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On Detecting Euclidean Space Homotopically among Topological Manifolds

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§ 1. Introduction

It is known that there exist uncountably many topologically distinct contractible open topological manifolds of any given dimension $\geqq 3$ [4]. Our problem is to give a supplementary homotopy theoretic condition which guarantees that a contractible open topological manifold is euclidean space.

Main Theorem 1.1. *Suppose M^n is a contractible metrizable topological n-manifold without boundary, which is 1-LC at infinity. If $n \geqq 5$, then M^n is homeomorphic to euclidean n-space R^n.*

A generalization of this result is proved in § 4. It attempts (with only limited success) to characterize homotopically topological open collar neighborhoods.

The condition 1-*LC* at ∞ is a weak form of the condition (*C*) that there be arbitrarily large compacta K in M so that every map of a circle into $M-K$ is contractible to a point in $M-K$.

Definitions 1.2. A *neighborhood of infinity* (∞) in a Hausdorff space X is a subset N such that the closure of $X-N$ is compact. X is said to be 1-*LC at* ∞ (1-*locally connected at infinity*) if for any neighborhood U of ∞ there exists a smaller neighborhood V of ∞ such that any continuous map of the circle into V is contractible in U.

The reader can easily check that the property of being 1-*LC* at ∞ is invariant under proper homotopy equivalence[1]. Hence Theorem 1.1 qualifies as a *homotopy theoretic* characterization of euclidean space of dimension $\geqq 5$ among all topological manifolds. It is unknown whether proper homotopy equivalence of topological manifolds preserves the stronger condition (*C*) mentioned above.

Luft [6] treated Theorem 1.1 under the assumption of (*C*). The following amusing corollary cannot be proved using Luft's version.

[1] i.e. homotopy equivalence in the category of proper continuous maps. A map is proper if the preimage of each compact set is compact.

Corollary 1.3. *Let M^n, $n \geqq 5$, be an oriented metrizable topological n-manifold without boundary. Suppose there exists a proper degree 1 map $f: R^n \to M^n$. Then M^n is homeomorphic to R^n.*

This result has been proved by Epstein and the author for the differentiable and piecewise linear categories. Here the proof is similar. It uses Theorem 1.1 and the fact that any proper degree 1 map induces a surjection of homology groups and of fundamental group. For details see [9, § 2.7].

Note that Corollary 1.3 includes

Corollary 1.4. *Let M^n be as in* 1.3. *Suppose M^n is proper homotopy equivalent to R^n. Then M^n is homeomorphic to R^n.*

It is conceivable that metrizable topological manifolds are all triangulable as piecewise linear manifolds, for which all the above results were established in [13] and [9]. If so, may this article represent a small step towards a proof of triangulability!

The remainder of this paper is organized as follows: § 2 proves an engulfing lemma; § 3 uses it to prove Theorem 1.1; § 4 generalizes Theorem 1.1; and § 5 establishes a general position lemma used in § 2. A proof of Theorem 1.1 under the extra hypothesis (C) is contained in the brief § 3 alone — i.e. *it doesn't require the result of* § 2 (see Remark 3.2).

This article was written in spring 1967 after I was able to circumvent a fault in a manuscript version of Luft's article [6]. It was revived (in spite of the incompleteness of my results in § 4) when Newman pointed out that a fault persists in Luft's published version. In the proof of Theorem 3.2 of [6] (= my Theorem 1.1 with condition (C)) the property (c) on p. 197 of [6] generally cannot be satisfied. By work independent of Luft or me, Newman [17] had constructed a (rather complicated) proof of Theorem 1.1 with condition (C).

Here is some terminology we will employ. The *support* of a map $h: M \to M$ of a space M to itself is the closure in M of $\{x \in M;\, h(x) \neq x\}$. An *automorphism* of M is a homeomorphism of M onto itself. A *compact automorphism* of M is an automorphism with compact support.

§ 2. Two Set Engulfing of a 2-Complex

This section is devoted to an engulfing lemma that allows us to work with manifolds 1-LC at infinity.

Proposition 2.1. *Let M^n be a metrizable topological n-manifold, $n \geqq 5$, without boundary and $O_0 \subset M^n$ an open set equipped with a p.l. manifold structure. Let $P \subset O_0$ be a possibly infinite subpolyhedron of O_0 closed in M and of dimension $\leqq 2$. Let V and U be connected open subsets of M with $V \subset U$ such that $P - V$ is compact and*

(a) $\pi_1(M, V)=0$.

(b) $\pi_2(U, V)$ *maps (by inclusion) onto* $\pi_2(M, V)$.

Then there exists a homeomorphism h *of* M *onto itself such that* $h(U)\supset P$ *and* h *fixes all points outside some compact set.*

This result should be a prototype of a general theorem for engulfing with a nest $U\supset V\supset W\supset\cdots$ of many open sets instead of with two open sets $U\supset V$ as in this proposition or with one open set as in ordinary engulfing [14]. Such a theorem exists in p.l. manifolds, and can replace engulfing with U when (M, U) is not sufficiently connected to apply Stallings' engulfing theorem. See [7] where a special case is proved.

The proof of Proposition 2.1 requires a re-examination of the proof of the topological engulfing theorem. Since two expositions of this exist already [8, 3] we chose to break the proof (unnecessarily) into three parts of increasing difficult: $n\geqq 7$, $n=6$, $n=5$. The only unfamiliar idea in the proof appears already for $n\geqq 7$ so that the reader familiar with engulfing may be content with this case. Again, this case is relatively simple and so should be instructive for a reader who wants to understand topological engulfing.

Proof of Proposition 2.1. To begin, allow any $n\geqq 5$. Let $P_0\subset P$ be a closed subpolyhedron such that $P_0\subset V$ but $Q=\text{closure}(P-P_0)$ is compact. Let $\delta Q=Q\cap P_0$.

Since $\pi_1(M, V)=1$ and $\pi_2(U, V)\xrightarrow{\text{onto}}\pi_2(M, V)$ there is a deformation of $P\hookrightarrow M$ fixing P_0 to a map into U. This can be regarded as a map

$$f\colon\ P\times 0\cup Q\times I\to M,\qquad I=[0,1],$$

such that $f|P\times 0$ gives the inclusion $P\hookrightarrow M$, $f|\delta Q\times I$ is $\delta Q\times I\to\delta Q\hookrightarrow M$, and $f(Q\times 1)\subset U$. Note that $f(X)\subset U$ where $X=P_0\times 0\cup\delta Q\times I\cup Q\times 1$ (cf. Fig. 1).

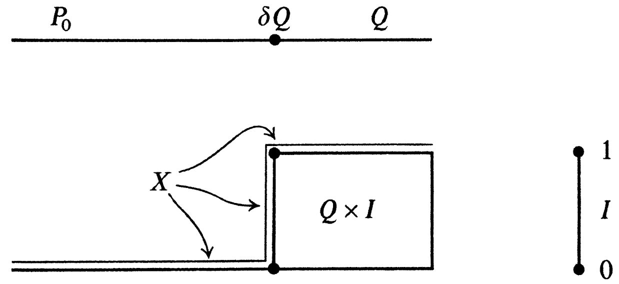

Fig. 1

Remark. In obtaining f we have met the one small point where the proof of 2.1 is genuinely a refinement of the usual topological engulfing argument. From this point we engulf as usual. We continue only to convince the reader that this is possible. It should become clear that 2.1 remains valid if, for example, one replaces (a), (b) by the assumption that f exists as above, and replaces $\dim P \leqq 2$ by the condition that $\dim P \leqq n-3$ and

$$2\dim P + 2 - n \leqq k, \quad \text{where} \quad \pi_i(M, U) = 0, \quad i \leqq k.$$

Note that (a) implies that $\pi_1(M, U)$ is trivial. Hence this is a generalization of 2.1 even for $n=5$.

A *chart* in M will always mean an open p.l. n-ball imbedded as an open subset of M. Since P is closed there is an atlas $\mathscr{A}$ of M by charts $B \subset M$ such that

(i) Either $B \cap P = \emptyset$ or $B \subset O_0$.

(ii) If $B \subset O_0$ the p.l. structure on the chart B is inherited from O_0.

Choose an integer N so large and a triangulation of Q so fine that if $\sigma_1, \sigma_2, \ldots, \sigma_k$ is a list, in any order of non-decreasing dimension, of all the closed simplices lying in Q but not in δQ, then for all σ_i and all j, $0 \leqq j \leqq N-1$, $f\left(\sigma_i \times \left[\frac{j}{N}, \frac{j+1}{N}\right]\right)$ lies in at least one chart of $\mathscr{A}$. Order the subpolyhedra $\sigma_i \times \left[\frac{j}{N}, \frac{j+1}{N}\right]$ lexicographically on $(N-j, i)$, $1 \leqq N-j \leqq N$, $1 \leqq i \leqq k$, and to save notation relabel the resulting sequence $Y_1, Y_2, \ldots, Y_{Nk}$. For each Y_i, $1 \leqq i \leqq Nk$, select a chart in $\mathscr{A}$ containing $f(Y_i)$ and call it O_i. Note that, if O_i meets P, $O_i \subset O_0$ and O_i inherits its p.l. structure from O_0.

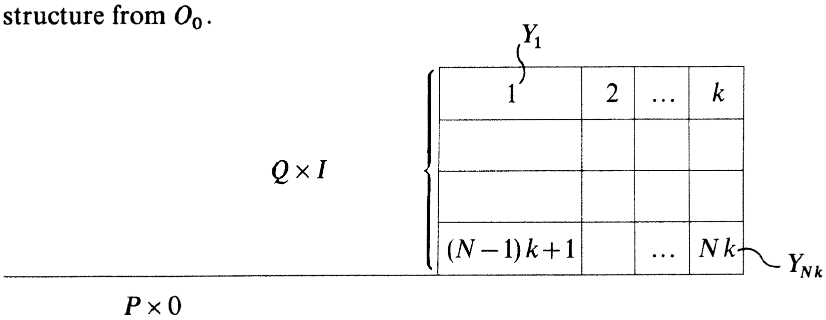

Fig. 2

Write X_i for $X \cup Y_1 \cup \cdots \cup Y_i$ and set $X_0 = X$. Note that X_{i+1} collapses cellularly over Y_{i+1} to X_i.

Fix i, $0 \leqq i \leqq Nk$, and suppose for an induction on i that there is a compact automorphism h_i of M and a map f_i: $P \times 0 \cup Q \times I \to M$ with the following properties

1) $f_i | P \times 0 = f | P \times 0$.

2) For all j, $f_i(Y_j) \subset O_j$.

3) *If* $n \geqq 7$, $h_i(U) \supset f_i(X_i)$, where $X_i = X \cup Y_1 \cup \cdots \cup Y_i$.

If $n = 6$ *or* 5, $h_i(U) \supset f_i(\delta X_i)$ where δX_i *is by definition the frontier of* X_i *in* $P \times [-1, 0] \cup Q \times I$, or, equivalently, δX_i is the union of $X_i \cap (P \times 0)$ with the frontier of X_i in $P \times 0 \cup Q \times I$.

This is trivially true for $i = 0$. We proceed to construct h_{i+1} and f_{i+1} with similar properties for the integer $i+1$ to complete the induction. This done, we can put $h = h_{Nk}$. Then $h(U) \supset P$ by 1) and 3) so that the proposition will be established.

We will say that a map f_i': $P \times 0 \cup Q \times I \to M$ is (i-) *admissible* if it satisfies 1), 2), 3) with f_i' in place of f_i.

Using a p.l. isomorphism $O_{i+1} \cong R^n$ assign to O_{i+1} a linear structure. We can regard the open subset $f_i^{-1}(O_{i+1}) \subset P \times 0 \cup Q \times I$ as a polyhedron (usually non-compact). Find an i-admissible approximation f_i' to f_i that gives a p.l. map $f_i^{-1}(O_{i+1}) \to O_{i+1} \cong R^n$ and coincides with f_i outside $f_i^{-1}(O_{i+1})$; also choose a triangulation of $f_i^{-1}(O_{i+1})$ such that f_i' is linear on simplices. (Here is one method: Begin with any triangulation of $f_i^{-1}(O_{i+1})$ such that $f_i | P \times 0 \cap f^{-1}(O_{i+1})$ is linear on simplices. When this triangulation is suitably subdivided the unique map $f^{-1}(O_{i+1}) \to O_{i+1} \cong R^n$ coincident with f_i on vertices and linear on simplices will provide f_i'; and the subdivided triangulation can be the one chosen.) Since there is an elementary cellular collapse from Y_{i+1} to $Y_{i+1} \cap X_i$ in $f_i^{-1}(O_{i+1})$ — viz. a collapse across a p.l. cell — we can further subdivide $f_i^{-1}(O_{i+1})$ so that Y_{i+1} collapses simplicially to $Y_{i+1} \cap X_i$ — say across simplices $\sigma_s, \sigma_{s-1}, \ldots, \sigma_1$ in this order (see Zeeman [16, Chapter 3, Lemma 13]). Now the general position Lemma 5.1 (in appendix) shows that we can alter f_i' slightly on $f_i^{-1} O_{i+1}$ so that as a map to O_{i+1} it is in general position, is linear on simplices, and remains admissible.

Recall that the double point set of a continuous map $g: A \to B$ is the closure in A of the set of all $x \in A$ such that $g(x) = g(y)$ for some $y \neq x$, $y \in A$. It is denoted $S(g)$.

The Case $n \geqq 7$. Suppose now that $n \geqq 7$ (or $\dim Q \leqq 1$). Then general position means that $S(f_i') \cap Y_{i+1}$ has dimension $\leqq 3+3-7 = -1$ — i.e. that it is empty. In other words Y_{i+1} is imbedded and its image doesn't meet that of its complement. Thus it is an easy matter to make $h_i U$ successively engulf the linearly imbedded simplices $f_i'(\sigma_1), f_i'(\sigma_2), \ldots, f_i'(\sigma_s)$ by means of automorphisms of M having compact support in $O_{i+1} - f_i'(X_i)$. Hence the induction can be completed if $n \geqq 7$.

The Cases $n = 6$ and $n = 5$. If $n = 6$ or 5, $\dim(S(f_i') \cap X_{i+1})$ is $\leqq 0$ or $\leqq 1$ respectively, and in either case the above engulfing process is obstructed.

We use a method of Stallings to exploit the fact that it is quite enough to engulf the image of the 2-skeleton of the 3-complex X_{i+1}. For a given (closed) simplex σ of X_{i+1} let $\hat{\sigma}$ be the boundary $\partial\sigma$ in case $\dim\sigma=3$, and let $\hat{\sigma}=\sigma$ in case dim $\sigma=0$, 1 or 2. Suppose for a subsidary induction on $\sigma_1, \dots, \sigma_j, \dots, \sigma_s$ that there exist a compact automorphism θ of M^n and an admissible map f_i'' such that

(a) *There is a neighborhood* N *of* $f_i'(Y_{i+1})$ *such that* $f_i''^{-1}N=f_i'^{-1}N$ *and the two maps coincide there.*

(b) $\theta h_i U\supset f_i''(\delta X_i\cup\hat{\sigma}_1\cup\hat{\sigma}_2\cup\cdots\cup\hat{\sigma}_j)$.

This is so for $j=0$ as $h_i(U)\supset f_i'(\delta X_i)$. When it is established for $j=s$ we will clearly have $\theta h_i U\supset f_i''(\delta X_{i+1})$ and then we will be able to put $h_{i+1}=\theta h_i$, $f_{i+1}=f_i''$ to complete the main induction. Hence the theorem will be proved when we have completed the subsidary induction.

If $n=6$, $S(f_i''|\delta X_i\cup\hat{\sigma}_1\cup\cdots\cup\hat{\sigma}_j)\cap\sigma_{j+1}$ has dimension $\leqq 2+3-6=-1$ i.e. is empty (cf. (a)). Hence we can complete the subsidary induction by pushing $\theta h_i U$ across the linearly imbedded simplex $f_i''(\sigma_{j+1})=f_i'(\sigma_{j+1})$ by a compact automorphism of O_{i+1} that fixes $f_i''(\delta X_i\cup\hat{\sigma}_1\cup\cdots\cup\hat{\sigma}_j)$. Note that in this case one can keep $f_i''=f_i'$ for all j.

The Case $n=5$ (concluded). There remains the case $n=5$. The set $S(f_i''|\delta X_i\cup\hat{\sigma}_1\dots\cup\hat{\sigma}_j)\cap\sigma_{j+1}$ (call it S) has dimension $\leqq 2+3-5=0$ and consists of a finite number of points in the interior of σ_{j+1}. Let the collapse across σ_{j+1} be from the face τ opposite vertex v. Then σ_{j+1} is the join $\tau * v$ and $(\partial\tau)*v\subset\delta X_i\cup\hat{\sigma}_1\cup\cdots\cup\hat{\sigma}_j$. Let $\bar{S}\subset\sigma_{j+1}$ be the union of all segments through points of S parallel to the segment from v to the barycenter of τ.

Assertion 2.2. *There exists a compact automorphism* $\bar{\theta}$ *of* M *and an admissible map* $\bar{f}_i''$ *satisfying:*

($\bar{\text{a}}$) *The inductive assumption* (a) *with* $\bar{f}_i''$ *in place of* f_i''.

($\bar{\text{b}}$) $\bar{\theta} h_i U\supset\bar{f}_i''(\delta X_i\cup\hat{\sigma}_1\cup\cdots\cup\hat{\sigma}_j\cup\bar{S})$.

Granting this, note that $\sigma_{j+1}=\tau * v$ collapses to $\partial\tau * v\cup\bar{S}$ which has image in $\bar{\theta} h_i U$ and that by ($\bar{\text{a}}$) this collapse crosses no singularity of $\bar{f}_i''$. Recalling that $f_i'\,\sigma_{j+1}=\bar{f}_i''\,\sigma_{j+1}$ is linearly imbedded in O_{i+1}, we can find another compact automorphism $\hat{\theta}$ of M so that

$$\hat{\theta} h_i U\supset\bar{f}_i''(\delta X_i\cup\hat{\sigma}_1\cup\cdots\cup\hat{\sigma}_j\cup\sigma_{j+1})$$

which (a fortiori) completes the subsidary induction on j.

Assertion 2.2 will be proved using the following version of Newman's engulfing. This version is in fact Newman's main result [8, Theorem 6] first stripped of unneeded generality then subjected to some generalization that we will explain presently. Using the above arguments as a guide a patient reader could provide a proof himself (especially for the case $p=1$).

Engulfing Theorem 2.3 (Newman). *Consider the following data:* M^n, $n \geqq 5$ *a topological n-manifold (without boundary) endowed with a metric d;* V *an open subset of* M *such that* $\pi_i(M, V)=0$ *for* $0 \leqq i \leqq p$; $f\colon \Gamma \to M$ *a proper continuous map to M of a possibly noncompact polyhedron* Γ *having dimension* $\leqq n-3$; $H \subset \Gamma$ *a closed subpolyhedron such that* $H \cap f^{-1}(M-V)$ *is compact and has a neighborhood in H of dimension* $\leqq p$; $U \subset M$ *an open subset having a p.l. manifold structure;* $L \subset \Gamma$ *a closed subpolyhedron such that* $f|L$ *is a p.l. map into* U; ε *a real number* >0.

With this data, there exists a homeomorphism h of M onto itself that fixes all points outside some compact set and there exists a continuous proper map $g\colon \Gamma \to M$ *such that the following hold:*

A) $h(V) \supset g(H)$.

B) $g|L = f|L$.

C) $d(g(x), f(x)) < \varepsilon$ *for all* $x \in \Gamma$.

Explanatory Remarks. (1) The case where $\Gamma = H = L$ corresponds closely to Stallings' p.l. engulfing theorem [13]. This simple case is the only one used in later sections.

(2) In spite of the fact that Newman assumes $\pi_i(M)=0$, $i \leqq p$ and $\pi_i(V)=0$, $i \leqq p-1$, the only connectivity assumption needed in his proof is $\pi_i(M, V)=0$, $0 \leqq i \leqq p$. The stronger assumption would suffice for the proof of Theorem 1.1 but not for its generalization Theorem 4.1 below.

(3) Again, Newman assumes $\dim \Gamma \leqq p$, but his proof applies with our assumptions on dimension.

(4) Newman assumes that $f|L$ is an embedding. We do not. Again one can observe that Newman's proof still applies. But we more cautiously deduce the theorem as we state it – denote it (T) – from the seemingly weaker result – denote it (T_0) – having the extra assumption that $f|L$ is an embedding. We do this by forming a quotient polyhedron Γ' with PL quotient map $q\colon \Gamma \to \Gamma'$, and forming a continuous map $f'\colon \Gamma' \to M$ so that the following conditions are verified: $f|L = f' q|L$; $d(f(x), f' q(x)) < \varepsilon/2$ for all $x \in \Gamma$; and $f'|(qL)$ is a PL embedding. Now (T) follows from (T_0) applied by substituting $(M^n, V, \Gamma', qH, qL, f', \varepsilon/2, p)$ for the data $(M^n, V, \Gamma, H, L, f, \varepsilon, p)$ of (T_0). It remains to check that $\Gamma \xrightarrow{q} \Gamma' \xrightarrow{f'} M$ exists as stated. Here is one proof. Write A for $f^{-1} U \subset \Gamma$. Choose triangulations of A and U so that L is a subcomplex of A, $f|L$ is a simplicial map $L \to U$, and each simplex of U has diameter $< \varepsilon/4$. By the method of relative simplicial approximation [15], find a subdivision A' of A mod L and a map $g\colon \Gamma \to M$ such that $g|L = f|L$,

$g|A'\colon A'\to U$ is simplicial, and $d(g(x), f(x))<\varepsilon/2$ for all $x\in\Gamma$. Define a formal simplicial complex B by identifying two vertices v_1, v_2 of A' whenever v_1, v_2 are in L and $f(v_1)=f(v_2)$. The simplicial quotient map $q'\colon A'\to B$ is a simplicial isomorphism outside the star StL of L in A'. Also as $g|StL$ is simplicial there is a unique map $g'\colon B\to M$ with $g\,q'=g'$. This g' embeds L simplicially. Now glue $\Gamma-StL$ to B by the p.l. homeomorphism $A'-StL\xrightarrow{q'}B-q'StL$ to define the polyhedron Γ'. The factorization $\Gamma\xrightarrow{q}\Gamma'\xrightarrow{f'}M$ of f, in which q derives from q' and f' derives from g' clearly has the required properties.

Here are the substitutions into Theorem 2.3 needed to establish Assertion 2.2:

$$(M^n\mapsto M^5),\quad (V\mapsto \theta h_i U),\quad (p\mapsto 1),\quad (f\mapsto f_i''),$$
$$(\Gamma\mapsto P\times 0\cup Q\times I),\quad (H\mapsto \delta X_i\cup\hat\sigma_1\cup\cdots\cup\hat\sigma_j\cup\bar S),$$

also $(L\mapsto f_i'^{-1}B\cup P\times 0$, where B is a compact polyhedral neighborhood of $f_i'(Y_{i+1})$ contained in the neighborhood N of condition (a) on which f_i' and f_i'' coincide), $(U\mapsto O_0$ if $O_{i+1}\subset O_0$; otherwise $U\mapsto$ disjoint union of an open subset of O_{i+1} containing B and an open subset of O_0 containing P). Furthermore ε is to be chosen so small that

(i) the map $P\times 0\cup Q\times I\to M$ provided by g in the conclusion of Theorem 2.3 and to be called $\bar f_i''$ must be i-admissible,

(ii) there is a neighborhood $\bar N\subset B$ of $f_i'(X_{i+1})=f_i''(X_{i+1})$ such that $\bar f_i''^{-1}(\bar N)$ is exactly $f_i'^{-1}(\bar N)$ — not larger.

Condition (ii) along with $g|L=f|L$ implies that $g=\bar f_i''$ satisfies ($\bar{\text{a}}$). Define $\bar\theta$ to be $h\theta$ where h is the automorphism of M^5 provided by Theorem 2.3, and θ is the automorphism in (b). Then $h(V)\supset g(H)$ becomes ($\bar{\text{b}}$) and so Assertion 2.2 is established. The tedious case $n=5$ of Proposition 2.1 is finally complete. Thus 2.1 is fully proved.

§ 3. The Proof of Theorem 1.1

We begin with the known fact [6] that there are two open sets O_1, O_2 in M^n each homeomorphic to R^n such that $O_1\cup O_2=M^n$. For later use, endow O_2 with a p.l. manifold structure (from R^n).

Let C be any compact set in M. We will show that C is contained in a ball. It is known that this implies the proposition [1]. The ball will in fact be the image of O_1 under an automorphism of M.

Let $U\subset M$ be an open neighborhood of ∞ such that $U\cap C=\emptyset$. As M is contractible, U has just one component having noncompact closure, and that component is a connected neighborhood of ∞. See [9, §1]. Hence we can and do assume that U is connected.

As M is $1-LC$ at ∞ there exists a neighborhood V of ∞, $V \subset U$, such that any loop in V is contractible in U. Again we can assume V is connected. Then inclusion induces the zero map $\pi_1 V \to \pi_1 U$. As M is 2-connected the boundary map $\pi_2(M, V) \xrightarrow{\partial} \pi_1(V)$ is an isomorphism. Hence the commutative diagram with exact row

$$\begin{array}{ccccc} \pi_2(U,V) & \longrightarrow & \pi_1(V) & \longrightarrow & \pi_1(U) \\ {\scriptstyle j}\big\downarrow & \overset{\cong}{\nearrow}{\scriptstyle \partial} & & & \\ \pi_2(M,V) & & & & \end{array}$$

shows that the map j (from inclusion) is surjective. Also

$$0 = \pi_1(M) \to \pi_1(M, V) \to \pi_0(V) = 0$$

shows that $\pi_1(M, V) = 0$. We shall presently apply our engulfing Result 2.1.

Let L be a closed infinite polyhedral neighborhood in the p.l. manifold O_2 of the closed subset $O_2 - O_1$ of O_2. We provide that L form a closed subset of M. This is possible because $O_2 - O_1 = M - O_1$ is closed in M and so has a neighborhood $A \subset O_2$ that is closed in M (since M is a normal space). L is necessarily closed in M if we choose it so small that it lies in A. Next fix a triangulation of O_2 such that L is a subcomplex.

Now Proposition 2.1 provides an automorphism h of M with compact support such that $h(U)$ contains the 2-skeleton $L^{(2)}$ of L. It is vital here that, because L is closed in M, the part of L outside V is compact.

Remark 3.1. If the hypothesis (C) of the introduction holds we can assume that U is simply connected. Then, as $\pi_i(M) = 0$, $i \leqq 2$, the engulfing theorem of Newman 2.3 with Γ, H, L all equal $L^{(2)}$ provides this h. All mention of V and of Proposition 2.1 becomes superfluous.

Let X be a compactum containing the support of h and $M - U$. Let $St(L) \subset O_2$ be the subcomplex formed of all closed simplices that meet L. Let $\bar{X}$ be the compact enlargement of X formed by adding to X all the simplexes of $St(L)$ that meet the compactum $X \cap St(L)$.

Since the interior $\mathring{L}$ of L in M contains $M - O_1$, one has $M = \mathring{L} \cup O_1$. Hence one can find a locally flat n-dimensional disc $D \subset O_1$ so large that

$$\mathring{L} \cup \mathring{D} \supset \bar{X}.$$

Next find a finite subcomplex $L' \subset L$ so large that

$$\mathring{L}' \cup \mathring{D} \supset \bar{X}.$$

At this point Fig. 3 should help to fix in the reader's mind the inclusions among the sets mentioned above.

Let $L'_{(n-3)}$ denote the union of all closed simplices of the first barycentric subdivision of L' that do not intersect the 2-skeleton $L'^{(2)} = L^{(2)} \cap L'$. This $L'_{(n-3)}$ is called the *dual* $(n-3)$-*skeleton* of L'. Apply Newman's Engulfing Theorem 2.3 (taking Γ, H, L all equal $L'_{(n-3)} - D$)

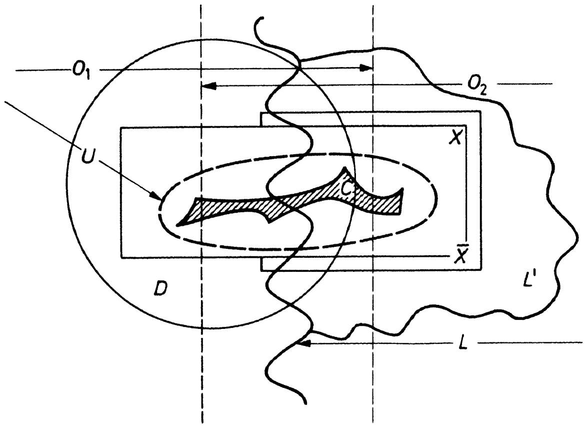

Fig. 3

to find a compact automorphism g_0 of $M-D$ such that $g_0(O_1-D)$ contains $L'_{(n-3)}-D$. Clearly g_0 extends to an automorphism g of M with the same support.

Recall that $h(U) \supset L^{(2)} \supset L'^{(2)}$. Now the stretching process of Bing and Stallings [14, § 8.1] can be applied in O_2 to provide an automorphism θ of M such that

(i) $\theta\, h(U) \cup g(O_1) \supset L'$,

(ii) $\theta(\Delta) = \Delta$ for each simplex Δ of O_2,

(iii) θ fixes all points outside the compact set $St(L') \subset O_2$.

Here is a rough explanation of the process. Every simplex σ of the barycentric subdivision of L' is uniquely the join of a simplex of the subdivision in $L'^{(2)}$ and a simplex of the subdivision in $L'_{(n-3)}$. For each such simplex σ, θ stretches $h(U)$ along the join parameter to cover $\sigma - g(O_1)$. Outside L', θ is a stretching between the 2-skeleton $O_2^{(2)} \supset L'^{(2)}$ and the "dual $(n-3)$ skeleton" $O_{2(n-3)} \supset L'_{(n-3)}$, but the stretching is attenuated away from L' and becomes the identity outside $St(L')$. Precise formulas can easily be adapted from [15, § 8.1].

From (ii), (iii) and the definition of $\bar{X}$, it follows that $\theta(M-\bar{X}) = M-\bar{X}$. Note that we cannot assert that $\theta(M-X) = M-X$. This explains why $\bar{X}$ was introduced. Now we have

$$\theta\, h(U) \supset \theta\, h(M-X) = \theta(M-X) \supset \theta(M-\bar{X}) = M-\bar{X}. \qquad (*)$$

Combining (∗) and $g(O_1) \supset D$ with (i) gives

$$\theta h(U) \cup g(O_1) \supset (M - \overline{X}) \cup D \cup L' = M.$$

Hence

$$U \cup (\theta h)^{-1} g(O_1) = (\theta h)^{-1} M = M.$$

As $U \cap C = \emptyset$, the ball $(\theta h)^{-1} g(O_1)$ must contain C. This ball completes the proof of Theorem 1.1.

§ 4. A Generalization

We say that π_1 is *essentially constant at* ∞ in a manifold M if there is a sequence of connected open neighborhoods of ∞ in M: $U_1 \supset U_2 \supset U_3 \supset \cdots$ with $\bigcap_i \text{closure}(U_i) = \emptyset$ such that the corresponding sequence of fundamental groups $\pi_1 U_1 \xleftarrow{f_1} \pi_1 U_2 \xleftarrow{f_1} \cdots$ induces isomorphisms $\text{Image}(f_1) \xleftarrow{\cong} \text{Image}(f_2) \xleftarrow{\cong} \cdots$. If in addition $\text{Image}(f_i) \cong \pi_1 M$ by inclusion we say that $\pi_1(\infty) \cong \pi_1 M$ (by inclusion). It turns out that the above statements are unaffected by change of base points and connecting base paths. Also they are in an obvious sense preserved under proper homotopy equivalence – as was the condition 1-LC at ∞ of § 1. For a careful treatment of these notions see [10] or [11] (also [9]).

Theorem 4.1. *Let M^n be a metrizable topological n-manifold, $n \geqq 5$, such that the inclusion $bM \hookrightarrow M$ of the boundary is a homotopy equivalence*[2]*. Suppose that π_1 is essentially constant at ∞ in M and that $\pi_1(\infty) \cong \pi_1 M$ by inclusion. Then M is homeomorphic to $bM \times [0, 1)$ provided that one of the following (possibly unnecessary) hypotheses is verified*

(a) *M is 2-connected.*

(b) *$bM \times \mathbb{R}$ admits a p.l. manifold structure.*

Clearly this theorem generalizes Theorem 1.1. The following corollary generalizes Corollary 1.3. For proof see [9, § 2.8].

Corollary 4.2. *Suppose M is an oriented metrizable topological n-manifold $n \geqq 5$, with nonempty simply-connected boundary bM. If there is a proper degree 1 map $(bM \times [0, 1), bM \times 0) \to (M, bM)$ then M is homeomorphic to $bM \times [0, 1)$, provided M is 2-connected or $bM \times R$ admits a p.l. manifold structure.*

Proof of 4.1. The proof occupies the remainder of this section. It is a generalization of the proof of Theorem 1.1 given in § 3. We present it in a sequence of assertions 4.3, 4.4, 4.5, 4.6, relying wherever possible on the argument of § 3. With no loss of generality assume M is connected.

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[2] The fact that $bM \hookrightarrow M$ is a homotopy equivalence implies, via Poincaré duality, that the manifold M (if connected) has arbitrarily small *connected* open neighborhood of ∞. See [9, § 1.2] for a more general result.

Assertion 4.3. *M is the union of an open collar neighborhood O_1 of bM and an open subset O_2 that carries a p.l. manifold structure.*

Proof of 4.3. Let U_1 be an open collar neighborhood of bM in M. Such exists by [2]. If (a) holds let U_2 be an open ball in $int\, M = M - bM$. If (b) holds let U_2 be U_1. In either case $\pi_i(M, U_2) = 0$, $i = 0, 1$. On the other hand $\pi_i(M, U_1) = 0$, for all i. At this point a standard engulfing argument (cf. [6, Lemma 3.1], § 3, or [3]) proves

Assertion 4.4. *Let $A_1 \subset U_1$ be a closed collar of bM. Let $A_2 \subset U_2$ be respectively another closed collar of bM or a closed disc according as* (a) *or* (b) *holds in* 4.1. *If $K \subset M$ is any compactum, there exist compact automorphisms h_1, h_2 of M fixing pointwise A_1, A_2 respectively such that*

$$h_1(U_1) \cup h_2(U_2) \supset K.$$

Using 4.4 in the infinite stretching process of Stallings on infinitely nested copies of U_1, U_2 one can next construct embeddings f_1, f_2 of U_1, U_2 onto the sets O_1, O_2 wanted in 4.3. See [6, Theorem 3.1]. Assertion 4.3 is now justified. In summary, the Proof of 4.3 is a natural generalization of the proof in [6] that an open manifold of dimension ≥ 5 is the union of two open balls.

Following the argument of § 3 we now set about proving

Assertion 4.5. *If $C \subset M$ is a given compactum, there exists an open collar neighborhood of bM that contains C.*

Then there also exists a closed (sub-)collar N of bM containing C. Next apply this fact to $M - \mathring{N}$ (which is homeomorphic to M). Iterating we decompose M into a sequence of closed collars and obtain thus a homeomorphism of M with $bM \times [0, 1] \cup bM \times [1, 2] \cup \cdots = bM \times [0, \infty)$ as 4.1 requires.

Thus 4.5 implies 4.1. To prove 4.5 we use the argument given in § 3, starting with O_1 and O_2 as provided by Assertion 4.3. Just one part of that argument offers difficulty in this new situation, namely the proof of

Assertion 4.6. *If $C \subset M$ is a given compactum, there exist open neighborhoods $U \supset V$ of ∞ in M such that $C \cap U = \emptyset$ and*

1) $\pi_1(M, V) = 0$.

2) *Inclusion induces a surjection $\pi_2(U, V) \to \pi_2(M, V)$.*

Thus Theorem 1.1 is established when we establish this (nontrivial!) assertion. Hypotheses (a), (b) of 4.1 are irrelevant in 4.6.

Proof of 4.6. Let U be any connected open neighborhood of ∞ in M so that $U \cap C = \emptyset$. Since π_1 is essentially constant at ∞ and $\pi_1(\infty) \cong \pi_1 M$ there exist connected open neighborhoods $V \subset U$ of ∞ so small that

A) *The map* $i\colon \pi_1 V \to \pi_1 U$ *induced by inclusion has image* $\operatorname{Im}(i) \cong \pi_1 M$ *(by inclusion).*

Next observe that if K is a compact set in M there exists a larger compactum $L \subset M$ such that the inclusion $bM \cup K \hookrightarrow bM \cup L$ is homotopic fixing bM to a map into bM. This is so because bM is a strong deformation retract of M.

It is thus possible to find a connected neighborhood V of ∞ so small that one has simultaneously the conditions:

A) *(stated above),*

B) $bM \cup K \hookrightarrow bM \cup L$ *is homotopic fixing* bM *to a map into* bM.

We assert that 1) and 2) in 4.6 hold for this choice of U, V. As for 1) it is an immediate consequence of the exact sequence

$$\pi_1 V \xrightarrow{j} \pi_1 M \to \pi_1(M, V) \to \pi_0 V = 0$$

and condition A). The verification of 2) will require Poincaré duality. It will be broken into four steps.

$\tilde{M} \xrightarrow{p} M$ will denote the universal covering space of M, and for $X \subset M$ $\tilde{X}$ will denote $p^{-1}(X)$. Čech homology theory with integer coefficients is used throughout.

Step 1. *Inclusion induces a surjection*

$$H_2(\tilde{U}, \tilde{V}) \to H_2(\tilde{M}, \tilde{V}).$$

For the proof we need

Lemma 4.7. *Suppose the commutative square of continuous maps of Hausdorff spaces*

$$\begin{array}{ccc} D & \xrightarrow{\bar{f}} & B \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

gives the fiber product (=pull-back) of f *and* g. *If* f *is proper, then* $\bar{f}$ *is also proper.*

Proof of 4.7. For the proof we can assume we have the standard fiber product $D = \{(a, b) \in A \times B \mid f(a) = g(b)\}$, with $\bar{f}(a, b) = b$ and $\bar{g}(a, b) = a$. If K is a compactum and f is proper consider $\bar{f}^{-1}(K)$. It is a closed subset of $S = D \cap (f^{-1}(gK) \times K)$. But $f^{-1}(gK)$ is compact as f is proper, and $D \subset A \times B$ is closed as C is Hausdorff. Thus S is compact and hence $\bar{f}^{-1}(K)$ is compact as required.

Proof of Step 1. We will establish the equivalent statement that $H_2(\tilde{M}, \tilde{V}) \to H_2(\tilde{M}, \tilde{U})$ is zero. Define $K = M - U$, $L = M - V$. Poincaré duality in the simply connected (hence orientable) manifold $\tilde{M}$ gives a commutative diagram:

$$\begin{array}{ccc} H_2(\tilde{M}, \tilde{M}-\tilde{L}) & \xrightarrow{i_*} & H_2(\tilde{M}, \tilde{M}-\tilde{K}) \\ \cong \big\downarrow \text{P.D.} & & \cong \big\downarrow \text{P.D.} \\ H_c^{n-2}(b\tilde{M} \cup \tilde{L}, b\tilde{M}) & \xrightarrow{j^*} & H_c^{n-2}(b\tilde{M} \cup \tilde{K}, b\tilde{M}). \end{array}$$

H_c^* is integral cohomology with compact supports. The vertical arrows are Poincaré duality isomorphisms essentially in the form given in Theorem 10, p. 342 in [12]. In fact, this theorem is stated for manifolds without boundary. So it is to be applied after adding to M an open collar along bM. The maps i_* and j^* are induced by inclusions. The commutativity of the square is a naturality property of the Poincaré duality isomorphism.

Now B) provides a homotopy $f: (bM \cup K) \times I \to bM \cup L$, fixing bM pointwise, of the inclusion $bM \cup K \hookrightarrow bM \cup L$ to map into bM. This f is proper, simply because K is compact. Let $\bar{f}: (b\tilde{M} \cup \tilde{K}) \times I \to b\tilde{M} \cup \tilde{L}$ be the (unique) homotopy, covering f, of the inclusion j: $b\tilde{M} \cup \tilde{K} \hookrightarrow b\tilde{M} \cup \tilde{L}$. Then $\bar{f}$ is a bundle map, which means that the evident commutative square

$$\begin{array}{ccc} \cdot & \xrightarrow{\bar{f}} & \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow{f} & \cdot \end{array}$$

is a fiber product. Thus Lemma 4.7 tells us that $\bar{f}$ is a proper homotopy. Now properly homotopic maps induce the same map of cohomology with compact supports. It follows that $j^* = \bar{f}_0^* = \bar{f}_1^*$ factors through $H_c^{n-2}(bM, bM) = 0$, i.e. $j^* = 0$. This shows $i_* = 0$, which completes Step 1.

Step 2. *$H_2(\tilde{U}) \to H_2(\tilde{M})$ is onto.*

Proof of Step 2. Use Step 1 and the map of exact sequences corresponding to $(\tilde{U}, \tilde{V}) \hookrightarrow (\tilde{M}, \tilde{V})$.

Remark. In the proofs of Steps 1 and 2 the behavior of π_1 at ∞ has been irrelevant.

Step 3. *$\pi_2 V \to \pi_2 M$ is onto.*

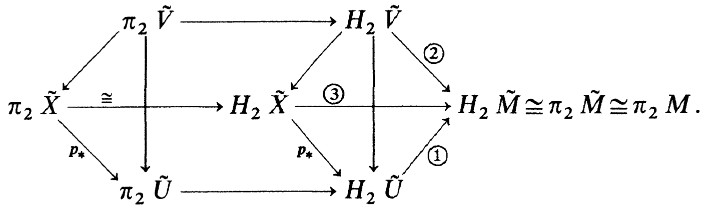
Proof of Step 3. For each element of the kernel of i: $\pi_1 V \to \pi_1 U$ attach to V a 2-cell killing this element. This produces $X \supset V$ such that inclusion $V \hookrightarrow U$ extends to a continuous map p: $X \to U$. As $\pi_1 V$ is countable X is nothing worse than a separable ANR — see [5]. Consider the commutative diagram

$$\begin{array}{ccccc} V & \hookrightarrow & U & \hookrightarrow & M \\ & \searrow & \uparrow p & \nearrow q & \\ & & X & & \end{array}$$

$\pi_1 X \cong p_* \pi_1 X \cong \operatorname{Im}(i)$ by the construction of X and p. Hence condition A) says that $q_*: \pi_1 X \xrightarrow{\cong} \pi_1 M$.

Since $\pi_1 U \to \pi_1 M$ and $\pi_1 V \to \pi_1 M$ are onto, $\tilde{U}$ and $\tilde{V}$ must be connected and it makes sense to talk of $\pi_2 \tilde{U} (\cong \pi_2 U)$ and $\pi_2 \tilde{V} (\cong \pi_2 V)$.

Next consider the following commutative diagram

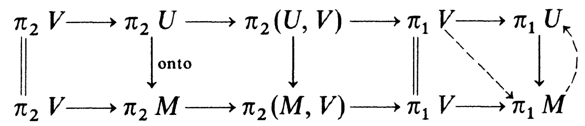


Unlabelled arrows arise from inclusion or else are Hurewicz homomorphisms. $\tilde{X} \supset \tilde{V}$ is the covering of X induced from $\tilde{U} \to U$ by $p: X \to U$, or equivalently from $\tilde{M} \to M$ by $q: X \to M$. Since $q_*: \pi_1 X \xrightarrow{\cong} \pi_1 M$, we see $\tilde{X}$ is simply connected and so $\pi_2 \tilde{X} \cong H_2 \tilde{X}$.

We have to show that the composition $\pi_2 \tilde{U} \to \pi_2 \tilde{M}$ is onto. Now ① is onto by Step 2. But U can be an arbitrarily small neighborhood of ∞. Hence ② is also onto. Thus ③ is onto, and the wanted conclusion follows. This establishes Step 3.

Step 4. *$\pi_2(U, V) \to \pi_2(M, V)$ is onto.*

Use the diagram with exact rows:



together with Condition A) which says that $\pi_1 V \to \pi_1 U$ factors through $\pi_1 M$ (dotted arrows).

§ 5. Appendix on General Position

Let $f: K \to M^n$ be a p.l. map from a simplicial complex to a p.l. n-manifold M without boundary. The map f is said to be in *general position* if:

1) f embeds each simplex of K.

2) If σ_1, σ_2 are simplices of K, then $f\sigma_1 \cap f\sigma_2 - f(\sigma_1 \cap \sigma_2)$ has dimension $\leqq \dim \sigma_1 + \dim \sigma_2 - n$.

We have used in § 2 the following easy general position lemma. For the sake of completeness we include a proof.

Lemma 5.1. *Let K be a countable simplicial complex of dimension $\leqq n$, and let $L \subset K$ be a full subcomplex. Let $f: K \to R^n$ be a map linear on simplices that embeds L. Let $\varepsilon: K \to R$ be a positive continuous function. Then there exists a map $g: K \to R^n$ in general position such that*

1) *g is linear on the simplices.*

2) $g|L=f|L$.

3) *For all* $x\in K$, $\|g(x)-f(x)\|<\varepsilon(x)$.

Remark. The assumption that $f|L$ is an embedding can be replaced by the assumption that $f|L$ is in general position.

Proof of 5.1. Here is the construction of g. Set $g|L=f|L$. Let $v_1, v_2, v_3, \ldots$ be the vertices of K outside L. We define g on these vertices by induction and extend to K by linearity. When $g(v_i)$ has been defined for $i<k$, let $g(v_k)$ be a point in the open ball of radius $\varepsilon(v_k)$ about $f(v_k)$ such that $g(v_k)$ lies in none of the countably many affine subspaces spanned by sets of $\leqq n$ points among the vertices of $g(L)$ and $g(v_1), \ldots, g(v_{k-1})$.

Clearly g satisfies conditions 1), 2), 3).

It is also clear that g imbeds each simplex of K. Small Greek letters will denote (closed) simplices of K and the corresponding small latin letters will denote their dimensions. It remains to show that for any two simplices σ_1, σ_2 of K.

$$\dim(g\,\sigma_1\cap g\,\sigma_2)\leqq\max\bigl(\dim(\sigma_1\cap\sigma_2), s_1+s_2-n\bigr). \qquad (\S)$$

As L is a full subcomplex of K, $\alpha_1=\sigma_1\cap L$, $\alpha_2=\sigma_2\cap L$ are simplices. Write $\gamma=\alpha_1\cap\alpha_2$ and express α_1, α_2 as joins

$$\alpha_1=\beta_1*\gamma, \qquad \alpha_2=\beta_2*\gamma.$$

Then write $\sigma_1\cap\sigma_2$ as a join $\gamma*\xi$ and σ_1, σ_2 as joins

$$\sigma_1=\alpha_1*\xi*\tau_1, \qquad \sigma_2=\alpha_2*\xi*\tau_2.$$

This situation is schematically represented in Fig. 4

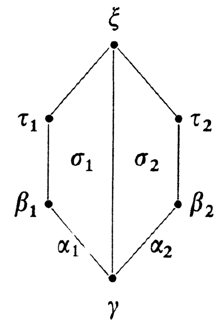


Fig. 4

We identify R^n with a hyperplane in R^{n+1} not containing the origin. If α is any simplex of K, $\bar{\alpha}$ will denote the $a+1$ dimensional vector subspace of R^{n+1} that is generated by $g(\alpha)$.

If $\bar\sigma_1+\bar\sigma_2=R^{n+1}$ — i.e. if no proper affine subspace of R^n contains $g\,\sigma_1\cup g\,\sigma_2$ — then $\dim(g\,\sigma_1\cap g\,\sigma_2)\leqq\dim(\bar\sigma_1\cap\bar\sigma_2)-1=s_1+s_2-n$ and (§) holds.

On the other hand, if $\bar\sigma_1+\bar\sigma_2\neq R^{n+1}$ then, by the construction of g, the vertices of $g\,\sigma_1$ and $g\,\sigma_2$ outside $g\,L$ i.e. those of $g\,\tau_1$, $g\,\xi$, $g\,\tau_2$ span a nonsingular simplex Σ in R^n such that

$$\bar\Sigma\cap(\bar\alpha_1+\bar\alpha_2)=0\in\mathbb{R}^{n+1}$$

where $\bar\Sigma=\bar\tau_1+\bar\xi+\bar\tau_2$ is the vector subspace generated by Σ in R^{n+1}. It follows that the segments in R^n from the points of Σ to the points of $g\,\alpha_1\cup g\,\alpha_2$ form the abstract join $\Sigma*[g\,\alpha_1\cup g\,\alpha_2]$. By looking at this join we conclude that $g\,\sigma_1=g(\tau_1*\xi*\alpha_1)=g(\tau_1*\xi)*g\,\alpha_1$ meets $g\,\sigma_2=g(\tau_2*\xi*\alpha_2)=g(\tau_2*\xi)*g\,\alpha_2$ in $g\,\xi*[g\,\alpha_1\cap g\,\alpha_2]=g\,\xi*g\,\gamma=g(\xi*\gamma)=g(\sigma_1\cap\sigma_2)$ which again proves (§).

References

1. Brown, M.: The monotone union of open n-cells is an open n-cell. Proc. Amer. Math. Soc. **12**, 812–814 (1961).
2. — Locally flat embeddings of topological manifolds. Ann. of Math. **75**, 331–341 (1962).
3. Connell, E. H.: A topological h-cobordism theorem. Illinois J. Math. **11**, 300–309 (1967).
4. Glaser, L. C.: Uncountably many contractible open 4-manifolds. Topology **6**, 37–42 (1967).
5. Hanner, O.: Some theorems on absolute neighborhood retracts. Ark. Math. **1**, 389–408 (1950).
6. Luft, E.: On contractible topological manifolds. Invent. Math. **4**, 192–201 (1967).
7. McMillan, D. R., and E. C. Zeeman: On contractible open manifolds. Proc. Cambridge Phil. Soc. **58**, 221–224 (1962).
8. Newman, M. H. A.: The engulfing theorem for topological manifolds. Ann. of Math. **84**, 555–571 (1966).
9. Siebenmann, L. C.: On detecting open collars. To appear in Trans. Amer. Math. Soc.
10. — doctoral dissertation, Princeton University, 1965, Dissertation Abstracts, 27 (June 1966) pp. 2044–2045 (Order No. 66-5012), University Microfilms Ltd., 300 N. Zeeb Rd., Box 1307, Ann Arbor, Michigan, USA.
11. — (version of [10] to appear).
12. Spanier, E.: Algebraic Topology. New York: McGraw-Hill 1966.
13. Stallings, J.: The piecewise linear structure of euclidean space. Proc. Cambridge Philos. Soc. **58**, 481–488 (1962).
14. — On topologically unknotted spheres. Ann. of Math. **77**, 490–503 (1963).
15. Zeeman, E. C.: Relative simplicial approximation. Proc. Camb. Phil. Soc. **60**, 39–43 (1964).
16. — Seminar on combinatorial topology (mimeographed notes), 1963, I.H.E.S., at Bures-sur-Yvette, France.
17. Newman, M. H. A.: The open topological n-cell. Preprint mimeographed at U. of Warwick.

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