SPINES AND SPINELESSNESS

BY

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A fundamental question in topology is the following: when can the embedding problem for manifolds be reduced to a problem in homotopy theory? This problem is especially interesting in codimension two, because of the possibility of non-locally flat points and the connection with knot theory. The new methods of [2] and [3] permit a systematic study of this problem.

An important case of this problem (the "semi-local" case) is a question of the existence and classification of spines. Let W^m be a compact piecewise linear (P.L.) manifold with boundary. A SPINE of W is a P.L. embedding $\varphi : M^n \to W^m$, M a closed P.L. manifold, so that φ is a homotopy equivalence. For example, W could be the total space of a bundle over M. Questions about the existence and classification of spines in codimension two will be the topic of this paper[1]. The results described here are part of some joint work with S. E. Cappell.

If W^m, $m = n + 2$, has any chance to have a codimension two spine, it must have the homotopy type of a closed P. L. n-manifold.

THEOREM 1. *Let $h : M^n \to W^{n+2}$, $n \geq 3$, be a homotopy equivalence, M and W compact, orientable P.L. manifolds, M closed. If n is even, assume that W is simply-connected. Then h is homotopic to a P.L. embedding.*

In general, a P.L. embedding $\varphi : M \to W$, homotopic to h, will not be locally flat. For example, let $X(W) \in H^2(W)$ be the restriction of the Poincaré dual of $h_*[M]$, $[M]$ an orientation class of M, and let ξ be the $SO(2)$-bundle with Euler class $h^*X(W)$. Let

$$L(h) = L(M)L(\xi) - h^*L(W),$$

[1] In codimension ≥ 3, all P.L. embeddings are locally flat and one has an existence theorem of Browder-Haefliger-Casson-Sullivan. For codimension 1, one has the theorem of Hollingsworth and Galewski [5].

$L(M) = L(\text{tangent bundle of } M) =$ Total Hirzebruch L-genus of M, for example. Thus

$$L(h) = 1 + L_1(h) + L_2(h) + \dots ,$$

where $L_i(h) \in H^{4i}(M;Q)$. Let

$$D : H^j(M;Q) \to H_{n-j}(M;Q)$$

be Poincaré duality.

THEOREM 2. *The homology classes $DL_i(h)$ must be represented by cycles of the subcomplex (in some triangulation) of non-locally flat points of φ, φ any P.L. embedding homotopic to h.*

Under suitable (mild) restrictions on the cohomology of M, there exists an embedding with non-locally flat points of lowest dimension consistent with this requirement.

Sometimes one can change the manifold M so as to reduce the set of non-locally flat points. One needs at least a homotopy equivalence $f : M' \to M$ so that $L(h \circ f)$ becomes as trivial as possible, and this is all one needs in many cases. In general, one needs f with a certain normal invariant determined by $h : M \to W$. The existence of homotopy equivalences with given normal invariants can be attacked using surgery theory [1] [8]. For example, if W^{n+2} is simple connected and has the homotopy type of an n-manifold, then it has a locally flat spine, n odd, and a spine that has at most one non-locally flat point, n even[1]. (If $n \neq 4$, one need only suppose W a finite Poincaré complex of dimension n, since any such is homotopy equivalent to a manifold [1].)

On the other hand, there are many examples where any spine must be very far from locally flat. For example, there exists W^{n+2}, with a torus $T^n = S^1 \times \dots \times S^1$ as spine, so that any spine $\varphi : T^n \to W^{n+2}$ will have non-locally flat points of dimension $(n-2)$.

These results are all proven in [3] by combining the theory of homology equivalent manifolds [2] with some pure P.L. topology and some homotopy theory (similar to that used by Sullivan in his "characteristic variety theorem"). Here we will discuss in more detail the following result, which is in strong contrast with Theorem 1.

THEOREM 3. *Let M^n be a closed connected P.L. manifold. Suppose $\pi_1 M$ is a finite group that has a central subgroup with a non-trivial abelian quotient (e.g., $\pi_1 M$ non-trivial abelian). Assume $n \geq 4$ is even. Then there exist infinitely many manifolds W, simply homotopy equivalent to M, with $\chi(W) = 0$, that have no spines whatsoever!*

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[1]A result of Kato-Matsumoto. The most conceptually direct proof of this result is to apply the codimension 2 splitting principle of [2, §8].

Thus, in even dimensions, one finds totally spineless manifolds of the right homotopy type as soon as the fundamental group becomes non-trivial. One can conjecture that this result holds for every finite fundamental group. An elaboration of the proof to be outlined below will also give some examples of total spinelessness for infinite fundamental groups. Of course, if W^{n+2} has not even the homotopy type of a P.L. n-manifold, it will fail to have a spine; examples of this type are quite easy to construct. Also, it is not hard to construct examples with $\chi(W) \neq 0$.

To construct the examples of Theorem 3, we first define an invariant. Let W^{n+2} be a compact oriented P.L. manifold, and

$$h : M^n \to W^{n+2}$$

a (simple) homotopy equivalence, M is closed oriented P.L. manifold. We suppose that $\chi(W) = 0$. (Actually, for the next part of the discussion, M need only be an oriented finite Poincaré complex.) Then there exists a map

$$f : (W,\partial W) \to (M\times D^2, M\times S^1) ,$$

which has the following properties (compare [3, 1.6]).

(i) f has degree one and induces an isomorphism on homology groups with local coefficients in $Z\pi_1 W$; and

(ii) $f\circ h$ is homotopic to the inclusion $M \subset M \times D^2$.

(It follows that if h is a simple homotopy equivalence, then f is a simple homology equivalence over $Z\pi_1 W$.) Furthermore, if $\pi_1 W$, or even just its abelianization, is finite, then f is unique up to homotopy.

Let $\frac{1}{2}D^2 \subset D^2$ be the disk of radius $\frac{1}{2}$. We may assume that f is transverse regular to M, and that $f|f^{-1}(M\times\frac{1}{2}D^2)$ is a bundle map. Let $b : \nu_W \to \xi$, $\nu_W =$ normal bundle of W, be a stable bundle map covering the homology equivalence f. Let V be the closure of $W - f^{-1}(M\times\frac{1}{2}D^2)$, and write $M \times S^1 \times [0,1] = M \times (D^2-\frac{1}{2}D^2)^-$, so that $M \times S^1 = M \times S^1 \times 0$. Then we have a normal map $(f|V,b|V)$,

$$f|V : (V,\partial_- W = \partial W,\partial_+ V) \to M \times S^1 \times ([0,1],0,1) ,$$

which induces a simple homology equivalence over $Z\pi_1 M$ on $\partial_- W$.

Let $\overline{\Phi}$ be the diagram

$$\begin{array}{ccc} Z[\pi_1(M\times S^1)] & \xrightarrow{id} & Z[\pi_1(M\times S^1)] \\ \downarrow id & & \downarrow \\ Z[\pi_1(M\times S^1)] & \to & Z\pi_1 M \end{array} , \quad \text{(ZG=integral group ring of G.)}$$

where the unlabelled maps are induced by the projection on M. Then by [2, §3], the homology surgery obstruction

$$\sigma(f|V,b|V) \in \Gamma_{n+2}(\overline{\Phi}) = \Gamma^s_{n+2}(\overline{\Phi}) \ ,$$

$$\text{or} \quad \sigma(f|V,b|V) \in \Gamma^h_{n+2}|\overline{\Phi})$$

if h was not a simple homotopy equivalence, is defined.

Let $\pi_1 W = \pi$. Then $h_* : \pi_1 M \to \pi$ induces a map $\overline{\Phi} \to \phi_\pi$, ϕ_π the diagram

$$\begin{array}{ccc} Z[\pi \times Z] & \to & Z[\pi \times Z] \\ \downarrow & & \downarrow \\ Z[\pi \times Z] & \to & Z[\pi] \ , \end{array}$$

and so also a map

$$h_* : \Gamma_{n+2}(\overline{\Phi}) \to \Gamma_{n+2}(\phi_\pi) \ .$$

Definition. $\alpha(W) = h_*(\sigma(f|V,b|V)) \in \Gamma_{n+2}(\phi_\pi)$.
(If W has only the homotopy type of a P.L. manifold, we get $\alpha^h(W) \in \Gamma^h_{n+2}(\phi_\pi)$. If both are defined, then $\alpha^h(W)$ is clearly the image of $\alpha(W)$.)

Proposition. *The invariant $\alpha(W)$ (resp $\alpha^h(W)$) depends only upon W and not upon the choice of a simple homotopy equivalence (homotopy equivalence) to an n-dimensional manifold or Poincaré complex.*

This is not hard to check. This invariant can be thought of as an obstruction to the existence of a locally flat spine.

Note. If the assumption $\chi(W) = 0$ is dropped, one obtains an invariant in a Γ-group of a diagram

$$\phi_{\pi,\chi(W)} : \quad \begin{array}{ccc} Z[\pi_1 S(\xi)] & \to & Z[\pi_1 S(\xi)]^\cdot \\ \downarrow & & \downarrow \\ Z[\pi_1 S(\xi)] & \to & Z\pi_1 W \ , \end{array}$$

ξ an $SO(2)$-bundle over W with Euler class $\chi(W)$.

Digression. Suppose W^{n+2} has the simple homotopy type of the finite Poincaré complex X^n. Let ξ be as above. Let $f : X \to W^{n+2}$ be a simple homotopy equivalence. Then $\eta = f^*(\nu_W \oplus \xi)$ is a reduction of the Spivak normal fiber space of X to a linear bundle. By transversality (see [1] [8]), this determines a surgery obstruction $\sigma(W) \in L^s_n(\pi)$ (or $L^h_n(\pi)$ if one drops the adjective "simple"), the Wall group of π. This vanishes if and only if X is simple homotopy equivalent to a manifold in a way that induces the same reduction of η to a linear bundle. By inducing over the circle bundle $S(\xi)$ (e.g., take $\times S^1$ if $\chi(W) = 0$), we obtain $p^!_\xi(\sigma(W)) \in L_{n+1}(\pi_1(S(\xi))$. There is also a natural homeomorphism [2, §3]

$$\partial : \Gamma_{n+2}(\phi_{\pi,\chi(W)}) \to L_{n+1}(\pi_1(S(\xi)) \ .$$

PROPOSITION. *$p_\xi^!(\sigma(W)) = \partial\alpha(W)$.*

COROLLARY. *If $\chi(W) = 0$ and $\partial\alpha(W) = 0$, then for $n \geq 5$, W^{n+2} has the homotopy type of a closed P.L. n-manifold.*

The corollary follows from the proposition as $p_\xi^! : L_n^h(\pi) \to L_{n+1}^s(\pi\times Z)$ is a monomorphism if ξ is trivial [7]. One can make a more careful analysis of when W^{n+2} has the (simple) homotopy type of a manifold, using surgery and the theory of homology equivalences of [2], and give many examples which are not of the homotopy type of any manifold.

To construct our examples for Theorem 3, we use the next result:

THEOREM 4. *Let M^n, $n \geq 5$, be a closed orientable P.L. manifold, with $\pi = \pi_1 M$ finite. Let $\gamma \in \Gamma_{n+2}(\Phi_\pi)$, with $\partial\gamma = 0$. Then there exists a compact orientable P.L. manifold W^{n+2}, simple homotopy equivalent to M, with $\chi(W) = 0$, and $\alpha(W) = \gamma$.*

Similarly, one can realize elements in $\Gamma_{n+2}^h(\Phi_\pi)$ by (n+2)-manifolds homotopy equivalent to W. The idea of the proof is as follows: From [2, §3], we have the exact sequence

$$L_{n+2}^s(\pi\times Z) \to \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi) \xrightarrow{i_*} \Gamma_{2+n}(\Phi_\pi) \to L_{n+1}^s(\pi\times Z) \ .$$

Therefore $\gamma = i_*\gamma_1$. Let

$$j_* : \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi) \to L_{n+2}^s(\pi)$$

be the natural map. Since $L_{n+2}^s(\pi\times Z) \to L_{n+2}^s(\pi)$ is surjective, by functoriality, we may suppose that $j_*\gamma_1 = 0$. Hence $\gamma_1 = \partial_1\gamma_2$,

$$\partial_1 : \Gamma_{n+3}^s\begin{pmatrix} Z[\pi\times Z] & \to & Z\pi \\ \downarrow & & \downarrow \\ Z\pi & \to & Z\pi \end{pmatrix} \to \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi) \ ,$$

again by [3, §3].

By the realization theorem [2, 3.4], γ_1 can be realized as the homology surgery obstruction of a normal cobordism of the identity of $M \times D^2$ to a simple $Z\pi$-homology equivalence

$$h : (W,\partial W) \to (M\times D^2, M\times S^1) \ ,$$

that induces isomorphisms of fundamental groups. In particular, W is simple homotopy equivalent to M (but ∂W is not necessarily homotopy equivalent to $M \times S^1$). Using various naturality, additivity, and cobordism invariance properties of homology surgery obstructions, one can show that $\alpha(W) = \gamma$.

PROPOSITION 5. *Let $\varphi : M^n \to W^{n+2}$ be a P.L. embedding of M as a spine. Let W' be a regular neighborhood of $\varphi(M)$. Then $\alpha(W') = \alpha(W)$.*

This can be proven using the definition, additivity properties of homology surgery obstructions, and Poincaré duality.

Next, recall from [3] the classifying space $BSRN_2$ for oriented codimension two regular neighborhoods, and, more especially, the fiber G_2/RN_2 of the natural map

$$BSRN_2 \xrightarrow{X} BSO_2$$

for the associated SO(2)-bundle. A mapping $M \to G_2/RN_2$ gives an embedding $M \subset W^{n+2}$ as a spine, so that W is actually a regular neighborhood of M in itself, with $X(W) = 0$. Hence $\alpha(W) \in \Gamma_{n+2}(\phi_{\pi_1 M})$ and $\alpha^h(W)$ are defined, $\pi_1 M$ finite. Given a map $M \xrightarrow{g} K(\pi,1)$, $g_*\alpha(W) \in \Gamma_{n+2}(\phi_\pi)$. In this way we obtain a homomorphism (Ω_n = oriented bordism).

$$\sigma = \sigma_n : \Omega_n\ (G_2/RN_2 \times K(\pi,1)) \to \Gamma^e_{n+2}(\phi_\pi), \quad e = s,h$$

Further, if π is trivial, σ is just the splitting invariant defined in [3, §2]. Therefore, if $\tilde{\Gamma}_{n+2}(\phi_\pi)$ is the quotient of $\Gamma_{n+2}(\phi_\pi)$ by the image of $\Gamma_{n+2}(\phi_0)$ under the natural map induced by the inclusion of the trivial group in π, then σ induces

$$\tilde{\sigma} : \Omega_n\ (G_2/RN_2 \times (K(\pi,1),pt)) \to \tilde{\Gamma}^e_{n+2}(\phi_\pi), \quad e = s,h\ .$$

PROPOSITION 6. *If π is finite, then $\Omega_n(G_2/RN_2 \times (K(\pi,1),pt))$ is a torsion group.*

PROOF. Apply the Künneth formula for homology and the spectral sequence relating homology and oriented cobordism.

PROPOSITION 7. *Let π be a finite group. Let n be even. Suppose π has a central subgroup with abelian quotient. Then there is an element $\gamma \in \Gamma^s_{n+2}(\phi_\pi)$, with $\partial\gamma = 0$, whose image in $\tilde{\Gamma}_{n+2}(\phi_\pi)$ has infinite order.*

This will imply Theorem 3. For, by Theorem 4, we can construct W_k, homotopy equivalent to a given M, with $\chi(W_k) = 0$ and $\alpha(W) = k\gamma$. By Proposition 6 and Proposition 5, if W_k had a spine, then the image of $k\gamma$ would have to have finite order modulo $\Gamma^h_{n+2}(\phi_0)$. But by [2, Appendix I], $\Gamma^h_{n+2}(\phi_0)$ and $\Gamma^s_{n+2}(\phi_2)$ are isomorphic modulo 2-groups.

To prove Proposition 7, recall the exact sequence

$$L_{n+2}(Z[\pi\times Z]) \to \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi) \xrightarrow{\delta} \Gamma_{n+2}(\phi_\pi)\ .$$

Let $j_* : \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi) \to L_{n+2}(\pi)$ be the natural map. We will construct an element

$$\rho \in \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi)$$

with the following properties:

(i) ρ is of infinite order;

(ii) $j_*\rho = 0$; and

(iii) $\beta_*\rho = 0$, where β_* is induced by the map $\beta : \phi_\pi \to \phi_0$ induced by $\pi \times Z \to Z$.

The image of ρ in $\Gamma_{n+2}(\phi_\pi)$ will be the desired element. For suppose $\delta(k\rho) \in \Gamma_{n+2}(\phi_0) \subset \Gamma_{n+2}(\phi_\pi)$ (inclusion via the natural map).

We have the diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_{n+2}(Z) & \xrightarrow{\varepsilon} & \Gamma_{n+2}(Z[Z]\to Z) & \to & \Gamma_{n+2}(\phi_0) & \to & L_{n+1}(Z) \\
\downarrow\uparrow\overline{\beta}_* & & \downarrow\uparrow & & \downarrow\uparrow\beta_* & & \downarrow\uparrow \\
L_{n+2}(\pi\times Z) & \to & \Gamma_{n+2}(Z[\pi\times Z]\to Z\pi) & \to & \Gamma_{n+2}(\phi_\pi) & \to & L_{n+1}(Z\times\pi)
\end{array}$$

The rows are exact, by [2, §3], the maps from top to bottom, which can be thought of as inclusions, are induced by the natural induction $Z \subset \pi \times Z$, and the maps the other way by projection $\pi \times Z \to Z$. So we must have that

$$k\rho = \tau + \mu ,$$

$\tau \in \Gamma_{n+2}(Z[Z] \to Z)$, and μ in the image of $L_{n+2}(\pi\times Z)$, assuming $\delta(k\rho) \in \Gamma_{n+1}(\phi_0)$. Now,

$$0 = \beta_*(k\rho) = \tau + \varepsilon\overline{\beta}_*\mu , \quad \text{by (ii)}.$$

So τ is in the image of ε, and hence $\delta(k\rho) = 0$. Therefore $k\rho$ is the image of an element in $L_{n+2}(Z\times\pi)$ whose image $L_{n+2}(\pi)$ is trivial, by (ii). By [7], this element is in the image of the map

$$L^h_{n+1}(\pi) \to L^s_{n+2}(\pi\times Z)$$

given by taking products with S^1. But $L^h_{n+1}(\pi)$ is a torsion group, n even (see [9], for example), contradicting (i).

To construct ρ, we have two cases:

CASE 1. $n \equiv 2 \pmod 4$. If $g \in \pi$ and t is a generator of Z (multiplicatively), set

$$A_g = \begin{pmatrix} N(t+t^{-1}-2) & 1 \\ 1 & g+g^{-1}-2 \end{pmatrix},$$

a matrix over $Z[\pi\times Z]$, N to be specified later. A_g is easily seen to represent an element

$$\rho_g \in \Gamma_{n+2}(Z[\pi\times Z] \to Z\pi)$$

that satisfies (ii) and (iii). We want to show it has infinite order.

Suppose first that π has a surjective homomorphism $\omega : \pi \to Z_p$ to a cyclic group; this will be the case if π has non-trivial abelian quotient. The induced homomorphism on Γ-groups carries ρ_g to $\rho_{\omega(g)}$, so it will suffice in this case to take $\pi = Z_p$ and g a generator. Let ζ be a primitive p^{th} root, and consider the homomorphism

$$Z[Z_p \times Z] \to Q(\zeta) \subset C = \text{complex numbers}$$

that carries g and t to ζ. Thus a Hermitian form over $Z[Z_p \times Z]$ gives a Hermitian form over the complex numbers, which will be non-degenerate for a form representing an element of $\Gamma_{n+2}(Z[Z\times Z_p] \to Z[Z_p])$. Using the fact that our homomorphism factors through the semi-simple ring $Q[Z_p\times Z_p]$ which still has an augmentation map to Q (unlike $Q(\zeta)$), one can show that a form representing zero in $\Gamma_{n+2}(Z[Z_p\times Z] \to Z[Z_p])$ becomes a hyperbolic form over the complex numbers. Hence from the signature map for Hermitian forms over C, we obtain a homomorphism

$$\Gamma_{n+2}(Z[Z_p \times Z] \to Z[Z_p]) \to Z .$$

If $(\zeta + \zeta^{-1} -2)N > 1$, this invariant will have value (-2) on the element ρ_g, which will therefore have infinite order.

In the case when π only has a central subgroup π', with non-trivial abelian quotient, one argues using the transfer homomorphism

$$\Gamma_{n+2}(Z[Z\times\pi] \to Z[\pi]) \to \Gamma_{n+2}(Z[Z\times\pi'] \to Z[\pi'])$$

to reduce to the preceding case.

CASE 2. $n \equiv 0(4)$. We argue similarly. Again it turns out to suffice to consider a cyclic group Z_p. If $p \neq 2$, we use the form

$$\begin{pmatrix} (g-g^{-1})N & 1 \\ -1 & t-t^{-1} \end{pmatrix},$$

g a generator of $\pi = Z_p$. Passing to $Q(\zeta) \subset C$, we obtain a skew Hermitian form, which, after multiplying by $\sqrt{-1}$, becomes the Hermitian form

$$\begin{pmatrix} -2N\sin\theta & i \\ -i & -2\sin\theta \end{pmatrix} .$$

If N is chosen so that $(4N\sin^2\theta - 1) > 0$, this form will have non-zero index.

For $p = 2$, we use

$$\begin{pmatrix} N(g+g^{-1})(t^{-1}-t) & 1 \\ -1 & t-t^{-1} \end{pmatrix} ,$$

and map $g \in Z_2$, the non-zero element, to (-1) and t to $\cos 2\pi/3 + i \sin 2\pi/3$. This mapping factors through $Q[Z_2 \times Z_3] = Q[Z_6]$, and we again see that the result will have non-trivial index for N large enough.

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