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OBSTRUCTIONS TO THE IMBEDDING OF A COMPLEX IN A EUCLIDEAN SPACE.

I. THE FIRST OBSTRUCTION

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1. Introduction

The purpose of this paper is to develop a theory for the obstructions to finding an imbedding of a complex in euclidean space, following the standard procedures of the obstruction theory for the homotopy problem.

In 1932, van Kampen (see [1]) stated the results for the imbedding of an n-complex in euclidean $2n$-space which, considering the fact that neither cohomology or local coefficients had yet been invented, are essentially the same as those contained in Part I of this paper. Unfortunately, his proof had a lacuna which he himself noted. Whitney, in 1944 (see [2]), while stating results only for manifolds, supplied the geometrical procedure which is needed to prove a more precisely stated version of van Kampen's theorem. Paragraphs 5 and 6 will be devoted to an exposition of Whitney's procedure in a form suitable for its application to the case of the second obstruction also.

Part I treats the first obstruction and serves as an introduction and model for the higher obstructions. The main results characterize the first obstruction to the imbedding of an n-complex, K, in euclidean $2n$-space in terms of the cohomology of the symmetric product of K minus the diagonal, and show that the vanishing of the first obstruction is necessary and sufficient for the existence of an imbedding of an n-complex in E^{2n} for $n \geqq 3$.

Part II discusses the higher obstructions and treats the second obstruction in detail. Its main result is the proof that for $n \geqq 5$ a necessary and sufficient condition for the existence of an imbedding of an n-complex in E^{2n-1} is the vanishing of the first and second obstructions.

It should be noted that while the restrictions on dimension for the validity of the imbedding theorems quoted above may not be necessary, removing them is probably not easy since the Poincaré conjecture that a compact simply connected 3-manifold is a three sphere follows from the above imbedding theorems for $n = 2$.

Part III contains computations for applying the imbedding theorems, examples, and some general discussion of related problems.

2. The deleted product

The deleted product X^* of a topological space X is the subset of the cartesian product of X with itself consisting of pairs of distinct points. The mapping

$T(x, y) = (y, x)$ of X^* into itself is then fixed point free. T will be called the antipodal map and the decomposition space under T will be denoted by $\bar{X}^*$. If K is a simplicial complex then K^* will denote the subcomplex of the cell complex, $K \times K$, consisting of products of pairs of simplices having no vertex in common. The antipodal map $T: K^* \to K^*$ is then a cell map and the induced map $T^\#$ on cochains has the property

$$T^\# f(\sigma^p \times \tau^q) = (-1)^{pq} f(\tau^q \times \sigma^p)$$

where $\sigma \times \tau$ has an orientation induced from the orientations of σ and τ. The cohomology of the decomposition complex $\bar{K}^*$ can, for our purposes, be most conveniently described in terms of equivariant cohomology on K^*. That is, we will consider the group $C^p(\bar{K}^*, I)$ of integral p-cochains on $\bar{K}^*$ to be the subgroup of $C^p(K^*, I)$ consisting of those cochains, f, with the property $T^\# f = f$. We will also need the cohomology of $\bar{K}^*$ with coefficients in a certain system of local coefficients that we will denote by I_T and call "twisted integer" coefficients. The group of cochains $C^p(\bar{K}^*, I_T)$ will be the subgroup of $C^p(K^*, I)$ of those cochains, f, satisfying $T^\# f = -f$. In both cases the following coboundary formula is well known:

$$\delta f(\sigma^p \times \tau^q) = f(\partial \sigma^p \times \tau^q) + (-1)^p f(\sigma^p \times \partial \tau^q).$$

Remark. In the symmetric product of K, $\bar{K}^*$ is the complement of a "nice" neighborhood of the diagonal. It should be noted that the homotopy type of $\bar{K}^*$ is not an invariant of the homotopy type of K. As a result, the homology groups of $\bar{K}^*$ (or of K^*) may serve to distinguish some spaces with the same homotopy type. This is the case, for example, with the point, the line segment, and the space that looks like a Y.

Lemma 2.1. *Let $| K |$ denote the underlying space of the complex K, then $| K^* |$ is a deformation retract of $| K |^*$ and $| \bar{K}^* |$ is a deformation retract of $| \bar{K} |^*$.*

Proof. If the vertices of K are indexed then each point p of K has barycentric coordinates $x_i(p)$ where $x_i(p)$ is a number between 0 and 1 and $\sum x_i(p) = 1$. The indices, i, for which $x_i(p) \neq 0$ are the indices of the vertices spanning the open simplex containing p. For a pair of distinct points, p, q, of K let $\beta(p, q)$ be the projection of p on the face of the simplex containing p spanned by those vertices, v_i, for which $x_i(p) > x_i(q)$. By projection we mean that the non-zero coordinates of $\beta(p, q)$ have the same ratios as the corresponding coordinates of p.

The required retraction $h: | K |^* \times I \to | K |^*$ can then be written

$$h(p, q, t) = ((1 - t)p + t\beta(p, q), \qquad (1 - t)q + t\beta(q, p)).$$

Since h commutes with T, both parts of the lemma follow.

As a corollary we have

Proposition 2.2. *The homology and cohomology groups of $\bar{K}^*$ are topological invariants of $| K |$.*

3. The first obstruction to an imbedding

An imbedding of the complex K in euclidean r-space, E^r, is a one-to-one, continuous map of K in E^r. Although in order to produce an imbedding we will have to use very smooth maps, for the purpose of defining the first obstruction it will be sufficient to require of a map only that it be proper in the following sense:

DEFINITION 3.1. A continuous map, f, of K into E^r will be called proper if $f(\sigma^{p-1})$ and $f(\tau^{r-p})$ are disjoint whenever σ and τ have no vertex in common.

This definition is designed to insure the existence of the intersection number of the singular simplices formed by the images of disjoint cells of K of complementary dimensions. We shall denote the intersection number of $f(\sigma^p)$ and $f(\tau^q)$ in E^r by $f(\sigma^p) \wedge f(\tau^q)$. Note that this presupposes a choice of orientation in E^r which we make once and for all. The following well known properties of the intersection number will be needed:

$$f(\sigma^p) \wedge f(\tau^q) = (-1)^{pq} f(\tau^q) \wedge f(\sigma^p)$$

$$f(\partial\sigma^p) \wedge f(\tau^q) = (-1)^p f(\sigma^p) \wedge f(\partial\tau^q).$$

For each proper map, f, of K into E^r we define a cochain, m_f^r of K^* with integral coefficients as follows:

$$m_f^r(\sigma^p \times \tau^q) = (-1)^q f(\sigma^p) \wedge f(\tau^q) \qquad (p + q = r).$$

LEMMA 3.2. $T^{\#} m_f^r = (-1)^r m_f^r$.

PROOF. Let $p + q = r$, and $m_f^r = m^r$

$$\begin{aligned} T^{\#} m^r(\sigma^p \times \tau^q) &= (-1)^{pq} m^r(\tau^q \times \sigma^p) \\ &= (-1)^{pq+p} f(\tau^q) \wedge f(\sigma^p) \\ &= (-1)^p f(\sigma^p) \wedge f(\tau^q) \\ &= (-1)^r m^r(\sigma^p \times \tau^q). \end{aligned}$$

Thus m_f^r is a cochain of $\bar{K}^*$ with integer coefficients if r is even, and with twisted integer coefficients if r is odd.

LEMMA 3.3. $\delta m_f^r = 0$.

PROOF. Let $p + q = r + 1$, $m^r = m_f^r$ then

$$\begin{aligned} \delta m^r(\sigma^p \times \tau^q) &= m^r(\partial\sigma^p \times \tau^q) + (-1)^p m^r(\sigma^p \times \partial\tau^q) \\ &= m^r(\partial\sigma^p \times \tau^q) + (-1)^{p+q-1} f(\sigma^p) \wedge f(\partial\tau^q) \\ &= (-1)^q f(\partial\sigma^p) \wedge f(\tau^q) + (-1)^{q-1} f(\partial\sigma^p) \wedge f(\tau^q) \\ &= 0. \end{aligned}$$

LEMMA 3.4. *Suppose that K_1 is a subcomplex of K_2, f_2 a proper map of K_2 in E^r, and f_1 is the restriction of f_2 to K_1. Let j be the inclusion of $\bar{K}_1^*$ in $\bar{K}_2^*$. Then $m_{f_1}^r = j^{\#} m_{f_2}^r$.*

The proof is a straightforward verification.

LEMMA 3.5. *The cohomology class of m_f^r in $H^r(\bar{K}^*)$ is independent of f.*

PROOF. Let f and g be two proper maps of K in E^r. The invariance of the intersection numbers under small deformation implies that any map, f', sufficiently close to f will have the property $m_{f'}^r = m_f^r$. Thus it is possible to find two maps, f' and g', such that:

(a) f' and g' are barycentric maps defined on the same subdivision, K' of K.

(b) $m_{f'}^r = m_f^r$ and $m_{g'}^r = m_g^r$.

(c) The images of the vertices of K' under both f' and g' are in general position together.

Then f' and g' together will define a barycentric mapping of a standard subdivision of $K' \times I$, f' mapping the vertices of $K' \times 0$, and g' mapping those $K' \times 1$. This map, h', of $K' \times I$ will be proper since the vertices go into general position. Applying Lemma 3.4 to the inclusions of $K \times 0$ and $K \times 1$ in $K \times I$, we have

$$m_{f'}^r = j_0^\# m_{h'}^r, \quad \text{and} \quad m_{g'}^r = j_1^\# m_{h'}^r$$

where j_0 and j_1 are the corresponding inclusions of the $\bar{K}^*$'s. But j_0 and j_1 are obviously homotopic, hence $m_{f'}^r$ is cohomologous to $m_{g'}^r$.

DEFINITION 3.6. The cohomology class of m_f^r (which by Lemma 5 is independent of f) will be called the first obstruction to the imbedding of K in E^r, and it will be denoted by m^r. If r is even, $m^r \in H^r(\bar{K}^*, I)$ and if r is odd, $m^r \in H^r(\bar{K}^*, I_T)$ where I denotes ordinary integer coefficients and I_T denotes twisted integer coefficients.

4. Characterizing the first obstruction

In this section we use a fixed ordering of the vertices of K. The cup product in $C(K)$ defined using this ordering induces a cup product on $C(K) \otimes C(K)$ by means of the formula

$$(u_1 \otimes v_1) \smile (u_2 \otimes v_2) = (-1)^{q_1 p_2}(u_1 \smile u_2) \otimes (v_1 \smile v_2)$$

with

$$u_i \in C^{p_i}(K), \qquad v_i \in C^{q_i}(K).$$

This cup product induces the following pairings for the cohomology of $\bar{K}^*$

$$H^p(\bar{K}^*, I) \quad \text{and} \quad H^q(\bar{K}^*, I) \quad \text{to} \quad H^{p+q}(\bar{K}^*, I)$$

$$H^p(\bar{K}^*, I) \quad \text{and} \quad H^q(\bar{K}^*, I_T) \quad \text{to} \quad H^{p+q}(\bar{K}^*, I_T)$$

$$H^p(\bar{K}^*, I_T) \quad \text{and} \quad H^q(\bar{K}^*, I_T) \quad \text{to} \quad H^{p+q}(\bar{K}^*, I).$$

We shall compute $m^r(K)$ by using particular maps of K in E^r. For this purpose we need the following.

DEFINITION 4.1. A G-curve in E^r is a curve any $r + 1$ of whose points span an r-simplex.

LEMMA 4.2. *If σ^p and τ^q are euclidean simplices with distinct vertices on a G-curve in E^r and $p + q = r$ then σ^p meets τ^q if and only if all the vertices of σ^p alternate with all the vertices of τ^q along the curve.*

The proof is left to the reader.

There is a G-curve in each E^n; for example, the curve parameterized by $(t, t^2, t^3 \cdots, t^n)$.

In the rest of this section, f will be a barycentric map of K in E^r which maps the vertices of K in order along a G-curve.

REMARK. If K is connected and not an interval or a single point, then K^* is connected and the map $K^* \to \bar{K}^*$ is a two sheeted covering.

THEOREM 4.3. *If K is connected and is neither an interval nor a point nor empty, then $m^1(K)$ is the non-trivial cohomology class with twisted integer coefficients that annihilates the image of $H_1(\bar{K}^*)$ in $H_1(\bar{K}^*)$.*

PROOF.

$$m_f^1((a_0, a_1) \times b_0) = \begin{cases} 1 \text{ if } a_0 < b_0 < a_1 \\ 0 \text{ otherwise} \end{cases}$$

$$m_f^1(a_0 \times (b_0, b_1)) = \begin{cases} -1 \text{ if } b_0 < a_0 < a_1 \\ 0 \text{ otherwise.} \end{cases}$$

Consider the mapping $f \times f$ of K^* into the (x, y) plane. An oriented 1-cell, γ, of K^* will be mapped into an oriented vertical or horizontal segment. $m_f^1(\gamma) = 0$ if $f \times f(\gamma)$ does not meet the line $x = y$; $m_f^1(\gamma) = 1$ if $f \times f(\gamma)$ crosses the line $x = y$ from the region $y > x$; and $m_f^1(\gamma) = -1$ if $f \times f(\gamma)$ crosses the line $x = y$ from the region $x > y$. Hence m_f^1 is zero on all 1-cycles of K^*. However on the cycle of $\bar{K}^*$ represented by a path from $a \times b$ to $b \times a$, m_f^1 is clearly plus one or minus one. Hence $m^1(K)$ is non-trivial and our theorem is proved.

THEOREM 4.4. $m^r(K) = (m^1(K))^r$ *(the exponent meaning product in the sense of the appropriate pairings described at the beginning of the section).*

PROOF. First note that both $m^r(K)$ and $(m^1(K))^r$ lie in $H^r(\bar{K}^*, I)$ for even r and both lie in $H^r(\bar{K}^*, I_T)$ for odd r.

Let m^{10} be the component of m_f^1 in $C^1(K) \otimes C^0(K)$ and m^{01} the component in $C^0(K) \otimes C^1(K)$. Then $m_f^1 = m^{10} + m^{01}$ and $m^{01} = -T^{\#}m^{10}$. It is easy to see by direct computation that $m^{10} \smile m^{10} = m^{01} \smile m^{01} = 0$. Namely

$$m^{10} \smile m^{10}((a_0, a_1, a_2) \times b_0) = m^{10}((a_0, a_1) \times b_0) \cdot m^{10}((a_1, a_2) \times b_0)$$

but the right side is zero unless b_0 lies between a_0 and a_1 and also between a_1 and a_2 which is impossible since the a's are in order.

It follows then that $(m_f^1)^r = m^{10} \smile m^{01} \smile m^{10} \smile \cdots + m^{01} \smile m^{10} \smile \cdots$ each term having r factors. When we evaluate $(m_f^1)^r$ on $(a_0, \cdots, a_p) \times (b_0, \cdots, b_q)$, we get zero unless the a's and b's interlace, furthermore the first term is zero unless the sequence of a's and b's starts with an a while the second term is non-zero only if the sequence starts with b. Thus we see that up to a possible difference in sign $(m_f^1)^r$ agrees with m_f^r. However since the second term is $(-1)^r T^{\#}$ of the first term, the possible difference in sign can depend only on r and not on the simplex in question. But $m^r(K)$ is a class of order 2 so the sign of the representative cochain makes no difference. Hence the theorem is proved.

5. Deformation cells

In this section and the following one we set up the geometrical apparatus for altering a mapping in order to remove some self-intersections. We lean heavily on the work of Whitney ([2] and [3]).

NOTATION. If A is a differentiable manifold and B a subset of A, we will denote by $T(A \mid B)$ the vector bundle over B of tangent vectors to A at points of B. In particular, for $x \in A$, $T(A \mid x)$ will denote the tangent space to A at x, while the tangent bundle to A, $T(A \mid A)$, will be abbreviated to $T(A)$. If B is a submanifold of A then $T(B)$ is a sub-bundle of $T(A \mid B)$ so that the quotient bundle $T(A \mid B)/T(B)$ is defined. It is the bundle whose fibre at $x \in B$ is the vector space $T(A \mid x)/T(B \mid x)$, and it will be denoted by $N(A, B)$. If A is a Riemannian manifold, $N(A, B)$ is naturally isomorphic to the sub-bundle of $T(A \mid B)$ which we will call $N(A \mid B)$ and which consists of those tangent vectors to A at points of B which are orthogonal to B.

When A is a Riemannian manifold and B is a submanifold of A with the property that its closure, $\bar{B}$, is a compact subset of a differentiable submanifold of A of the same dimension as B, then we can define, in the standard way, for sufficiently small ε, the tubular neighborhood, $W(B, \varepsilon)$, of B in A as follows.

For a tangent vector X to A, let $\exp(X)$ be the point of A (if it exists) at a distance equal to the length of X along a geodesic in the direction of X. Since $\bar{B}$ is compact, $\exp(X)$ is defined for each $X \in N(A \mid B)$ of length less than some positive number. It is well known that this mapping has non-zero Jacobian in some neighborhood of the zero cross-section of $N(A \mid B)$. The implicit function theorem, together with the compactness of $\bar{B}$, then asserts that for a suitably small ε, the exponential map when restricted to vectors of $N(A \mid B)$ of length less than ε is a homeomorphism. We will denote its image by $W(B, \varepsilon)$ (or $W(B)$ when the ε is fixed during the argument) and we denote by h_ε (or h) the inverse of the exponential map so that $h: W(B) \to N(A \mid B)$ is a differentiable homeomorphism. If π is the natural projection of $N(A \mid B)$ on B, then it is easy to see that $\pi h: W(B) \to B$ is a fibre map and that each fibre is a cell.

We note that if B is a C^∞-submanifold of A then h is a regular C^∞ homeomorphism.

Throughout the remainder of this section and the next, R will denote an n-dimensional, C^∞, Riemannian manifold; A and B will be C^∞-submanifolds of R of dimensions p and q respectively; and r will denote the integer $p + q - n$.

In the applications later, R will be an open subset of euclidean n-space, A and B will be images under a map, f, of a p-simplex and a q-simplex of K respectively. We define two $r+1$-cells, P and Q, in the unit ball, D, of euclidean $r+2$-space using the euclidean distance, d, in D as follows.

$$P = \{x \in D \mid d(x, (\tfrac{1}{2}, 0, 0 \cdots 0)) = 1\}$$

$$Q = \{x \in D \mid d(x, (-\tfrac{1}{2}, 0, 0, \cdots, 0)) = 1\}.$$

$P \cap Q$ is then an r-sphere in the interior of D.

DEFINITION 5.1. *A deformation cell* Γ, for the pair (A, B) of submanifolds of R is the image of a map $\gamma: D \to R$ which satisfies

(1) γ is a regular C^∞ homeomorphism of D into R (hence $\Gamma = \gamma(D)$ is a C^∞ submanifold)

(2) $\gamma(P) = A \cap \Gamma, \quad \gamma(Q) = B \cap \Gamma, \quad \gamma(P \cap Q) = A \cap B$

(3) $T(A \mid A \cap \Gamma) \cap T(\Gamma \mid A \cap \Gamma) = T(A \cap \Gamma)$
$T(B \mid B \cap \Gamma) \cap T(\Gamma \mid B \cap \Gamma) = T(B \cap \Gamma)$
$T(A \mid A \cap B) \cap T(B \mid A \cap B) = T(A \cap B)$.

REMARK. Condition (2) requires that $A \cap B$ shall be an r-sphere, while the last statement of condition (3) requires that A intersect B transversally. In particular for a point x of $A \cap B$, $T(A \mid x) + T(B \mid x)$ must equal $T(R \mid x)$ if a deformation cell is to exist.

LEMMA 5.2. *If $A \cap B$ is an r-sphere with $p, q \geqq 2r + 3$ such that $T(A \mid A \cap B) \cap T(B \mid A \cap B) = T(A \cap B)$ and if there is a continuous map, α, of D in R which satisfies*

(1) α maps $P \cap Q$ homeomorphically on $A \cap B$

(2) $\alpha(P) \subset A, \quad \alpha(Q) \subset B$.

Then there exists a deformation cell for (A, B).

PROOF. Let γ_1 be a regular C^∞ homeomorphism of $P \cap Q$ on $A \cap B$. The existence of the map α with the properties listed together with the inequalities relating p, q, and r enable us to use Whitney's Theorem 5 of [3] to extend γ_1 to a map, γ_2, of $P \cup Q$ into $A \cup B$ whose restrictions to P and Q are, $1-1$, regular C^∞ homeomorphisms into A and B respectively. By using the same theorem again, and the existence of the map, α, we may find an extension, γ_3, of γ_2 which is a C^∞ map of D into R. Using the hypothesis that $T(A \mid A \cap B) \cap T(B \mid A \cap B) = T(A \cap B)$ and the regularity of γ_2 on P and Q it is easy to check that γ_3 is regular at points of $P \cap Q$. Hence there is a neighborhood U of $A \cap B$ such that γ_3 is $1-1$ and regular on $\gamma_3^{-1}(U)$. Let U_1 be a neighborhood of $A \cap B$ whose closure lies in U. Then $d\gamma_3$ induces an isomorphism of $N(D \mid P \cap \gamma_3^{-1}(U_1))$ into $N(R, A \cap U_1)$. We want to extend this map to an isomorphism θ of $N(D \mid P)$ into $N(R, A)$ such that the following diagram is commutative

$$\begin{array}{ccc} N(D \mid P) & \xrightarrow{\ \theta\ } & N(R, A) \\ \downarrow & & \downarrow \\ P & \xrightarrow{\ g_3 \mid P\ } & A \end{array}$$

where the vertical arrows are the projections of the respective bundles. The obstruction to the existence of such a map θ lies in $\pi_r(S^{n-p-1})$ which is zero under our assumptions on dimensions. Hence such a θ exists, and by paragraph 6.7 of [4] it may be chosen to be a C^∞ map. By using the exponential maps in D and in R we may construct a $1 - 1$, regular C^∞ homeomorphism γ_4 of a neighborhood of P into R such that $\gamma_4 \mid A = \gamma_3 \mid A$ and $d\gamma_4 = d\gamma_3$ at points of $P \cap U_1$. Similarly we may construct γ_5 to be a $1 - 1$ regular C^∞ homeomorphism of a neighborhood of Q into R with $\gamma_5 \mid B = \gamma_3 \mid B$ and $d\gamma_5 = d\gamma_3$ at points of $Q \cap U_1$. Choose a covering of $P \cup Q$ by three open sets $\gamma_3^{-1}(U)$, Y, and Z such that $P \subset Y \cup \gamma_3^{-1}(U)$, $Q \subset Z \cup \gamma_3^{-1}(U)$, such that Y, Z and $\gamma_3^{-1}(U_1)$ are disjoint and such that γ_4 is defined in Y, γ_5 is defined in Z. Let λ be a C^∞ real valued function on D which is 1 in $\gamma_3^{-1}(U) - (Y \cup Z)$ and 0 outside $\gamma_3^{-1}(U)$. For $x, y \in R$ $0 \leqq a \leqq 1$, let $\varphi(x, y, a)$ be the point (if it exists) of R on the shortest geodesic from x to y such that $d(x, \varphi(x, y, a))/d(x, y) = a$, where d is the Riemannian distance. Then we may define γ_6 from a neighborhood of $P \cup Q$ into R by

$$\gamma_6(x) = \begin{cases} \varphi(\gamma_4(x), \gamma_3(x), \lambda(x)) & x \notin Z \\ \varphi(\gamma_5(x), \gamma_3(x), \lambda(x)) & x \notin Y. \end{cases}$$

It is straightforward to check that γ_6 is regular and $1 - 1$ in a closed neighborhood of $P \cup Q$, that γ_6 is an extension of γ_3, and that $d\gamma_6$ induces isomorphisms on $N(D \mid P)$ and $N(D \mid Q)$ into $N(R, A)$ and $N(R, B)$ respectively. Finally we may use Theorem 5 of [3] again to extend γ_6 to a $1 - 1$ regular homeomorphism, γ_7, of D in R. One may then check that $\Gamma = \gamma_7(D)$ is the required deformation cell for (A, B).

6. The bundle of a deformation cell

In this paragraph we will use the same notations as in paragraph 5. In particular A and B will be submanifolds of the n-dimensional Riemannian manifold R, with $\dim(A) = p$, $\dim(B) = q$ and $r = p + q - n$.

The following trivial lemma will clarify the next definition.

LEMMA 6.1. *If Γ is a deformation cell for (A, B) then for each $x \in A \cap B$*

$$T(A \mid x)/T(A \mid x) \cap T(\Gamma \mid x) \approx T(R \mid x)/T(B \mid x) + T(\Gamma \mid x).$$

PROOF.

$$\begin{aligned} T(A \mid x)/T(A \mid x) \cap T(\Gamma \mid x) &= T(A \mid x)/T(A \mid x) \cap (T(\Gamma \mid x) + T(B \mid x)) \\ &\approx T(A \mid x) + T(\Gamma \mid x) + T(B \mid x)/T(\Gamma \mid x) + T(B \mid x) \\ &\approx T(R \mid x)/T(\Gamma \mid x) + T(B \mid x). \end{aligned}$$

DEFINITION 6.2. When Γ is a deformation cell for (A, B) the vector bundle $V(A, B; \Gamma)$ is defined to be the bundle over the $r + 1$-sphere, $\Gamma \cap (A \cup B)$ obtained by taking the union of the bundles $N(A, \Gamma \cap A)$ and $T(R \mid \Gamma \cap B)/T(\Gamma \mid \Gamma \cap B) + T(B \mid \Gamma \cap B)$ and identifying corresponding fibres over points of $A \cap B$ by the second isomorphism theorem as in Lemma 6.1.

Definition 6.3. A normalizing map for a vector bundle is a map of the bundle into a vector space, L, which induces a linear isomorphism of each fibre onto L.

It is well known and easy to see that there exists a normalizing map for a vector bundle if and only if it is a product bundle, and that every bundle over a cell is a product bundle.

Definition 6.4. If Γ is a deformation cell for (A, B) and $g: N(R, \Gamma) \to L$ is a normalizing map for the normal bundle of Γ, then g will be called separating if L can be written as a direct sum $L = L_1 + L_2$ in such a way that g induces normalizing maps of $N(A, \Gamma \cap A)$ into L_1 and of $N(B, \Gamma \cap B)$ into L_2.

Lemma 6.5. *When $p, q \geqq 2r + 3$, $V(A, B; \Gamma)$ is a product bundle if and only if there exists a separating normalizing map for $N(R, \Gamma)$.*

Proof. If $g: N(R, \Gamma) \to L_1 + L_2$ is a separating normalizing map, then g induces a map, $g_x: T(R \mid x) \to L_1 + L_2$, for each $x \in \Gamma$, which has for its kernel $T(\Gamma \mid x)$. For $x \in A$, $g_x(T(A \mid x)) = L_1$. Let j be the projection of $L_1 + L_2$ on L_1. Then for $x \in B$, jg_x has $T(B \mid x)$ in its kernel since g_x maps $T(B \mid x)$ on L_2. Hence jg induces a normalizing map for each of the bundles $N(A, A \cap \Gamma)$ and $T(R \mid B \cap \Gamma)/T(B \mid B \cap \Gamma) + T(\Gamma \mid B \cap \Gamma)$ and thus also for the bundle $V(A, B; \Gamma)$. Since $V(A, B; \Gamma)$ has a normalizing map it is a product bundle.

Suppose now that $V = V(A, B; \Gamma)$ is a product bundle and that $g: V \to L_1$ is a normalizing map for it. We may consider g as a map from $T(A \mid \Gamma \cap A) \cup T(R) \mid \Gamma \cap B)$ into L_1. We will produce a separating normalizing map $\bar{g}$ for $N(R, \Gamma)$ in four steps.

(1) Extend g to a map g_1 of $T(R \mid \Gamma \cap (A \cup B))$ into L_1.

(2) Extend g_1 to a map g_2 of $T(R \mid \Gamma)$ into L_1 of maximal rank with $T(\Gamma)$ in the kernel.

(3) Find a normalizing map g_3 of the kernel of g_2 modulo $T(\Gamma)$ into L_2.

(4) The required separating normalizing map for $N(R, \Gamma) = T(R \mid \Gamma)/T(\Gamma)$ is $g_2 + g_3$.

(1) Extending g to g_1 is equivalent to finding complementary spaces for $T(A \mid x)$ in $T(R \mid x)$ when $x \in \Gamma \cap A$ which agree with the given ones, i.e. the kernels of g_x, for $x \in A \cap B$. But since the space of complementary spaces is contractible when $n > p + 1$, this can always be done.

(2) $T(R \mid \Gamma)/T(\Gamma)$ has fibres of dimension $n - r - 2$ while L_1 has dimension $p - r - 1$. The space of homomorphisms of maximal rank of an $n - r - 2$-dimensional space into one of dimension $p - r - 1$ has the same homotopy type as the Stiefel manifold, $V_{p-r-1,n-r-2}$, of $p - r - 1$-frames in $n - r - 2$-space. Since $(A \cap \Gamma) \cup (B \cap \Gamma)$ has the homotopy type of an $r + 1$-sphere, the obstruction to extending g_1 to a map g_2 lies in $\pi_{r+1}(V_{p-r-1,n-r-2})$. By hypothesis $2r + 2 < q$, which we can rewrite as

$$r + 1 < q - r - 1$$

or

$$r + 1 < n - p - 1$$

or

$$r + 1 < (n - r - 2) - (p - r - 1)$$

from which it follows ([4] p. 132) that $\pi_{r+1}(V_{p-r-1,n-r-2}) = 0$. Hence there exists the extension g_2 that we are looking for.

(3) The kernel of g_2 modulo $T(\Gamma)$ is a vector bundle over the cell, Γ, and hence it is a product bundle. Let g_3 be a normalizing map for it into L_2.

(4) The check that $g_2 + g_3:N(R, \Gamma) \to L_1 + L_2$ is a separating normalizing map for $N(R, \Gamma)$ is straightforward and completes the proof.

Recall from paragraph 5 that for a submanifold A of R, h is a C^∞ homeomorphism of a neighborhood of A with a neighborhood of the zero cross section in $N(R \mid A)$. Let π be the projection of $N(R \mid A)$ onto A.

DEFINITION 6.6. If $g:N(R \mid A) \to L$ is a normalizing map, F is an open subset of A, and U a neighborhood of the origin in L, then a map $\theta:F \times U \to R$ will be called a coordinatizing map for F with respect to g if

(1) θ is a regular C^∞ homeomorphism

(2) $\pi h\theta(a \times v) = a \qquad a \in F,\ v \in U$

(3) $gh\theta(a \times v) = v \qquad a \in F,\ v \in U$.

LEMMA 6.7. *If F is an open subset of the submanifold A of R whose closure, $\bar{F}$, is a compact subset of A and if g is a C^∞ normalizing map for $N(R \mid A)$ in L, then there exists a coordinatizing map for F with respect to g.*

PROOF. If we choose ε small enough so that $W(G, \varepsilon)$ is defined for some open neighborhood G of $\bar{F}$ and consider in $N(R \mid G)$ the set of vectors of length less than ε, then g maps this set onto a neighborhood of the origin in L. Since $\bar{F}$ is compact there is a neighborhood U of the origin in L such that for each $a \in F$, g maps the ε-ball of the fibre at a onto a set containing U. Let $\theta(a \times v)$ for $a \in F$, $v \in U$ be the unique point in $\exp(\pi^{-1}a) \cap \exp(g^{-1}v)$. Then properties (2) and (3) of the definitions of a coordinatizing map are automatically fulfilled. The fact that θ is a C^∞ mapping follows from the existence theorem for differential equations used in the definition of exp. That θ is $1 - 1$ is clear from the properties (2) and (3).

CONSTRUCTION 6.8. Given a coordinatizing map $\theta:F \times U \to L$ of F with respect to a normalizing map $g:N(R \mid A) \to L$, and a C^∞-isotopy, φ, of A which is the identity outside F we construct a C^∞-isotopy $\Phi = \Phi(\varphi, g)$ of R on itself with the following properties:

(1) Φ is an extension of φ, i.e. $\Phi(x, t) = \varphi(x, t)$ for $x \in A$

(2) Φ is the identity outside a neighborhood of F in R

(3) $gh\Phi(x, t) = gh(x)$ for $x \in W(A)$ and all t.

To define $\Phi(\varphi, g)$, let λ be a C^∞ real valued function on L such that

$$\begin{aligned} 0 \leqq \lambda(v) &\leqq 1 && v \in L \\ \lambda(v) &= 0 && v \notin U \\ \lambda(v) &= 1 && v \text{ in some neighborhood of } 0. \end{aligned}$$

Let

$$\Phi(x, t) = \begin{cases} x & x \notin \varphi(F \times U) \\ \theta(\varphi(a, \lambda(v)t), v) & x = \theta(a, v). \end{cases}$$

The check that Φ has properties (1) and (2) is routine; property (3) follows from the condition (3) in the definition of the coordinatizing map θ.

LEMMA 6.9. *If Γ is a deformation cell for (A, B) and g is a separating normalizing map for $N(R, \Gamma)$ into $L = L_1 + L_2$; then there is an ε small enough so that*

$$gh_\varepsilon(A \cap W(\Gamma, \varepsilon)) \cap gh_\varepsilon(B \cap W(\Gamma, \varepsilon))$$

contains only the zero element of L.

PROOF. Since g is separating, $g(T(A \mid \Gamma)) \subset L_1$, and $g(T(B \mid \Gamma)) \subset L_2$. Suppose that there were points $a_i \in A$, $b_i \in B$ arbitrarily close to Γ such that $gh_\varepsilon(a_i) = gh_\varepsilon(b_i)$. If a_i is not in Γ, gh_ε maps the geodesic from a_i to Γ into a straight line through the origin in a direction near L_1 while gh_ε maps the geodesic from b_i to Γ into a straight line whose direction is near L_2. If infinitely many $a_i \notin \Gamma$, these directions would have a limit direction lying both in L_1 and L_2 which is impossible. Hence there is an ε sufficiently small so that $a \in A \cap W(\Gamma, \varepsilon)$, $b \in B \cap W(\Gamma, \varepsilon)$ and $gh_\varepsilon(a) = gh_\varepsilon(b)$ imply $a \in \Gamma$, $b \in \Gamma$ and $gh_\varepsilon(a) = 0$.

Let $\psi: D \times I \to D$ be a C^∞ isotopy such that ψ_t is the identity on a neighborhood of the boundary of D and such that $\psi(P \times 1) \cap Q = \emptyset$.

LEMMA 6.10. *If Γ is a deformation cell for (A, B) which has a separating normalizing map g for $N(R, \Gamma)$ then $\Phi = \Phi(\gamma\psi\gamma^{-1}, g)$ is an isotopy of R which deforms A away from B if Φ is defined using a coordinatizing map θ for a sufficiently small neighborhood of Γ.*

PROOF. Φ only moves points of $W(\Gamma, \varepsilon)$ so that we need only show that $\Phi(A \times 1) \cap B \cap W(\Gamma, \varepsilon) = \emptyset$. Using property (3) of Φ in Construction 6.8 we have

$$gh\Phi(A \times 1) = gh(A) \text{ (for points in } W(\Gamma, \varepsilon)).$$

But if ε were chosen small enough, according to Lemma 6.9, $gh(A) \cap gh(B) = 0$, and hence a point of $\Phi(A \times 1) \cap B$ must lie in $\Phi(A \cap \Gamma \times 1) \cap B$. But since Φ is an extension of $\gamma\psi\gamma^{-1}$, $\Phi(A \cap \Gamma \times 1) \cap B = \gamma\psi\gamma^{-1}(A \cap \Gamma \times 1) \cap B = \gamma\psi(P \times 1) \cap B$. Since $\psi(P \times 1) \cap Q = \emptyset$, it follows that $\gamma\psi(P \times 1) \cap B = \emptyset$ and the lemma is proved.

7. The imbedding theorem

With the geometric machinery of paragraphs 5 and 6 at our disposal we are prepared to prove the imbedding theorem. For this purpose we consider the class, G, of "general" maps of K into E^n defined as follows.

DEFINITION 7.1. A general map $f: K \to E^n$ is one which satisfies

(1) for each $\sigma \in K$, $f \mid \sigma$ is a regular C^∞ homeomorphism.

(2) for each pair of open simplices σ^p and τ^q of K with $p + q \leqq n$, $f(\sigma^p)$ meets $f(\tau^q)$ transversally. For $p + q < n$ this means that $f(\sigma^p) \cap f(\tau^q)$ is empty, while for $p + q = n$, $f(\sigma^p) \cap f(\tau^q)$ consists of a finite number of points at each of which the tangent spaces to $f(\sigma^p)$ and $f(\tau^q)$ span the tangent space to E^n.

(3) If σ^p and τ^q are open simplices with $p + q = n$ then no point of $f(\sigma^p) \cap f(\tau^q)$ lies in the image of any other open simplex. If the dimension of K is less than n, then all barycentric maps in general position are in the class, G, of general maps.

THEOREM 7.1. *If* $\dim(K) \leqq n - 3$ *then a necessary and sufficient condition that there exist a general map f of K in* E^n *such that for all pairs of distinct open simplices,* σ^p, τ^q *of K, with* $p + q = n$, $f(\sigma^p) \cap f(\tau^q) = \emptyset$ *is that* $m^n(K) = 0$. *As a special case we have the imbedding theorem.*

THEOREM 7.2. *For* $n \geqq 3$ *a necessary and sufficient condition that there exist an imbedding of the n-dimensional complex K in* E^{2n} *is that* $m^{2n}(K) = 0$.

First we give an outline of the proof of Theorem 7.1, and then we prove the lemmas that fill in the details. The necessity that $m^n(K)$ be zero has already been shown.

(1) We start with a barycentric map f_1 of K in E^n in general position.

(2) We show that for any cochain, z, which represents $m^n(K)$ there is a general map f_2 such that $m^n{}_{f_2} = z$ (Lemma 7.3). If $m^n(K) = 0$, we choose f_2 so that $m^n{}_{f_2} = 0$.

(3) It may happen after the use of Lemma 7.3 on our original barycentric map f_1 to produce f_2, for a pair of open simplices σ^p and τ^q whose closures have a single vertex in common, that the algebraic intersection number of $f_2(\sigma^p)$ and $f_2(\tau^q)$, which we denote again by $f_2(\sigma^p) \wedge f_2(\tau^q)$, is not zero. If this happens, we introduce new intersections between $f_2(\sigma^p)$ and $f_2(\tau^q)$ so that we obtain a general map f_3 with the property that for any pair of distinct open simplices σ^p and τ^q with $p + q = n$ we have $f_3(\sigma^p) \wedge f_3(\tau^q) = 0$ (Lemma 7.4).

(4) Since the intersection number $f_3(\sigma^p) \wedge f_3(\tau^q) = 0$, and since f_3 is general $f_3(\sigma^p) \cap f_3(\tau^q)$ consists of a finite number of pairs of points (a_i, b_i) at which the intersection numbers are $+1$ and -1 respectively. We use Lemma 5.2 to provide us with deformation cells Γ_i which meet $f_3(K)$ only in $f_3(\sigma) \cup f_3(\tau)$; here we use the hypothesis, $\dim(K) \leqq n - 3$. We prove (Lemma 7.5), that there exist separating normalizing maps for these deformation cells and hence we may use Lemma 6.10 to remove these intersections.

LEMMA 7.3. *If* $\dim(K) < n - 1$ *then for any cochain, z, representing* $m^n(K)$ *there is a general map* $g: K \to E^n$ *such that* $m_g^n(K) = z$.

PROOF. Let $b(\sigma, \tau)$ be the equivariant cochain which is 1 on $\sigma \times \tau$, ± 1 on $\tau \times \sigma$ and zero elsewhere. It is clearly sufficient to prove that for any general map f there is a general map g such that

$$b(\sigma, \tau) = m_f^n - m_g^n .$$

To this end pick points a and b in $f(\sigma)$ and $f(\tau)$ respectively that are not in the image of any other simplex. Since the dimension of K is less than $n - 1$ there is an arc joining a to b which does not meet $f(K)$ elsewhere. We may use the trivial case of Lemma 5.2 with $r = -1$, $A = f(\sigma)$, $B = f(\tau)$ to find a deformation cell Γ which is then an arc meeting $f(K)$ only at a and b. Furthermore there is no difficulty in choosing Γ so that the tangent to Γ at a is not tangent to the image of any closed simplex having σ as a face and similarly at b for τ. The bundle $V(A, B; \Gamma)$ being a vector bundle over two points may be considered as a product bundle in two essentially different ways corresponding to a change of orientation of one of the fibres. Let h_1 and h_2 be two corresponding separating normalizing maps for $N(R, \Gamma)$. For sufficiently small ε, $W(\Gamma, \varepsilon)$ will meet the image of only those simplices which have σ or τ as a face. Consider the isotopy $\Phi = \Phi(\gamma\psi\gamma^{-1}, h_1)$ of Lemma 6.10 which is the identity outside $W(\Gamma, \varepsilon)$. Let Φ_t be the homeomorphism of E^n on itself corresponding to the parameter value t of the isotopy Φ. Define f_t by

$$f_t(x) = \begin{cases} \Phi_t f(x) & x \in f^{-1}(W(\Gamma, \varepsilon)) \cap \text{star } \sigma \\ f(x) & \text{otherwise.} \end{cases}$$

Since Φ_t is the identity on a neighborhood of the boundary of $W(\Gamma, \varepsilon)$, f_t is a C^∞ map.

Let $F: K \times I \to E$ be defined by $F(x, t) = f_t(x)$ and finally let $g = f_1$.

Then it is easy to show that

$$\delta b(\sigma, \tau) = ((-1)^{n+1} F(\sigma \times I) \wedge f(\tau))(m_g^n - m_f^n)$$

using the definition of m_f^n, the elementary properties of intersection numbers, and the fact that f agrees with g outside star $\sigma \times$ star $\tau \cup$ star $\tau \times$ star σ.

By construction the tangent spaces to $F(\sigma \times I)$ and $f(\tau)$ are independent at their point of intersection. Hence $F(\sigma \times I) \wedge f(\tau) = \pm 1$. Since the tangent space to $F(\sigma \times I)$ changes orientation when we base our construction on h_2 instead of h_1 we can achieve either sign. Finally it is clear that if our ε was chosen small enough g will be a general map.

A general map will be called radial near vertices if the image of each straight line from the vertex is straight in some neighborhood of the image of the vertex. Barycentric maps are radial near vertices.

LEMMA 7.4. *If f is a general map of K in E^n which is radial near vertices and σ^p, $\tau^q (p + q = n)$ are two open simplices whose closures have one vertex, v, in common then there is a general map, g, of K in E^n, radial near vertices such that $f(\sigma) \wedge f(\tau) - g(\sigma) \wedge g(\tau) = \pm 1$.*

PROOF. Let S be the surface of a sphere about v, small enough so that f is radial within S. Let $A = f(\sigma) \cap S$, $B = f(\tau) \cap S$, and $R = S$. Use Lemma 5.2 to find a deformation cell Γ for A, B so that Γ is an arc joining $f(\sigma)$ to $f(\tau)$ in S and not meeting $f(K)$ elsewhere. In S apply the construction of 7.3 to modify $f \mid f^{-1}(S)$ using Γ. Let Φ be the isotopy of S used in the construction. Extend Φ to an isotopy Φ' of E^n on itself equal to the identity outside a ball containing S by letting Φ' be linear on radial segments inside S and $\Phi' = \Phi'(\Phi, g)$ outside S where g is constant on outward unit normal vectors and $\Phi'(\Phi, g)$ is defined in construction 6.8. Let Φ_1' be the final value of Φ' and let

$$g(x) = \begin{cases} \Phi_1' f(x) & x \in \text{star } (\sigma) \\ f(x) & \text{otherwise.} \end{cases}$$

A check similar to that of Lemma 7.3 shows that $f(\sigma) \wedge f(\tau) - g(\sigma) \wedge g(\tau) = +1$ or -1 according to which separating normalizing map was used for Γ. Although Φ_1' is not necessarily differentiable at $f(v)$, g is general since there is no differentiability condition on g at v.

LEMMA 7.5. *If* $\dim(A) + \dim(B) = n$ *and* Γ *is a deformation cell for* (A, B) *and if the intersection numbers have opposite signs at the two points of intersection of* A *and* B *then there exists a separating normalizing map for* Γ.

PROOF. We need only show that $V(A, B; \Gamma)$ is a product bundle; since it is a vector bundle over a circle, it is sufficient to show that it is orientable.

Let W_A denote an orientation for the tangent spaces to A determined by the choice of a frame. If the tangent spaces to A and B intersect in the zero element at a point of $A \cap B$ let $W_A \times W_B$ be the orientation of the space they span, determined by taking first the chosen frame for A and then the frame for B to constitute a frame for the sum. If the tangent space to B is a subspace of the tangent space to A at a point there is a unique orientation, W_A/W_B, of the quotient space determined by taking a frame for A which extends one for B and letting the projection of the vectors of the A-frame which are not part of the B-frame, in order, be the frame for A/B.

Choose orientations W_R, W_A, W_B, W_Γ and $W_{A\cap\Gamma}$. Let a be the point of $A \cap B$ such that $W_A \times W_B = W_R$ and b the point of $A \cap B$ such that $W_A \times W_B = -W_R$. Orient $B \cap \Gamma$ so that $W_{A\cap\Gamma} \times W_{B\cap\Gamma} = W_\Gamma$ at a. Since

$$W_A = W_{A\cap\Gamma} \times W_A/W_{A\cap\Gamma} \qquad \text{at } A \cap \Gamma$$

$$W_B = W_{B\cap\Gamma} \times W_B/W_{B\cap\Gamma} \qquad \text{at } B \cap \Gamma$$

we have

$$W_A \times W_B = W_{A\cap\Gamma} \times W_A/W_{A\cap\Gamma} \times W_{B\cap\Gamma} \times W_B/W_{B\cap\Gamma} \qquad \text{at } A \cap B$$

or

$$W_A \times W_B = \pm W_{A\cap\Gamma} \times W_{B\cap\Gamma} \times W_B/W_{B\cap\Gamma} \times W_A/W_{A\cap\Gamma} \qquad \text{at } A \cap B$$

where the sign does not depend on the point of $A \cap B$ in question.

At a, $W_A \times W_B = W_R$ and $W_{A\cap\Gamma} \times W_{B\cap\Gamma} = W_\Gamma$ while at b, $W_A \times W_B = -W_R$ and $W_{A\cap\Gamma} \times W_{B\cap\Gamma} = -W_\Gamma$ since the intersection of $A \cap \Gamma$ with $B \cap \Gamma$ in Γ is algebraically zero. Thus we have

$$W_R = \pm W_\Gamma \times W_B/W_{B\cap\Gamma} \times W_A/W_{A\cap\Gamma} \qquad \text{at } A \cap B$$

the sign being independent of the point of $A \cap B$. This shows that the orientation $W_A/W_{A\cap\Gamma}$ agrees with $W_R/W_B/W_{B\cap\Gamma} \times W_\Gamma$ at b if it agrees at a. Hence $V(A, B; \Gamma)$ is orientable and the lemma is proved.

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