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DISRUPTION OF LOW-DIMENSIONAL HANDLEBODY THEORY BY ROHLIN'S THEOREM

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1. Introduction

The handlebody theory of high-dimensional manifolds developed by M. Morse, S. Smale, and A. Wallace† is one of the key methods for classifying manifolds. Unfortunately it has never functioned well in dimensions 4 or 5. This article shows that some of the most useful theorems of high-dimensional handlebody theory in fact fail in low dimensions. For one, the s-cobordism theorem is false in dimension 4 or 5. W.-C. Hsiang has also noticed this. I expect others have noticed various failures because it was for some time easy to "prove" the Hauptvermutung for manifolds by assuming valid in low dimensions enough results of high-dimensional handlebody theory. Upon the disproof of the Hauptvermutung ([19] and [20]) it remained for me to locate the counterexamples. I have not succeeded completely.

One of the most interesting features of the counterexamples established for DIFF (= differentiable C^∞) manifolds and PL (= piecewise linear) manifolds is this: Some and perhaps all are no longer counterexamples when viewed in the category of TOP (= topological) manifolds. There I know of just one failure: Some closed TOP manifolds of dimension 4 or 5 admit no TOP handle decomposition (see Section 5).

Rohlin observed long ago (cf. [16]) that Whitney's method of eliminating double points breaks down in dimension 4. This is the main obstacle to finding a 5-dimensional handlebody theory. Rohlin's observation and the results of this article as well as the failure of the Hauptvermutung are all traceable to

† And several others. Perhaps the theory began with H. Poincaré's description of a generic smooth real-valued function on a smooth manifold [44, sec. 2].

Rohlin's Theorem ([30] and [15]). *For any closed oriented* DIFF *or* PL *4-manifold* M^4 *with vanishing second Stiefel–Whitney class* $w_2(M)$, *the signature* (= *index*) $s(M)$ *of* M *is divisible by* 16.

The expected divisibility is 8 in the sense that, for Poincaré complexes in place of PL manifolds, a minimum signature of 8 is realized (by a complex arising from Milnor's plumbing [3]). Rohlin's theorem is undecided for topological manifolds.

In this article Rohlin's theorem intervenes as part of the analysis of homotopy tori due to Wall, to Hsiang and Shaneson, and to A. Casson. It is reviewed in Section 3. Results are collected in Section 2, proofs are given in Section 4, and Section 5 is devoted to manifolds admitting no handle decomposition.

This article is a by-product of joint work with R. Kirby ([20] and [21]), and in this sense he is joint author.

2. Statements

Here I describe failures of $\mathscr{C}_0$, the "*ribbon*" *conjecture*; $\mathscr{C}_1$, the *s-cobordism conjecture*; and $\mathscr{C}_2$, the *pseudoisotopy conjecture*, in versions known to be valid in dimensions ≥ 6. The subscripts 0, 1, and 2 are suggested by relations with the functions K_0, K_1, and K_2 of algebraic K-theory.

SOME NOTATION. If X is a manifold, ∂X denotes its boundary and int X denotes its interior. DIFF manifolds are allowed corners along the boundary. The symbol ∂ alone is used as an abbreviation of "its boundary" or "the boundary," also the plural. $I = [0, 1]$. T^n = the n-torus, the n-fold product of circles $S^1 = T^1$. We agree that $T^{n+1} = T^n \times S^1$, so that $T^n \supset T^{n-1} \supset \cdots \supset S^1$. S^1 is identified with R/Z (the real numbers modulo the integers) and $I/\partial I$ (the interval with boundary points identified). Then $T^n = R^n/Z^n$. $B^n = \{x \in R^n; |x| \leq 1\}$ is the smooth unit n-ball assigned a compatible PL structure, PL isomorphic to $I^n \subset R^n$.

Definition. A *ribbon*† W^n is a noncompact (connected) manifold (of category DIFF, PL, or TOP) such that (a) $\partial W \cong N \times R$ for some compact $(n-2)$-manifold N, and (b) there exists a finite complex K and a proper‡ map $r: K \times R \to W$, inducing an isomorphism $\pi_1 K \cong \pi_1 W$, such that r is a retraction in the category of proper homotopy classes of proper continuous maps.

Any ribbon has two ends. Call one ε. There is an invariant $\sigma(\varepsilon)$ defined in $\tilde{K}_0 Z[\pi_1(W)]$. It comes from Wall's finiteness obstruction [39] of any suitable closed neighborhood X of ε. Suitable here means that $W - X$ is a

† Called pseudoproduct in [37].

‡ *Proper* means that the preimage of each compactum is compact.

neighborhood of the other end and X is a submanifold. For more discussion see [33], [35], and [34a, sec. 4].

$\mathscr{C}_0(n)$: **Ribbon Conjecture (DIFF, PL, or TOP).** *A ribbon W^n of dimension n is isomorphic to $M \times R$ for some compact manifold M if and only if the invariant $\sigma(\varepsilon)$ of either end ε of W is zero.*

The conjecture is established for $n \geq 6$ in [33]. Browder first treated the simply connected case in [4]. The PL and TOP proofs are patterned on the original DIFF proof ([33] and [21]). Recall that, if $\pi_1 W$ is free abelian, then $K_0 Z[\pi_1 W] = 0$ by [1], so that no obstruction intervenes. For $n = 3$ the result has been proved by Husch and Price [14] with the extra provisos that $\partial W \neq 0$ or $\pi_1 W \neq Z_2$, and that W be irreducible (i.e., each embedded locally flat 2-sphere in W bounds 3-disk). Clearly $\mathscr{C}_0(3)$ implies the Poincaré conjecture.

Theorem 0. *In the* DIFF *and* PL *categories the conjecture $\mathscr{C}_0(n)$ is false for dimension* 5, 4, *or* 3. *More precisely, either $\mathscr{C}_0(n)$ fails for a ribbon which is proper homotopy equivalent to $S^3 \times S^1 \times R$ or $S^3 \times R$, or else the 3-dimensional Poincaré conjecture fails.*

Definition. Let W^n be a compact (connected) n-manifold and V, V' disjoint compact $(n-1)$-submanifolds of ∂W such that $\partial W - (\text{int } V \cup \text{int } V') \cong \partial V \times [0, 1] \cong \partial V' \times [0, 1]$. Suppose that $\partial V \subsetneq W$ and $\partial V' \subsetneq W$ are homotopy equivalences. Then $(W; V, V')$ is called an *h-cobordism* (mod boundary).

$\mathscr{C}_1(n)$: ***s*-Cobordism Conjecture (DIFF, PL, or TOP).** *An h-cobordism $(W; V, V')$ is a product cobordism, i.e., $(W; V, V') \cong V \times (I; 0, 1)$, if and only if the inclusion $V \subsetneq W$ is a simple homotopy equivalence.*

REMARK. It is useful to note that $(W; V, V') \cong V \times ([0, 1]; 0, 1)$ implies directly the existence of an isomorphism extending any given isomorphism $\partial W - (\text{int } V \cup \text{int } V') \cong \partial V \times [0, 1]$ that sends ∂V to $\partial V \times 0$ (cf. [34, prop. II]).

$\mathscr{C}_1(n)$ has been proved for $n \geq 6$ ([17] and [13]); the simply connected version is due to S. Smale, and the general case was first treated by B. Mazur. The PL and TOP proofs are again patterned on the original DIFF proof ([13] and [21]). Simple homotopy type of topological manifolds is defined in [20], and can equally well be defined by a handle decomposition whenever one exists [22]. Note that every homotopy equivalence $V \subsetneq W$ is simple if $\pi_1 W$ is free abelian, because then $\text{Wh}(\pi_1 W) = 0$ [1].

$\mathscr{C}_1(5)$ is valid if both $\pi_1 W = 0$ and there is an isomorphism $V \cong V'$ in the preferred homotopy class ([40] and [32]). $\mathscr{C}_1$ (3) is established with the same provisos as $\mathscr{C}_0(3)$ in [38] and [2].

Theorem 1. *In the* DIFF *and* PL *categories the conjecture $\mathscr{C}_1(n)$ is false for dimension $n = 5$ or 4 (one or both). To be more specific, fix $k = 0, 1$, or 2, and choose the* DIFF *or* PL *category. Then there exists an h-cobordism*

$(W; V, V')$ of the following description:

(a) *V is $B^k \times T^{4-k}$ or $B^k \times T^{3-k}$ (perhaps both possible).*

(b) *$V' \cong V$.*

(c) *$(W; V, V')$ is an invertible cobordism.*

(d) *There exists a topological homomorphism $e : V \times (I; 0, 1) \to (W; V, V')$ that gives an isomorphism $\partial(V \times I) \to \partial W$.*

(e) *$(W; V, V')$ is not a product cobordism. In fact, no finite odd covering† of it is a product cobordism.*

Let M be a manifold (DIFF, PL, or TOP) and $\alpha : I \times M \circlearrowleft$ an automorphism respecting $0 \times M$, $1 \times M$, and fixing $I \times \partial M$. α is called a *pseudoisotopy* (mod ∂).

$\mathscr{C}_2(n)$: **Pseudoisotopy Conjecture (DIFF, PL, or TOP).** *For any connected compact $(n-1)$-manifold M, such that $\pi_1(M)$ is free abelian the following holds. Each pseudoisotopy $\alpha : M \times I \circlearrowleft$ is isotopic fixing $\partial(I \times M)$ to an automorphism respecting the slices $t \times M$, $t \in I$, (i.e., to an isotopy). Equivalently, each automorphism of $I \times M$ fixing $I \times \partial M \cup 0 \times M$ is isotopic through such automorphisms to the identity.*

$\mathscr{C}_2(n)$ has been proved for $n \geq 6$ in case $\pi_1 M = 0$, by Cerf for DIFF ([6] and [7]) and by C. Morlet and Rourke [31] for PL. Armstrong's [45] may help prove $\mathscr{C}_2(n)$ for TOP. $\mathscr{C}_2(n)$, $n \geq 6$, is still undecided for $\pi_1 \neq 1$. It is known that, without the restriction that π_1 be free abelian, nontrivial obstructions come into play [36]. However, $\mathscr{C}_2(n)$, $n \geq 6$, cannot be far wrong, since

Proposition 2.1. *For $n \geq 6$ the following conjecture $\mathscr{C}'_2(n)$ holds true.*

$\mathscr{C}'_2(n)$: *Weak Pseudoisotopy Conjecture* (DIFF, PL, *or* TOP). With the data of $\mathscr{C}_2(n)$, for every automorphism $\alpha : I \times M \circlearrowleft$ fixing $I \times \partial M \cup 0 \times M$, there exists a finite covering $\overline{M}$ of M, even one of odd order, so that the automorphism $\bar{\alpha} : I \times M \circlearrowleft$ covering α and fixing $I \times \partial\overline{M} \cup 0 \times \overline{M}$ is isotopic, fixing $I \times \partial\overline{M} \cup 0 \times \overline{M}$ to the identity.

Parallel to Theorem 1 we have

Theorem 2. *In the* DIFF *and* PL *categories the conjecture $\mathscr{C}_2(n)$ is false for dimension $n = 5$ or 4 (one or both). To be more specific fix $j = 0$ or 1 and choose the* DIFF *or* PL *category. Then there exists an automorphism $\alpha : I \times M \circlearrowleft$.*

(a) *$M = B^j \times T^{4-j}$ or $B^j \times T^{3-j}$ (perhaps both possible).*

(b) *$\alpha | \partial(I \times M) = $ identity.*

(c) *(the empty condition).*

(d) *α is topologically isotopic, fixing $\partial(I \times M)$ to the identity.*

(e) *α is not isotopic (in the category) to* $\mathrm{id}|I \times M$*, fixing $I \times \partial M \cup 0 \times M$. In fact, no finite odd-order covering of α is isotopic to the identity in this way.*

† And no finite even covering either, if $k = 2$. See $n'(1)$ in Corollary 3.3 and the proof of Theorem 1. A similar remark applies to Theorem 2 if $j = 1$.

3. Results from Surgery

Transversality is used extensively in this section. In the PL category we agree to use Williamson's PL microbundle transversality [43], always with respect to an obvious PL microbundle in the target space.

Adopt the DIFF or the PL category and let X denote a compact n-manifold with boundary ∂X. We consider compact manifolds W equipped each with a homotopy equivalence $f : W \to X$ that gives an isomorphism on ∂.† Sullivan calls f a homotopy smoothing [resp. triangulation] of X in the DIFF [resp. PL] category. Another $f' : W' \to X$ is called equivalent if there exists an isomorphism $\psi : W \to W'$ such that $f'\psi = f$ on ∂W and $f'\psi$ is homotopic to f fixing boundary. The equivalence classes form a pointed set denoted $\mathscr{S}(X, \partial X)$ or simply $\mathscr{S}(X, \partial)$.

Theorem 3 (Wall [42] and Hsiang and Shaneson [12]). *Adopt the* PL *category. For $n + k \geq 5$ there is a bijection* $[*] : \mathscr{S}(B^k \times T^n, \partial) \cong H^{3-k} \times (T^n; Z_2)$.

Complement 3.1. *The bijection* $[*]$ *enjoys these naturality properties:*

n(1). *If $f : W \to B^k \times T^n$ is a homotopy triangulation and $p : B^k \times T^n \to B^k \times T^n$ is a covering map, a corresponding covering $\bar{f} : \overline{W} \to B^k \times T^n$ of f, $p\bar{f} = pf$, satisfies $[\bar{f}] = p^*[f]$. Hence if p has odd degree, or $k = 3$, then $[\bar{f}] = [f]$, since $p^* =$* id.

n(2). *Consider a homotopy triangulation $f : W \to B^k \times T^n$ that is h-transverse to $B^k \times T^{n-1}$. The prefix "h-" means‡ that the restriction $f_1 : f^{-1}(B^k \times T^{n-1}) \to B^k \times T^{n-1}$ of f is a homotopy equivalence. Then the class $[f_1] \in H^{3-k}(T^{n-1}; Z_2)$ is the restriction of $[f]$.*

From *n*(2) we get

Complement 3.2. *For any dimensions n and k, $[*] : \mathscr{S}(B^k \times T^n, \partial) \to H^{3-k}(T^n; Z_2)$ can be defined by letting $[f]$ be the restriction to T^n of $[f \times (\mathrm{id}|T^5)] \in H^{3-k}(T^{n+5}; Z_2)$. It continues to enjoy the naturality properties n*(1) *and n*(2), *but may not be an isomorphism.*

This is also clear from the definition of $[*]$ below.

The DIFF *Version*

Since every PL manifold has an essentially unique smoothing in dimensions ≤ 6 ([5] and [26]), the theorem holds for DIFF in dimensions 5 and 6. (By imitating the PL proof one avoids Cerf's difficult theorem $\Gamma_4 = 0$ [5].)

† In DIFF, when X has corners on ∂X, one must insist that $f|\partial X$ extends to an isomorphism of neighborhoods of the boundaries.

‡ The notion of h-transversality will be used repeatedly below.

REMARK.† The general result in DIFF is as follows for $n + k \geq 5$. By a classification theorem parallel to that of [20], the smoothness structures on the topological manifold $B^k \times T^n$ that extend the standard structure on ∂ are classified up to isotopy fixing ∂ by $[B^k \times T^n/\partial; \mathrm{TOP/O}]$. The forgetful map to $\mathscr{S}(B^k \times T^n, \partial)$ is an isomorphism. Thus we have

$$\mathscr{S}(B^k \times T^n, \partial) \cong [B^k \times T^n/\partial; \mathrm{TOP/O}] = \sum_i H^{i-k}(T^n; \pi_i(\mathrm{TOP/O})).$$

The PL result can also be stated

$$\mathscr{S}(B^k \times T^n, \partial) \cong [B^k \times T^n/\partial; \mathrm{TOP/PL}] = H^{3-k}(T^n; \pi_3(\mathrm{TOP/PL})),$$

since $\pi_i \mathrm{TOP/PL} = 0$ for $i \neq 3$ [20]. This gives a very convincing "explanation" of the naturality properties under Theorem 3.

In the PL or DIFF category, consider automorphisms $\alpha : B^k \times T^n \circlearrowleft$ of $B^k \times T^n$ fixing ∂ and homotopic‡ fixing ∂ to the identity. Agree that two such automorphisms α_0 and α_1 are equivalent if there exists an automorphism $\beta : I \times B^k \times T^n \circlearrowleft$ fixing $I \times \partial B^k \times T^n$ with $\beta | i \times B^k \times T^n = i \times \alpha_i, i = 0, 1$. β is called a pseudoisotopy mod ∂. The set of equivalence classes is denoted $\mathscr{A}(B^k \times T^n, \partial)$.

Given an automorphism $\alpha : B^k \times T^n \circlearrowleft$ and a homotopy fixing ∂, α_t, $0 \leq t \leq 1$, of $\alpha_0 = \mathrm{id}$ to $\alpha_1 = \alpha$, form $F : I \times B^k \times T^n \to I \times B^k \times T^n$ by setting $f(t, u, v) = (t, \alpha_t(u, v))$. This F gives an element of $\mathscr{S}(I \times B^k \times T^n) \cong \mathscr{S}(B^{k+1} \times T^n)$. The correspondence $\alpha \to F$ gives a well-defined map

$$\psi : \mathscr{A}(B^k \times T^n, \partial) \to \mathscr{S}(B^{k+1} \times T^n, \partial).$$

To see this one must note that any homotopy equivalence $B^{k+1} \times T^n \to B^{k+1} \times T^n$ fixing ∂ is homotopic fixing ∂ to an isomorphism.

Definition. The *composed map* [*] ψ is denoted

$$\langle * \rangle : \mathscr{A}(B^k \times T^n, \partial) \to H^{2-k}(T^n; Z_2).$$

If $n + k \geq 5$, ψ has an inverse ψ' defined as follows. Let $f : W \to I \times B^k \times T^n$ represent $x \in \mathscr{S}(I \times B^k \times T^n, \partial)$. Use the s-cobordism theorem to extend $f^{-1}|0 \times B^k \times T^n \cup I \times \partial B^k \times T^n$ to an isomorphism $g : I \times B^k \times T^n \to W$. Then let $fg|1 \times B^k \times T^n$ represent $\psi'(x)$. Thus we have from Theorem 3

Corollary 3.3. *There is a mapping*

$$\langle * \rangle : \mathscr{A}(B^k \times T^n) \to H^{2-k}(T^n; Z_2)$$

natural for coverings and restrictions:

† For proof see [21].
‡ This condition is redundant for $k \geq 2$, but not for $k = 0$ or 1.

n'(1). If $p : B^k \times T^n \circlearrowleft$ *is a covering map, and* $\langle\alpha\rangle$, $\langle\alpha'\rangle$ *are related by* $p\alpha' = \alpha p$ *(i.e.,* α' *covers* α*), then* $\langle\alpha'\rangle = p^*\langle\alpha\rangle$.

n'(2). If $\alpha : B^k \times T^n \circlearrowleft$ *represents* $\langle\alpha\rangle$ *and respects* $B^k \times T^{n-1}$, *then* $\langle\alpha | B^k \times T^{n-1}\rangle \in H^{2-k}(T^{n-1}; Z_2)$ *is the restriction of* $\langle\alpha\rangle$.

The map $\langle * \rangle$ *is bijective for* $n + k \geq 5$ *in the* PL *category and for* $n + k = 5$ *in the* DIFF *category.*

Discussion of the Proof of Theorem 3. The proof is a calculation that starts from an exact sequence of Sullivan–Wall [41, sec. 10]:

$$\cdots \to [\Sigma(N/\partial), G/\mathrm{PL}] \xrightarrow{\Theta'} L_{m+1}(\pi_1 N) \to \mathscr{S}(N, \partial) \to [(N/\partial), G/\mathrm{PL}] \xrightarrow{\Theta} L_m(\pi_1 N),$$

where $N = B^k \times T^n$, $m = n + k$. The maps Θ, Θ' turn out to be injective, so $\mathscr{S}(B^k \times T^n, \partial) = \mathrm{coker}\,(\Theta')$. Leaving the details aside, I will now give a definition of $[f]$ which makes evident the naturality properties of $[f]$ in Complement 3.1 and, more important, indicates the role of Rohlin's theorem.†

Adopt the DIFF or PL category and consider a compact oriented 3-manifold M^3 with $\pi_1 M$ free abelian. Let $f : W^{n+3} \to M^3 \times T^n$ be a homotopy smoothing or triangulation mod ∂ such that there exist (a) a manifold X with $w_1(X) = w_2(X) = 0$ forming a cobordism $(X; M^3 \times T^n, W^{n+3})$, (b) an isomorphism $\gamma : \delta X = X - \{\mathrm{int}\, M^3 \times T^n \cup \mathrm{int}\, W\} \to \partial(M^3) \times T^n \times I$ such that $\gamma(x) = (x, 0)$ for $x \in \partial M^3 \times T^n$ and $\gamma(x) = (f(x), 1)$ for $x \in \partial W$, and (c) a retraction $g : X \to M^3 \times T^n$ extending f and the projection $\delta X \to M^3 \times T^n$ provided by γ. *We will construct an invariant* $\theta_M(f) \in Z_2$ *of the class of* f *in* $\mathscr{S}(M \times T^n)$. Suppose first that $g|W = f$ can be made h-transverse to the submanifolds $M^3 \times T^{n-1} \supset M^3 \times T^{n-2} \supset \cdots \supset M^3 \times T^1 \supset M^3$. When this has been accomplished we can further deform g fixing ∂X to make g transverse to M^3. Then set $X^4 = g^{-1}(M^3)$ and define

$$\theta_M(f) = s(X^4)/8 \ (\mathrm{mod}\ 2).$$

Here $s(X^4)$ is the signature of the orientable 4-manifold X^4 with boundary. It can be defined as the algebraic signature of the intersection pairing $\Phi : H_2(X; Z) \otimes H_2(X, Z) \to Z$. This is divisible by 8 since $w_1 X = w_2 X = 0$ implies that $\Phi(x \otimes x)$ is even for all x [3]. To make f h-transverse one attempts to use repeatedly

Theorem F (Farrell [10]). *Let* $f' : W^n \to M^{n-1} \times S^1$ *be a homotopy equivalence of compact* DIFF *or* PL *manifolds that is an isomorphism on* ∂. *Then* f' *can be made h-transverse to* $M \times 0$ *by a homotopy fixing* ∂ *provided* $\pi_1 M$ *is free abelian and* $n > 6$.

REMARK. Farrell has shown this to be a formal consequence of $\mathscr{C}_0(n)$ supported by $\mathscr{C}_1(n + 1)$ and $\mathscr{C}_1(n)$. For the definition of θ_M it would suffice

† It is the one explained to Kirby and me by W. Browder. It may be explicit in [12].

to make h-regular some covering of f' induced from a covering $S^1 \to S^1$. For this $\mathscr{C}_0(n)$ suffices (see the proof of Theorem 0).

As Theorem F may fail for $n = 5$, or 4, we are obliged to proceed by artifice. Let $P^p = CP(2k)$, $p = 4k \geq 8$. Then make $\mathbf{g} = (\mathrm{id}|P) \times gh$-transverse on $\mathbf{W} = P \times W$ and (simply) transverse on $\mathbf{X} = P \times X$ to $P \times M^3$. Set $\mathbf{X}^{(4)} = \mathbf{g}^{-1}(P \times M^3)$ and define

$$\theta_M(f) = s(\mathbf{X}^{(4)})/8 \ (\mathrm{mod}\ 2).$$

This is consistent with our first attempt since $s(P \times X^4) = s(P) \times s(X^4) = s(X^4)$.

We now check that $s(\mathbf{X}^{(4)})/8$ is an integer and that $\theta_M(f)$ is well defined. Suppose the above process of definition is carried out twice in different ways, giving data distinguished by subscripts 0 and 1. Form Y^{n+4} from $X_0 \cup X_1 \cup \partial(X_0) \times I$, by identifying ∂X_0 in X_0 to $\partial X_0 \times 0$ and ∂X_1 in X_1 to $\partial X_0 \times 1$ in the natural way. We shall now define $\mathbf{h} : \mathbf{Y} = P \times Y^{n+4} \to P \times M^3 \times T^n$ extending $\mathbf{g}_0 \cup \mathbf{g}_1 : \mathbf{X}_0 \cup \mathbf{X}_1 \to P \times M^3 \times T^n$. On $P \times \{M^3 \times T^n \cup \partial(M^3 \times T^n) \times I\} \times I \subset P \times (\partial X_0) \times I$ we can let it be projection to $P \times M^3 \times T^n$. A homotopy mod ∂ from $\mathbf{g}_0|W$ to $\mathbf{g}_1|W$ further extends this over $\mathbf{W} \times I$ to give $\mathbf{h}$. Now $\mathbf{h}$ can be altered on $\mathrm{int}(\mathbf{W} \times I)$, by repeatedly applying Theorem F, so that $\mathbf{h}|\mathbf{W} \times I$ is h-regular to $P \times M^3 \times T^s$, $0 \leq s \leq n$. This leaves $\mathbf{h}|\mathbf{X}_0 \cup \mathbf{X}_1 = \mathbf{g}_0 \cup \mathbf{g}_1$ unchanged.

Consider the closed $(p + 4)$-manifold $\mathbf{Y}^{(4)} = \mathbf{h}^{-1}(P \times M^3)$. It is not hard to check that $s(\mathbf{Y}^{(4)}) = s(\mathbf{X}_0^{(4)}) + s(\mathbf{X}_1^{(4)})$. But by another application of transversality $\mathbf{Y}^{(4)}$ is orientably cobordant to $P \times Y^4$, where Y^4 is obtained by making $h = \mathbf{h}|Y^{n+4}$ transversal to $P \times M^3 \subset P \times M^3 \times T^n$. It is easily seen that $w_1(Y^4) = w_2(Y^4) = 0$. Hence Rohlin's theorem shows that $s(Y^4) \equiv 0 \ (\mathrm{mod}\ 16)$. Thus $s(\mathbf{X}_1^{(4)}) + s(\mathbf{X}_2^{(4)}) = s(\mathbf{Y}^{(4)}) = s(Y^4) \equiv 0 \ (\mathrm{mod}\ 16)$. From the case $\mathbf{X}_1^{(4)} = \mathbf{X}_2^{(4)}$ we see that $s(\mathbf{X}^{(4)})$ is divisible by 8, so that $\theta_M(f) = s(\mathbf{X}^{(4)})/8 \ (\mathrm{mod}\ 2)$ makes sense. Now $s(\mathbf{X}_1^{(4)})/8 \equiv s(\mathbf{X}_2^{(4)})/8 \ (\mathrm{mod}\ 2)$ by the same congruence. Hence $\theta_M(f)$ is a well-defined function of the equivalence class of f.

A further use of Theorem F shows that $\theta_M(f) = \theta_M(\gamma f)$ where $\gamma : M \times T^n \circlearrowleft$ permutes the factors of T^n; in other words, $\theta_M(f)$ is independent of the choice of the nest

$$M^3 \times T^{n-1} \supset M^3 \times T^{n-2} \supset \cdots \supset M^3 \times S^1 \supset M^3.$$

Given a homotopy triangulation $f : W \to B^k \times T^n$, we are now ready to define its cohomology class $[f] \in H^3(B^k \times T^n, \partial; Z_2)$ by specifying its effect on each generator of $H_3(B^k \times T^n, \partial; Z_2)$. There are $\binom{n}{3-k}$ of them: the fundamental classes $\{B^k \times T^{3-k}\}$ of $B^k \times T^{3-k}$, where T^{3-k} here ranges over all $\binom{n}{3-k}$ standard $(3-k)$-dimensional subtori. Fixing any one T^{3-k}, write $B^k \times T^n = (B^k \times T^{3-k}) \times T^n$ by reordering the factors of T^n and define

$$\langle [f], \{B^k \times T^{3-k}\}\rangle = \theta_{B^k \times T^{3-k}}(f).$$

Injectivity of Θ shows that $\theta_{B^k \times T^{3-k}}(f)$ can always be defined; i.e., a cobordism $(X; B^k \times T^n, W)$ with $w_1(X) = w_2(X) = 0$ can be found. In fact, X can be (stably) PL parallelizable. This completes the definition of $[f]$ and the discussion of Theorem 3.

4. Proofs

This section proves the results of Section 2 using the known results described in Section 3. The minimum required from the latter is the pair of invariants $[*]$, $\langle * \rangle$ and their naturality properties, together with the examples constructed in Section 5 (see Construction 5.5).

Proof of Theorem 0. Fix the DIFF [or PL] category and consider a homotopy smoothing [or triangulation] $g : X^n \to B^3 \times T^{n-3} \pmod{\partial}$, $n \geq 5$, such that $0 \neq [g] \in H^0(T^{n-3}; Z_2) = Z_2$. Let $f : W^n \to S^3 \times T^{n-3}$ be formed by welding g to the identity map $B^3 \times T^{n-3} \to B^3 \times T^{n-3}$ by identifying their restrictions to ∂. Now $\theta_{S^3}(f) = \theta_{B^3}(g) = [g] \neq 0$ in Z_2, according to the description of θ and $[*]$ in Section 3. We write $[f]$ for $\theta_{S^3}(f)$ since like θ_{B^3} it has the naturality properties $n(1)$, $n(2)$ of Complement 3.1. We have thus $f : W^n \to S^3 \times T^{n-3}$, a homotopy smoothing [or triangulation] with $[f] \neq 0$ and $n \geq 5$. Assuming $\mathscr{C}_0(5)$, we will find

{a} *such an f for $n = 4$.*

Choose $f : W^5 \to S^3 \times T^2$ with $[f] \neq 0$, and let $\bar{f} : \overline{W}^5 \to S^3 \times S^1 \times R$ be a proper homotopy equivalence covering the homotopy equivalence f. Applying $\mathscr{C}_0(5)$ we find that $\overline{W}^5 \cong M^4 \times R$ for a closed manifold M^4. We can now find a homotopy $\bar{f}_t$, $0 \leq t \leq 1$, of $\bar{f}$ so that $\bar{f}_1$ is transverse to $S^3 \times S^1 \times 0$ with $\bar{f}_1^{-1}(S^3 \times S^1 \times 0) = W^4$, and $\bar{f}_t$, $0 \leq t \leq 1$, fixes all points outside a compactum. Then $\bar{f}_t$ induces on any finite covering of large order $f' : W' \to S^3 \times S^1 \times S^1$ of f that is covered by $\bar{f}$, a homotopy f'_t, $0 \leq t \leq 1$, such that f'_1 is transverse to $S^3 \times S^1 \times 0$ at $f_1'^{-1}(S^3 \times S^1 \times 0) \cong W^4$ and $f'_1 | W^4 : W^4 \to S^3 \times S^1$ is a homotopy equivalence. By $n(2)$, $[f'_1 | W^4] = [f'_1]$, and, by $n(1)$, $[f'_1] = [f'] = [f] \neq 0$. Hence, $f'_1 | W^4$ completes step {a}.

Applying the above argument to $f : W^4 \to S^3 \times S^1$ with $[f] \neq 0$, we find that $\mathscr{C}_0(4)$ applied to a ribbon proper homotopy equivalent to $S^3 \times R$ allows one to find

{b} *such an* f *for* $n = 3$.
This $f : W^3 \to S^3$ is then a contradiction of the Poincaré conjecture. The proof of Theorem 0 is now complete.

REMARK. Since $J : \pi_3 SO \to \pi_3 SG$ is onto, W^3 (if it exists) bounds a parallelizable compact 4-manifold M^4. The definition of $[f]$ tells us that $[f] \equiv \sigma(M^4)/8 \pmod 2$. As $[f] \neq 0$, $\sigma(M^4)$ is not divisible by 16 and we conclude that W^3 is not even h-cobordant to S^3.

The proof of Theorem 1 will require

Lemma 4.1 (from [20]). *Given any self-homeomorphism* $f : B^k \times T^n \circlearrowleft$ *fixing* ∂ *and homotopic fixing* ∂ *to the identity, there exists a covering map* $p : B^k \times T^n \to B^k \times T^n$ *of odd finite degree and a self-homeomorphism* $\tilde{f} : B^k \times T^n \circlearrowleft$ *fixing* ∂ *and covering* f *(i.e.,* $p\tilde{f} = fp$*) such that* $\tilde{f}$ *is isotopic fixing* ∂ *to the identity.*

Proof of Lemma 4.1. For any compact topological manifold M with metric d there exists $\varepsilon > 0$ such that, for every self-homeomorphism $f : M \circlearrowleft$ with $d(f, \mathrm{id}) = \sup\{d(f(x), x) | x \in M\} < \varepsilon$ and $f|\partial M =$ identity, there is an isotopy of f to $\mathrm{id}|M$ fixing ∂M. This is a case of theorems of Černavskiĭ and Edwards–Kirby ([8] and [9]). Let the metric on $B^k \times R^n$ be inherited from $R^{k+n} = R^k \times R^n \supset B^k \times R^n$. Assign to $B^k \times T^n = B^k \times (R^n/Z^n)$ the quotient metric $d(x, y) = \inf\{d(x', y') | qx' = x, qy' = y\}$, $q : B^k \times R^n \to B^k \times T^n$ being the quotient map. Then choose $\varepsilon > 0$ to correspond as above to $M = B^k \times T^n$ with this metric.

Let $p_s = (\mathrm{id}|B^k) \times q_s : B^k \times T^n \to B^k \times T^n$, where $q_s(\theta_1, \ldots, \theta_n) = (s\theta_1, \ldots, s\theta_n)$, and let f_s be an automorphism of $B^k \times T^n$ such that $fp_s = p_s f_s$ and $f_s|\partial =$ identity. If $k \neq 0$, this defines f_s uniquely, but if $k = 0$, $\partial B^k \times T^n = \varnothing$, so that f_s is defined up to a covering translation only. To completely specify f_s when $k = 0$, choose a path γ from the zero point $0 \in T^n$ to $f(0) \in T^n$, and insist that γ lift to a path from 0 to $f_s(0)$.

Let $\tilde{f} : B^k \times R^n$ be the automorphism such that $\tilde{f}|\partial =$ identity, $q\tilde{f} = fq$, where $q : B^k \times R^n \to B^k \times T^n = B^k \times (R^n/Z^n)$ is the quotient map, and (in case $k = 0$) γ lifts to a path from $0 \in R^n$ to $\tilde{f}(0) \in R^n$.

Assertion. *If* $p : B^k \times T^n \to T^n$ *denotes projection, then* $d(pf_s, p) \equiv \sup\{d(pf_s(x), p(x)) | x \in B^k \times T^n\} \leq C/s$ *for some constant* C *independent of* s.

To check this, first observe that $d(\tilde{p}\tilde{f}, \tilde{p}) \equiv \sup\{d(\tilde{p}\tilde{f}(x), \tilde{p}(x)\}$ is finite, where $\tilde{p} : B^k \times R^n \to R^n$ is projection. For, as f_* fixes $\pi_1(B^k \times T^n)$, $\tilde{f}$ commutes with covering translations, and the value is $\sup\{d(\tilde{p}\tilde{f}(x), \tilde{p}(x)) | \tilde{p}(x) \in I^n\} = C < \infty$. Next observe that $d(pf_s, p) \leq d(\tilde{p}\tilde{f}_s, \tilde{p})$, where $\tilde{f}_s : B^k \times R^n \circlearrowleft$ is any covering of f_s. Now one choice of $\tilde{f}_s$ is $\tilde{f}_s(t, y) = (1/s)\tilde{f}(t, sy)$. This gives $d(pf_s, p) \leq d(\tilde{p}\tilde{f}_s, \tilde{p}) = (1/s)d(\tilde{p}\tilde{f}, \tilde{p}) = C/s$.

To complete the proof choose s odd so that $C/s < \varepsilon/2$. Let $\{\varepsilon\} : R^k \to R^k$ denote multiplication by ε. Then define f'_s on $(\varepsilon/2)B^k \times T^n \subset B^k \times T^n$ to be

$(\{\varepsilon/2\} \times \mathrm{id}|T^n) \circ f_s \circ (\{2/\varepsilon\} \times \mathrm{id}|T^n)$. Elsewhere in $B^k \times T^n$ let it be the identity. Then f'_s is isotopic to f_s fixing ∂ (increase ε to 2) and $d(f'_s, \mathrm{id}|B^k \times T^n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so f'_s is isotopic to $\mathrm{id}|B^k \times T^n$, fixing ∂. This completes the lemma.

Proof of Theorem 1. Fix a nonzero element x in $H^{2-k}(T^{5-k}; Z_2)$ such that the restriction of x to T^{2-k} (where $T^{n+1} = T^n \times S^1$) is nonzero. By Corollary 3.3, there exists an automorphism $f : B^k \times T^{5-k} \rightleftarrows$ fixing ∂ and homotopic fixing ∂ to the identity, such that x is the obstruction $\langle f \rangle$ to finding a pseudo-isotopy from f fixing ∂ to $\mathrm{id}|B^k \times T^{5-k}$. Supposing $\mathscr{C}_1(5)$ and $\mathscr{C}_1(4)$ we shall repeatedly rechoose f until f respects each of the three nested submanifolds $B^k \times T^{2-k} \subset B^k \times T^{3-k} \subset B^k \times T^{4-k}$, and $f|B^k \times T^{2-k} = $ identity. In view of Corollary 3.3, this shows that $x = \langle f \rangle$ has restriction zero on T^{2-k}, a contradiction.

Step (a): *Making f respect $B^k \times T^{4-k}$.* Write $T^{5-k} = T^{4-k} \times S^1$, $S^1 = (I/\partial I)$. By passing if necessary to a finite covering f_s, s odd, of f we can rechoose f so that f is topologically isotopic fixing ∂ to $\mathrm{id}|B^k \times T^{5-k}$, and, in addition, $f(B^k \times T^{4-k} \times \frac{1}{2})$ and $B^k \times T^{4-k} \times 0$ are disjoint (this uses Lemma 4.1).

Their complement in $B^k \times T^{5-k}$ has two components. Let W and W' be their closures, W the one containing $B^k \times T^{4-k} \times t$, for small $t > 0$. Set $c = (W; V, V') = (W; B^k \times T^{4-k} \times 0, f(B^k \times T^{4-k} \times \frac{1}{2}))$. Then $c' = (W'; V', V)$ is its inverse; i.e., the compositions cc' and $c'c$ are product cobordisms. In particular, c is an h-cobordism. Properties (a), (b), and (c) are clearly verified.

We now check (d). Lift f to an automorphism $\bar{f}$ of the covering $B^k \times T^{4-k} \times R$, $S^1 = R/Z$, with $\bar{f}(B^k \times T^{4-k} \times \frac{1}{2}) \subset B^k \times T^{4-k} \times I$. Now c lifts to an isomorphic cobordism $\bar{c} = (\bar{W}; B^k \times T^{4-k} \times 0, \bar{f}(B^k \times T^{4-k} \times \frac{1}{2}))$. The isotopy f_t, $0 \le t \le 1$, of f fixing ∂ to the identity, lifts to an isotopy $\bar{f}_t$ of $\bar{f}$, fixing ∂, to the identity. (If $k = 0$, we may have to rechoose f_t to get $\bar{f}_1 = $ identity rather than a covering translation.) Find an integer $j > 0$ so large that $\bar{f}_t(B^k \times T^{4-k} \times (-\infty, -j])$ is disjoint from $\bar{f}(B^k \times T^{4-k} \times \frac{1}{2})$ for all $t \in [0, 1]$. The isotopy extension theorem of Lees [24] and Edwards–Kirby [9] then provides an isotopy g_t, $0 \le t \le 1$, of $\bar{f}$ such that $g_t = \bar{f}_t$ on $B^k \times T^{4-k} \times (-\infty, -j]$ and $g_t = \bar{f}$ on $B^k \times T^{4-k} \times [\frac{1}{2}, \infty)$. Then g_1 is identity on $B^k \times T^{4-k} \times -j$ and so gives a homeomorphism $B^k \times T^{4-k} \times ([-j, \frac{1}{2}]; -j, \frac{1}{2}) \to (B^k \times T^{4-k} \times [-j, 0] \cup \bar{W}; B^k \times T^{4-k} \times -j, \bar{f}(B^k \times T^{4-k} \times \frac{1}{2}))$, which is an isomorphism on all boundaries. Since the right-hand side is isomorphic to $\bar{c}$ and to $c = (W; V, V')$, (d) is established.

If $c = (W; V, V')$ defined above is a product cobordism, it is an easy matter to construct an isotopy of f fixing ∂ to make f respect $B^k \times T^{4-k} \times 0$. The isotopy is written in terms of a suitable product structure on c [see the remark below $\mathscr{C}_1(n)$ in Section 2] and a suitable collaring of ∂W in $B^k \times T^{5-k}$. I leave the reader to supply it.

We have now established:

Assertion a. *$\mathscr{C}_1(5)$ applied to a cobordism satisfying* (a),..., (d) *implies that f can be chosen to respect $B^k \times T^{4-k}$.*

So we can embark on

Step (b)*: Making f respect $B^k \times T^{3-k} \subset B^k \times T^{4-k}$ as well as $B^k \times T^{4-k}$.* Consider $f|B^k \times T^{4-k}$. Note that a homotopy of f, fixing ∂, to the identity provides one of $f|B^k \times T^{4-k}$ to the identity (by projection). We now argue as for step (a), with the following two refinements. Each time we pass to a finite covering f' of odd order of $f|B^k \times T^{4-k}$ we simultaneously pass to such a covering of f which restricts to f'. When $f|B^k \times T^{4-k}$ is isotoped to respect $B^k \times T^{3-k}$, extend the isotopy to an isotopy of f. This allows us to prove

Assertion b. *When f respects $B^k \times T^{4-k}$, the conjecture $\mathscr{C}_1(4)$ applied to a cobordism satisfying* (a),..., (d) *implies that f can be chosen to respect $B^k \times T^{3-k}$ as well as $B^k \times T^{4-k}$.*

Step (c)*: Making f respect $B^k \times T^{2-k}$ as well as $B^k \times T^{3-k}$ and $B^k \times T^{4-k}$.* Since $f|B^k \times T^{3-k}$ is homotopic, fixing ∂, to the identity one can use three-dimensional methods to find an isotopy of f fixing ∂ to the identity. However, for us it is simpler to use once again the argument for step (a). It establishes

Assertion c. *Step* (c) *can be completed by an application of $\mathscr{C}_1(3)$ to an invertible cobordism $c = (W^3; V, V')$, where $V \cong B^k \times T^{2-k}$.*

Now $\mathscr{C}_1(3)$ is known for h-cobordisms (W^3, V, V') such that W is irreducible (i.e., every locally flat 2-sphere in W^3 bounds a 3-ball) and $V \not\cong P^2(R)$ ([38] and [2]). But $W \subset B^k \times T^{2-k} \times I$ as c is invertible; so W is irreducible. Thus step (c) can always be completed.

Step (d)*: After step* (c)*, f can be made the identity on $B^k \times T^{2-k}$ by an isotopy fixing ∂.* This is elementary 2-dimensional topology. With step (d) we get the contradiction $x|T^{2-k} = \langle f \rangle | T^{2-k} = 0$ using the naturality of $\langle f \rangle$ under restriction (Corollary 3.3). Thus $\mathscr{C}_1(5)$ in Assertion a or $\mathscr{C}_1(4)$ in Assertion b does not apply, even after passage to a finite covering of odd order. This completes the proof of Theorem 1.

Since it mimics the preceding proof of Theorem 1, we now undertake

Proof of Theorem 2. Fix the DIFF or PL category. Let $k = j + 1$ be 1 or 2, and fix an element $x \in H^{2-k}(T^{5-k}; Z_2)$ with nonzero restriction to $T^{2-k} \subset T^{5-k}$. We consider automorphisms $f: B^k \times T^{5-k} \circlearrowleft$ fixing ∂ and homotopic mod ∂ to the identity, such that $\langle f \rangle = x \in H^{2-k}(T^{5-k}, Z_2)$.

Supposing $\mathscr{C}_2(5)$ and $\mathscr{C}_2(4)$ we shall be able to carry out steps (a), (b), (c), and (d) of the preceding proof to get the contradiction $x|T^{2-k} = 0$.

Step (a): *Making f respect $B^k \times T^{4-k}$*. In the PL case identity B^k with $I \times B^j$, $k = j + 1$ so that f can be regarded as an automorphism of $I \times B^j \times T^{-k}$.

In the DIFF case first use an isotopy fixing ∂ to make f = identity near ∂. Then identify B^k with $(I \times B^j)'$, $k = j + 1$, where $(I \times B^j)'$ denotes $I \times B^j$ with its corners rounded. This makes of f an automorphism of $I \times B^j \times T^{5-k}$ as well as $B^k \times T^{5-k}$, just as in the PL case.

In either case, after passage to a finite covering of odd order, f is topologically isotopic fixing ∂ to the identity and one can apply $\mathscr{C}'_2(5)$ to $f: I \times (B^j \times T^{4-j}) \circlearrowleft$ to make f respect the slices $t \times (B^j \times T^{4-j})$, for $t \in I$. Note that if we set $\alpha = f$, then α satisfies conditions (a), (b), (c), and (d) of Theorem 2.

Writing $T^{5-k} = T^{4-k} \times S^1$, $S^1 = (I/\partial I)$, we arrange by passing to a finite covering of odd order that $f(B^k \times T^{4-k} \times \frac{1}{2})$ and $B^k \times T^{4-k} \times 0$ are disjoint. Their complement in $B^k \times T^{5-k}$ has two components. Let W be the closure of the one containing $B^k \times T^{4-k} \times t$ for small $t > 0$. Since f still respects slices $t \times (B^j \times T^{5-k})$ for $t \in I$, the isotopy extension theorem ([24] and [9]) shows there exists an automorphism $g: I \times (B^j \times T^{5-k}) \circlearrowleft$ preserving slices, extending $f|I \times B^j \times T^{4-k} \times \frac{1}{2}$, and fixing $I \times \partial B^j \times T^{5-k}$, $0 \times B^j \times T^{5-k}$, and $I \times B^j \times 0$. Then $f|B^k \times T^{4-k} \times [0, \frac{1}{2}]$ provides a product structure for $(W; B^k \times T^{4-k} \times 0, f(B^k \times T^{4-k} \times \frac{1}{2}))$. Now step (a) can be completed as for Theorem 1. So we have

Assertion a. $\mathscr{C}'_2(5)$ *applied to an automorphism satisfying* (a),..., (d) *implies that step* (a) *can be completed.*

Step (b): *Making f respect $B^k \times T^{3-k}$ as well as $B^k \times T^{4-k}$*. Following the argument for step (a) one gets

Assertion b. $\mathscr{C}'_2(4)$ *applied to an automorphism satisfying* (a),..., (d) *implies that step* (b) *can be completed.*

As steps (c) and (d) can always be completed, the proof of Theorem 2 is now concluded just as Theorem 1 was.

We now return to the proof of the weak pseudoisotopy conjecture in dimension ≥ 6, which we have just disproved in dimension 4 or 5.

Proof of Proposition 2.1. It consists of an induction on $r = \text{rank}(\pi_1 M_1)$ starting from the established case $r = 0$ and using the s-cobordism theorem. Choose $f: M \to S^1$ to give a surjection of fundamental groups; make f transverse to a point $*$ in S^1; and change f by surgery with 1- and 2-handles to arrange that $\pi_1(f^{-1}(*)) \cong \text{kernel}\{f_* : \pi_1 M \to \pi_1 S^1\} \cong Z^{r-1}$ by inclusion. The infinite cyclic cover M' of M induced by f from $\exp: R^1 \to S^1$ is a doubly infinite composition of copies W_i of a cobordism W from $f^{-1}(*)$ to itself. Thus $M' = \cdots \cup W_{-1} \cup W_0 \cup W_1 \cup \cdots$, where $W_i \cap W_{i+1} = V_i$ is a copy of $f^{-1}(*)$. Van Kampen's theorem shows that $\pi_1 W_1 \cong \pi_1 M' \cong Z^{r-1}$.

Let $\alpha' : I \times M' \rightleftarrows$ be the covering of $\alpha : I \times M \rightleftarrows$ that fixes $I \times \partial M' \cup 0 \times M'$. Now choose k so large that $\alpha'(I \times V_0) \cap I \times (V_{-k} \cup V_{k+1}) = \varnothing$. Passing to an odd finite covering of α we arrange that $k = 1$. Then $\alpha'(I \times V_0)$ splits $I \times \{W_0 \cup W_1 \cup W_2\}$ into two codimension-zero submanifolds X_0, X_1, which by a purely geometrical argument are h-cobordisms with initial ends $0 \times W_0$ and $0 \times (W_1 \cup W_2)$, respectively. By the s-cobordism theorem, each has a product structure extending $\alpha'|I \times V_0$ and $\mathrm{id}|I \times V_{-1}$ or $\mathrm{id}|I \times V_2$. Using $\mathscr{C}'(n)$ for $\pi_1 = Z^{r-1}$, we find an isotopy of this product structure fixing $I \times (V_{-1} \cup V_2)$ to $\mathrm{id}|W_0 \cup W_1 \cup W_2$ at least after passage to an odd finite covering. This isotopy repeated on $I \times (W_i \cup W_{i+1} \cup W_{i+2})$, $i = 0, \pm 3, \pm 6, \ldots$ deforms α' to make $\alpha'|I \times V_i =$ identity and covers an isotopy of the evident 3-fold covering α_3 of α. A second application of the inductive hypothesis to $\alpha'|I \times (W_1 \cup W_2 \cup W_3)$ gives an isotopy of some odd finite covering of α_3 to the identity. This completes the inductive proof.

All the results of Section 2 are now established.

5. Manifolds That Cannot Be Handled

The failures of $\mathscr{C}_0, \mathscr{C}_1$, and $\mathscr{C}_2$ are intimately related to

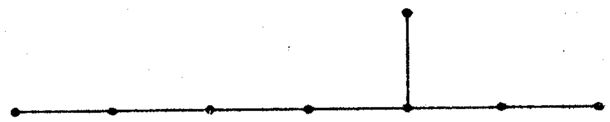
Theorem 5.1. *There exists a closed orientable manifold M of dimension 4 or 5, with $w_2 = 0$, that admits no topological handle decomposition.*

REMARK. Handle decompositions always exist for TOP manifolds of dimension ≥ 6 ([22] and [21]). Classically they exist for all DIFF and PL manifolds.

The proof of Theorem 5.1 would be clear if we knew the statement of Rohlin's theorem to be false for topological manifolds. For suppose M^4 is a counterexample. It cannot be given a PL manifold structure because it could then be given a smoothness structure by a result ($\Gamma_3 = 0$) of Smale and Munkres (see [26]). Further, M cannot have a handle decomposition because every four-dimensional handlebody admits a PL structure. This is proved by induction over handles. Consider $M_0^4 \cup H$, where M_0 has a PL manifold structure and $H = D^k \times R^{4-k}$ (where $D^k = [-1, 1]^k \subset R^k$) is an open handle attached to ∂M_0^4 by a topological embedding $h : \partial D^k \times R^{4-k} \to \partial M_0$. Think of H as a closed handle $D^k \times D^{4-k}$ with an extra collar. We want a PL structure on $M_0 \cup D^k \times D^{4-k}$. Moise has shown [24] that there is a topological isotopy h_t, $0 \leq t \leq 1$, of h fixing $\partial D^k \times (R^{4-k} - 2D^{4+k})$ to an embedding h_1 PL near $\partial D^k \times D^{4-k}$. Using this isotopy to alter the PL structure of M_0 on a collar of ∂M_0 we give $M_0 \cup (D^k \times D^{4-k})$ PL structure that coincides with the standard one on $D^k \times D^{4-k}$.

REMARK. The above argument makes Moise's results seem anomalous. However they can be obtained [21] by the same handle-straightening method [20] that has settled so many questions in dimension ≥ 5. To assure that no irreducible 3-manifolds occur among the "torus problems" encountered, it seems best to work with handles so small that their images are irreducible and use the Novikov embedding as in [23] in place of the immersion called α in [20]. Then $\mathscr{C}_0(3)$, proved in [14], should be used in constructing and solving the "torus problems." A proof of Lemma 1 of [23], now needed for their construction, can be extracted from [34, sec. 4] (or [9, sec. 8] or again [34a, secs. 5.2, 5.3]).

There is a near failure of the statement of Rohlin's theorem for TOP that yields Theorem 5.1. Let P^4 be Milnor's plumbing [3] of eight copies of the unit tangent disk bundle of S^2 according to the scheme



∂P^4 is the binary dodecahedral space, an integral homology sphere.

Let X^4 be the double $D(P^4)$ of P^4 with one of the copies of P^4 collapsed to a point $*$. This X^4 is an orientable integral homology manifold. Further, $w_2(X) = 0$ since $X - * = \operatorname{int} P$ is parallelizable, and the signature $s(X)$ of X is 8. (Contrast Rohlin's theorem.)

Proposition 5.2. *There exists a closed topological manifold W homotopy equivalent to $X^4 \times T^n$, if $n \geq 1$.*

Proof of Proposition 5.2. Deleting from X^4 and $D(P^4)$ the interior $\dot{B}$ of a smooth 4-disk, one obtains a compact homology manifold X_0^4 with boundary S^3 and a degree 1 map

$$f:(M, S^3) \to (X_0, S^3).$$

Aiming to do surgery we find a (stable, trivial) vector bundle ξ over X_0 extending $\tau(B^3_-)$, where B^3_- is the lower hemisphere of $S^3 \subset X_0$, and a (stable) bundle map $\varphi : \tau(M) \to \xi$ extending $\mathrm{id}|\tau(B^3_-)$. This is easy since $D(P^4)$ is parallelizable on the complement of a point.

Consider the problem of doing surgery on (f, φ) changing M, f, and φ (not X_0 or ξ) away from B^3_- to make f a homotopy equivalence of pairs. In view of Rohlin's theorem, anyone who can do this has disproved the Poincaré conjecture! This being undecided, multiply the problem with the n-torus T^n, $n \geq 2$. Then the surgery can be done because the total dimension is $n \geq 6$ and surgery is allowed over a piece of the boundary of the target with the same fundamental group as of the whole. This purely geometrical result [39, sec. 3.3] is a foundation stone of Wall's theory of nonsimply connected surgery.

After surgery we have a homotopy equivalence of triads

$$f:(V;\partial_- V,\partial_+ V)\to(X_0;B^3_-,B^3_+)\times T^n,$$

where $\partial V=\partial_- V\cup\partial_+ V,\partial_- V=B^3_-\times T^n$, and $g|\partial_- V=\mathrm{id}|B^3_-\times T^n$. Hence $g|\partial_+ V$ gives a homotopy smoothing of $B^3_+\times T^n\,(\mathrm{mod}\,\partial)$. As such, it is homotopic mod ∂ to a homeomorphism ($n\geq 2$) (see [18], [20], and [21]). For $n\geq 3$ this was directly proved: Apply the s-cobordism theorem to get an automorphism; then apply Lemma 4.1, using the fact that $g|\partial_+ V$ is equivalent to any of its finite coverings. (Each finite covering of g is a solution of the same surgery problem, but this solution is unique [39, sec. 3.3.1].) To define $W^{4+n}\simeq X^4\times T^n$ attach $B^4\times T^n$ to V^{4+n} by a homeomorphism of $S^3\times T^n$ homotopic to $g|\partial$. This establishes Theorem 5.1 if $n\geq 2$. To find $W^5\simeq X^4\times S^1$, start with W^{4+2} and split infinite cyclic covering using $\mathscr{C}_0(6)$ for TOP.†

REMARK. The elementary construction of W^{4+n}, $n\geq 3$, extends to the case $n=2$ as follows. We need only show that the interior of a collar neighborhood of ∂V^6 in V^6 is homeomorphic to $S^3\times T^2\times R$. A covering trick reduces this to finding a homeomorphism of $\partial V^6\times S^1$ to $S^3\times T^2\times S^1$. This last problem is solved directly, as in the body of the proof of Proposition 5.2.

To complete the proof of Theorem 5.1 we make

Assertion 5.3. *If $W^5\simeq X^4\times S^1$ of Proposition 5.2 (or even its ∞-cyclic covering $\overline{W}$), has a handle decomposition, then Rohlin's theorem fails for* TOP.

Proof of Assertion. Suppose that $\overline{W}$ has a handle decomposition, for example one covering a handle decomposition of W. Choose a compactum K so large that $\overline{W}-K$ has two unbounded components; i.e., K separates the ends of $\overline{W}$. Let H be a finite subhandlebody of $\overline{W}$ (= a union of handles of W that forms a handlebody) so large that $H\supset K$ and hence H separates the ends of $\overline{W}$.

Then clearly ∂H also separates the ends. Now use homology and Poincaré duality to check that a compact connected codimension 1 submanifold without boundary M separates the ends iff $H_{n-1}(M)\to H_{n-1}(\overline{W})$ is nonzero. Also, verify that, if no component of ∂H separated the ends, ∂H could not separate the ends. Hence some component M^4 of ∂H separates the two ends of $\overline{W}$. We will show that this M^4 violates Rohlin's theorem.

From the construction of $W^{4+n}\simeq X^4\times T^n$ it is clear that $w_2(W)=0$. (Alternatively use the Wu formulas.) Since M^4 has a trivial normal bundle in $\overline{W}$, we also have $w_2(M)=0$.

† See Remark 5.4 below.

The image of the fundamental class $[M^4] \in H_4(M^4; Q)$ is a generator μ of $H_4(\overline{W}; Q)$. Novikov observes [29] that $s(M)$ coincides with the signature of the quadratic form $\langle x \cup x, \mu \rangle$ on $H^2(\overline{W}; Q)$. As $\overline{W} \simeq X$, this signature is ± 8. Thus M^4 would violate Rohlin's theorem for TOP. This completes the proof of Theorem 5.1.

REMARK 5.4. (a) To eliminate $\mathscr{C}_0(6)$ for TOP from the proof of 5.1 and 5.2, proceed as follows. In [21], a direct argument, using 4.1 and recalled in 5.2, will be sharpened to yield a homeomorphism $\partial V^6 \approx S^3 \times T^2$. Thus an infinite cyclic covering $\overline{V}^6 \simeq X_0 \times S^1 \times R^1$ of V^6 has $\partial \overline{V}^6 \approx S^3 \times S^1 \times R^1$. Now find a closed neighborhood U^6 of (just) one end of $\overline{V}^6$, having compact smooth frontier $\mathrm{Fr}(U)$ in $\overline{V}^6$. Arguments of [33] applied almost verbatim let one alter U by finitely many embedded smooth handle surgeries along ∂U [not just $\mathrm{Fr}(U)$] until $\partial U \subsetneq U$ is a homotopy equivalence. The new ∂U is easily seen to be the interior of a compact TOP manifold $V^5 \simeq X_0 \times S^1$ with $\partial V^5 \approx S^3 \times S^1$. This improves the construction of $W^5 \simeq X \times S^1$ in 5.2. (*Added in proof.*)

(b) Alternatively, Novikov's observation can be replaced by using $\mathscr{C}_0(9)$ in TOP to express $\overline{W}^5 \times CP(2)$ as $N^8 \times R$. Then N^8 is cobordant to $M^4 \times CP(2)$, so that X, N^8, $M^4 \times CP(2)$, M^4 all have the same signature, ± 8.

Construction 5.5. In the proof of Proposition 5.2 we incidentally constructed the exotic element of $\mathscr{S}(B^3 \times T^n, \partial) \cong Z_2$, $n \geq 2$. It is straightforward to calculate its invariant as $s(X^4)/8 \pmod 2$. The target cobordism $(X_0; B^3_-, B^3_+)$ used in Proposition 5.2 can be described as $B^3 \times (I; 0, 1) \# X$ provided we trim off the interior of a small tubular neighborhood in X_0 of $S^2 = B^3_- \cap B^3_+$. Here $\#$ indicates a connected sum avoiding boundary and the singularity of X. Similarly, form $B^k \times T^{3-k} \times (I; 0, 1) \# X$. It can be used to build an element of $\mathscr{S}(B^k \times T^n, \partial)$, $k + n \geq 6$, with a prescribed invariant $x \in H^3(B^k \times T^n, \partial; Z_2)$ as follows. For each of the $\binom{n}{3-k}$ standard $(3-k)$-subtori T_i^{3-k} of T^n form a cobordism

$$\{B^k \times T^{3-k} \times (I; 0, 1) \# X_i\} \times T^{n+k-3},$$

where X_i is X or S^4 according as x evaluated on $B^k \times T_i^{3-k}$ yields 1 or 0. The two ends are naturally identified to $B^k \times T^n$, so as to send T^{3-k} to T_i^{3-k}. Now compose these cobordisms in any order to get $(Z; \partial_- Z, \partial_+ Z)$. Much as in Proposition 5.2 one can do surgery to find a compact smooth $(n+k)$-manifold V and a homotopy equivalence $f: (V, \partial V) \to (Z, \partial Z)$ that gives a homotopy smoothing $f: \partial_+ V = f^{-1}\partial_+ Z \to \partial_+ Z \cong B^k \times T^n \pmod \partial$ and a diffeomorphism elsewhere on ∂V. I leave the reader to check that $[f | \partial_+ V] = x$.

DISCUSSION. I have left open the question whether the misbehavior of handlebody theory demonstrated in this article occurs in dimension 4 or 5 or both. It deserves attention.

When this has been answered one should inquire precisely which lemmas of smooth and PL handlebody theory fail.

Most of the results of this article would be neatly explained if the following two conjectures hold:

(I) $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$ are all valid for TOP manifolds in all dimensions.

(II) Isotopy classes of PL structures on any topological manifold M are in bijective correspondence with equivalence classes of nonstable reductions of their tangent microbundle to PL microbundle. (This makes sense for manifolds with boundary.)

Conjecture (I) is perhaps too optimistic, as it includes the classical Poincaré conjecture. Conjecture (II) is known ([20] and [27]) if $\dim(M)$ and $\dim(\partial M)$ are $\neq 4$. R. Lashof has nearly proved it for open connected 4-manifolds (these proceedings). See also Morlet [28].

References

1. H. Bass, A. Heller, and R. Swan, *The Whitehead group of a polynomial extension*, Publ. Inst. Hautes Etudes Sci. 22 (1964).
2. E. M. Brown, *Unknotting in $M^2 \times I$*, Trans. Amer. Math. Soc. 123 (1966), 480–505.
3. W. Browder, *Surgery on simply connected manifolds* (to be published in book form).
4. W. Browder, *Structures on $M \times R$*, Proc. Cambridge Phil. Soc. 61 (1965), 337–345.
5. J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$) (Springer-Verlag Lecture Notes, Vol. 53), Springer, Berlin, 1968.
6. J. Cerf, *Isotopie et pseudo-isotopie*, Proceedings of the International Congress of Mathematics, Moscow, 1966.
7. J. Cerf (to appear in Publ. Inst. Hautes Etudes Sci.).
8. A. V. Černavskiĭ, *Local contractibility of the homeomorphism group of a manifold*, Doklady 9 (1968), 1171–1174.
9. R. D. Edwards and R. C. Kirby, *Deformations of spaces of embeddings*, Ann. of Math. (to appear).
10. F. T. Farrell, *The obstruction to fibering a manifold over a circle*, Thesis, Yale University, 1967 (available from University Microfilms, Ann Arbor, Mich.).
11. M. W. Hirsch, *On smoothing manifolds and maps*, Bull. Amer. Math. Soc. 69 (1963), 352–356.
12. W.-C. Hsiang and J. L. Shaneson, *Fake tori, the annulus conjecture and the conjectures of Kirby*, Proc. Nat. Acad. Sci. U.S. 62 (1969), 687–691.

12a. W.-C. Hsiang and J. L. Shaneson, *Fake tori*, these proceedings.

13. J. F. P. Hudson, *Piecewise Linear Topology*, W. A. Benjamin, New York, 1969.
14. L. S. Husch and T. Price, *Finding a boundary for a 3-manifold*, Ann. of Math. (to appear).

15. M. Kervaire an… *Rohlin*, Proceed… 1958, pp. 454–4…
16. M. Kervaire an… 47 (1961), 1651-…
17. M. Kervaire, *L…* (1965), 31–42.
18. R. C. Kirby, *Le…* California at L…
19. R. C. Kirby an… *triangulation co…*
20. R. C. Kirby ar… *Hauptvermutun…*

21a. R. C. Kirby a… *manifold struct…*

21b. R. C. Kirby an… *manifold struct…*

21c. R. C. Kirby a… preparation).

22. R. C. Kirby and … Soc. 16 (1969), 8…
23. R. Lashof and … Soc. 75 (1969), 7…
24. J. Lees, *Immersi…* 75 (1969), 529–5…
25. J. Milnor, *Lectu…* informal course …
26. J. Munkres, *Con…* 183–191.
27. E. Moise, *Affine…*
28. C. Morlet, *Haupt…* p. 362.
29. S. P. Novikov, … *Pontrjagin*, Sov. …
30. V. A. Rohlin, *A…* (1952), 221–224.
31. C. P. Rourke, *Em…*
32. J. L. Shaneson, … $S^1 \times S^3$, Bull. A…
33. L. C. Siebenman… *dimension greate…* University Micro…
34. L. C. Siebenmann…

34a. L. C. Siebenmar… 45 (1970), 1-48.

15. M. Kervaire and J. Milnor, *Bernoulli numbers, homotopy groups and a theorem of Rohlin*, Proceedings of the International Congress of Mathematicians, Edinburgh, 1958, pp. 454–458.
16. M. Kervaire and J. Milnor, *On 2-spheres in 4-manifolds*, Proc. Nat. Acad. Sci. U.S. 47 (1961), 1651–1657.
17. M. Kervaire, *Le théoreme de Barden–Mazur–Stallings*, Comment. Math. Helv. 40 (1965), 31–42.
18. R. C. Kirby, *Lectures on triangulation of manifolds* (mimeographed), University of California at Los Angeles, 1969.
19. R. C. Kirby and L. C. Siebenmann, *For manifolds the Hauptvermutung and the triangulation conjecture are false*, Notices Amer. Math. Soc. 16 (1969), 695.
20. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. 75 (1969), 742–749.
21a. R. C. Kirby and L. C. Siebenmann, *Deformation of smooth and piecewise-linear manifold structures*, Ann. of Math. (to appear).
21b. R. C. Kirby and L. C. Siebenmann, *Classification of smooth and piecewise-linear manifold structures* (to appear).
21c. R. C. Kirby and L. C. Siebenmann, sequel to [20] containing applications (in preparation).
22. R. C. Kirby and L. C. Siebenmann, *Foundations of topology*, Notices Amer. Math. Soc. 16 (1969), 848.
23. R. Lashof and M. Rothenberg, *Triangulation of manifolds, I*, Bull. Amer. Math. Soc. 75 (1969), 750–754.
24. J. Lees, *Immersions and surgeries on topological manifolds*, Bull. Amer. Math. Soc. 75 (1969), 529–534.
25. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University, preliminary informal course notes, 1965.
26. J. Munkres, *Concordance of differentiable structures*, Michigan Math. J. 14 (1967), 183–191.
27. E. Moise, *Affine structures on 3-manifolds*, Ann. of Math. 56 (1952), 96–114.
28. C. Morlet, *Hauptvermutung et triangulation des variétés*, Sém. Bourbaki, 1968–1969, p. 362.
29. S. P. Novikov, *Homotopic and topological invariance of certain rational classes of Pontrjagin*, Sov. Math. Doklady 6 (1965), 854–857.
30. V. A. Rohlin, *A new result in the theory of 4-dimensional manifolds*, Doklady 8 (1952), 221–224.
31. C. P. Rourke, *Embedded handle theory, concordance and isotopy*, these proceedings.
32. J. L. Shaneson, *Embeddings of spheres in codimension two and h-cobordisms of* $S^1 \times S^3$, Bull. Amer. Math. Soc. 74 (1968), 972–974.
33. L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Thesis, Princeton University, 1965 (available from University Microfilms, Ann Arbor, Mich.); to appear in revised version.
34. L. C. Siebenmann, *Invertible cobordisms*, Arch. Math. 19 (1968), 528–535.
34a. L. C. Siebenmann, *A total Whitehead torsion obstruction*, Comment. Math. Helv. 45 (1970), 1–48.

35. L. C. Siebenmann, *The structure of tame ends*, Notices Amer. Math. Soc. 13 (1966), 862.
36. L. C. Siebenmann, *Torsion invariants for pseudo-isotopes on closed manifolds*, Notices Amer. Math. Soc. 14 (1967), 942.
37. L. C. Siebenmann, *On the homotopy type of compact topological manifolds*, Bull. Amer. Math. Soc. 74 (1968), 738–742.
38. J. Stallings, *On fibering certain 3-manifolds*, in *Topology of 3-Manifolds and Related Topics* (edited by M. K. Fort, Jr.), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 95–100.
39. C. T. C. Wall, *Finiteness conditions for CW complexes II*, Proc. Roy. Soc. A295 (1966), 129–139.
40. C. T. C. Wall, *Topology of smooth manifolds*, J. London Math. Soc. 40 (1965), 1–20.
41. C. T. C. Wall, *Surgery of compact manifolds*, mimeographed notes, University of Liverpool (to appear in book form).
42. C. T. C. Wall, *On homotopy tori and the annulus theorem*, Bull. London Math. Soc. 1 (1969).
43. J. R. L. Williamson, *Cobordisms of combinatorial manifolds*, Ann. of Math. 83 (1966), 1–33.
44. H. Poincaré, *Cinquième complément à l'analysis situs*, Rendiconti del Circolo matematico di Palermo 18 (1904), 45–110; also *Oeuvres de H. Poincaré*, Vol. VI, Gauthier-Villars, Paris, 1953, pp. 435–498.
45. M. A. Armstrong, *Collars and concordances of topological manifolds*, Comment. Math. Helv. 45 (1970), 119–128.

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