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ARE NONTRIANGULABLE MANIFOLDS TRIANGULABLE?

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The awkward title refers to the following awkward situation. R. Kirby and the author [4, 5] have shown that in each dimension ≥ 5 there exist closed topological manifolds that admit no piecewise linear† manifold structure (= a piecewise-linearly compatible atlas of charts to euclidean space), and hence admit no triangulation as a combinatorial manifold (= a simplicial complex in which the star of each vertex is piecewise linearly homeomorphic to a simplex). Is it, however, possible that some, or perhaps every, such topological (metrizable) manifold can be triangulated as a simplicial complex?

This note exhibits strong connections of this question with the classical Poincaré conjecture (that every closed manifold homotopy equivalent to S^3 is homeomorphic to S^3) and the conjecture that the suspension $\Sigma^{n-3}M^3$ of a 3-manifold cannot be homeomorphic to S^n if $\pi_1 M \neq 1$. The latter has been studied by Glaser [2].

Theorem A (Generalizing [1]). *If M^3 is a homotopy 3-sphere, then the double suspension $\Sigma^2 M^3$ of M^3 is homeomorphic to S^5 by a homeomorphism sending the suspension circle to $S^1 \subset S^5$.*

Proof. For any space X, $\Sigma^2 X$ is the join $X * S^1$ and so can be expressed as the quotient of $X \times D^2$ under identification of $X \times \partial D^2$ to $\partial D^2 = S^1$ by projection. Let $r: R^2 \to \operatorname{int} D^2 = \{x \in R^2 : \|x\| < 1\}$ be a ray-preserving homeomorphism.

For any homotopy 3-sphere M^3, $M^3 \times T^2 = M^3 \times S^1 \times S^1$ is homeomorphic to $S^3 \times T^2$, in virtue of being homotopy equivalent to $S^3 \times T^2$. This is proved in [4] and [5].

† [15] is a reference for piecewise linear topology.

Let $H : M^3 \times T^2 \to S^3 \times T^2$ be a homeomorphism so that $h_* : \pi_1(M^3 \times T^2) \to \pi_1(S^3 \times T^2)$ commutes with projection to $\pi_1(T^2)$. Then any homeomorphism $h' : M^3 \times R^2 \to S^3 \times R^2$ covering h satisfies

$$\|p_2(x) - p_2h'(x)\| < \text{constant},$$

for all $x \in M^3 \times R^2$. Here p_2 denotes projection to R^2. Hence

$$f_0 = \{(\mathrm{id}|S^3) \times r\} \circ h \circ \{(\mathrm{id}|M) \times r^{-1}\}$$

$$f_0 : M^3 \times \operatorname{int} D^2 \to S^3 \times \operatorname{int} D^2$$

extends to a homeomorphism $f : M^3 * S^1 \to S^3 * S^1 = S^5$ that is the identity on the suspension circle. (This is a version of a key argument in [4].)

REMARK. If M^3 is the boundary of a contractible (PL) manifold W^4, then $W^4 \times T^2$ is (PL) homeomorphic to $D^4 \times T^2$ by the s-cobordism theorem, and a homeomorphism $M^3 \times T^2 \approx S^3 \times T^2$ follows without appeal to [4] and [5].

For a smooth homology 3-sphere M (= a closed smooth 3-manifold with $H_*(M; Z) \simeq H_*(S^3; Z)$) there is the following interesting invariant in Z_2 for which no algorithm is known in terms of a handle decomposition of M. Since the J-homeomorphism

$$\pi_3 SO \to Z_{24} = \pi_3 G = \pi_{3+k}(S^k), \qquad k \text{ large},$$

is onto, M is the boundary of a parallelizable smooth oriented compact 4-manifold P^4. The signature $s(P^4)$ of P^4 is always divisible by 8. It is divisible by 16 if M is diffeomorphic to S^3 by a theorem of Rohlin [6]. Thus $s(P^4)/8$ (mod 2) is an invariant of M. Call it $\alpha(M)$. Since the smooth, piecewise linear (PL), and topological (TOP) classifications of 3-manifolds coincide, α is a topological invariant defined for any homology 3-sphere.

Lemma 1. *If M is a homotopy 3-sphere, then $M \times T^n$, $n > 2$, is PL homeomorphic to $S^3 \times T^n$ if and only if $\alpha(M) = 0$.*

Proof. Once M is oriented we have a natural homotopy equivalence $f : M \times T^n \to S^3 \times T^n$. In Section 3 of [8] is given an invariant $\theta_{S^3}(f) \in Z_2$ of the class of f on the set $\mathscr{S}(S^3 \times T^n)$ of homotopy triangulations of $S^3 \times T^n$. It is easily seen that $\alpha(M) = \theta_{S^3}(f)$. If $\theta_{S^3}(f) = 0$, f is homotopic to a PL homeomorphism by a result of C. T. C. Wall, and W.-C. Hsiang and J. Shaneson. See the references in [8].

Suppose there exists a PL homeomorphism $g : M \times T^n \to S^3 \times T^n$. Then $f' = f \circ g^{-1} : S^3 \times T^n \to S^3 \times T^n$ has zero normal invariant in $[S^3 \times T^n; G/PL]$ since f has. This implies that f', and hence f, is homotopic to a PL homeomorphism as follows. It suffices to show this in case f' fixes $\pi_1(T^n) = Z^n$, and preserves orientation. Since $T^n = K(Z^n, 1)$, the component of f' on T^n then is homotopic to the projection. The component of f on S^3 is a map

$h : T^n \to G_4$. Consider the fibration sequence

$$\tilde{\mathrm{P}}\mathrm{L}_4 \to G_4 \to G_4/\tilde{\mathrm{P}}\mathrm{L}_4 \simeq G/\mathrm{PL}.$$

Recall that $G_4/\tilde{\mathrm{P}}\mathrm{L}_4 \simeq G/\mathrm{PL}$ by stabilization (see [7]). Since f' has normal invariant zero, h becomes contractible in G/PL and hence is already contractible in $G_4/\tilde{\mathrm{P}}\mathrm{L}_4$. Thus h lifts to $\tilde{\mathrm{P}}\mathrm{L}_4$, which shows that f' can be realized by a PL automorphism of $S^3 \times T^n$, in fact one extending over $D^4 \times T^n$.

Lemma 2. *If $\alpha(M) \neq 0$, then $M^3 \times R^2$ is not* PL *homeomorphic to $S^3 \times R^2$.*

Proof. Such a PL homeomorphism would easily imply that $M^3 \times S^1$ is h-cobordant to $S^3 \times S^1$. Hence $M^3 \times T^2$ would be PL homeomorphic to $S^3 \times T^2$ by the s-cobordism theorem. Hence $\alpha(M) = 0$ by Lemma 1, a contradiction.

Consider the following hypothesis (H). It asserts that the classical Poincaré conjecture fails dramatically.

(H) *There exists a homotopy 3-sphere M that bounds a compact smooth parallelizable 4-manifold with signature ± 8.*

Theorem B. *If* (H) *holds, every connected orientable* (metrizable) *topological 5-manifold W without boundary can be triangulated as a simplicial complex.*

The largest part of the proof is

Lemma 3. *Supposing* (H), *one can find a polyhedral structure Σ on $S^1 \times R^4$ (hence a homeomorphism with a simplicial complex) so that the points near which $(S^1 \times R^4)_\Sigma$ is not a* PL *manifold form a circle C homologous to $S^1 \times 0$, and $\Sigma | S^1 \times R^4 - C$ is not isotopic to the standard* PL *manifold structure, i.e., corresponds to the nonzero element of*

$$H^3(S^1 \times R^4 - C; Z_2) = Z_2.$$

NOTATIONS. Write $cX = [0, 1] \times X/0 \times X$ *for the cone on X and* $c'X = [0, 1) \times X/0 \times X$ *for the open cone on X*. The symbol $\cong$ denotes PL homeomorphism, and $\approx$ denotes ordinary homeomorphism.

Proof of Lemma 3. Let M^3 be a combinatorial homotopy 3-sphere provided by (H). Note that $c'M^3$ has a natural triangulation making $c'M - *$ (where $*$ = cone point) a PL manifold.

Now $S^1 \times c'M^3$ is a subset of $S^1 * M^3$ which the homeomorphism of Theorem A sends onto a neighborhood of S^1 in S^5. By engulfing we deduce a homeomorphism

$$\theta : S^1 \times c'M^3 \to S^1 \times R^4.$$

Let θ impose a (polyhedral) PL structure Σ on $S^1 \times R^4$. Let C be the circle $\theta(S^1 \times *)$, which we can (and do) arrange to coincide with $S^1 \times 0$.

Then $(S^1 \times R^4 - C)_\Sigma$ is not even PL homeomorphic to $S^1 \times R^4 - C \cong S^1 \times S^3 \times R$ since a PL universal covering of it is $R^1 \times (c'M^3 - *) \cong R^2 \times M^3$, which is not PL homeomorphic to $R^2 \times S^3$ by Lemma 2. This completes the proof of Lemma 3.

Proof of Theorem B. (a) *Compact case.* We can assume that the single obstruction ([4] and [5]) $k(W) \in H^4(W; Z_2)$ to imposing a PL manifold structure on W is nonzero. Represent the Poincaré dual of $k(W)$ by a map of a circle S into W. The map can be a locally flat embedding. This follows from Homma's method [3]; alternatively, one could use the results of [4].

A neighborhood of S is triangulable as a PL manifold, there being no obstruction to this, and we can assume S is PL embedded in it. Then S has an open tubular neighborhood in W. Since W is orientable we can identify this neighborhood with $S^1 \times R^4$ sending S to $S^1 \times O$.

Now consider $(S^1 \times R^4)_\Sigma$ and the contained circle C provided by Lemma 3. $W^5 - C$ admits a PL manifold structure σ since the obstruction to this, the restriction of $k(W)$, is zero.

Now $H^3(S^1 \times R^4 - C; Z_2) = Z_2$ implies [4, 5] that $S^1 \times R^4 - C$ has exactly two concordance or isotopy classes of PL structures. But $\sigma | S^1 \times R^4 - C$ is not of the standard class; else σ would extend over W. Nor is $\Sigma | S^1 \times R^4 - C$ by Lemma 3. Hence $\sigma | S^1 \times R^4 - C$ and $\Sigma | S^1 \times R^4 - C$ are ($\varepsilon -$) isotopic. This shows that there exists a small isotopy of σ after which σ and Σ agree on $S^1 \times R^4 - C$. This means that W has a (polyhedral) PL structure, and hence is triangulable as a simplicial complex.

(b) *Noncompact case.* The obstruction $k(W) \in H^4(W; Z_2)$ has its Poincaré dual $\mathrm{D}k(W)$ in $H_1^{\mathrm{LC}}(W; Z_2)$, where H^{LC} here indicates homology based on locally finite but possibly infinite singular chains. Every 1-dimensional homology class is represented by a proper map into W of a combinatorial 1-manifold S, a countable union of circles and lines. The circles can be replaced by lines with the help of a locally finite family of paths in W one from each circle to ∞.

As under (a) we can arrange that S is locally flatly embedded and equipped with tubular neighborhood $S \times R^4 \subset W$. And $W - S$ can be given a PL manifold structure σ. The rest of the proof continues to imitate (a) using the universal covering of the structure Σ provided by Lemma 3.

REMARK. Orientability was not used in the noncompact case. However, in the compact nonorientable case S may be orientation reversing. One could deal with this if some M^3 provided by (H) had an orientation-reversing homeomorphism, or more generally if $M \# M$ (M oriented) could bound a contractible manifold.

Added March 1970: It is now thought that the arguments of [2] are not decisive and leave open, for example, the possibility

(H′) *There exists a (integral) homology 3-sphere M such that $\pi_1 M \neq 0$, $\Sigma^2 M$ is homeomorphic to S^5, and M bounds a compact smooth parallelizable 4-manifold with signature ± 8.*

Thus Theorem B should be strengthened by

Assertion. *Theorem* B *is valid assuming* (H′) *in place of* (H).

To justify this it clearly suffices to give

Proof of Lemma 3 *assuming* (H′) *in place of* (H). We will lean on arguments given in Theorems C and A′ below.

Let M^3 be as provided by (H′). Then $\Sigma^2 M$ homeomorphic to S^5 implies that $S^1 \times cM$, $S^1 \times \Sigma M$, and $R \times \Sigma M$ are topological manifolds. Thus $R \times \Sigma M \approx R \times S^4$ ($\approx$ denoting homeomorphism) by an argument recalled in the proof of A′ below. This implies $S^1 \times \Sigma M \approx S^1 \times S^4$ (see [14]).

Let $p \in \Sigma M$ be a point distinct from the suspension points. Then $S^1 \times (\Sigma M - p) \approx S^1 \times (S^4 - \infty) \approx S^1 \times R^4$ since homotopic locally flat circles in $S^1 \times S^4$ are ambient isotopic [3]. We deduce a composed embedding h,

$$h : S^1 \times cM \subsetneq S^1 \times (\Sigma M - p) \xrightarrow{\approx} S^1 \times R^4.$$

Define $C = h(S^1 \times *)$ where $*$ = cone point. Let M be triangulated and define Σ on $h(S^1 \times cM)$ to make h a PL embedding. Since $H^*(S^1 \times R^4, C) = 0$, there is no obstruction to extending over $S^1 \times R^4 - C$ the PL manifold structure given by Σ near C in $S^1 \times R^4 - C$. Thus Σ is extended over all of $S^1 \times R^4$ and it remains only to check that $\Sigma | S^1 \times R^4 - C$ is not the standard PL manifold structure. But this is implied by the last assertion in the proof of C below.

The converse to Theorem B can be proved in complete generality. I believe it is known to M. Cohen and D. Sullivan.

Theorem C. *If a topological manifold W^n (without boundary), $n \geq 5$, is triangulated as a simplicial complex, but admits no* PL *manifold structure, then there exists a homology 3-sphere M^3 with $\alpha(M^3) = 1$ such that the suspension $\Sigma^{n-3} M^3$ is homeomorphic to S^n.*

Proof. The triangulation of W gives a PL manifold structure to the complement of the $(n-k)$-skeleton $W^{(n-k)}$, $k \leq 4$. This is because the link of a $(n-k)$-simplex in the first barycentric derived subdivision W' has the homology of S^{k-1}. So one can check inductively that the links are PL spheres until $k = 4$, when they will be combinatorial manifolds that are homology 3-spheres, but perhaps not 3-spheres.

Let σ denote the natural PL structure on $W - W^{(n-4)}$. Fix an open $(n-4)$-simplex $A \subset W^{(n-4)}$.

Assertion. *The structure σ can be extended over A if and only if $\alpha(M) = 0$, where M is the link of A in W'.*

The assertion implies the theorem. Supposing $\alpha(M) = 0$ for every link M of every $(n-4)$-simplex A, we can extend σ over $W - W^{(n-5)}$. Then the obstruction to further extending σ over W lies in $H^4(W, W - W^{(n-5)}; Z_2) = 0$. So W has a PL manifold structure, against hypothesis.

Proof of Assertion. Since the star of the closed simplex $\bar{A}$ in W' is $\bar{A} * M$, the open star of A in W' is PL homeomorphic to $R^{n-4} \times c'M$, where $c'M$ denotes then open cone on M, and in such a way that A corresponds to $R^{n-4} \times \{*\}$ ($*$ = cone point). This shows, incidentally, that $\Sigma^{n-3}M^3$ is a manifold. As it is homotopy equivalent to S^n, $n \geq 5$, it is homeomorphic to S^n.

The assertion can now be restated as

Assertion. *The natural* PL *manifold structure σ on $R^{n-4} \times \{c'M - *\}$ extends over $R^{n-4} \times c'M$ iff $\alpha(M) = 0$.*

Let P^4 be a compact smooth parallelizable 4-manifold with boundary M, and consider $X = P^4 \cup c(M)$, where the base of the cone is identified with $M = \partial P$.

Since the obstruction to extending σ is the primary obstruction $k(R^{n-4} \times X) \in H^4(X; Z_2) = Z_2$, to imposing a PL manifold structure on $R^{n-4} \times X$, the assertion follows from

Lemma 4. $k(R^{n-4} \times X) = s(X)/8 \pmod 2$.

Proof. Identify R^{n-4} with an open disk in T^{n-4} about $O \in T^{n-4}$. There is an evident tangent topological microbundle map $\tau(T^{n-4} \times X) \to \tau(T^{n-4} \times X)|O \times X$ (even if X is not a manifold). The target is microidentical to $\tau(R^{n-4} \times X)|O \times X$. Since k is the obstruction to lifting a tangent bundle classifying map to B_{TOP} up to B_{PL}, it follows that $k(T^{n-4} \times X) \in H^4(T^{n-4} \times X; Z_2)$ is the pullback by projection to X of $k(R^n \times X) \in H^4(X; Z_2)$. In particular, the one obstruction vanishes iff the other does.

Let X_0 be X with the interior of a PL disk in $\mathrm{int}(P^4) \subset X$ removed. We identify the boundary of the disk with S^3. Split S^3 into hemispheres B^3_- and B^3_+. Now there is no obstruction to extending the standard structure on $R^{n-4} \times B^3_-$ to all of $R^{n-4} \times X_0$. The paragraph above shows that the same holds with T^{n-4} in place of R^{n-4}. Write σ for such a structure on $T^{n-4} \times X$.

Restricted to the boundary, σ provides a homotopy triangulation modulo boundary

$$f = \text{"identity"} : (T^{n-4} \times B^3_+)_\sigma \to T^{n-4} \times B^3_+.$$

The invariant $[f] \in Z_2$ of the class of f in $\mathcal{S}(T^{n-4} \times B^3_+, \partial)$ as described in [8, sec. 3] is clearly $s(X)/8 \pmod 2$. So now it suffices to show that σ can be extended over $T^{n-4} \times X$ iff $[f] = 0$. Extendability easily implies that $[f] = 0$ [8, sec. 3]. On the other hand, $[f] = 0$ implies that $\sigma/T^{n-4} \times S^3$ is isotopic to the standard structure [5] and hence σ extends over $T^{n-4} \times X$.

This completes the proof of Lemma 4 and, with it, the proof of Theorem C.

The following theorem is a corollary of Theorem A that may prove worthwhile. It was suggested to me by results of M. Cohen [9] and by efforts of L. C. Glaser to prove the

Conjecture (Generalizing one studied in [2]). *In any topological manifold (without boundary) that is triangulated as a simplicial complex, the link of each simplex is homotopy equivalent to a sphere of appropriate dimension.*

Theorem A′ (Added November 1969). *Let X be a connected simplicial homotopy n-manifold,* i.e., *a connected simplicial complex such that the link of each j-simplex is homotopy equivalent to an $(n-j-1)$-sphere. Then X is a topological manifold provided that the dimension of X is not* 4.

Proof of A′. Observe first that the link M^k of each $(n-k-1)$-sphere σ is itself a homotopy manifold (of dimension k). In fact, the link in M^k of a simplex τ of M^k coincides with the link in X^n of the simplex which is the join $\sigma * \tau$.

Since the star of σ in X^n is the join $\sigma * M^k$, an open neighborhood of the (open) simplex σ in X is PL homeomorphic to $c'(M^k) \times R^{n-k-1}$, where, as usual, $c'(M^k)$ means the open cone on M^k. Clearly such open sets cover X^n.

As noted for Theorem C, M^k is PL homeomorphic to S^k if $k = 0, 1$, or 2. And M^3 is a PL manifold so that $c'(M^3) \times R \approx R^5$ by Theorem A ($\approx$ denoting homeomorphism). This shows, for one thing, that, for $k = 4$, $M^4 \times R$ is a topological manifold.

Assertion. $M^4 \times R \approx S^4 \times R$; *hence* $c'(M^4) \approx R^5$.

For a proof observe that (M^4-point) $\times R \approx R^5$ by topological engulfing [12]. Then note that the six-line argument under 4.5 in [1] reduces the assertion to the fact that isolated nonlocally flat points of codimension-one embeddings do not exist [11]. See also [13].

Inductively assume Theorem A′ for dimensions $< n$ and ≥ 5, where $n \geq 5$. (A' is trivial for $n \leq 3$.) Then for each simplex σ of X, the link M^k of σ in X satisfies

$$c'(M^k) \times R^{n-k-1} \approx R^n$$

—trivially for $k = 0, 1, 2$; by $c'(M^3) \times R \approx R^5$ for $k = 3$; by $c'(M^4) \approx R^5$ for $k = 4$; and by a homeomorphism $M^k \approx S^k$ for $5 \leq k \leq n-1$. The last homeomorphism comes from the topological Poincaré theorem of Newman [12], since by inductive hypothesis each M^k, $5 \leq k \leq n-1$, is a topological manifold. Thus X^n is covered by copies of R^n, as required.

Complement to Theorem A′. *If X^n is allowed a formal boundary* [i.e., a subcomplex that is a simplicial homotopy $(n-1)$-manifold without formal boundary, consisting of simplices whose links in X^n are contractible rather than homotopy equivalent to a sphere], *then X^n is a topological n-manifold* *with boundary, provided that $n \neq 4, 5$, or $n = 5$ and the formal boundary is known to be a topological manifold.*

The proof is similar and uses the fact that $\Sigma^2 M^3 \approx D^5$ *for every homotopy 3-disk*, which is proved like Theorem A.

M. Cohen gives some results related to the above in [10].

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