, let k be large and choose an embedding of M into $K \times R^k$ which is homotopic to $(g,0)$, g a homotopy inverse of h. Let v be the normal bundle of this embedding, and let $E(v)$ be its total space. Then this embedding extends to a diffeomorphism c of $E(v)$ onto $K \times R^k$ (see [28]. In the general non-simply connected case one must use engulfing or the differentiable "weak h-cobordism theorem" infinitely iterated). Let $\eta(h)$ be the homotopy class in $[M, F/O]$, i.e., the isotopy class of fibre homotopy trivializations of v, represented by the composite

$$E(v) \xrightarrow{c} K \times R^k \xrightarrow{h \times \mathrm{id}} M \times R^k .$$

It is not hard to see that $\eta(h)$ depends only upon the concordance class of h, and so we get a map

$$\eta: hS(M) \longrightarrow [M, F/O] .$$

Moreover, $\eta(h)$ comes from $[M, F]$ if and only if v is trivial.

Our goal is to prove that if M is a closed, orientable five-manifold, with fundamental group Z then η is a monomorphism carrying $hS(M)$ bijectively onto

$$S(M) = \ker\big([M, F/O] \longrightarrow [M, BO]\big) .$$

Note that $[M, F/O]/S(M)$ is isomorphic to $J(M)$, the fibre-homotopy equivalence classes of vector bundles over M.

THEOREM 6.1. *Let $h: K \to M$ be a homotopy equivalence of smooth orientable 5-manifolds with free abelian fundamental group. Then h is tangential*; i.e., *$h^*\nu(M)$ is equivalent to $\nu(K)$, the stable normal bundle of K.*

PROOF. It suffices to show that $\xi = \tau K \oplus h^*(\nu(M))$ is trivial. Stiefel-Whitney classes are homotopy invariants of manifolds and $W^2(\nu(K)) = W^2(K)$ by the Whitney product formula. Hence $W^2(\xi) = 0$. Since $H^4(M; Z)$ has no

2-torsion by Poincaré duality, $p_1(\nu(M)) = -p_1(M)$. By [6] or [21], the L-genus $L_k(N^{4k+1})$ of smooth, closed, orientable $(4k+1)$-manifolds is a homotopy invariant. Hence $p_1(K) = h^* p_1(M)$. Hence since $H^4(K; Z)$ has no 2-torsion, $p_1(\xi) = 0$. (In general, if p is the total Pontrjagin class, $p(\mu \oplus \pi) - p(\mu)p(\pi)$ has order 2. See [16].)

Since ξ is orientable and $W^2(\xi) = 0$, ξ can be framed over the 2-skeleton. The obstruction to extending this framing over the 3-skeleton lies in $H^3(M, \pi_2(SO)) = 0$. The obstruction to extending to a framing over the 4-skeleton vanishes because $p_1(\xi) = 0$, by [9, Lem. 1.1]. Finally, a framing over the 4-skeleton extends over M because $H^5(M; \pi_4(SO)) = 0$.

THEOREM 6.2. *Let M be a closed, connected, orientable smooth five-manifold with $\pi_1 M = Z$. Let η: $hS(M) \to [M, F/O]$ be as defined above. Then*

$$\text{Image}\,\eta = S(M) = \text{kernel}\,([M, F/O] \longrightarrow [M, BO]) .$$

PROOF. That Image $\eta \subseteq S(M)$ follows from Theorem 6.1. On the other hand, there is, for any smooth closed manifold M^n of dimension at least five an exact sequence of pointed sets

$$hS(M) \xrightarrow{\eta} [M, F/O] \xrightarrow{s} L_n^s(\pi_1 M, wM) .$$

This is due to Sullivan [28, § II]. (See also [32].) The map s is defined as follows. If $\beta \in [M, F/O]$, choose a representative fibre-homotopy trivialization H: $E(v) \to M \times R^k$, k large and v a k-dimensional vector bundle over M. Let $v(\beta) = \nu(M) \oplus \xi$, where $\xi \oplus v$ is trivial. Since H is a proper map, we can take H to be transverse to $M \times 0$ and such that $K = H^{-1}(M \times 0)$ is a closed submanifold of $E(v)$. Let φ be the composite

$$K \xrightarrow{c} E(v) \xrightarrow{p(v)} M ,$$

$p(v)$ the projection map of v. The map H pulls back a framing of M in $M \times R^k$ to a framing of K in $E(v)$. Using this framing we can get a stable equivalence of $\nu(K)$ with $\nu(E(v)) \mid K$. But $\tau(E(v))$ is just $p(v)^*(\tau M \oplus v)$, and so we get a stable equivalence of $\nu(K)$ with $\varphi^*(\nu(M) \oplus \xi)$ and from this in turn we can find a stable framing F of $\tau K \oplus \varphi^* v(\beta)$. We define

$$s(\beta) = \theta(K, \varphi, F) .$$

It is not hard to see that s is really well-defined. The proof that $s^{-1}(0) =$ Image η is just a version of a standard argument about framed modifications of submanifolds.

Thus to complete the proof of Theorem 6.2, it suffices to show that s vanishes on $S(M)$, M a closed, connected orientable five-manifold with $\pi_1 M =$

Z. But if $\beta \in S(M)$, $v(\beta) = \nu(M)$. Hence we need only prove the following lemma.

LEMMA 6.3. *Let M be a smooth closed connected 5-manifold with fundamental group Z and stable normal bundle v. Let (N, φ, F) represent an element of $B_5(M, v)$. Then $\theta(N, \varphi, F) = 0$.*

PROOF. Let CP^2 be complex projective space of two dimensions over the complex numbers. Let u be the stable normal bundle of CP^2, and let G be a framing of $\tau(CP^2) \oplus u$. Then $F \times G$ is a framing of $\tau(N \times CP^2) \oplus (\eta \times \mathrm{id})^*(v \times u) = (\tau N \oplus \varphi^* v) \times (\tau(CP^2) \oplus u)$, and

$$(N \times CP^2, \varphi \times \mathrm{id}, F \times G)$$

represents an element of $B_9(M \times CP^2, v \times u)$. By [32, Th. 9.9], $\theta(N, \varphi, F) = 0$ if and only if $\theta(N \times CP^2, \varphi \times \mathrm{id}, F \times G) = 0$.

If the universal covering space of M is of the homotopy type of a finite complex, we can choose a fibration g of $M \times CP^2$ over S^1 and then show that $\alpha_g(N \times CP^2, \varphi \times \mathrm{id}, F \times G) = 0$; this would suffice. Since we are not assuming this however, we have to work a little harder. Let $g: M \times CP^2 \to S^1$ be a smooth map which induces an isomorphism of fundamental groups and which has regular value z. Let $L = g^{-1}(z)$. By the arguments of [2, 3.1 and 3.2], we can assume that L (and therefore also $M \times CP^2 - L$) is connected and simply-connected. Let $\bar{\varphi}$ be homotopic to φ and such that $\bar{\varphi}$ is transverse to L, and let $P = (\bar{\varphi})^{-1}L$. Then let $\psi: P \to L$ be the restriction of $\bar{\varphi}$. Then $\theta(P, \psi, F \times G \mid P) = \theta$ is an element of $L_8(e) = Z$; in fact it is well-known that (up to sign)

$$\theta = 1/8\ (\mathrm{index}\, P - \mathrm{index}\, L)\ .$$

However, $v \times u$ is the stable normal bundle of $M \times CP^2$, and so

$$(\varphi)^*\big(p_i(M \times CP^2)\big) = p_i(N \times CP^2)\ ,$$

where p_i now denotes the i^{th} rational Pontrjagin class. Letting i denote the appropriate inclusion maps,

$$\psi^*(p_i L) = \psi^* i^* p_i(M \times CP^2) = i^*(\bar{\varphi})^*\big(p_i(M \times CP^2)\big) = i^*\big(p_i(N \times CP^2)\big) = p_i(P)\ .$$

Thus by the Hirzebruch formula, index P = index L and so $\theta = 0$. It now follows by Lemma 3.1 that $(N \times CP^2, \varphi \times \mathrm{id}, F \times G)$ is cobordant to (N', φ', F'), where φ' is transverse to L and where if P' is the inverse image of L under φ', φ' restricts to a homotopy equivalence of P' with L. Now $\theta(N'_{P'}, \varphi'_L, H_L)$ is an element of $L_9(e) = 0$. Hence, using Lemma 5.3 we can find finally (Q, φ'', E) cobordant to $(N \times CP^2, \varphi \times \mathrm{id}, F \times G)$ such that φ'' is transverse to L and such that if R is the inverse image of L under φ'', $\varphi'' \mid R: R \to L$ and the map $\varphi''_L: Q_R \to (M \times CP^2)_L$ are homotopy equivalences. But in this case φ'' itself is easily seen to be a homotopy equivalence. Thus

$$\theta(N \times CP^2, \varphi \times \text{id}, F \times G) = \theta(Q, \varphi'', E) = 0 .$$

Remark. By an argument similar to the argument in the last paragraph of the preceding proof, one can show that if M is a closed smooth $(4k + 1)$-manifold, $k \geqq 2$, with fundamental group Z and stable normal bundle v, then for every (N, φ, F) representing an element of $B_{4k+1}(M, v)$, $\theta(N, \varphi, F) = 0$.

Next we try to show that $\eta: hS(M) \to S(M)$ is a monomorphism if M is a smooth, closed, connected orientable five-manifold with fundamental group Z.

If M^n is any smooth closed manifold, we say that two tangential simple homotopy equivalences $h: K \to M$ and $h': K' \to M$ are tangentially cobordant if there is a cobordism W of K with K' and a map $r: W \to M$ such that $r \mid K$ and $r \mid K'$ are homotopic to h and h', respectively, and such that $r^*(\tau M) \oplus \theta^1$ is equivalent to τW.

LEMMA 6.4. *Let M^n be a closed, connected smooth manifold, and let h and h' be tangential homotopy smoothings of M. Then h and h' are tangentially cobordant if and only if $\eta(h) = \eta(h')$.*

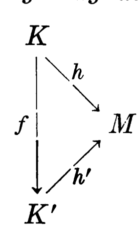
Remark. One can define tangential cobordism for any two homotopy smoothings by replacing the stable tangent or normal bundle of M by a suitable vector bundle in the same fibre-homotopy class. The analogous result to 6.4 holds. However, we do not need this here.

The proof of Lemma 6.4, or even the more general version for arbitrary homotopy smoothings, is straightforward. We remark only that if two maps of M into F/O are represented by $h_1: E(v_1) \to M \times R^k$ and $h_2: E(v_2) \to M \times R^k$, k very large, then a homotopy between them is a proper map

$$H: E(v_1) \times I \longrightarrow M \times R^k$$

such that $H(x, 0) = h_1(x)$, and such that there is a vector bundle equivalence $\mu: E(v_2) \to E(v_1)$ with $H(\mu(x), 1) = h_2(x)$.

THEOREM 6.5. *Let M be a smooth, closed, orientable five-manifold with fundamental group Z. Let h and h' be tangentially cobordant homotopy smoothings. Then h and h' are concordant*; i.e., *there is a diffeomorphism $f: K \to K'$ such that the following diagram commutes up to homotopy.*



PROOF. Let (W, r) be a tangential cobordism, and let $f: W \to [0, 1]$ be a Morse function [17]. Let $\varphi = (r, f)$. Choose a framing F of $\tau W \oplus \varphi^*(\nu(M) \times I)$ and let $\theta = \theta(W, \varphi, F)$. If $\theta = 0$, then we can modify φ by surgery to get a homotopy equivalence, i.e., (W, φ, F) is cobordant to (W', φ', F') with φ' a homotopy equivalence. In this case 6.5 follows from the s-cobordism theorem.

Suppose then that $\theta \neq 0$ in $L_6(Z) = Z_2$. By Theorem 5.1, the inclusion induced map $L_6(e) \to L_6(Z)$ is an isomorphism. On the other hand there is a map $g: S^3 \times S^3 \to S^6$ and a framing F_1 of $\tau(S^3 \times S^3) \oplus g^*\nu(S^6)$ such that $\theta(S^3 \times S^3, g, F_1)$ is the non-zero element of $L_6(e)$, (see [20]). It follows easily that, by taking connected sum in the interior of W with this surgery problem, we get a new problem with invariant zero. That is, if we take $W'' = W \# (S^3 \times S^3)$, $\varphi'' = \varphi \# g: W'' \to (M \times I) \# S^6 = M \times I$, and $F'' = F \# F_1$, then $\theta(W'', \varphi'', F'') = 0$. Now perform surgery to get a homotopy equivalence and apply the s-cobordism theorem, as above.

Remark. In the last proof, one could also proceed by first observing that the non-zero element of $L_6(Z)$ is represented by (H, λ, u), where H is free over $Z[Z]$ on two generators e and f, $\lambda(e, f) = 1$, $\lambda(e, e) = \lambda(f, f) = 0$, and $\mu(e) = \mu(f) = 1 \in Z_2 \subseteq \Lambda/I$. (See § 1.) If $\theta(W, \varphi, F)$ above is not zero, one can suppose that φ is 3-connected and that (W, φ, F) has associated (-1)-hermitian form (H, λ, u), by [31]. Choose a regular framed immersion of S^3 in W representing e. After adding a single self-intersection locally, one obtains a new immersion of S^3 in W which is regularly homotopic to an embedding of S^3 in W. This of course destroys the framing, but since $\pi_2(SO(3)) = 0$, one can find a new framing and perform surgery. Using the arguments of [10] or of [31, Th. 3.3], it follows easily that the result is a homotopy equivalence.

THEOREM 6.6. *Let M be a closed, connected, orientable smooth five-manifold with fundamental group Z. Then*

$$\eta: hS(M) \longrightarrow [M, F/O]$$

is a monomorphism with image $S(M) = \ker([M, F/O] \to [M, BO])$.

PROOF. Immediate from 6.2, 6.4, and 6.5.

Next we give a way of estimating the size of $[M, F/O]$. Let $\iota \in H^2(F/O; Z_2)$ be the non-zero element. If $\eta \in [M, F/O]$, let $\gamma_M(\eta) = \eta^*(\iota) \in H^2(M; Z_2)$. This defines a homomorphism $\gamma_M: [M, F/O] \to H^2(M; Z_2)$, since ι is primitive.

PROPOSITION 6.7. *Let M be a smooth, closed, connected five-manifold with fundamental group Z^k, the free abelian group on k generators. Then $\gamma_M \mid S(M)$ is a monomorphism.*

PROOF. Choose a handle decomposition of M with exactly k four-handles.

This is possible by combining [31, Cor. 5.1.3] with some standard observations. Let $M(i)$ denote the union of the handles of dimension at most i. Suppose $b: M \to F/O$ represents an element of $S(M)$ such that $b^*(\iota) = 0$ in $H^2(M; Z_2)$. Then $(b \mid M(2))^*\iota = 0$ in $H^2(M(2); Z_2)$ and F/O is a $K(Z_2, 2)$ as far as $M(2)$, which has the homotopy type of a two-complex, is concerned. Also, $\pi_3(F/O) = 0$. So by the homotopy extension property and the covering homotopy property of $F \to F/O$, we can assume that $b(M(3))$ is a point. Let $X = M(4)/M(3)$, and let $\pi: M(4) \to X$ be the quotient map. Then X has the homotopy type of the one-point union of k 4-spheres, and it is straightforward to check that $\pi_*: H_4(M(4); Z) \to H_4(X; Z)$ is an isomorphism.

Now $b \mid M(4)$ has a factorization $M(4) \xrightarrow{\pi} X \xrightarrow{c} F/O$. Let $j: F/O \to BO$ be the natural map. Then $(j \circ c)_*: H_4(X; Z) \to H_4(BO; Z)$ is zero. Since the Hurewicz map $\pi_4(BO) \to H_4(BO)$ is a monomorphism (see [16] or apply [9, Lem. 1.1]) and since $j_*: \pi_4(F/O) \to \pi_4(BO)$ is a monomorphism, $c_*: \pi_4 X \to \pi_4(F/O)$ vanishes. Hence c is null-homotopic and therefore so is $b \mid M(4)$. Since $\pi_5(F/O) = 0$, Proposition 6.7 now follows.

Remark. $\gamma_M(S(M))$ is not always all of $H^2(M; Z_2)$. For example, if $M = S^1 \times CP^2$, $S(M) = 0$.

COROLLARY 6.8. *If M is a smooth, orientable, closed five-manifold with fundamental group Z, then the set $hS(M)$ is finite and is bounded in size by the number of elements of $H^2(M; Z_2)$.*

In particular $hS(S^1 \times S^4) = 0$, and so every manifold of the same homotopy type as $S^1 \times S^4$ is diffeomorphic to $S^1 \times S^4$. In [25], we saw this implied the next two results.

COROLLARY 6.9. *Any h-cobordism of $S^1 \times S^3$ with itself is a product.*

COROLLARY 6.10. *Let $\varphi: S^3 \to S^5$ be a smooth embedding. Then φ is (ambient) isotopic to the standard inclusion $S^3 \subseteq S^5$ if and only if $S^5 - \varphi S^3$ has the homotopy type of a circle.*

In a future paper, we intend to give more precise information on the classification of 5-manifolds with fundamental group Z. We conclude this section with the following result.

THEOREM 6.11. (Hauptvermutung for 5-manifolds with $\pi_1 = Z$.) *Let M be a smooth closed orientable 5-manifold with $\pi_1 M = Z$. Let $h: K \to M$, K a smooth manifold, be a topological homeomorphism. Then h is homotopic to a diffeomorphism.*

PROOF. By [28, Th. H] (see also [23]), $\eta(h)$ is in the kernel of the natural map $[M, F/O] \to [M, F/PL]$. Since $\pi_i(PL/O) = 0$ for $i \leqq 5$, it follows that

$\eta(h) = 0$. Now apply Theorem 6.6.

Remarks. (1) Theorem 6.6 has a PL-analogue. It is proven using PL non-simply-connected surgery instead of smooth surgery. From the PL analogue of Theorem 6.6 it follows immediately that if $\eta(h) = 0$, h is homotopic to a PL-equivalence.

(2) $\pi_i(PL/O)$ is also known as Γ_i, the concordance classes of smoothings of S^i. (See, for example, Lashof and Rothenberg, *Microbundles and Smoothing*, Topology 3 (1965), 357–388.)

7. Five-manifolds with fundamental group $Z \oplus Z$ or $Z \oplus Z_2$

In this section we briefly indicate what can be obtained from the methods of § 6 for manifolds with fundamental group $Z \oplus Z$ or $Z \oplus Z_2$. See [7] for further results. If $M^n, n \geqq 5$, is a smooth, closed, connected n-manifold, then, as suggested by 6.4, there is an action of $L^s_{n+1}(\pi_1 M, wM)$ on $hS(M)$ such that $\eta(x) = \eta(y)$ in $[M, F/O]$ if and only if x and y are in the same orbit. (See, e.g., [32].) Suppose first that M is a closed, connected smooth five-manifold with $\pi_1 M = Z \oplus Z_2$. If $wM \neq 0$, assume that w does not vanish on an element of order two. Note that $\mathrm{Wh}\,(Z_2) = 0$. Hence by using Theorem 5.1, Proposition 4.1, and [33, Lem. 2], one can show that the inclusion induced map $L_6(e) \rightarrow L^s_6(Z \oplus Z_2, wM)$ is an isomorphism. Using this fact as in § 6 one can prove the following.

THEOREM 7.1. *Let M^5 be a closed connected smooth manifold with $\pi_1 M = Z \oplus Z_2$. If M is non-orientable, assume that $w(M)$ is non-zero on an element of order two. Then*

$$\eta\colon hS(M) \longrightarrow [M, F/O]$$

is a monomorphism.

For fundamental group $Z \oplus Z$, we stay with the orientable case. Then by using Theorem 5.1, a theorem of Rohlin, and the results (e.g., 6.1 and 6.7) and methods of § 6, and the existence of an almost four-parallelizable manifold of index 16, one can prove, for example, the following.

THEOREM 7.2. *Let $M = S^3 \times S^1 \times S^1$. Then $hS(M)$ is a set of four elements, and there exists a closed manifold of the same homotopy type as M, but not diffeomorphic to M.*

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