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ON THE SIGNATURE OF KNOTS AND LINKS(1)

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Abstract. In 1965, K. Murasugi introduced an integral matrix M of a link and defined the signature of the link by the signature of $M+M'$. In this paper, we study some basic properties of the signature of links. We also describe the effect produced on the signature of a knot contained in a solid torus by a further knotting of the solid torus.

1. **Introduction.** A *link l of multiplicity μ* is the union of μ ordered, oriented and pairwise disjoint polygonal simple closed curves l_i in the oriented 3-sphere S^3. In particular, if $\mu=1$, it is called a *knot*. Two links l and l' of multiplicity μ are said to be of the *same link type* if there exists an orientation preserving homeomorphism h of S^3 onto itself such that $h|l_i$ is an orientation preserving homeomorphism of l_i onto l'_i, $i=1, 2, \ldots, \mu$.

Now let l be a link and L a projection of l, that is, the image of l under a regular projection. In 1965, K. Murasugi associated an integral matrix M to L, called the *L-principal minor of l*, and he showed that the signature of $M+M'$ is an invariant of the link type of l [8]. The signature of $M+M'$ will be called the *signature* of the link l and denoted by $\sigma(l)$.

H. F. Trotter [12] in 1962 and J. W. Milnor [6] in 1968 also defined the signature of a knot in different ways, and D. Erle [1] showed that Milnor's definition is equivalent to Trotter's. The author of the paper proved in [11] that for the case of a knot Murasugi's definition is equivalent to Trotter's.

In §2 we will state some known results which will be used in later sections.

Let l be a link of multiplicity μ and $\Delta_l(t)$ the reduced Alexander polynomial of l. In §3 we first prove that if $\Delta_l(-1)\neq 0$, then $\sigma(l)$ is even or odd according as μ is odd or even (Corollary 2). Furthermore we will show that if $\Delta_l(t)\neq 0$, then the absolute value of $\sigma(l)$ is not greater than the degree of $\Delta_l(t)$ (Theorem 3). In [8] Murasugi showed that if k is a knot, then $|\Delta_k(-1)|\equiv 1$ or 3 (mod 4) according as $\sigma(k)\equiv 0$ or 2 (mod 4). We will generalize this result to the case of a link by using the Hosokawa polynomial (Theorem 4).

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Let k be a knot, V a tubular neighborhood of k and V^* a trivial solid torus in S^3. Let f: $V^* \to V$ be a faithful homeomorphism of V^* onto V. Let l^* be a knot in V^* and $l = f(l^*)$. Then l is homologous to some multiple of k, say $l \sim nk$, in V. In §4 we will prove that

$$\begin{aligned} \sigma(l) &= \sigma(l^*) && \text{if } n \text{ is even,} \\ &= \sigma(l^*)+\sigma(k) && \text{if } n \text{ is odd} \end{aligned}$$

(Theorem 9).

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2. **Preliminaries.** Let l be a link, L a projection of l and M the L-principal minor of l. We first state the following two results proved by Murasugi [8]:

(a) nullity $(M+M')$ is an invariant of the link type of l. nullity $(M+M')+1$ is called the *nullity* of l and denoted by $n(l)$.

(b) Let $\Delta_l(t)$ be the reduced Alexander polynomial of l. Then

$$\Delta_l(t) = \pm t^{\lambda} \det(M - tM') \quad \text{for some integer } \lambda. \tag{2.1}$$

Now let l be a link of multiplicity μ and F an orientable surface in S^3 whose boundary is l. Let T: $F \to S^3 - F$ be a translation in the positive normal direction of F and $\alpha_1, \alpha_2, \ldots, \alpha_{2h+\mu-1}$ a homology basis on F, where h denotes the genus of F. The matrix

$$\|\operatorname{Link}(T\alpha_i, \alpha_j)\|_{i,j=1,2,\ldots,2h+\mu-1}$$

is called a *Seifert matrix of l with respect to F* [5].

Let L be a projection of a link l. The orientable surface with boundary l which is constructed by using L as shown in §1 of [9] is called the *orientable surface associated with L.*

THEOREM A. *If l is a link with a connected and special projection L, then the L-principal minor of l is a Seifert matrix of l with respect to the orientable surface associated with L.*

For the definition of a special projection, see Definition 3.1 of [8]. The detailed proof of Theorem A may be found in §2 of [11]. We note that every link can be deformed isotopically to a link with a connected and special projection.

Finally let k be a knot and V a Seifert matrix of k. It was shown in [12] that the signature of $V+V'$ is an invariant of the knot type of k. Further, as a consequence of Theorem A, it was proved in [11] that

THEOREM B. *The signature $\sigma(k)$ of k is equal to the signature of $V+V'$.*

Throughout the paper $\boldsymbol{Z}$ and $\boldsymbol{Q}$ will denote the ring of integers and the field of rational numbers respectively.

3. Some properties of the signature.

THEOREM 1. *If l is a link of multiplicity μ, then $\sigma(l) \equiv \mu - n(l)$ (mod 2).*

Proof. First we deform l isotopically to a link l_0 which has a connected and special projection L_0. Let M be the L_0-principal minor of l_0. It follows from Theorem A that $M+M'$ is a $(2h+\mu-1)\times(2h+\mu-1)$ matrix, where h is the genus of the orientable surface associated with L_0. Since l and l_0 belong to the same link type, we have $\sigma(l) = \text{signature}\,(M+M')$ and $n(l) = \text{nullity}\,(M+M')+1$.

Now $M+M'$ is congruent over $\boldsymbol{Q}$ to a diagonal matrix with r positive, s negative and $n(l)-1$ zero entries on the diagonal. Since $2h+\mu-1 = r+s+n(l)-1$ and $\sigma(l) = r-s$, it follows that

$$\begin{aligned} \sigma(l) &= \mu - n(l) + 2h - 2s \\ &\equiv \mu - n(l) \pmod 2. \end{aligned}$$

COROLLARY 2. *If l is a link of multiplicity μ such that $\Delta_l(-1) \neq 0$, then $\sigma(l)$ is even or odd according as μ is odd or even.*

Proof. (2.1) implies that $\Delta_l(-1) \neq 0$ if and only if $n(l) = 1$. Hence Corollary 2 is an immediate consequence of Theorem 1.

THEOREM 3. *If l is a link with $\Delta_l(t) \neq 0$, then $|\sigma(l)| \leqq$ the degree of $\Delta_l(t)$.*

Proof. Let M be the L-principal minor of l and m the number of rows of M. Since $\Delta_l(t) \neq 0$, (2.1) implies $\det(M - tM') \neq 0$.

Now if M is a singular matrix, it is congruent over $\boldsymbol{Q}$ to the matrix

$$\begin{bmatrix} 0 & 0 \\ a & M_1 \end{bmatrix},$$

where M_1 is $(m-1)\times(m-1)$ and a is $(m-1)\times 1$. Moreover, $\det(M-tM') \neq 0$ yields that a has at least one nonzero entry. Therefore M is congruent over $\boldsymbol{Q}$ to the matrix

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & M_2 \end{bmatrix},$$

where M_2 is $(m-2)\times(m-2)$ and q is $(m-2)\times 1$. It is easy to show that $\tilde{M}+\tilde{M}'$ is congruent over $\boldsymbol{Q}$ to the direct sum of $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and M_2+M_2' and that $\tilde{M}-t\tilde{M}'$ is equivalent over the rational group ring of $H=(t\colon\ )$ (in the sense of Fox [2]) to the direct sum of $\left[\begin{smallmatrix} 0 & -t \\ 1 & 0 \end{smallmatrix}\right]$ and $M_2 - tM_2'$.

From this it follows that

$$\text{signature}\,(M+M') = \text{signature}\,(M_2+M_2')$$

and

$$\det(M-tM') = c_1 t \det(M_2 - tM_2')$$

for some nonzero rational number c_1.

Since $\det(M_2-tM_2')\neq 0$, if M_2 is singular, by applying the preceding argument to M_2 we can obtain an $(m-4)\times(m-4)$ matrix M_3 such that

$$\text{signature}\,(M_2+M_2') = \text{signature}\,(M_3+M_3')$$

and

$$\det(M_2-tM_2') = c_2t\det(M_3-tM_3')$$

for some nonzero rational number c_2.

By repeating this process several times, if necessary, we can show that there exists a nonsingular $(m-2n)\times(m-2n)$ matrix N for some n such that

$$\sigma(l) = \text{signature}\,(N+N')$$

and

$$\Delta_l(t) = ct^{\lambda}\det(N-tN')$$

for some integer λ and some nonzero rational number c.

It is clear that

$$|\sigma(l)| \leqq \text{the number of rows of } N. \tag{3.1}$$

Since the constant term and the leading term of $\det(N-tN')$ are $\det N$ and $\pm\det N$ respectively, these are nonzero rational numbers. Hence we have

$$\begin{aligned}\text{the degree of } \Delta_l(t) &= \text{the degree of } \det(N-tN') \\ &= \text{the number of rows of } N.\end{aligned} \tag{3.2}$$

Theorem 3 is a consequence of (3.1) and (3.2), which completes the proof.

Let l be a link of multiplicity μ and $l_1, l_2, \ldots, l_\mu$ the components of l. If $\mu\geqq 2$, we define the matrix $A=\|l_{ij}\|_{i,j=1,2,\ldots,\mu}$ by the formula

$$l_{ij} = \text{Link}\,(l_i, l_j) \quad \text{for } i\neq j, \qquad l_{ii} = -\sum_{j=1;\,j\neq i}^{\mu} l_{ij}.$$

Let A_i be the matrix obtained from A by deleting the ith row and the ith column. Clearly $\det A_i$ does not depend on the choice of i or the order of $l_1, l_2, \ldots, l_\mu$. Then we define

$$\begin{aligned} D(l) &= 1 &&\text{if } \mu = 1,\\ &= \det A_i &&\text{if } \mu \geqq 2.\end{aligned} \tag{3.3}$$

It is easy to see that $D(l)$ is an invariant of the link type of l.

The polynomial

$$\nabla_l(t) = \Delta_l(t)/(1-t)^{\mu-1} \tag{3.4}$$

is called the *Hosokawa polynomial* of l [4].

THEOREM 4. *Let l be a link of multiplicity μ. If $\nabla_l(-1)\neq 0$ and $\sigma(l)=\varepsilon m$, where $\varepsilon=\pm l$ and $m\geqq 0$, then*

$$|\nabla_l(-1)| \equiv \varepsilon^m(-1)^{(m-u+1)/2}D(l) \pmod 4.$$

Proof. We may assume that l has a connected and special projection L. Let M be the L-principal minor of l, F the orientable surface associated with L and h the genus of F. Then by Theorem A M is a Seifert matrix of L with respect to some homology basis on F.

Now it follows from the argument given in §3 of [4] that we can choose a homology basis on F with respect to which the Seifert matrix V of l is of the form

$$V = \begin{bmatrix} C & B' \\ B & A_1 \end{bmatrix},$$

where C is a $2h\times 2h$ matrix such that $C-C'$ is the direct sum of h copies of $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and A_1 is a $(\mu-1)\times(\mu-1)$ matrix appearing in (3.3). Therefore there exists a unimodular integral matrix P such that $PMP'=V$.

Let

$$f(t) = \det(V-tV')/(1-t)^{\mu-1} = \begin{vmatrix} C-tC' & B' \\ (1-t)B & A_1 \end{vmatrix}.$$

Note that since A_1 is symmetric, each of the last $\mu-1$ columns of $V-tV'$ has a factor $1-t$.

By (2.1) and (3.4) we have $f(t)=\pm t^{\lambda}\nabla_l(t)$ for some integer λ. Since $f(t)$ is a polynomial of degree at most $2h$ and satisfies the condition $f(t)=t^{2h}f(t^{-1})$, we may put

$$f(t) = c_0+c_1t+\cdots+c_{2h}t^{2h},$$

where $c_i=c_{2h-i}$ for $0\leqq i\leqq h-1$. Using the facts $\det(C-C')=1$ and $\det A_1=D(l)$, we obtain

$$f(1) = D(l) = 2\sum_{i=0}^{h-1} c_i+c_h.$$

From this and the fact $(-1)^i-(-1)^h\equiv 0 \pmod 2$ it follows that

$$\begin{aligned} f(-1) &= 2\sum_{i=0}^{h-1}(-1)^ic_i+(-1)^hc_h \\ &= (-1)^hD(l)+2\sum_{i=0}^{h-1}\{(-1)^i-(-1)^h\}c_i \qquad (3.5)\\ &\equiv (-1)^hD(l) \pmod 4. \end{aligned}$$

Since $\nabla_l(-1)\neq 0$ and $\sigma(l)=\varepsilon m$, for some unimodular rational matrix, Q, $QP(M+M')P'Q'=Q(V+V')Q'$ is a diagonal matrix with diagonal entries

$$(a_1,\ldots,a_n,-a_1',\ldots,-a_n',\varepsilon b_1,\ldots,\varepsilon b_m),$$

where a_i, a'_j and b_k are positive rational numbers. Therefore we have

$$\begin{aligned} f(-1) &= \det(M+M')/2^{\mu-1} \\ &= (-1)^n \varepsilon^m a_1 \cdots a_n\, a'_1 \cdots a'_n\, b_1 \cdots b_m/2^{\mu-1} \end{aligned}$$

and

$$\operatorname{sign} f(-1) = (-1)^n \varepsilon^m. \tag{3.6}$$

Since $2n+m=2h+\mu-1$, it follows from (3.5) and (3.6) that

$$\begin{aligned} |\nabla_l(-1)| &= |f(-1)| = f(-1)\cdot \operatorname{sign} f(-1) \\ &\equiv \varepsilon^m(-1)^{(m-\mu+1)/2} D(l) \pmod 4. \end{aligned}$$

Thus the proof is completed.

COROLLARY 5. *Let k be any knot. If $|\sigma(k)|=2m$, then $|\Delta_k(-1)|\equiv(-1)^m \pmod 4$.*

This corollary was first proved by Murasugi (Theorem 5.6 in [8]).

Now it follows from Corollary 5 and Theorem 1 of [6] that

$$\begin{gathered} |\Delta_k(-1)| \equiv (-1)^m \pmod 4 \text{ if and only if } |\sigma(k)| \equiv 2m \pmod 4, \\ \text{and if } |\Delta_k(-1)| = 1 \text{ then } \sigma(k) \equiv 0 \pmod 8. \end{gathered} \tag{3.7}$$

Moreover the condition (3.7) is the best possible in the following sense:

THEOREM 6. *Let m and n be nonnegative integers. Then there exists a knot k such that*

(1) $|\Delta_k(-1)|=4m+1$ *and* $|\sigma(k)|=8n$,

(2) $|\Delta_k(-1)|=8m+5$ *and* $|\sigma(k)|=8n+4$,

(3) $|\Delta_k(-1)|=4m+3$ *and* $|\sigma(k)|=4n+2$.

REMARK. The existence of a knot k such that

(4) $|\Delta_k(-1)|=8m+1$ $(m>0)$ and $|\sigma(k)|=8n+4$

can be proved for the case $m\equiv \pm 1 \pmod 3$. The case $m\equiv 0 \pmod 3$ still remains open, though the affirmative answer is expected.

Proof of Theorem 6. We will use the following facts (a)–(e):

(a) If $k=k_1 \# k_2$ is the composition of two knots k_1 and k_2, then $\sigma(k) =\sigma(k_1)+\sigma(k_2)$ and $|\Delta_k(-1)|=|\Delta_{k_1}(-1)|\cdot|\Delta_{k_2}(-1)|$.

(b) If k^{-1} is the mirror image of a knot k, then $\sigma(k^{-1})=-\sigma(k)$ and $|\Delta_{k^{-1}}(-1)| =|\Delta_k(-1)|$.

Let $K(p,q)$ denote the torus knot of type (p,q).

(c) $\sigma(K(5,3))=8$ and $|\Delta_{K(5,3)}(-1)|=1$.

(d) $\sigma(K(2p+1,2))=2p$ and $|\Delta_{K(2p+1,2)}(-1)|=2p+1$.

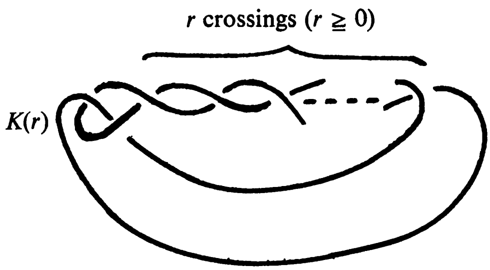

FIGURE 1

Let $K(r)$ be the knot shown in Figure 1.
(e) $\sigma(K(r))=0$ or 2 according as r is even or odd and $|\Delta_{K(r)}(-1)|=2r+1$.

Now let k_1, k_2 and k_3 be knots defined by

$$k_1 = K(2m) \# K(5,3) \# \cdots \# K(5,3) \quad (n \text{ times});$$

$$k_2 = K(8m+5,2) \# K(5,3)^{\varepsilon} \# \cdots \# K(5,3)^{\varepsilon}$$
$$(|n-m| \text{ times}), \quad \text{where } \varepsilon = \operatorname{sign}(n-m);$$

$$k_3 = K(2m+1) \# K(5,3) \# \cdots \# K(5,3) \quad (s \text{ times}) \quad \text{if } n = 2s,$$
$$= K(2m+1)^{-1} \# K(5,3) \# \cdots \# K(5,3) \quad (s+1 \text{ times}) \quad \text{if } n = 2s+1.$$

Then, by using (a)–(e), we can show easily that k_i satisfies the condition (i) of Theorem 6 for $i=1, 2, 3$.

4. **Knots in solid tori.** Let U and V be solid tori in S^3. A homeomorphism of U onto V is called *faithful* if it preserves the orientations induced by the orientation of S^3 in U and V and if it carries a longitude of U onto that of V [3].

LEMMA 7. *If $f\colon U \to V$ is a faithful homeomorphism, then*

$$\operatorname{Link}(\alpha, \beta) = \operatorname{Link}(f(\alpha), f(\beta))$$

for any pair of disjoint 1*-cycles α and β in* Int U.

Proof. Let q be a longitude of U. Then α is homologous to some multiple of q, say $\alpha \sim mq$, in U and there exists a 2-chain C in U such that $\alpha - mq = \partial C$. Using the fact that f is faithful, we obtain
(1) $f(\alpha)-mf(q)=\partial f(C)$,
(2) $f(q)$ is a longitude of V,
(3) $S(C,\beta)=S(f(C),f(\beta))$,
where $S(A^2, B^1)$ denotes the intersection number of a 2-chain A^2 and a 1-chain B^1 in S^3.

Since β lies in Int U and q bounds a 2-chain Q in $S^3-\operatorname{Int} U$, it follows that

Link $(q, \beta)=S(Q, \beta)=0$. Applying the same argument to $f(q)$ and $f(\beta)$, we can show that Link $(f(q), f(\beta))=0$. Therefore we obtain

$$\begin{aligned}\operatorname{Link}(\alpha, \beta) &= \operatorname{Link}(\alpha-mq, \beta) = S(C, \beta)\\ &= S(f(C), f(\beta))\\ &= \operatorname{Link}(f(\alpha)-mf(q), f(\beta))\\ &= \operatorname{Link}(f(\alpha), f(\beta)).\end{aligned}$$

LEMMA 8. *Let q be a longitude of a solid torus U and Q a 2-chain in $S^3-\operatorname{Int} U$ such that $\partial Q=q$. If α is a 1-cycle in* Int *U and β is a 1-cycle in S^3-U with $S(\beta, Q)=0$, then* Link $(\alpha, \beta)=0$.

Proof. Since $\alpha \sim mq$ in U for some integer m, there exists a 2-chain C in U such that $\alpha=mq+\partial C=\partial(mQ+C)$. Therefore we obtain

$$\operatorname{Link}(\alpha, \beta) = S(mQ+C, \beta) = mS(Q, \beta)+S(C, \beta) = 0.$$

Let k be a knot in S^3, V a tubular neighborhood of k and V^* a tubular neighborhood of a trivial knot k^*. Let $f\colon V^* \to V$ be a faithful homeomorphism of V^* onto V, l^* a knot contained in Int V^* and $l=f(l^*)$. Then $l^* \sim nk^*$ in Int V^* for some integer n.

THEOREM 9.

$$\begin{aligned}\sigma(l) &= \sigma(l^*) && \textit{if } n \textit{ is even,}\\ &= \sigma(l^*)+\sigma(k) && \textit{if } n \textit{ is odd.}\end{aligned}$$

Proof. There is no loss of generality in supposing that n is nonnegative. Note that the signature does not depend on the choice of orientation for a knot.

Case I. $n=0$.

Since l^* is nullhomologous in V^*, l^* bounds an orientable surface F^* in Int V^*. Let h be the genus of F^* and $\alpha_1^*, \alpha_2^*, \ldots, \alpha_{2h}^*$ a homology basis on F^*. Clearly $F=f(F^*)$ is an orientable surafce contained in Int V whose boundary is l, and $\alpha_1=f(\alpha_1^*), \alpha_2=f(\alpha_2^*), \ldots, \alpha_{2h}=f(\alpha_{2h}^*)$ is a homology basis on F. Let $T^*\colon F^* \to S^3-F^*$ and $T\colon F \to S^3-F$ be translations in the positive normal direction of F^* and of F respectively. We may assume that $T^*\alpha_i^*$ and $T\alpha_i$ are contained in Int V^* and in Int V respectively. Since $T\alpha_i \sim f(T^*\alpha_i^*)$ in $V-F$, it follows from Lemma 7 that

$$\begin{aligned}\operatorname{Link}(T^*\alpha_i^*, \alpha_j^*) &= \operatorname{Link}(f(T^*\alpha_i^*), \alpha_j)\\ &= \operatorname{Link}(T\alpha_i, \alpha_j).\end{aligned}$$

Therefore we have shown that a Seifert matrix $\|\operatorname{Link}(T^*\alpha_i^*, \alpha_j^*)\|_{i,j=1,2,\ldots,2h}$ of l^* coincides with a Seifert matrix $\|\operatorname{Link}(T\alpha_i, \alpha_j)\|_{i,j=1,2,\ldots,2h}$ of l. This implies immediately that $\sigma(l)=\sigma(l^*)$.

Case II. n is a positive integer.

First we will construct orientable surfaces F^* and F bounded by l^* and l respectively. Our construction is the same as the one given in §4 of [10], but for the sake of completeness we include it here.

We choose n pairwise disjoint longitudes $q_1^*, q_2^*, \ldots, q_n^*$ on the boundary of V^* such that q_i^* is homologous to k^* in V^*, $i=1, 2, \ldots, n$. Since $l^* \sim nk^* \sim \sum_{i=1}^n q_i^*$ in V^*, there exists an orientable surface F_0^* in V^* with $F_0^* = l^* - \sum_{i=1}^n q_i^*$. Let $F_1^*, F_2^*, \ldots, F_n^*$ be pairwise disjoint 2-cells in $S^3 - \operatorname{Int} V^*$ with $\partial F_i^* = q_i^*$, $i=1, 2, \ldots, n$. Then $F^* = F_0^* \cup \bigcup_{i=1}^n F_i^*$ is an orientable surface in S^3 whose boundary is l^*.

Let $F_0 = f(F_0^*)$ and $q_i = f(q_i^*)$ for $i=1, 2, \ldots, n$. Then F_0 is an orientable surface contained in V whose boundary is $l - \sum_{i=1}^n q_i$. Since q_1 is a longitude of V, it bounds an orientable surface F_1 in $S^3 - \operatorname{Int} V$. Without loss of generality we may assume that $q_1, q_2, \ldots, q_n$ are ordered in the positive normal direction of F_1. By pushing F_1 isotopically in its positive normal direction we obtain an orientable surface F_2 in $S^3 - \operatorname{Int} V$ which is parallel to F_1 and whose boundary is q_2. Similarly, by pushing F_2 isotopically in its positive normal direction, we obtain an orientable surface F_3 in $S^3 - \operatorname{Int} V$ with $\partial F_3 = q_3$ which is parallel to F_2 and intersects neither F_1 nor F_2. By continuing this process we finally obtain n pairwise disjoint orientable surfaces $F_1, \ldots, F_n$ in $S^3 - \operatorname{Int} V$ with $\partial F_i = q_i$, $i=1, 2, \ldots, n$. Clearly $F = F_0 \cup \bigcup_{i=1}^n F_i$ is an orientable surface bounded by l.

Let g be the genus of F^* and let

$$\alpha_1^*, \alpha_2^*, \ldots, \alpha_{2g}^* \tag{4.1}$$

be a homology basis on F^*. Since $F_1^*, \ldots, F_n^*$ are 2-cells, we may assume that these basis elements are lying on F_0^*. Therefore $\alpha_1 = f(\alpha_1^*)$, $\alpha_2 = f(\alpha_2^*), \ldots, \alpha_{2g} = f(\alpha_{2g}^*)$ are lying on F_0. Let h be the genus of F_1 and $\alpha_1^1, \alpha_2^1, \ldots, \alpha_{2h}^1$ a homology basis on F_1. For $\nu = 2, 3, \ldots, n$, we choose $\alpha_1^\nu, \alpha_2^\nu, \ldots, \alpha_{2h}^\nu$ as a homology basis on F_ν, where α_i^ν is the image of α_i^1 under the above described isotopy which carries F_1 to F_ν. Then it is easy to show that

$$\alpha_1^1, \alpha_2^1, \ldots, \alpha_{2h}^1; \quad \ldots; \quad \alpha_1^n, \alpha_2^n, \ldots, \alpha_{2h}^n; \quad \alpha_1, \alpha_2, \ldots, \alpha_{2g} \tag{4.2}$$

is a homology basis on F.

Now we want to consider the Seifert matrix of l with respect to the homology basis (4.2) on F. Let $T^*: F^* \to S^3 - F^*$ and $T: F \to S^3 - F$ be translations in the positive normal direction of F^* and of F respectively. Let

$$A^* = \|\operatorname{Link}(T^*\alpha_i^*, \alpha_j^*)\|_{i,j=1,2,\ldots,2g}$$

be the Seifert matrix of l^* with respect to homology basis (4.1) on F^*. Then the argument in Case I implies easily that

$$\|\operatorname{Link}(T\alpha_i, \alpha_j)\|_{i,j=1,2,\ldots,2g} = A^*. \tag{4.3}$$

Since α_s is a 1-cycle in V and $T\alpha_i^\nu \cap F_1 = \varnothing$, it follows from Lemma 8 that

$$\text{(4.4)} \qquad \mathrm{Link}\,(T\alpha_i^\nu, \alpha_s) = 0 \quad \text{for } 1 \leqq \nu \leqq n;\ 1 \leqq i \leqq 2h;\ 1 \leqq s \leqq 2g.$$

Since $\alpha_j^\nu \sim T\alpha_j^\nu$ in $S^3 - V$, we have Link $(T\alpha_r, \alpha_j^\nu) =$ Link $(T\alpha_r, T\alpha_j^\nu)$. Applying Lemma 8 to $T\alpha_r$ and $T\alpha_j^\nu$, we obtain

$$\text{(4.5)} \qquad \mathrm{Link}\,(T\alpha_r, \alpha_j^\nu) = 0 \quad \text{for } 1 \leqq \nu \leqq n;\ 1 \leqq j \leqq 2h;\ 1 \leqq r \leqq 2g.$$

If $\nu \geqq \mu$, we have $\alpha_j^\nu \sim \alpha_j^\mu$ in $S^3 - T\alpha_i^\nu$ (see Figure 2(a)), from which we obtain Link$(T\alpha_i^\nu, \alpha_j^\mu) =$ Link$(T\alpha_i^\nu, \alpha_j^\nu)$. If $\nu < \mu$, $T\alpha_i^\nu \sim T^{-1}\alpha_i^\mu$ in $S^3 - \alpha_j^\mu$, where $T^{-1}: F \to S^3 - F$ is a translation in the negative normal direction of F (see Figure 2(b)). From this we obtain

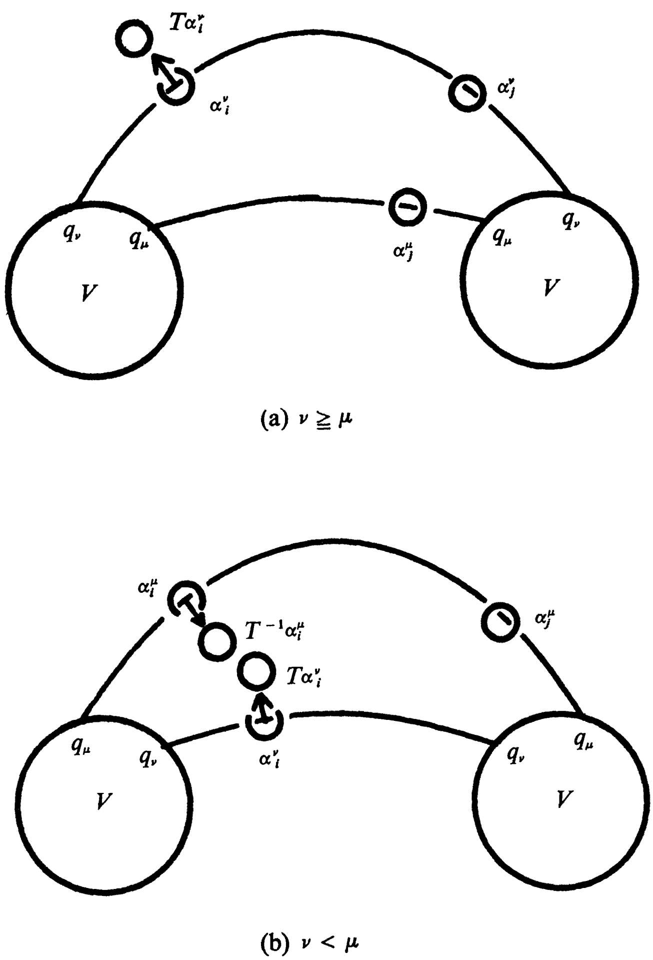

FIGURE 2

Link $(T\alpha_i^\nu, \alpha_j^\mu) = $ Link $(T^{-1}\alpha_i^\mu, \alpha_j^\mu) = $ Link $(\alpha_i^\mu, T\alpha_j^\mu)$. Therefore we have shown that

$$\begin{aligned} \text{Link}\,(T\alpha_i^\nu, \alpha_j^\mu) &= \text{Link}\,(T\alpha_i^\nu, \alpha_j^\nu) \quad \text{if } \nu \geqq \mu, \\ &= \text{Link}\,(\alpha_i^\mu, T\alpha_j^\mu) \quad \text{if } \nu < \mu. \end{aligned} \tag{4.6}$$

Since q_1 and k belong to the same knot type, the matrix

$$B = \|\text{Link}\,(T\alpha_i^1, \alpha_j^1)\|_{i,j=1,2,\ldots,2h}$$

can be considered as a Seifert matrix of k. By the construction of $\alpha_1^\nu, \alpha_2^\nu, \ldots, \alpha_{2h}^\nu$ it is clear that

$$\text{Link}\,(T\alpha_i^\nu, \alpha_j^\nu) = \text{Link}\,(T\alpha_i^1, \alpha_j^1)$$

for $\nu = 1, 2, \ldots, n$. From this and (4.6) we have

$$\begin{aligned} \|\text{Link}\,(T\alpha_i^\nu, \alpha_j^\mu)\|_{i,j=1,2,\ldots,2h} &= B \quad \text{if } \nu \geqq \mu, \\ &= B' \quad \text{if } \nu < \mu. \end{aligned} \tag{4.7}$$

Let A be the Seifert matrix of l with respect to the homology basis (4.2) on F. Then (4.3)–(4.7) show that A is a matrix of the following form:

$$A = \begin{bmatrix} \tilde{B} & 0 \\ 0 & A^* \end{bmatrix},$$

where $\tilde{B} = \|B_{\nu\mu}\|_{\nu,\mu=1,2,\ldots,n}$,

$$\begin{aligned} B_{\nu\mu} &= B \quad \text{if } \nu \geqq \mu, \\ &= B' \quad \text{if } \nu < \mu. \end{aligned}$$

To calculate signature $(A+A')$, we will make use of the facts that $A-A'$ is the matrix $S = \|S(\alpha_i^1, \alpha_j^1)\|_{i,j=1,2,\ldots,2h}$ of intersection numbers of α_i^1 and α_j^1 on F_1 [5] and that S is a unimodular skew symmetric matrix [9]. In $A+A'$, we subtract the 2nd row block from the 1st and the 2nd column block from the 1st; the 3rd row block from the 2nd and the 3rd column block from the 2nd; ...; the nth row block from the $(n-1)$th and the nth column block from the $(n-1)$th, which shows that $A+A'$ is congruent over $\boldsymbol{Z}$ to the following matrix:

$$\begin{array}{c} \quad n \text{ blocks} \\ n \text{ blocks} \left[\begin{array}{ccccc|c} 0 & -S & & & 0 & \\ S & 0 & -S & & & \\ & \ddots & \ddots & \ddots & & 0 \\ 0 & & S & 0 & -S & \\ & & & S & B+B' & \\ \hline & & 0 & & & A^*+A^{*\prime} \end{array}\right]. \end{array} \tag{4.8}$$

Furthermore, by adding the 1st row block to the 3rd and the 1st column block to the 3rd; the new 3rd row block to the 5th and the new 3rd column block to the 5th; . . ., it can be shown that (4.8) is congruent over $\boldsymbol{Z}$ to the matrix

$$(4.9)\quad \begin{array}{ll} \tilde{S} \oplus \cdots \oplus \tilde{S} \oplus \bar{S} \oplus (A^*+A^{*\prime}) \quad (n/2-1 \text{ copies}) & \text{if } n \text{ is even,} \\ \tilde{S} \oplus \cdots \oplus \tilde{S} \oplus (B+B') \oplus (A^*+A^{*\prime}) \quad ((n-1)/2 \text{ copies}) & \text{if } n \text{ is odd,} \end{array}$$

where

$$\tilde{S} = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix} \quad \text{and} \quad \bar{S} = \begin{bmatrix} 0 & -S \\ S & B+B' \end{bmatrix}.$$

Since $\bar{S}$ is congruent to $\tilde{S}$ over $\boldsymbol{Z}$ and signatue $\tilde{S}=0$, it follows from (4.9) that

$$\begin{aligned} \sigma(l) &= \sigma(l^*) && \text{if } n \text{ is even,} \\ &= \sigma(l^*)+\sigma(k) && \text{if } n \text{ is odd.} \end{aligned}$$

This completes the proof of Theorem 9.

REMARK 1. In [10] H. Seifert showed that $\Delta_l(t)=\Delta_{l^*}(t)\Delta_k(t^n)$, where $\Delta_l(t)$, $\Delta_{l^*}(t)$ and $\Delta_k(t)$ are the Alexander polynomials of l, l^* and k.

REMARK 2. Let $M_2(l)$, $M_2(l^*)$ and $M_2(k)$ be the 2-fold branched covering spaces of l, l^* and k. Then it follows from (4.9) that

$$\begin{aligned} H_1(M_2(l)) &= H_1(M_2(l^*)) && \text{if } n \text{ is even,} \\ &= H_1(M_2(l^*)) \oplus H_1(M_2(k)) && \text{if } n \text{ is odd,} \end{aligned}$$

where the coefficients of these homology groups are the integers.

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