Approximating L^2-signatures by their compact analogues

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Abstract

Let Γ be a group together with a sequence of normal subgroups $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ of finite index $[\Gamma : \Gamma_k]$ such that $\bigcap_k \Gamma_k = \{1\}$. Let (X, Y) be a (compact) $4n$-dimensional Poincaré pair and $p : (\overline{X}, \overline{Y}) \to (X, Y)$ be a Γ-covering, i.e. normal covering with Γ as deck transformation group. We get associated Γ/Γ_k-coverings $(X_k, Y_k) \to (X, Y)$. We prove that

$$\mathrm{sign}^{(2)}(\overline{X}, \overline{Y}) = \lim_{k\to\infty} \frac{\mathrm{sign}(X_k, Y_k)}{[\Gamma : \Gamma_k]},$$

where sign or $\mathrm{sign}^{(2)}$ is the signature or L^2-signature, respectively, and the convergence of the right side for any such sequence $(\Gamma_k)_{k\geq 1}$ is part of the statement.

If Γ is amenable, we prove in a similar way an approximation theorem for $\mathrm{sign}^{(2)}(\overline{X}, \overline{Y})$ in terms of the signatures of a regular exhaustion of $\overline{X}$.

Our results are extensions of Lück's approximation results for L^2-Betti numbers [10, Theorem 0.1].

Key words: L^2-signature, signature, covering with residually finite deck transformation group, amenable exhaustion.
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0 Introduction

Throughout most of this paper we will use the following conventions. We fix a group Γ, together with a sequence of normal subgroups $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ of finite index $[\Gamma : \Gamma_k]$ such that $\bigcap_k \Gamma_k = \{1\}$. (Provided that Γ is countable, Γ is residually finite if and only if such a sequence $(\Gamma_k)_{k\geq 1}$ exists.) Moreover, given a Γ-covering $p : \overline{X} \to X$, i.e. a normal covering with Γ as group of deck transformations, we will denote the associated Γ/Γ_k-coverings by $X_k := \overline{X}/\Gamma_k \to X$ and for a subspace $Y \subset X$ let $\overline{Y} \subset \overline{X}$ and $Y_k \subset X_k$ be the obvious pre-images of Y.

One of the main results of the paper is

0.1 Theorem. *Let (X, Y) be a $4n$-dimensional Poincaré pair. Then the sequence $(\mathrm{sign}(X_k, Y_k)/[\Gamma : \Gamma_k])_{k\geq 1}$ converges and*

$$\lim_{k\to\infty} \frac{\mathrm{sign}(X_k, Y_k)}{[\Gamma : \Gamma_k]} = \mathrm{sign}^{(2)}(\overline{X}, \overline{Y}).$$

Some explanations are in order. An l-dimensional *Poincaré pair* (X, Y) is a pair of finite CW-complexes (X, Y) with connected X together with a so called fundamental class $[X, Y] \in H_l(X; \mathbb{Q})$ such that for the universal covering, and hence for any Γ-covering $p : \overline{X} \to X$, the Poincaré $\mathbb{Q}\Gamma$-chain map induced by the cap product with (a representative of) the fundamental class

$$\cdot \cap [X, Y] : C^{l-*}(\overline{X}, \overline{Y}) \to C_*(\overline{X})$$

is a $\mathbb{Q}\Gamma$-chain homotopy equivalence. Because we are working with free finitely generated left $\mathbb{Q}\Gamma$-chain complexes, this is the same as saying that the induced map in homology is an isomorphism. One usually also requires that Y itself is a l-dimensional Poincaré space (using the corresponding definition where the second space is empty) with $\partial[X, Y] = [Y]$, although this is not really necessary for our applications. Here $C_*(\overline{X})$ is the cellular (left) $\mathbb{Q}\Gamma$-chain complex and $C^{l-*}(\overline{X}, \overline{Y})$ is the dual $\mathbb{Q}\Gamma$-chain complex $\hom_{\mathbb{Q}\Gamma}(C_{l-*}(\overline{X}, \overline{Y}), \mathbb{Q}\Gamma)$. This is canonically a right $\mathbb{Q}\Gamma$-chain complex. Throughout the paper, we deal with left modules. We turn any right $\mathbb{Q}\Gamma$-module into a left $\mathbb{Q}\Gamma$-module using the involution of $\mathbb{Q}\Gamma$ induced by $\Gamma \ni g \mapsto g^{-1}$.

Examples for Poincaré pairs are given by a compact connected topological oriented manifold X with boundary Y or merely by a rational homology manifold.

The Poincaré duality chain map of a $4n$-dimensional Poincaré pair (X, Y) induces an isomorphism $H^p(X, Y; \mathbb{C}) \to H_{4n-p}(X; \mathbb{C})$. If we compose the inverse with the map induced in cohomology by the inclusion $X \hookrightarrow (X, Y)$ and with the natural isomorphism $H^p(X; \mathbb{C}) \cong H_p(X; \mathbb{C})^*$ to the dual space $H_p(X; \mathbb{C})^*$ of $H_p(X; \mathbb{C})$, we get in the middle dimension $2n$ a homomorphism

$$A : H_{2n}(X; \mathbb{C}) \to H_{2n}(X; \mathbb{C})^*$$

which is selfadjoint. The signature of the (oriented) pair (X, Y) is by definition the signature of the (in general indefinite) form A, i.e. the difference of the number of positive and negative eigenvalues of the matrix representing A (after choosing a basis for $H_{2n}(X, \mathbb{C})$ and the dual basis for $H_{2n}(X)^*$).

The L^2-signature on $(\overline{X}, \overline{Y})$ is defined similarly, but one has to replace homology by L^2-homology. L^2-homology and L^2-cohomology groups in this paper are always reduced, i.e. we divide by the *closure* of the image of the differential to remain in the category of Hilbert modules. We get then an operator $A : H_{2n}^{(2)}(\overline{X}) \to H_{2n}^{(2)}(\overline{X})$ (using the natural isomorphism of a Hilbert space with its dual space). The L^2-homology is a Hilbert module over the von Neumann algebra $\mathcal{N}\Gamma$ and A is a selfadjoint bounded Γ-equivariant operator. Hence $H_{2n}^{(2)}(\overline{X})$ splits orthogonally into the positive part of A, the negative part of A and the kernel of A. The difference of the $\mathcal{N}\Gamma$-dimensions of the positive part and the negative part is by definition the *L^2-signature*.

All this can also be reformulated in terms of cohomology instead of homology, which is convenient e.g. when dealing with de Rham cohomology.

An analogue of Theorem 0.1 for L^2-Betti numbers has been proved by Lück [10, Theorem 0.1].

If X is a smooth closed manifold, Atiyah's L^2-index theorem [1, (1.1)] shows that the signature is multiplicative under finite coverings and that $\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X_k)/[\Gamma : \Gamma_k]$ holds for $k \geq 1$.

There are Poincaré spaces $X = (X, \emptyset)$ for which the signature is not multiplicative under coverings by [15, Example 22.28], [25, Corollary 5.4.1]). There are also compact smooth manifolds with boundary with the same property, see [3, Proposition 2.12] together with the Atiyah-Patodi-Singer index theorem [2, Theorem 4.14]. This shows that Atiyah's L^2-signature theorem does not generalize to these situations.

Our result says for these cases that the signature is multiplicative at least approximately. For closed topological manifolds, it is known that the signature is multiplicative under finite coverings [19, Theorem 8]. In a companion [13, Theorem 0.2] to this paper, we prove the following theorem, this way apparently filling a gap in the literature:

0.2 Theorem. *Let M be a closed topological manifold with normal covering $\overline{M} \to M$. Then*

$$\operatorname{sign}^{(2)}(\overline{M}) = \operatorname{sign}(M).$$

There, we also discuss to what extend Theorem 0.2 can be true for Poincaré duality spaces $X = (X, \emptyset)$. We show [13] that Theorem 0.2 for Poincaré duality spaces $X = (X; \emptyset)$ is implied by the L-theory isomorphism conjecture or by (a strong form of) the Baum-Connes conjecture provided that Γ is torsion-free.

Dodziuk-Mathai [9, Theorem 0.1] give an analog of Lück's approximation theorem [10, Theorem 0.1] for L^2-Betti numbers to Følner exhaustions of amenable covering spaces.

Along the same lines, we compute the L^2-signature using a Følner exhaustion in Theorem 0.4, proved in Section 2.1. The relevant definition is the following:

0.3 Definition. Let X be a connected compact smooth Riemannian manifold possibly with boundary ∂X and $\overline{X} \to X$ be a Γ-covering for some amenable group Γ. We lift the metric on X to $\overline{X}$ and use this metric to measure the volume of submanifolds (open as well as of codimension 1) of $\overline{X}$.

Let $X_1 \subset X_2 \subset \dots \overline{X}$ with $\bigcup_{k\in\mathbb{N}} X_k = \overline{X}$ be an exhaustion of $(\overline{X}, \partial\overline{X})$ by smooth submanifolds with boundary (where we don't make any assumptions about the intersection of ∂X_k and $\partial\overline{X}$). Set $Y_k := \partial X_k - (\partial X_k \cap \partial\overline{X})$ (i.e. $\partial X_k = Y_k \cup (\partial X_k \cap \partial\overline{X})$). The exhaustion is called *regular* if it has the following properties:

(1) $\operatorname{area}(Y_k)/\operatorname{vol}(X_k) \xrightarrow{k\to\infty} 0$;

(2) The second fundamental forms of ∂X_k in $\overline{X}$ and each of their covariant derivatives are uniformly bounded (independent of k);

(3) The boundaries ∂X_k are uniformly collared and the injectivity radius of ∂X_k is uniformly bounded from below (always uniformly in k).

Regular exhaustions were introduced in [8, p. 152]. The existence of such an exhaustion is equivalent to amenability of Γ provided that the total space $\overline{X}$ is connected.

0.4 Theorem. *In the situation of Definition 0.3 (in particular we require that X is connected) we get*

$$\lim_{k\to\infty} \frac{\operatorname{sign}(X_k, \partial X_k)}{\operatorname{vol}(X_k)} = \frac{\operatorname{sign}^{(2)}(\overline{X}, \overline{Y})}{\operatorname{vol}(X)},$$

where the convergence of the left hand side is part of the assertion.

The assumption that the base space X is connected is necessary.

For smooth manifolds with boundary, the L^2-signature of course is defined in terms of the intersection pairing on L^2-homology.

On the other hand, there is the L^2-index of the signature operator with Atiyah-Patodi-Singer boundary conditions. The latter is computed in [14, Theorem 1.1] in terms of the L^2-η-invariant and a local integral. It is a non-trivial assertion that the L^2-index of the signature operator really gives the (co)homologically defined L^2-signature. This is proved in [13, Theorem 3.2], using [24].

In the following Theorem 0.5 we give a combinatorial version of Theorem 0.4. The notion of a balanced exhaustion will be explained in Definition 2.47.

0.5 Theorem. *Assume that X is a compact simplicial complex triangulating a rational homology manifold with boundary the subcomplex ∂X. Assume $\overline{X}$ is a normal covering of X with amenable covering group Γ. Let $X_1 \subset X_2 \subset \dots$ be subcomplexes forming a balanced amenable exhaustion of $\overline{X}$ by rational homology manifolds (with boundaries Y_k). If X is a homology manifold, such an exhaustion does always exist. Then*

$$\lim_{k\to\infty} \frac{\operatorname{sign}(X_k, Y_k)}{|X_k|} |X| = \operatorname{sign}^{(2)}(\overline{X}, \partial\overline{X}).$$

For more information about approximation results see for instance [11, Chapter 13].

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Organization of the paper: We will prove convergence of the signature for coverings in Section 1, and in Section 2 the statement about amenable exhaustions.

1 Residual convergence of signatures

This section is devoted to the proof of Theorem 0.1.

1.1 Abstract $\mathbb{Q}\Gamma$-chain complexes

Let C_* be a finitely generated based free $4n$-dimensional left $\mathbb{Q}\Gamma$-chain complex.

1.1 Definition. C_* being finitely generated based free means that each chain module C_p is of the shape $\mathbb{Q}\Gamma^r = \oplus_{i=1}^r \mathbb{Q}\Gamma$ for some integer $r \geq 0$.

We define its *dual* $\mathbb{Q}\Gamma$*-chain complex* C^{4n-*} as follows. It has by definition as p-th chain module C_{4n-p} and its p-th differential $c^{4n-p} : C^{4n-p} \to C^{4n-(p-1)}$ is given by the adjoint $\left(c_{2d-(p-1)} : C_{2d-(p-1)} \to C_{2d-p}\right)^*$. The adjoint $f^* : \mathbb{Q}\Gamma^s \to \mathbb{Q}\Gamma^r$ of a $\mathbb{Q}\Gamma$-map $f : \mathbb{Q}\Gamma^r \to \mathbb{Q}\Gamma^s$ is given by (right multiplication with) the matrix $A^* \in M(s, r, \mathbb{Q}\Gamma)$ if f is given by right multiplication with the matrix $(A_{ij}) \in M(r, s, \mathbb{Q}\Gamma)$ and $A^*_{i,j} = \overline{A_{j,i}}$ for $\overline{\sum_{w\in\Gamma} \lambda_w \cdot w} := \sum_{w\in\Gamma} \lambda_w \cdot w^{-1}$. Note that by this definition f^* is a left $\mathbb{Q}\Gamma$-module map.

Identify $\hom_{\mathbb{Q}\Gamma}(\mathbb{Q}\Gamma^r, \mathbb{Q}\Gamma)$ (homomorphisms which commute with the left $\mathbb{Q}\Gamma$-module structure) with $\mathbb{Q}\Gamma^r$ using the canonical basis on $\mathbb{Q}\Gamma^r$ by sending ϕ to the vector $(\overline{\phi(e_i)})_{i=1,\ldots,r}$. This is an isomorphism of *left* $\mathbb{Q}\Gamma$-modules using our convention for the left $\mathbb{Q}\Gamma$-module structure on $\hom_{\mathbb{Q}\Gamma}$. Then f^* corresponds to $\hom_{\mathbb{Q}\Gamma}(f, \mathrm{id}_{\mathbb{Q}\Gamma})$.

Given a $\mathbb{Q}\Gamma$-chain map $f_* : C^{4n-*} \to C_*$, define its *adjoint* $\mathbb{Q}\Gamma$-chain map $f^{4n-*} : C^{4n-*} \to C_*$ as given by left multiplication with the adjoint of the matrix representing f_*.

1.2 Definition. Define the finitely generated $4n$-dimensional Hilbert $\mathcal{N}\Gamma$-chain complex $C_*^{(2)}$ by $l^2(\Gamma) \otimes_{\mathbb{Q}\Gamma} C_*$ and the finitely generated based free $4n$-dimensional $\mathbb{Q}[\Gamma/\Gamma_k]$-chain complex $C_*[k]$ by $\mathbb{Q}[\Gamma/\Gamma_k] \otimes_{\mathbb{Q}\Gamma} C_*$. Notice that $(C_*^{(2)})^{4n-*}$ is the same as $(C^{4n-*})^{(2)}$ and will be denoted by $C_{(2)}^{4n-*}$ and similarly for $C_*[k]$.

If $f_* \colon C^{4n-*} \to C_*$ is a $\mathbb{Q}\Gamma$-chain map, define $f_*^{(2)} \colon C_{(2)}^{4n-*} \to C_*^{(2)}$ as given by left multiplication with the matrix representing f_*, i.e. $f_*^{(2)} = \mathrm{id}_{l^2(\Gamma)} \otimes f_*$. In a similar way we define $f_*[k] \colon C^{4n-*}[k] \to C_*[k]$.

Let $f_* : C^{4n-*} \to C_*$ be a $\mathbb{Q}\Gamma$-chain map such that f_* and its dual f^{4n-*} are $\mathbb{Q}\Gamma$-chain homotopic. Then both $H_{2n}^{(2)}(f_*^{(2)})$ and $H_{2n}(f_*[k])$ are selfadjoint. We want to define the signature of such a chain complex.

1.3 Definition. Given a selfadjoint map $g : V \to V$ of Hilbert $\mathcal{N}\Gamma$-modules and an interval $I \subset \mathbb{R}$, let $\chi_I(g)$ be the map obtained from g by functional calculus for the characteristic function $\chi_I : \mathbb{R} \to \mathbb{R}$ of I. Define

$$\begin{aligned} b_+^{(2)}(g) &:= \operatorname{tr}_{\mathcal{N}\Gamma}(\chi_{(0,\infty)}(g)); \qquad b_-^{(2)}(g) := \operatorname{tr}_{\mathcal{N}\Gamma}(\chi_{(-\infty,0)}(g)); \\ b^{(2)}(g) &:= \dim_{\mathcal{N}\Gamma}(\ker(g)) = \operatorname{tr}_{\mathcal{N}\Gamma}(\chi_{\{0\}}(g)); \\ \operatorname{sign}^{(2)}(g) &:= b_+^{(2)}(g) - b_-^{(2)}(g). \end{aligned}$$

If $h : W \to W$ is a selfadjoint endomorphism of a finite-dimensional Hermitian complex vector space, define analogously

$$\begin{aligned} b_+(h) &:= \operatorname{tr}_{\mathbb{C}}(\chi_{(0,\infty)}(h)); \qquad b_-(h) := \operatorname{tr}_{\mathbb{C}}(\chi_{(-\infty,0)}(h)); \\ b(h) &:= \dim_{\mathbb{C}}(\ker(h)) \ = \ \operatorname{tr}_{\mathbb{C}}(\chi_{\{0\}}(h)); \\ \operatorname{sign}(h) &:= b_+(h) - b_-(h). \end{aligned}$$

Of course, $\operatorname{sign}(h)$ is the difference of the number of positive and of negative eigenvalues of h (counted with multiplicity).

1.4 Definition. Let $f_* : C^{4n-*} \to C_*$ be a $\mathbb{Q}\Gamma$-chain map such that f_* and its dual f^{4n-*} are $\mathbb{Q}\Gamma$-chain homotopic. Then both $H_{2n}^{(2)}(f_*^{(2)})$ and $H_{2n}(f_*[k])$ of Definition 1.2 are selfadjoint. Using Definition 1.3 define

$$\begin{aligned} b_{2n\pm}^{(2)}(f_*^{(2)}) &:= b_\pm^{(2)}(H_{2n}^{(2)}(f_*)); & b_{2n\pm}(f_*[k]) &:= b_\pm^{(2)}(H_{2n}(f_*[k])); \\ b_{2n}^{(2)}(f_*^{(2)}) &:= b^{(2)}(H_{2n}^{(2)}(f_*)); & b_{2n}(f_*[k]) &:= b(H_{2n}(f_*[k])); \\ \operatorname{sign}^{(2)}(f_*^{(2)}) &:= \operatorname{sign}^{(2)}(H_{2n}^{(2)}(f_*)); & \operatorname{sign}(f_*[k]) &:= \operatorname{sign}(H_{2n}(f_*[k])). \end{aligned}$$

1.2 The $\mathbb{Q}\Gamma$-chain complex of a Poincaré pair

A classical result proved e.g. in [13], or (with much more information) in [16, 17] says that, given a $4n$-dimensional Poincaré pair (X, Y) with Γ-covering $\overline{X} \to X$, the composition of the Poincaré $\mathbb{Q}\Gamma$-chain map $- \cap [\overline{X}, \overline{Y}] : C^{4n-*}(\overline{X}, \overline{Y}; \mathbb{Q}) \to C_*(\overline{X}; \mathbb{Q})$ with the $\mathbb{Q}\Gamma$-chain map induced by the inclusion yields a $\mathbb{Q}\Gamma$-chain map

$$f_*\colon C^{4n-*}(\overline{X}, \overline{Y}; \mathbb{Q}) \to C_*(\overline{X}, \overline{Y}; \mathbb{Q})$$

of finitely generated based free $4n$-dimensional $\mathbb{Q}\Gamma$-chain complexes such that f_* is $\mathbb{Q}\Gamma$-chain homotopic to f^{4n-*}.

The normal subgroups $\Gamma_k \subset \Gamma$ correspond to Γ/Γ_k-coverings (X_k, Y_k) of (X, Y) as explained in the introduction.

Use Definition 1.2 and Definition 1.4 to define

$$
\begin{aligned}
b^{(2)}_{2n\pm}(\overline{X},\overline{Y}) &:= b^{(2)}_{2n\pm}(f^{(2)}_*); & b_{2n\pm}(X_k,Y_k) &:= b_{2n\pm}(f_*[k]);\\
b^{(2)}_{2n}(\overline{X},\overline{Y}) &:= b^{(2)}_{2n}(f^{(2)}_*); & b_{2n}(X_k,Y_k) &:= b_{2n}(f_*[k]);\\
\operatorname{sign}^{(2)}(\overline{X},\overline{Y}) &:= \operatorname{sign}^{(2)}(f^{(2)}_*); & \operatorname{sign}(X_k,Y_k) &:= \operatorname{sign}(f_*[k]).
\end{aligned}
$$

Note that $C^{4n-*}[k]$ and $C_*[k]$ are the cellular $\mathbb{Q}[\Gamma/\Gamma_k]$-cochain and chain complexes of (X_k,Y_k), and $f[k]$ its Poincaré duality map. Therefore the definitions above coincide with the usual definitions of Betti numbers and signature for the compact Poincaré duality pairs (X_k,Y_k).

1.3 The proof of Theorem 0.1

Theorem 0.1 is an immediate consequence of

1.5 Theorem. *Let $f_*\colon C^{4n-*}(\overline{X},\overline{Y};\mathbb{Q}) \to C_*(\overline{X},\overline{Y};\mathbb{Q})$ be the $\mathbb{Q}\Gamma$-chain map introduced above. Then*

$$b^{(2)}_{2n\pm}(f^{(2)}_*) = \lim_{k\to\infty} \frac{b_{2n\pm}(f_*[k])}{[\Gamma:\Gamma_k]}.$$

The proof of Theorem 1.5 is split into a sequence of lemmas.

1.6 Lemma. *Let $A : l^2(\Gamma)^n \to l^2(\Gamma)^n$ be a selfadjoint Hilbert $\mathcal{N}\Gamma$-module morphism. Let $q_j : \mathbb{R}\to\mathbb{R}$ be a sequence of measurable functions converging pointwise to the function q such that $|q_j(x)| \le C$ on the spectrum of A, where C does not depend on j. Then*

$$\operatorname{tr}_{\mathcal{N}\Gamma}(q_j(A)) \xrightarrow{j\to\infty} \operatorname{tr}_{\mathcal{N}\Gamma}(q(A)).$$

Proof. By the spectral theorem, $q_j(A)$ converges strongly to $q(A)$. Moreover, $\|q_j(A)\| \le C$ for $j\in\mathbb{Z}$. By [7, p. 34] $q_j(A)$ converges ultra-strongly and therefore ultra-weakly to $q(A)$. Since $l^2(\Gamma)^n$ is a finite Hilbert-$\mathcal{N}\Gamma$-module $\mathbf{1} : l^2(\Gamma)^n \to l^2(\Gamma)^n$ is of Γ-trace class. Normality of the Γ-trace implies the conclusion (compare [7, Proposition 2 on p. 82] or [21, Theorem 2.3(4)]). $\square$

Let $\Delta_p := c_{p+1}\circ c^*_{p+1} + c^*_p \circ c_p : C_p \to C_p$ be the *combinatorial Laplacian* on $\overline{X}$, where we abbreviate $C_p := C_p(\overline{X},\overline{A};\mathbb{Q})$. Using a cellular basis of C_p coming from $C_p(\overline{X},\overline{Y};\mathbb{Z})$ this is given by a matrix over $\mathbb{Z}\Gamma$. Then $\Delta^{(2)}_p = c^{(2)}_{p+1}{c^{(2)}_{p+1}}^* + {c^{(2)}_p}^* c^{(2)}_p : C^{(2)}_p \to C^{(2)}_p$ is the Laplacian of $C^{(2)}_*$ and $\Delta_p[k] = c_{p+1}[k]c_{p+1}[k]^* + c_p[k]^*c_p[k]$ is the Laplacian on $C_p[k]$, i.e. the cellular Laplacian of X_k. Let $f_* : C^{4n-*} \to C_*$ be homotopic to its adjoint as introduced in the beginning of this section. The next lemma follows from [10, Lemma 2.5].

1.7 Lemma. *There is $K\ge 1$ such that for all $k \ge 1$*

$$\|\Delta^{(2)}_{2n}\|, \|\Delta_{2n}[k]\|, \|f^{(2)}_{2n}\|, \|f_{2n}[k]\| \le K.$$

1.8 Definition. In the sequel we write

$$\mathrm{tr}_k := \frac{\mathrm{tr}_{\mathbb{Q}}}{[\Gamma : \Gamma_k]}; \quad \dim_k := \frac{\dim_{\mathbb{Q}}}{[\Gamma : \Gamma_k]}; \quad \mathrm{sign}_k := \frac{\mathrm{sign}_{\mathbb{Q}}}{[\Gamma : \Gamma_k]},$$

and denote by $\mathrm{pr}_{2n}^{(2)} : C_{2n}^{(2)} \to C_{2n}^{(2)}$ and $\mathrm{pr}_{2n}[k] : C_{2n}[k] \to C_{2n}[k]$ the orthogonal projection onto the kernel of $\Delta_{2n}^{(2)}$ and $\Delta_{2n}[k]$, respectively.

1.9 Definition. For each $\epsilon > 0$ fix a polynomial $p^\epsilon(x) \in \mathbb{R}[x]$ with real coefficients satisfying $p^\epsilon(0) = 1$, $0 \leq p^\epsilon(x) \leq 1 + \epsilon$ for $|x| \leq \epsilon$ and $0 \leq p(x) \leq \epsilon$ for $\epsilon \leq |x| \leq K$ (where K is the constant of Lemma 1.7). Such polynomials exist by the Weierstrass approximation theorem [18, Theorem 7.26].

1.10 Lemma. *For each p and k we have*

$$\dim_k C_p[k] = \dim_{\mathcal{N}\Gamma} C_p^{(2)}(\overline{X}),$$

and hence in particular

$$\lim_{k\to\infty} \dim_k C_p[k] = \dim_{\mathcal{N}\Gamma} C_p^{(2)}(\overline{X}).$$

Proof. For every k, $\dim_k C_p(X_k)$ is equal to the number of p-cells in X, and the same is true for $\dim_{\mathcal{N}\Gamma} C_p^{(2)}(\overline{X})$. □

1.11 Lemma. *For $\mathbb{Q}\Gamma$-linear maps $h_1, \ldots, h_d : \mathbb{Q}\Gamma^r \to \mathbb{Q}\Gamma^r$ and a polynomial $p(x_1, \ldots, x_d)$ in non-commuting variables $x_1, \ldots, x_d$ we have*

$$\mathrm{tr}_{\mathcal{N}\Gamma}(p(h_1^{(2)}, \ldots, h_d^{(2)})) = \lim_{k\to\infty} tr_k \left(p(h_1[k], \ldots, h_d[k])\right).$$

Proof. By linearity it suffices to prove this for monomials $p = x_{i_1} \ldots x_{i_d}$, and since the h_j are not assumed to be different, without loss of generality we can assume $p = x_1 \ldots x_d$. The proof of [10, Lemma 2.6] applies and shows that there is $L > 0$ such that $\mathrm{tr}_{\mathcal{N}\Gamma}(h_1^{(2)} \circ \cdots \circ h_d^{(2)}) = \mathrm{tr}_k(h_1[k] \circ \cdots \circ h_d[k])$ for $k \geq L$. □

The lemma is formulated in a way that it can be applied if the assignment $h \to h[k]$ is not a homomorphism. This is unnecessary here, but will be needed in Section 2.

1.12 Lemma. *There is a constant $C_1 > 0$ (independent of k) such that for $0 < \epsilon < 1$ and $k \geq 1$*

$$\mathrm{tr}_k \left(\chi_{(0,\epsilon]}(\Delta_{2n}[k])\right) \leq \frac{C_1}{-\ln(\epsilon)}. \tag{1.13}$$

Proof. This is part of the conclusion of[10, Lemma 2.8]. □

1.14 Lemma. *There is a constant $C > 0$ (independent of k) such that for all $k \geq 1$ and $0 < \epsilon < 1$*

$$0 \leq \operatorname{tr}_k \left(|p^\epsilon(\Delta_{2n}[k]) - \operatorname{pr}_{2n}[k]| \right) \leq C \cdot \epsilon + \frac{C}{-\ln(\epsilon)}.$$

Recall that p^ϵ was fixed in Definition 1.9, and $\operatorname{pr}[k]$ *is defined in Definition 1.8. Moreover we have*

$$\lim_{\epsilon \to 0} \operatorname{tr}_{\mathcal{N}\Gamma} \left(\left| p^\epsilon(\Delta_{2n}^{(2)}) - \operatorname{pr}_{2n}^{(2)} \right| \right) = 0.$$

Proof. First observe that by our construction $p^\epsilon(\Delta_{2n}[k]) - \operatorname{pr}_{2n}[k]$ is non-negative since $0 \leq p^\epsilon - \chi_{\{0\}}$ on the spectrum of $\Delta_{2n}[k]$. We also have $p^\epsilon - \chi_{\{0\}} \leq \epsilon + \chi_{(0,\epsilon]}$ on the spectrum of the operators. Since the trace is positive, we get

$$0 \leq \operatorname{tr}_k(p^\epsilon(\Delta_{2n}[k]) - \operatorname{pr}_{2n}[k]) \quad \leq \quad \epsilon \operatorname{tr}_k(\operatorname{id}_{C_{2n}[k]}) + \operatorname{tr}_k(\chi_{(0,\epsilon]}(\Delta_{2n}[k])).$$

Now the first inequality follows from Lemma 1.10 and Lemma 1.12. The second one follows from

$$\operatorname{tr}_{\mathcal{N}\Gamma} \left(p^\epsilon(\Delta_{2n}^{(2)}) - \operatorname{pr}_{2n}^{(2)} \right) \quad \leq \quad \epsilon \operatorname{tr}_{\mathcal{N}\Gamma} \left(\operatorname{id}_{C_{2n}^{(2)}} \right) + \operatorname{tr}_{\mathcal{N}\Gamma} \left(\chi_{(0,\epsilon]}(\Delta_{2n}^{(2)}) \right)$$

and the fact that because of Lemma 1.6 $\lim_{\epsilon \to 0} \operatorname{tr}_{\mathcal{N}\Gamma} \left(\chi_{(0,\epsilon]}(\Delta_{2n}^{(2)}) \right) = 0$. $\square$

We also cite the following result [10, Theorem 2.3]:

1.15 Theorem. *The normalized sequence of Betti numbers converges, i.e. for each p*

$$\lim_{k \to \infty} \dim_k(\ker(\Delta_p[k])) = \dim_{\mathcal{N}\Gamma} \ker(\Delta_p^{(2)}).$$

For the proof of Theorem 1.5 eventually we want to approximate $\chi_{(a,b)}$ by polynomials. Next we check that for a fixed polynomial we can replace $\operatorname{pr}_{2n}[k]$ in the argument by $p^\epsilon(\Delta_{2n}[k])$.

1.16 Lemma. *Fix a polynomial $q \in \mathbb{R}[x]$. Then we find a constant $D > 0$ (independent of k) such that for all $k \geq 1$ and $0 < \epsilon < 1$*

$$\begin{aligned} &| \operatorname{tr}_k \left(q \left(p^\epsilon(\Delta_{2n}[k]) \circ f_{2n}[k] \circ p^\epsilon(\Delta_{2n}[k]) \right) \right) - \\ &\qquad \operatorname{tr}_k \left(q \left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k] \right) \right) | \leq D \cdot \epsilon + \frac{D}{-\ln(\epsilon)}. \end{aligned}$$

Moreover, we have

$$\lim_{\epsilon \to 0} \operatorname{tr}_{\mathcal{N}\Gamma} \left(q \left(p^\epsilon(\Delta_{2n}^{(2)}) \circ f_{2n}^{(2)} \circ p^\epsilon(\Delta_{2n}^{(2)}) \right) \right) = \operatorname{tr}_{\mathcal{N}\Gamma} \left(q \left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)} \right) \right).$$

Proof. By linearity it suffices to prove the statement for all monomials $q(x) = x^n$. Obviously it suffices to consider $n \geq 1$. In the sequel we abbreviate $x = p^\epsilon(\Delta_{2n}[k])$, $f = f_{2n}[k]$ and $y = \operatorname{pr}_{2n}[k]$. Notice that $\|x\| \leq (1+\epsilon)$, $\|f\| \leq K$ and $\|y\| \leq 1$ holds for the constant K appearing in Lemma 1.7. We estimate using the trace property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and the trace estimate $|\operatorname{tr}(AB)| \leq \|A\| \cdot \operatorname{tr}(|B|)$ (which also holds for the normalized traces tr_k and for $\operatorname{tr}_{\mathcal{N}\Gamma}$ by [7, p. 106] since all the traces we are considering are normal),

$$\begin{aligned}
&|\operatorname{tr}_k\left((p^\epsilon(\Delta_{2n}[k]) \circ f_{2n}[k] \circ p^\epsilon(\Delta_{2n}[k]))^n\right) \\
&\qquad - \operatorname{tr}_k\left((\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k])^n\right)| \\
&= |\operatorname{tr}_k\left(xfxxfx\ldots xfx - yfyyfy\ldots yfy\right)| \\
&= |\operatorname{tr}_k\left((x-y)fxxfx\ldots xfx + yf(x-y)xfx\ldots xfx\right. \\
&= \qquad\qquad \left. + \, yfy(x-y)fxxfx\ldots fx + \ldots + yfyyfy\ldots yf(x-y)\right)| \\
&\leq 2n \cdot (1+\epsilon)^{2n-1} \cdot K^n \cdot \operatorname{tr}(|x-y|) \\
&= 2n \cdot (1+\epsilon)^{2n-1} \cdot K^n \cdot \operatorname{tr}_k\left(|p^\epsilon(\Delta_{2n}[k]) - \operatorname{pr}_{2n}[k]|\right).
\end{aligned}$$

Exactly the same reasoning applies if $\Delta_{2n}[k]$ and $\operatorname{pr}_{2n}[k]$ is replaced by $\Delta_{2n}^{(2)}$ and $\operatorname{pr}_{2n}^{(2)}$, respectively, to give the corresponding estimate in this case.

The assertion of the lemma now follows from Lemma 1.14. □

1.17 Lemma. *Fix a polynomial $q(x) \in \mathbb{R}[x]$. Then*

$$\lim_{k\to\infty} \operatorname{tr}_k\left(q\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right) = \operatorname{tr}_{\mathcal{N}\Gamma}\left(q\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right).$$

Proof. Fix $\delta > 0$. By Lemma 1.16 we find $\epsilon > 0$ such that for all $k \geq 1$

$$\begin{aligned}
&\big|\operatorname{tr}_k\left(q\left(p^\epsilon(\Delta_{2n}[k]) \circ f_{2n}[k] \circ p^\epsilon(\Delta_{2n}[k])\right)\right) - \\
&\qquad\qquad \operatorname{tr}_k\left(q\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right)\big| \leq \delta/3;
\end{aligned}$$

$$\begin{aligned}
&\big|\operatorname{tr}_{\mathcal{N}\Gamma}\left(q\left(p^\epsilon(\Delta_{2n}^{(2)}) \circ f_{2n}^{(2)} \circ p^\epsilon(\Delta_{2n}^{(2)})\right)\right) - \\
&\qquad\qquad \operatorname{tr}_{\mathcal{N}\Gamma}\left(q\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right)\big| \leq \delta/3.
\end{aligned}$$

Hence it suffices to show for each fixed ϵ

$$\begin{aligned}
&\lim_{k\to\infty} \operatorname{tr}_k\big(q\big(p^\epsilon(c_{p+1}[k]c_{p+1}[k]^* + \\
&\qquad c_p[k]^*c_p[k]) \circ f_{2n}[k] \circ p^\epsilon(c_{p+1}[k]c_{p+1}[k]^* + c_p[k]^*c_p[k])\big)\big) \\
&= \operatorname{tr}_{\mathcal{N}\Gamma}\big(q\left(p^\epsilon(c_{p+1}^{(2)}{c_{p+1}^{(2)}}^* + {c_p^{(2)}}^* c_p^{(2)}) \circ f_{2n}^{(2)} \circ p^\epsilon(c_{p+1}^{(2)}{c_{p+1}^{(2)}}^* + {c_p^{(2)}}^* c_p^{(2)})\right)\big).
\end{aligned}$$

Since q and p^ϵ are fixed, we deal with a fixed polynomial expression in c_p, c_p^*, c_{p+1}, c_{p+1}^*, and f_{2n}. Therefore the last claim follows from Lemma 1.11. This finishes the proof of Lemma 1.17. □

1.18 Lemma. *We have for $a, b \in \mathbb{R}$ with $a < b$*

$$\operatorname{tr}_{\mathcal{N}\Gamma}\left(\chi_{(a,b)}\left(H_p^{(2)}(f_*^{(2)})\right)\right) \leq \liminf_{k\to\infty} \operatorname{tr}_k\left(\chi_{(a,b)}\left(H_p(f_*[k])\right)\right).$$

Proof. We approximate $\chi_{(a,b)}$ by polynomials. Namely, for $0 < \epsilon < (b-a)/2$ and K as above let $q^\epsilon \in \mathbb{R}[x]$ be a polynomial with

$$\begin{aligned} -1 \leq q^\epsilon(x) \leq \chi_{(a,b)}(x) & \qquad \text{for } |x| \leq K; \\ q^\epsilon(x) \geq \chi_{(a,b)}(x) - \epsilon & \qquad \text{for } x \in [-K,a] \cup [a+\epsilon, b-\epsilon] \cup [b,K]. \end{aligned}$$

Under the identification of $\operatorname{im}(\operatorname{pr}_{2n}[k])$ and $H_p(C_*[k])$ coming from the (combinatorial) Hodge decomposition the operator $\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]$ restricted to $\operatorname{im}(\operatorname{pr}_{2n}[k])$ becomes $H_p(f_*[k])$ which is selfadjoint because of $f_* \simeq f^{4n-*}$. Hence $\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]$ and also the operator $q^\epsilon(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k])$ are selfadjoint. Exactly the same is true on the L^2-level and we conclude

$$\begin{aligned} \operatorname{tr}_k\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right) &= \operatorname{tr}_k\left(\chi_{(a,b)}\left(H_p(f_*[k])\right)\right); & (1.19) \\ \operatorname{tr}_{\mathcal{N}\Gamma}\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right) &= \operatorname{tr}_{\mathcal{N}\Gamma}\left(\chi_{(a,b)}\left(H_p^{(2)}(f_*^{(2)})\right)\right). & (1.20) \end{aligned}$$

Positivity of the trace and $q^\epsilon(x) \leq \chi_{(a,b)}(x)$ for all x in the spectrum of $\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]$ implies

$$\operatorname{tr}_k\left(q^\epsilon\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right) \leq \operatorname{tr}_k\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right).$$

Note that for fixed q^ϵ the left hand side converges for $k \to \infty$ by Lemma 1.17. For the right hand side this is not clear, but in any case we get

$$\begin{aligned} &\operatorname{tr}_{\mathcal{N}\Gamma}\left(q^\epsilon\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right) \\ &\qquad \leq \liminf_{n\to\infty} \operatorname{tr}_k\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right). \qquad (1.21) \end{aligned}$$

On the spectrum of the operator in question, the functions q^ϵ are uniformly bounded and converge pointwise to $\chi_{(a,b)}$ if $\epsilon \to 0$. By Lemma 1.6

$$\lim_{\epsilon\to 0} \operatorname{tr}_{\mathcal{N}\Gamma}\left(q^\epsilon\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right) = \operatorname{tr}_{\mathcal{N}\Gamma}\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right).$$

Since inequality (1.21) holds for arbitrary $\epsilon > 0$, we conclude

$$\begin{aligned} &\operatorname{tr}_{\mathcal{N}\Gamma}\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}^{(2)} \circ f_{2n}^{(2)} \circ \operatorname{pr}_{2n}^{(2)}\right)\right) \\ &\qquad \leq \liminf_{k\to\infty} \operatorname{tr}_k\left(\chi_{(a,b)}\left(\operatorname{pr}_{2n}[k] \circ f_{2n}[k] \circ \operatorname{pr}_{2n}[k]\right)\right). \end{aligned}$$

Now the claim follows from (1.19) and (1.20). □

1.22 Lemma. *Let $f_* : C_* \to D_*$ be a $\mathbb{Q}\Gamma$-chain map of finitely generated based free $\mathbb{Q}\Gamma$-chain complexes. Then we get for all p*

$$\lim_{k\to\infty} \dim_k\left(\ker\left(H_p(f_*[k])\right)\right) = \dim_{\mathcal{N}\Gamma}\left(\ker\left(H_p^{(2)}(f_*^{(2)})\right)\right).$$

Proof. We can assume without loss of generality that C_* and D_* are $(p+1)$-dimensional. Consider the long exact sequence of left $\mathbb{Q}\Gamma$-chain complexes $0 \to D_* \to \operatorname{cone}(f_*)_* \to \Sigma C_* \to 0$, where $\operatorname{cone}(f_*)_*$ is the mapping cone of f_* and ΣC_* the suspension of C_*. It is a split exact sequence in each dimension and thus remains exact after applying $l^2(\Gamma) \otimes_{\mathbb{Q}\Gamma} -$. The weakly exact long homology sequence yields a weakly exact sequence of Hilbert $\mathcal{N}(\Gamma)$-modules

$$0 \to H_{p+2}^{(2)}(\operatorname{cone}(f_*)_*^{(2)}) \to H_{p+1}^{(2)}(C_*^{(2)}) \xrightarrow{H_{p+1}^{(2)}(f_*^{(2)})} H_{p+1}^{(2)}(D_*^{(2)})$$

$$\to H_{p+1}^{(2)}(\operatorname{cone}(f_*)_*^{(2)})) \to \ker(H_p(f_*^{(2)})) \to 0.$$

This implies

$$\begin{aligned} &\dim_{\mathcal{N}\Gamma}\left(\ker(H_p(f_*^{(2)}))\right) \\ &= \dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(\operatorname{cone}(f_*)_*^{(2)})\right) - \dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(D_*^{(2)})\right) \\ &\quad + \dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(C_*^{(2)})\right) - \dim_{\mathcal{N}\Gamma}\left(H_{p+2}^{(2)}(\operatorname{cone}(f_*)_*^{(2)})\right). \end{aligned} \tag{1.23}$$

Analogously we get

$$\begin{aligned} &\dim_k\left(\ker(H_p(f_*[k]))\right) \\ &= \dim_k\left(H_{p+1}(\operatorname{cone}(f_*[k])_*)\right) - \dim_k\left(H_{p+1}(D_*[k])\right) \\ &\quad + \dim_k\left(H_{p+1}(C_*[k])\right) - \dim_k\left(H_{p+2}(\operatorname{cone}(f_*[k])_*)\right). \end{aligned} \tag{1.24}$$

We conclude from Theorem 1.15

$$\dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(\operatorname{cone}(f_*)_*^{(2)})\right) = \lim_{k\to\infty} \dim_k\left(H_{p+1}(\operatorname{cone}(f_*[k])_*)\right); \tag{1.25}$$

$$\dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(D_*^{(2)})\right) = \lim_{k\to\infty} \dim_k\left(H_{p+1}(D_*[k])\right); \tag{1.26}$$

$$\dim_{\mathcal{N}\Gamma}\left(H_{p+1}^{(2)}(C_*^{(2)})\right) = \lim_{k\to\infty} \dim_k\left(H_{p+1}(C_*[k])\right); \tag{1.27}$$

$$\dim_{\mathcal{N}\Gamma}\left(H_{p+2}^{(2)}(\operatorname{cone}(f_*)_*^{(2)})\right) = \lim_{k\to\infty} \dim_k\left(H_{p+2}(\operatorname{cone}(f_*[k])_*)\right). \tag{1.28}$$

Now the claim follows from equations (1.23)–(1.28). □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 0.1. We get from Lemma 1.18 and Lemma 1.22

$$b_{2n+}^{(2)}(g_*^{(2)}) \leq \liminf_{k\to\infty} \frac{b_{2n+}(g_*[k])}{[\Gamma : \Gamma_k]}; \qquad b_{2n-}^{(2)}(g_*^{(2)}) \leq \liminf_{k\to\infty} \frac{b_{2n-}(g_*[k])}{[\Gamma : \Gamma_k]};$$

$$b_p^{(2)}(g_*^{(2)}) = \lim_{k\to\infty} \frac{b_p(g_*[k])}{[\Gamma : \Gamma_k]}.$$

Since

$$\begin{aligned}
b_{2n+}^{(2)}(g_*^{(2)}) + b_{2n-}^{(2)}(g_*^{(2)}) + b_{2n}^{(2)}(g_*^{(2)}) &= \dim_{\mathcal{N}\Gamma}\left(C_{2n}^{(2)}\right);\\
\frac{b_{2n+}(g_*[k])}{|\Gamma/\Gamma_k|} + \frac{b_{2n-}(g_*[k])}{|\Gamma/\Gamma_k|} + \frac{b_{2n}(g_*[k])}{|\Gamma/\Gamma_k|} &= \dim_k(C_{2n}[k]);\\
\dim_{\mathcal{N}\Gamma}\left(C_{2n}^{(2)}\right) = \lim_{k\to\infty}\dim_k(C_{2n}[k]) &= \frac{\dim_{\mathbb{Q}}(C_{2n}[k])}{[\Gamma:\Gamma_k]},
\end{aligned}$$

Theorem 1.5 and thus Theorem 0.1 follow from Lemma 1.10. □

1.4 Further remarks

1.29 Remark. Theorem 0.1 can be applied to a $4n$-dimensional Riemannian manifold X with boundary Y. In this case, the Atiyah-Patodi-Singer theorem [2, Theorem 4.14] and [5, (0.9)] and the L^2-signature theorem of [13] imply

$$\begin{aligned}
\frac{\operatorname{sign}(X_k,\partial X_k)}{\operatorname{vol}(X_k)} &= \frac{1}{\operatorname{vol}(X_k)}\cdot\int_{X_k} L(X_k) + \frac{\eta(\partial X_k)}{\operatorname{vol}(X_k)} + \frac{1}{\operatorname{vol}(X_k)}\cdot\int_{\partial X_k}\Pi_L(\partial X_k),\\
\frac{\operatorname{sign}^{(2)}(\overline{X},\overline{\partial X})}{\operatorname{vol}(X)} &= \frac{1}{\operatorname{vol}(X)}\cdot\int_X L(X) + \frac{\eta^{(2)}(\partial\overline{X})}{\operatorname{vol}(X)} + \frac{1}{\operatorname{vol}(X)}\cdot\int_{\partial X}\Pi_L(\partial X).
\end{aligned}$$

Here $L(X_k)$ and $L(X)$ denote the Hirzebruch L-polynomial, and $\Pi_L(\partial X_k)$ and $\Pi_L(\partial X)$ are a local correction terms which arises because the metric is not a product near the boundary. Being local expressions, the first and the third summand does not depend on k. It follows that the sequence of η-invariants converges. In fact, even without the assumption that Y^{4n-1} is a boundary of a suitable manifold X, in [24, Theorem 3.12] it is proved

$$\lim_{k\to\infty}\frac{\eta(Y_k)}{[\Gamma:\Gamma_k]} = \eta^{(2)}(\overline{Y}).$$

Key ingredients are on the one hand the analysis of Cheeger-Gromov in [5, Section 7] of the formulas (2.24) and (2.25) (which holds for operators different from the signature operator). We present similar considerations in Section 2.1. The second key ingredient is Lück's approximation result for L^2-Betti numbers [10, Theorem 0.1] (which is special to the Laplacian, the square of the signature operator).

1.30 Remark. The normalized signatures $\frac{\operatorname{sign}(X_k,Y_k)}{|\Gamma/\Gamma_k|}$ are the L^2-signatures $\operatorname{sign}^{(2)}(X_k,Y_k)$ of the Γ/Γ_k-coverings $(X_k,Y_k)\to(X,Y)$. With this reformulation, one may ask whether Theorem 0.1 holds if Γ/Γ_k is not necessarily finite.

This is indeed the case if the groups Γ/Γ_k belong to a large class of groups $\mathcal{G}$ defined in [20, Definition 1.11].

The corresponding question for L^2-Betti numbers is answered affirmatively in [20, Theorem 6.9] whenever $\Gamma/\Gamma_k\in\mathcal{G}$. As just mentioned, Theorem 0.1 extends to this situation as well, and the proof we have given is formally unchanged, using the generalization of Lemma 1.7 and Lemma 1.11 given in [20, Lemma 5.5 and 5.6]. It only remains to establish Lemma 1.12, which is not done in [20]. We do this in the following Lemma 1.31, which applies because of [20, 6.9] and because of Lemma 1.7.

1.31 Lemma. *If* $\|\Delta[X_k]\| \leq K$ *and*

$$\ln \det{}'_{(2)}(\Delta[X_k]) := \int_{0^+}^{\infty} \ln(\lambda)\ dF_{\Delta[X_k]}(\lambda) \geq 0 \tag{1.32}$$

then

$$\operatorname{tr}_k(\chi_{(0,\epsilon]}(\Delta[X_k])) \leq \frac{d \cdot \ln(K)}{-\ln(\epsilon)}. \tag{1.33}$$

Here $F_{\Delta[X_k]}(\lambda) := \operatorname{tr}_k(\chi_{[0,\lambda]}(\Delta[k]))$ *is the spectral density function of the operator* $\Delta[k]$ *computed using* tr_k *instead of* $\dim_{\mathbb{C}}$*, and* $d = F_{\Delta[k]}(K)$ *is the number of rows (and columns) of the matrix* Δ*.*

Proof. We argue as follows (with $F := F_{\Delta[k]}$):

$$\begin{aligned}\int_{0^+}^{\infty} \ln(\lambda)\ dF(\lambda) &= \int_{0^+}^{\epsilon} \ln(\lambda)\ dF(\lambda) + \int_{\epsilon}^{\|\Delta[k]\|} \ln(\lambda)\ dF(\lambda) \\ &\leq \ln(\epsilon)\ \underbrace{(F(\epsilon) - F(0))}_{=\operatorname{tr}_k(\chi_{(0,\epsilon]}(\Delta[k]))} + \ln(\|\Delta[k]\|)F(\|\Delta[k]\|).\end{aligned}$$

For $0 < \epsilon < 1$, using the bound $\|\Delta[k]\| \leq K$ of the generalization of Lemma 1.7, Inequality (1.32) immediately gives (1.33). $\square$

2 Amenable convergence of signatures

2.1 Analytic version

In this subsection we want to prove Theorem 0.4. We will use the following notion of manifold with bounded geometry (compare e.g. [12, Definition 2.24]).

2.1 Definition. A Riemannian manifold (M, g) (the boundary may or may not be empty) is called a manifold of *bounded geometry* if *bounded geometry constants* C_q for $q \in \mathbb{N}$ and $R_I, R_C > 0$ exist, so that the following holds:

(1) The geodesic flow of the unit inward normal field induces a diffeomorphism of $[0, 2R_C) \times \partial M$ onto its image, the geodesic collar;

(2) For $x \in M$ with $d(x, \partial M) > R_C/2$ the exponential map $T_xM \to M$ is a diffeomorphism on $B_{R_I}(0)$;

(3) The injectivity radius of ∂M is bigger than R_I;

(4) For every $q \in \mathbb{N}$ we have $\left|\nabla^i R\right| \leq C_k$ and $\left|\nabla_\partial^i l\right| \leq C_l$ for $0 \leq i \leq q$, where R is the curvature tensor of M, l the second fundamental form tensor of ∂M, and ∇^i and ∇_∂^i are the covariant derivatives of M and ∂M.

By [22, Theorem 2.4] this is equivalent to [12, Definition 2.24].

Every compact manifold, or more generally every covering of a compact manifold, is a manifold with bounded geometry.

We now repeat a few well known facts about manifolds of bounded geometry.

2.2 Proposition. *Let M be a compact smooth Riemannian manifold. There is a constant $A > 0$, depending only on the bounded geometry constants and the dimension of M, such that*

$$\begin{aligned} |\exp(-t\Delta_p(M))(x,x)| &\leq A && \text{for } t \geq 1, x \in M; \\ b_p(M) &\leq A \operatorname{vol}(M); \\ b_p(M, \partial M) &\leq A \operatorname{vol}(M), \end{aligned}$$

where the Laplacian can be taken with either relative or absolute boundary conditions.

Proof. The first inequality is proved in [12, Theorem 2.35]. The claim for the Betti numbers is a consequence of the fact that the Betti number $b_p(M)$ or $b_p(M, \partial M)$ can be written as $\lim_{t\to\infty} \int_M \operatorname{tr}_x \exp(-t\Delta_p(M))(x,x)\ dx$ for the Laplacian with absolute or relative boundary conditions, respectively. □

2.3 Theorem. *Let M, N be Riemannian manifolds without boundary which are of bounded geometry and with a fixed set of bounded geometry constants. Let U be an open subset of M which is isometric to a subset of N (which we identify with U). For $R > 0$ set*

$$U_R := \{x \in U|\ d(x, M - U) \geq R \text{ and } d(x, N - U) \geq R\}.$$

Let $D[M]$ and $D[N]$ be the (tangential) signature operators on M and N, respectively; and similarly $\Delta[M]$ and $\Delta[N]$ the Laplacian (on differential forms). Let $e^{-t\Delta}(x,y)$ and $De^{-tD^2}(x,y)$ be the integral kernels (which are smooth) of the operators $e^{-t\Delta}$ and De^{-tD^2}. Then there are constants $C_1, C_2 > 0$ which depend only on the dimension and the given bounded geometry constants such that for $t > 0$, $x \in U_R$

$$\left| e^{-t\Delta[M]}(x,x) - e^{-t\Delta[N]}(x,x) \right| \leq C_1 \cdot e^{-R^2 C_2/t}; \tag{2.4}$$

$$\left| D[M]e^{-tD[M]^2}(x,x) - D[N]e^{-tD[N]^2}(x,x) \right| \leq C_1 \cdot e^{-R^2 C_2/t}. \tag{2.5}$$

Proof. This follows by a standard argument of Cheeger-Gromov-Taylor [6] from unit propagation speed and local elliptic estimates (here the bounded geometry constants come in). A detailed account is given in the proof of [12, Theorem 2.26] which yields immediately (2.4). Replacing $\sqrt{\Delta}$ by D (which is possible since we are looking for manifolds without boundary, so that we do not have to worry about the non-locality of boundary conditions and therefore have unit propagation speed for D, too), the proof also applies to the tangential signature operator to give (2.5). □

2.6 Proposition. *Let M^m be a manifold of bounded geometry with fixed bounded geometry constants and with $\partial M = \emptyset$. Let D be the (tangential) signature operator on M. Then there is a function $A : [0,\infty) \to (0,\infty)$ which depends only on the bounded geometry constants and the dimension m, such that for $T \geq 0$*

$$\left|\operatorname{tr}_x\left(De^{-tD^2}(x,x)\right)\right| \leq A(T) \cdot t^{1/2} \qquad \text{for } 0 \leq t \leq T, x \in M.$$

Proof. One can use the proof of [14, Lemma 3.1.1 on p. 324] (where a slightly different statement is proved). The proposition is also implicit in [5, Proof of Theorem 0.1 on p. 140]. The proof uses the cancellation of the coefficients of negative powers of t in the local asymptotic expansion due to Bismut and Freed [4, Theorem 2.4] and a localization argument based on elliptic estimates (here the local geometry comes in), together with the finite propagation speed method of Cheeger-Gromov-Taylor [6]. □

We fix the following notation.

2.7 Notation. In the situation of Definition 0.3 put for $r \geq 0$

$$U_r(Y_k) := \{x \in \overline{X};\ d(x, Y_k) \leq r\},$$

where two points $y, z \in \overline{X}$ have distance $d(y,z) = d$ if there is a geodesic of length d in $\overline{X}$ joining y and z and $d = \infty$ if there is no such geodesic. In particular $d(y,z) < \infty$ implies that y and z lie in the same path component of $\overline{X}$. Let $\mathcal{F}$ be a (compact) connected simplicial fundamental domain for X in $\overline{X}$ such that $\mathcal{F} \cap \partial\overline{X}$ is a fundamental domain for ∂X. (We can construct $\mathcal{F}$ as a union of lifts of the top-dimensional simplices in a smooth triangulation of X and achieve $\mathcal{F}$ to be connected, since X is connected by assumption.) For $r \geq 0$ let $N_k(r)$ be the number of translates of $\mathcal{F}$ contained in $X_k - U_r(Y_k)$ and $n_k(r)$ the number of translates of $\mathcal{F}$ which have a non-trivial intersection with $U_r(Y_k)$. Set $N_k := N_k(0)$; $n_k := n_k(0)$.

The next lemma shows that our Definition 0.3 of a regular exhaustion coincides with the one given by Dodziuk and Mathai [8], with one exception: we require a lower bound on the injectivity radius of the boundaries ∂X_k and control of the covariant derivatives of the second fundamental form, what they seem to have forgotten (but also use).

2.8 Lemma. *If $(X_k)_{k\geq 1}$ is a regular exhaustion of $\overline{X}$ as in Definition 0.3, then for each $r \geq 0$*

$$\lim_{k\to\infty} \frac{\operatorname{vol}(U_r(Y_k))}{\operatorname{vol}(X_k)} = 0.$$

Proof. To obtain this we discretize: Choose $\epsilon > 0$ such that 4ϵ is smaller than the injectivity radius, and choose sets of points $P_k \subset Y_k$ such that the balls of radius ϵ around $x \in P_k$ are mutually disjoint, but the balls of radius 4ϵ are a covering of Y_k. Because of bounded geometry (compare the proof of [23, Lemma 1.2 in Appendix 1]), we find $c_1, c_2 > 0$ independent of k such that

$$c_1 |P_k| \leq \operatorname{area}(Y_k) \leq c_2 |P_k|.$$

The triangle inequality implies $U_r(Y_k) \subset \bigcup_k B_{r+4\epsilon}(x_k)$. Therefore

$$\mathrm{vol}(U_r(Y_k)) \leq C_{r+4\epsilon}\,|P_k| \leq C_{r+4\epsilon} c_1^{-1}\,\mathrm{area}(Y_k),$$

where $C_{r+4\epsilon}$ is a uniform upper bound for the volume of balls of radius $r+4\epsilon$ in $\overline{X}$ which exists because of bounded geometry. Since we have by assumption $\lim_{k\to\infty} \frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)} = 0$, Lemma 2.8 follows. □

2.9 Lemma. *If $(X_k)_{k\geq 1}$ is a regular exhaustion of $\overline{X}$ as in Definition 0.3, then*

$$\lim_{k\to\infty} \frac{\mathrm{area}(\partial X_k \cap \partial\overline{X})}{\mathrm{vol}(X_k)} = \frac{\mathrm{area}(\partial X)}{\mathrm{vol}(X)}.$$

Proof. Obviously $\mathrm{vol}(\mathcal{F}) = \mathrm{vol}(X)$ and $\mathrm{area}(\mathcal{F} \cap \partial\overline{X}) = \mathrm{area}(\partial X)$. Recall that $\mathcal{F}$ is connected. Suppose that $\mathcal{F} \cap X_k \neq \emptyset$ and $\mathcal{F} \not\subset X_k - U_r(Y_k)$. Then, for each r, $\mathcal{F}$ must intersect $U_r(Y_k)$ because otherwise we can find a path in $\mathcal{F}$ connecting a point in X_k to a point in $X - X_k$ and this path must meet Y_k. Hence we get for $r \geq 0$, using Notation 2.7

$$N_k(r)\cdot \mathrm{vol}(X) \leq \mathrm{vol}(X_k) \leq (N_k(r) + n_k(r))\cdot \mathrm{vol}(X); \tag{2.10}$$

$$N_k(r)\cdot \mathrm{area}(\partial X) \leq \mathrm{area}(\partial X_k \cap \partial\overline{X}) \leq (N_k(r) + n_k(r))\cdot \mathrm{area}(\partial X). \tag{2.11}$$

If follows that

$$\frac{N_k \cdot \mathrm{area}(\partial X)}{(N_k + n_k)\cdot \mathrm{vol}(X)} \leq \frac{\mathrm{area}(\partial X_k \cap \partial\overline{X})}{\mathrm{vol}(X_k)} \leq \frac{(N_k + n_k)\cdot \mathrm{area}(\partial X)}{N_k\,\mathrm{vol}(X)}. \tag{2.12}$$

Since $\mathcal{F} \cap U_r(Y_k) \neq \emptyset$ implies $\mathcal{F} \subset U_{r+\mathrm{diam}(\mathcal{F})}(Y_k)$, we have $n_k(r)\cdot\mathrm{vol}(X) \leq \mathrm{vol}(U_{r+\mathrm{diam}(\mathcal{F})}(Y_k))$. Therefore (2.10) implies

$$\frac{n_k(r)}{n_k(r) + N_k(r)} = \frac{n_k(r)\,\mathrm{vol}(X)}{(n_k(r) + N_k(r))\,\mathrm{vol}(X)} \leq \frac{\mathrm{vol}(U_{r+\mathrm{diam}(\mathcal{F})}(Y_k))}{\mathrm{vol}(X_k)}.$$

From Lemma 2.8 we conclude

$$\lim_{k\to\infty} \frac{n_k(r)}{N_k(r)} = 0. \tag{2.13}$$

Now Lemma 2.9 follows from (2.12) and (2.13). □

2.14 Theorem. *If $(X_k)_{k\geq 1}$ is a regular exhaustion of $\overline{X}$ as in Definition 0.3, then, using Notation 2.7,*

$$\lim_{k\to\infty} \frac{b_p(\partial X_k)}{N_k} = \lim_{k\to\infty} \frac{b_p(\partial X_k)\cdot\mathrm{vol}(X)}{\mathrm{vol}(X_k)} = b_p^{(2)}(\partial\overline{X}).$$

Proof. Let $V_k \subset \partial X_k \cap \partial\overline{X}$ be the union of translates $g\mathcal{F} \cap \partial\overline{X}$ for $g \in \Gamma$ such that $g\mathcal{F} \subset X_k - Y_k$. The number of these translates $g\mathcal{F} \cap \partial\overline{X}$ is just N_k. The number $\dot{N}_{m,\delta}$ of "boundary pieces" appearing in [9] is bounded by $C_\delta \cdot n_k$ for a constant C_δ which does not depend on k. Because of Inequality (2.13), $(V_k)_{\geq k}$ is a regular exhaustion of $\partial\overline{X}$ in the sense of [9] by (2.13). We conclude from [9, Theorem 0.1]

$$\lim_{k\to\infty} \frac{b_p(V_k)}{N_k} = b_p^{(2)}(\partial\overline{X}). \tag{2.15}$$

We can thicken V_k inside of $\partial\overline{X}$ to a regular neighborhood V_k'. From Proposition 2.2 we obtain a constant A independent of k such that

$$\begin{aligned} b_p(\partial X_k - \mathrm{int}(V_k'), \partial V_k') &\leq A \cdot \mathrm{vol}(\partial X_k - \mathrm{int}(V_k')) \\ &\leq A \cdot (\mathrm{vol}(Y_k) + n_k \cdot \mathrm{vol}(\partial\overline{X} \cap \mathcal{F})). \end{aligned} \tag{2.16}$$

We have by excision $b_p(\partial X_k, V_k') = b_p(\partial X_k - \mathrm{int}(V_k'), \partial V_k')$ and by homotopy invariance $b_p(V_k) = b_p(V_k')$. From (2.16) and the long exact homology sequence of the pair $(\partial X_k, V_k)$ we conclude

$$|b_p(\partial X_k) - b_p(V_k)| \leq 2A \cdot (\mathrm{vol}(Y_k) + n_k \cdot \mathrm{vol}(\partial\overline{X} \cap \mathcal{F})). \tag{2.17}$$

We get from (2.10) and (2.13) (since $\mathrm{vol}(Y_k)/\,\mathrm{vol}(X_k) \xrightarrow{k\to\infty} 0$ by assumption) that

$$\lim_{k\to\infty} \frac{2A\left(\mathrm{vol}(Y_k) + n_k \cdot \mathrm{vol}(\partial\overline{X} \cap \mathcal{F})\right)}{N_k} = 0. \tag{2.18}$$

We conclude from (2.15) and (2.17) and (2.18) that

$$\lim_{k\to\infty} \frac{b_p(\partial X_k)}{N_k} = b_p^{(2)}(\partial\overline{X}). \tag{2.19}$$

Now Theorem 2.14 follows from (2.10), (2.13) and (2.19).

∎

Remember that the Atiyah-Patodi-Singer index theorem [2, Theorem 4.14] and [5, (0.9)] and its L^2-version (compare e.g. [13]) imply for manifolds as in Definition 0.3

$$\begin{aligned} \frac{\mathrm{sign}(X_k, \partial X_k)}{\mathrm{vol}(X_k)} &= \frac{1}{\mathrm{vol}(X_k)} \cdot \int_{X_k} L(X_k) + \frac{\eta(\partial X_k)}{\mathrm{vol}(X_k)} + \frac{1}{\mathrm{vol}(X_k)} \cdot \int_{\partial X_k} \Pi_L(\partial X_k), \\ \frac{\mathrm{sign}^{(2)}(\overline{X}, \overline{\partial X})}{\mathrm{vol}(X)} &= \frac{1}{\mathrm{vol}(X)} \cdot \int_X L(X) + \frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} + \frac{1}{\mathrm{vol}(X)} \cdot \int_{\partial X} \Pi_L(\partial X). \end{aligned}$$

Here $L(X_k)$ and $L(X)$ denote the Hirzebruch L-polynomial, and $\Pi_L(\partial X_k)$ and $\Pi_L(\partial X)$ are local correction terms which arises because the metric is not a product near the boundary. We want to show that each of the individual summands converges for $k \to \infty$ to the corresponding term for $\overline{X}$.

The L-polynomial is given in terms of the curvature, Π_L in terms of the second fundamental form, therefore both are uniformly bounded independent of k by some constant C.

Moreover, because these are local expressions, the integral over each translate of the connected fundamental domain $\mathcal{F}$ which is contained in $X_k - Y_k$ coincides with the corresponding integral on X or ∂X. Then by splitting the domain of integration appropriately (as done in the proofs above), and using Notation 2.7,

$$\begin{aligned} \left| \int_{X_k} L(X_k) - N_k \cdot \int_X L(X) \right| &\leq n_k \cdot \operatorname{vol}(X) \cdot C; &(2.20)\\ \left| \int_{\partial X_k} \Pi_L(\partial X_k) - N_k \cdot \int_{\partial X} \Pi_L(\partial X) \right| &\leq \operatorname{area}(Y_k) \cdot C \\ &\quad + n_k \cdot \operatorname{area}(\partial X) \cdot C. &(2.21) \end{aligned}$$

We conclude from (2.10) and (2.20)

$$\begin{aligned} &\left| \frac{1}{\operatorname{vol}(X_k)} \cdot \int_{X_k} L(X_k) - \frac{1}{\operatorname{vol}(X)} \cdot \int_X L(X) \right| \\ &\leq \left| \left(\frac{1}{\operatorname{vol}(X_k)} - \frac{1}{N_k \cdot \operatorname{vol}(X)} \right) \cdot \int_{X_k} L(X_k) \right| \\ &\quad + \frac{1}{N_k \cdot \operatorname{vol}(X)} \cdot \left| \int_{X_k} L(X_k) - N_k \int_X L(X) \right| \\ &\overset{(2.20)}{\leq} \left| \frac{1}{\operatorname{vol}(X_k)} - \frac{1}{N_k \operatorname{vol}(X)} \right| \cdot \operatorname{vol}(X_k) \cdot C + \frac{n_k}{N_k} \cdot C \overset{(2.10)}{\leq} 2C \cdot \frac{n_k}{N_k}. \end{aligned} \qquad (2.22)$$

We conclude from (2.10), (2.11), (2.21) and Lemma 2.9

$$
\begin{aligned}
&\left|\frac{1}{\mathrm{vol}(X_k)}\cdot\int_{\partial X_k}\Pi_L(\partial X_k)-\frac{1}{\mathrm{vol}(X)}\cdot\int_{\partial X}\Pi_L(\partial X)\right|\\
&\leq\left|\left(\frac{1}{\mathrm{vol}(X_k)}-\frac{1}{N_k\,\mathrm{vol}(X)}\right)\cdot\int_{\partial X_k}\Pi_L(\partial X_k)\right|\\
&\qquad\qquad+\frac{1}{N_k\,\mathrm{vol}(X)}\cdot\left|\int_{\partial X_k}\Pi_L(\partial X_k)-N_k\int_{\partial X}\Pi_L(\partial X)\right|\\
&\overset{(2.21)}{\leq}\left|\frac{1}{\mathrm{vol}(X_k)}-\frac{1}{N_k\,\mathrm{vol}(X)}\right|\cdot\mathrm{area}(\partial X_k)\cdot C\\
&\qquad\qquad+\frac{\mathrm{area}\,Y_k}{N_k\,\mathrm{vol}(X)}\cdot C+\frac{n_k}{N_k}\cdot\frac{C\cdot\mathrm{area}(\partial X)}{\mathrm{vol}(X)}\\
&\overset{(2.10)}{\leq} C\cdot\left|\frac{1}{(n_k+N_k)\cdot\mathrm{vol}(X)}-\frac{1}{N_K\cdot\mathrm{vol}(X)}\right|\cdot\big(\mathrm{area}(Y_k)+\mathrm{area}(\partial X_k\cap\partial\overline{X})\big)\\
&\qquad\qquad+\frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)}\cdot\frac{C\cdot(N_k+n_k)}{N_k}+\frac{n_k}{N_k}\cdot\frac{C\cdot\mathrm{area}(\partial X)}{\mathrm{vol}(X)}\\
&\overset{(2.11)}{\leq} C\cdot\frac{n_k}{N_k\cdot(n_k+N_k)\cdot\mathrm{vol}(X)}\cdot(\mathrm{area}(Y_k)+(n_k+N_k)\cdot\mathrm{area}(\partial X))\\
&\qquad\qquad+\frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)}\cdot\frac{C\cdot(N_k+n_k)}{N_k}+\frac{n_k}{N_k}\cdot\frac{C\cdot\mathrm{area}(\partial X)}{\mathrm{vol}(X)}\\
&\overset{(2.10)}{\leq}\frac{n_k}{N_k}\cdot\frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)}\cdot\frac{1}{\mathrm{vol}(X)}+\frac{n_k}{N_k}\cdot\frac{\mathrm{area}(\partial X)}{\mathrm{vol}(X)}\\
&\qquad\qquad+\frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)}\cdot\frac{C\cdot(N_k+n_k)}{N_k}+\frac{n_k}{N_k}\cdot\frac{C\cdot\mathrm{area}(\partial X)}{\mathrm{vol}(X)}.
\end{aligned}
\tag{2.23}
$$

Since $\lim_{k\to\infty}\frac{\mathrm{area}(Y_k)}{\mathrm{vol}(X_k)}=0$ by assumption, from (2.13), (2.22) and (2.23) follows

$$
\begin{aligned}
\lim_{k\to\infty}\left|\frac{1}{\mathrm{vol}(X_k)}\cdot\int_{X_k}L(X_k)-\frac{1}{\mathrm{vol}(X)}\cdot\int_X L(X)\right| &= 0;\\
\lim_{k\to\infty}\left|\frac{1}{\mathrm{vol}(X_k)}\cdot\int_{\partial X_k}\Pi_L(\partial X_k)-\frac{1}{\mathrm{vol}(X)}\int_{\partial X}\Pi_L(\partial X)\right| &= 0.
\end{aligned}
$$

It remains to consider the eta-invariants. Because of their non-local nature this is the most difficult task. The strategy of the proof of the next proposition is similar to the proof of Remark 1.29.

We first recall a few facts about the η-invariant. Let D be the tangential signature operator of a $4n-1$-dimensional Riemannian manifold M, and $\overline{M}$ a Γ-covering with lifted signature operator $\bar{D}$. Then

$$
\eta(M)=\frac{1}{\Gamma(1/2)}\int_0^\infty t^{-1/2}\,\mathrm{tr}(De^{-tD^2})\,dt. \tag{2.24}
$$

and

$$\eta^{(2)}(\overline{M}) := \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2}\, \mathrm{tr}_{\mathcal{N}\Gamma}(\overline{D}e^{-t\overline{D}^2})\, dt, \tag{2.25}$$

where (with a fundamental domain $\mathcal{F}$ of the covering $\overline{M} \to M$)

$$\mathrm{tr}_{\mathcal{N}\Gamma}(\overline{D}e^{-t\overline{D}^2}) = \int_{\mathcal{F}} \mathrm{tr}_x \left((\overline{D}e^{-t\overline{D}^2})(x,x) \right) dx; \tag{2.26}$$

$$\mathrm{tr}(De^{-tD^2}) = \int_M \mathrm{tr}_x \left((De^{-tD^2})(x,x) \right) dx. \tag{2.27}$$

Following Cheeger and Gromov [5, Section 7] we give an a priori estimate for the large time part of the integrand defining the η-invariant. First observe that for $x \neq 0$ we have the following inequality of functions:

$$\left| \int_T^\infty x t^{-1/2} e^{-tx^2}\, dt \right| = e^{-Tx^2} \int_T^\infty |x|\, t^{-1/2} e^{-x^2(t-T)}\, dt$$

Making the substitution $t = (u\,|x|^{-2} + T)$ we obtain

$$\begin{aligned}
\left| \int_T^\infty x t^{-1/2} e^{-tx^2}\, dt \right| &= e^{-Tx^2} \int_0^\infty |x|\, e^{-u} (u\,|x|^{-2} + T)^{-1/2}\, |x|^{-2}\, du \\
&= e^{-Tx^2} \int_0^\infty e^{-u} (u + T\,|x|^2)^{-1/2}\, du \\
&\leq e^{-Tx^2} - \chi_{\{0\}} \cdot \sqrt{\pi},
\end{aligned}$$

where we used $\int_0^\infty u^{-1/2} e^{-u}\, du = \sqrt{\pi}$ and for the last inequality that $T\,|x|^2 \geq 0$. For $x = 0$ obviously $\int_T^\infty x t^{-1/2} e^{-tx^2}\, dt = 0$. Hence we get for all $x \in \mathbb{R}$

$$\left| \int_T^\infty x t^{-1/2} e^{-tx^2}\, dt \right| \leq \left(e^{-Tx^2} - \chi_{\{0\}}(x) \right) \cdot \sqrt{\pi},$$

where $\chi_{\{0\}}$ is the characteristic function of the set $\{0\}$. Applying the functional calculus with $x = \overline{D}$ we get

$$\left| \int_T^\infty t^{-1/2}\, \mathrm{tr}_{\mathcal{N}\Gamma}(\overline{D}e^{-t\overline{D}^2})\, dt \right| \leq \sqrt{\pi} \cdot \mathrm{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}} - \mathrm{pr}_{\ker \overline{\Delta}}), \tag{2.28}$$

and analogously with $x = D$

$$\left| \int_T^\infty t^{-1/2}\, \mathrm{tr}(De^{-tD^2})\, dt \right| \leq \sqrt{\pi} \cdot \mathrm{tr}(e^{-T\Delta} - \mathrm{pr}_{\ker \Delta}). \tag{2.29}$$

2.30 Proposition. *If $(X_k)_{k \geq 1}$ is a regular exhaustion of $\overline{X}$ as in Definition 0.3 then*

$$\lim_{k \to \infty} \frac{\eta(\partial X_k)}{\mathrm{vol}(X_k)} = \frac{\eta^{(2)}(\partial \overline{X})}{\mathrm{vol}(X)}.$$

Proof. In the sequel $D[k]$ or $\overline{D}$ is the (tangential) signature operator and $\Delta[k]$ or $\overline{\Delta}$ is the differential form Laplacian on ∂X_k or $\partial\overline{X}$, respectively. Fix $\epsilon > 0$. Choose T such that

$$\left|\operatorname{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}} - \operatorname{pr}_{\ker\overline{\Delta}})\right| \leq \frac{\Gamma(1/2)\cdot\operatorname{vol}(X)\cdot\epsilon}{8\sqrt{\pi}}. \tag{2.31}$$

Put $\partial X_k^R := \bigcup_{g\in G \text{ s.t. } U_R(g\mathcal{F})\subset X_k}(g\mathcal{F}\cap\partial\overline{X})$. By Theorem 2.3 for the given $T > 0$ and $\epsilon > 0$ we find $R > 0$ independent of k such that for $0 \leq t \leq T$ and $x \in \partial X_k^R$.

$$\left|\operatorname{tr}_x(D[k]e^{-tD[k]^2}(x,x)) - \operatorname{tr}_x(\overline{D}e^{-t\overline{D}^2}(x,x))\right| \leq \frac{\Gamma(1/2)\cdot\operatorname{vol}(X)\cdot\epsilon}{4\sqrt{T}\cdot\operatorname{vol}(\partial X)}; \tag{2.32}$$

$$\left|\operatorname{tr}_x(e^{-t\Delta[k]}(x,x)) - \operatorname{tr}_x(e^{-t\overline{\Delta}}(x,x))\right| \leq \frac{\Gamma(1/2)\cdot\operatorname{vol}(X)\cdot\epsilon}{8\sqrt{\pi}\cdot\operatorname{vol}(\partial X)}. \tag{2.33}$$

Notice that $U_R(g\mathcal{F}) \subset X_k \iff \mathcal{F} \subset X_k - U_R(Y_k)$. Hence ∂X_k^R consists of $N_k(R)$ translates of $\mathcal{F}\cap\partial\overline{X}$. This implies $\operatorname{vol}(\partial X_k^R) = N_k(R)\cdot\operatorname{vol}(\partial X)$. From Proposition 2.2 and (2.33) we get for a constant A_1 independent of k (using the fact that $\overline{\Delta}$ and its kernel are Γ-equivariant)

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(e^{-T\Delta[k]})}{N_k(R)} - \operatorname{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}})\right| \\
&= \left|\frac{1}{N_k(R)}\cdot\int_{\partial X_k}\operatorname{tr}_x(e^{-T\Delta[k]}(x,x))\,dx - \int_{\mathcal{F}\cap\partial\overline{X}}\operatorname{tr}_x(e^{-T\overline{\Delta}}(x,x))\,dx\right| \\
&\leq \left|\frac{1}{N_k(R)}\cdot\int_{\partial X_k^R}\operatorname{tr}_x(e^{-T\Delta[k]}(x,x)) - \operatorname{tr}_x(e^{-T\overline{\Delta}}(x,x))\,dx\right| \\
&\qquad + \left|\frac{1}{N_k(R)}\cdot\int_{\partial X_k-\partial X_k^R}\operatorname{tr}_x(e^{-T\Delta[k]}(x,x))\,dx\right| \\
&= \frac{\Gamma(1/2)\cdot\operatorname{vol}(X)\cdot\epsilon\cdot\operatorname{vol}(\partial X_k^R)}{N_k(R)\cdot 8\sqrt{\pi}\cdot\operatorname{vol}(\partial X)} + \frac{A_1\cdot\operatorname{vol}(\partial X_k - \partial X_k^R)}{N_k(R)} \\
&\leq \frac{\Gamma(1/2)\cdot\operatorname{vol}(X)\cdot\epsilon}{8\sqrt{\pi}} + \frac{A_1\cdot\operatorname{vol}(\partial X_k - \partial X_k^R)}{N_k(R)}.
\end{aligned} \tag{2.34}$$

We conclude from (2.28), (2.29), (2.31) and (2.34) (using $\overline{D}^2 = \overline{\Delta}$ and $D[k]^2 = \Delta[k]$)

$$\begin{aligned}
&\left|\frac{1}{N_k(R)}\cdot\frac{1}{\Gamma(1/2)}\cdot\int_T^\infty t^{-1/2}\operatorname{tr}(D[k]e^{-tD[k]^2})\,dt\right. \\
&\qquad\left. - \frac{1}{\Gamma(1/2)}\cdot\int_T^\infty t^{-1/2}\operatorname{tr}_{\mathcal{N}\Gamma}(\overline{D}e^{-t\overline{D}^2})\right| \\
&\leq \frac{1}{N_k(R)}\cdot\frac{\sqrt{\pi}}{\Gamma(1/2)}\cdot\operatorname{tr}(e^{-T\Delta[k]} - \operatorname{pr}_{\ker\Delta[k]})
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \operatorname{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}} - \operatorname{pr}_{\ker \overline{\Delta}}) \\
\leq \quad & \frac{2\sqrt{\pi}}{\Gamma(1/2)} \cdot \operatorname{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}} - \operatorname{pr}_{\ker \overline{\Delta}}) \\
& +\frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \left| \frac{1}{N_k(R)} \cdot \operatorname{tr}(e^{-T\Delta[k]}) - \operatorname{tr}_{\mathcal{N}\Gamma}(e^{-T\overline{\Delta}}) \right| \\
& +\frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \left| \operatorname{tr}_{\mathcal{N}\Gamma}(\operatorname{pr}_{\ker \overline{\Delta}}) - \frac{1}{N_k(R)} \operatorname{tr}(\operatorname{pr}_{\ker \Delta[k]}) \right| \\
\leq \quad & \frac{2 \operatorname{vol}(X) \cdot \epsilon}{8} + \frac{\operatorname{vol}(X) \cdot \epsilon}{8} + \frac{\sqrt{\pi} \cdot A_1 \cdot \operatorname{vol}(\partial X_k - \partial X_k^R)}{\Gamma(1/2) \cdot N_k(R)} \\
& +\frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \left| \operatorname{tr}_{\mathcal{N}\Gamma}(\operatorname{pr}_{\ker \overline{\Delta}}) - \frac{1}{N_k(R)} \operatorname{tr}(\operatorname{pr}_{\ker \Delta[k]}) \right|. \qquad (2.35)
\end{aligned}
$$

From (2.26), (2.27), (2.32), and Proposition 2.6 we obtain a constant A_2 independent of k such that the following holds:

$$
\begin{aligned}
& \left| \frac{1}{N_k(R)} \cdot \frac{1}{\Gamma(1/2)} \cdot \int_0^T t^{-1/2} \operatorname{tr}(D[k] e^{-tD[k]^2}) \, dt \right. \\
& \qquad \left. - \frac{1}{\Gamma(1/2)} \int_0^T t^{-1/2} \operatorname{tr}_{\mathcal{N}\Gamma}(\overline{D} e^{-t\overline{D}^2}) \right| \\
= \quad & \left| \frac{1}{N_k(R)} \cdot \frac{1}{\Gamma(1/2)} \cdot \int_0^T t^{-1/2} \int_{\partial X_k} \operatorname{tr}_x(D[k] e^{-tD[k]^2}(x,x)) \, dx \, dt \right. \\
& \qquad \left. - \frac{1}{\Gamma(1/2)} \cdot \int_0^T t^{-1/2} \int_{\mathcal{F} \cap \partial \overline{X}} \operatorname{tr}_x(\overline{D} e^{-t\overline{D}^2}(x,x)) \, dx \, dt \right| \\
\leq \quad & \frac{1}{N_k(R)} \cdot \frac{1}{\Gamma(1/2)} \cdot \qquad (2.36) \\
& \left| \int_0^T t^{-1/2} \int_{\partial X_k^R} \operatorname{tr}_x(D[k] e^{-tD[k]^2}(x,x)) - \operatorname{tr}_x(\overline{D} e^{-t\overline{D}^2}(x,x)) \, dx \, dt \right| \\
& \qquad + \left| \int_0^T t^{-1/2} \int_{\partial X_k - \partial X_k^R} \operatorname{tr}_x(D[k] e^{-tD[k]^2}(x,x)) \, dx \, dt \right| \\
\leq \quad & \frac{1}{N_k(R)} \int_0^T t^{-1/2} \Big(\frac{\operatorname{vol}(X) \cdot \epsilon}{4\sqrt{T} \cdot \operatorname{vol}(\partial X)} \cdot \operatorname{vol}(\partial X_k^R) \qquad (2.37) \\
& \qquad + \frac{A_2}{\Gamma(1/2)} \cdot t^{1/2} \cdot \operatorname{vol}(\partial X_k - \partial X_k^R) \Big) \, dt. \\
\leq \quad & \frac{\operatorname{vol}(X) \cdot \epsilon}{8} + \frac{A_2 \cdot T}{\Gamma(1/2)} \cdot \frac{\operatorname{vol}(\partial X_k - \partial X_k^R)}{N_k(R)}. \qquad (2.38)
\end{aligned}
$$

We conclude from (2.24), (2.25), (2.35) and (2.38)

$$\begin{aligned}
&\left|\eta^{(2)}(\partial\overline{X}) - \frac{1}{N_k(R)} \cdot \eta(\partial X_k)\right| \\
&\leq \quad \frac{\mathrm{vol}(X)\cdot\epsilon}{8} + \frac{A_2 \cdot T}{\Gamma(1/2)} \cdot \frac{\mathrm{vol}(\partial X_k - \partial X_k^R)}{N_k(R)} + \frac{3\,\mathrm{vol}(X)\cdot\epsilon}{8} + \\
&\qquad \frac{\sqrt{\pi}\cdot A_1 \cdot \mathrm{vol}(\partial X_k - \partial X_k^R)}{\Gamma(1/2)\cdot N_k} \\
&\qquad + \frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \left|\mathrm{tr}_{\mathcal{N}\Gamma}(\mathrm{pr}_{\ker\overline{\Delta}}) - \frac{1}{N_k(R)} \cdot \mathrm{tr}(\mathrm{pr}_{\ker\Delta[k]})\right|.
\end{aligned} \tag{2.39}$$

We get from (2.10)

$$\begin{aligned}
&\left|\frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} - \frac{\eta(\partial X_k)}{\mathrm{vol}(X_k)}\right| \\
&\leq \quad \frac{N_k(R)}{\mathrm{vol}(X_k)} \cdot \left|\eta^{(2)}(\partial\overline{X}) - \frac{1}{N_k(R)}\eta(\partial X_k)\right| + \left|\left(\frac{N_k(R)}{\mathrm{vol}(X_k)} - \frac{1}{\mathrm{vol}(X)}\right)\cdot\eta^{(2)}(\partial\overline{X})\right| \\
&\leq \quad \frac{1}{\mathrm{vol}(X)} \cdot \left|\eta^{(2)}(\partial\overline{X}) - \frac{1}{N_k(R)}\eta(\partial X_k)\right| + \frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} \cdot \frac{n_k(R)}{N_k(R)}.
\end{aligned} \tag{2.40}$$

We conclude from (2.39) and (2.40)

$$\begin{aligned}
&\left|\frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} - \frac{\eta(\partial X_k)}{\mathrm{vol}(X_k)}\right| \\
&\leq \quad \frac{\epsilon}{8} + \frac{A_2 \cdot T}{\mathrm{vol}(X)\cdot\Gamma(1/2)} \cdot \frac{\mathrm{vol}(\partial X_k - \partial X_k^R)}{N_k(R)} + \frac{3\epsilon}{8} \\
&\qquad + \frac{\sqrt{\pi}\cdot A_1}{\mathrm{vol}(X)\cdot\Gamma(1/2)} \cdot \frac{\mathrm{vol}(\partial X_k - \partial X_k^R)}{N_k} \\
&\qquad + \frac{\sqrt{\pi}}{\mathrm{vol}(X)\cdot\Gamma(1/2)} \cdot \left|\mathrm{tr}_{\mathcal{N}\Gamma}(\mathrm{pr}_{\ker\overline{\Delta}}) - \frac{1}{N_k(R)} \cdot \mathrm{tr}(\mathrm{pr}_{\ker\Delta[k]})\right| \\
&\qquad + \frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} \cdot \frac{n_k(R)}{N_k(R)}.
\end{aligned} \tag{2.41}$$

Recall that ∂X_k^R consists of $N_k(R)$ translates of $\mathcal{F}\cap\partial\overline{X}$. The same arguments as above (using (2.13)) imply

$$\lim_{k\to\infty} \frac{\mathrm{vol}(\partial X_k - \partial X_k^R)}{N_k} \quad = \quad 0. \tag{2.42}$$

We get from Theorem 2.14

$$\lim_{k\to\infty} \frac{\mathrm{tr}(\mathrm{pr}_{\ker\Delta[k]})}{N_k(R)} = \lim_{k\to\infty} \frac{b_*(\partial X_k)}{N_k(R)} = b_*^{(2)}(\partial\overline{X}) = \mathrm{tr}_{\mathcal{N}\Gamma}(\mathrm{pr}_{\ker\overline{\Delta}}) \tag{2.43}$$

(with the convention $b_* := \sum_{p\geq 0} b_p$). From (2.13), (2.41), (2.42) and (2.43) we get the existence of a constant $K(\epsilon)$ such that for all $k \geq K(\epsilon)$

$$\left| \frac{\eta^{(2)}(\partial\overline{X})}{\mathrm{vol}(X)} - \frac{\eta(\partial X_k)}{\mathrm{vol}(X_k)} \right| \quad \leq \quad \epsilon. \tag{2.44}$$

Now Proposition 2.30 follows. This finishes the proof of Theorem 0.4. □

2.45 Remark. In the proof of Proposition 2.30, we were using general properties of the traces and of generalized Dirac operators, which apply not only to the tangential signature operator, together with one additional ingredient, the convergence result for the kernel of the tangential signature operator of Theorem 2.14.

Using the symmetry of the tangential signature operator one can restrict to $(2n-1)$-forms on the boundary, as explained in [3, Proposition 4.20]. In particular, (2.43) has to hold only for b_{2n-1}, and we still could prove Proposition 2.30 (compare also [5, 24].

2.2 Combinatorial version

In this subsection, we prove a combinatorial version of Theorem 0.4, namely Theorem 0.5 which we have stated in the introduction. It uses the more algebraic techniques employed in Section 1 rather than the heat kernel analysis of Subsection 2.1. This way, the result applies to triangulated rational homology manifolds (with boundary).

Throughout this subsection, let X be a compact triangulated rational homology manifold with boundary L, and of dimension $4n$. Let $\overline{X}$ be a regular covering of X with finitely generated amenable covering group Γ.

We start by describing the type of exhaustion we are going to use.

2.46 Definition. Let $\mathcal{F}$ be a fundamental domain for the covering $\overline{X} \to X$, i.e. $\mathcal{F}$ contains exactly one lift of each top-dimensional simplex of X. For each simplex σ in X choose a lift $\overline{\sigma}$ in $\mathcal{F}$. Let S be a finite system of generators of Γ. It gives rise to a left invariant word metric on Γ. For a subcomplex $Z \subset \overline{X}$ and $R > 0$, define

$$U_R(Z) := \bigcup_{\sigma \text{simplex of } X} \{\gamma\overline{\sigma} \mid \gamma \in \Gamma \text{ and } \exists \gamma_1 \in \Gamma \text{with } d(\gamma-1,\gamma) < R, \gamma_1\overline{\sigma} \cap Z \neq \emptyset\}.$$

This depends on the choice of S as well as the lifts $\overline{\sigma}$. For each $g \in \Gamma$, $U_R(gZ) = gU_R(Z)$.

2.47 Definition. For a simplicial complex Y let $|Y|$ be the total number of simplices of Y. Similarly, for any subset W of a simplicial complex which is a union of open simplices, $|W|$ is the number of open simplices in W.

A sequence $X_1 \subset X_2 \subset \dots \overline{X}$ of finite subcomplexes is called an *amenable exhaustion* if $\bigcup_{k\in\mathbb{N}} X_k = \overline{X}$ and if for each $R > 0$

$$\frac{|U_R(X_k)|}{|X_k|} \xrightarrow{k\to\infty} 1.$$

It is called a *balanced* exhaustion, if for each orbit $\Gamma\overline{\sigma}$ of simplices in $\overline{X}$

$$\frac{|X_k \cap \Gamma\overline{\sigma}|}{|X_k|} \xrightarrow{k\to\infty} \frac{1}{|X|}.$$

Denote

$$\mathrm{tr}_k := \frac{tr_{\mathbb{C}}}{|X_k|}\,|X|\,; \qquad \dim_k := \frac{\dim_{\mathbb{C}}}{|X_k|}\,|X|\,.$$

Before proving Theorem 0.5, we investigate the relation between the Poincaré duality maps of one homology manifold being a codimension-zero subcomplex of another homology manifold. As an illustration we consider the following diagram. Let $U \subset M$ be codimension zero submanifold with boundary ∂U of a compact manifold M. For the moment assume ∂M is empty.

$$\begin{tikzcd}
H^p(M) \arrow[d, "\cap[M]"] & H^p(M, M-U) \arrow[l] \arrow[r, "\cong"] & H^p(U, \partial U) \arrow[d, "\cap[U,\partial U]"] \\
H_{n-p}(M) \arrow[d, "="] & & H_{n-p}(U) \arrow[d] \\
H_{n-p}(M) \arrow[r] & H_{n-p}(M, M-U) & H_{n-p}(U, \partial U). \arrow[l, "\cong"']
\end{tikzcd}$$

In this diagram, the maps without labels are induced by inclusions and the isomorphisms are given by excision. The diagram commutes because the fundamental class of M is mapped to the fundamental class of $(U, \partial U)$ under the composition of the maps in the lowest row (for $p = 0$). A corresponding result holds if M itself has a boundary.

Because we have to apply this in the L^2-setting, we give a chain-level description of this diagram. For this, let (X, L) be an n-dimensional pair of simplicial complexes triangulating an oriented rational homology n-manifold X with boundary L. Let $\overline{X}, \overline{L}$ be the lifted triangulation of a normal covering of X ($\overline{L}$ is the inverse image of L in $\overline{X}$) with covering group Γ. Without loss of generality we assume X and $\overline{X}$ are connected (we can deal with one component of X at a time, and then the L^2-signature is unchanged if we consider only one component of $\overline{X}$).

We first want to describe the (simplicial) L^2-chain- and cochain complexes of $\overline{X}$. Set $\pi := \pi_1(X)$. We have by definition

$$\begin{aligned}
C^*_{(2)}(\overline{X}, \overline{L}) &= \hom_{\mathbb{Z}\pi}(C_*(\tilde{X}, \tilde{L}), l^2(\Gamma)), \\
C^*_{(2)}(\overline{X}) &= \hom_{\mathbb{Z}\pi}(C_*(\tilde{X}), l^2(\Gamma)), \text{ and} \\
C_*^{(2)}(\overline{X}) &= l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C_*(\tilde{X}).
\end{aligned} \tag{2.48}$$

Here $\tilde{X}$ is the induced triangulation of the universal covering of X, $\tilde{L}$ is the inverse image of L in $\tilde{X}$, and we always use the simplicial (co)chain complexes.

2.49 Convention. There are canonical identifications of the simplicial L^2-chain and L^2-cochain complexes $C_p^{(2)}(\overline{X})$, $C^p_{(2)}(\overline{X})$ with the spaces of L^2-summable functions on the set of p-dimensional simplices of $\overline{X}$, and of $C_p^{(2)}(\overline{X},\overline{L})$ and $C^p_{(2)}(\overline{X},\overline{L})$ with L^2-summable functions on the set of p-dimensional simplices of $\overline{X}$ which do not belong to $\overline{L}$, respectively.

These identifications are used in the sequel.

We write L^2-functions on the set of simplices of $\overline{X}$ as formal sums $\sum \lambda_\sigma \sigma$. Then the identification of cochains with L^2-functions is the anti-linear isomorphism given by $a \mapsto \sum_{\sigma\in\overline{X}} \langle 1, a(\tilde{\sigma})\rangle \sigma$, where $\tilde{\sigma}$ is an arbitrary lift of σ to the universal covering. Note that there is a given projection $p\colon \tilde{X} \to \overline{X}$ since $\overline{X}$ is a connected normal covering of X. The identification of chains with L^2-functions on the set of simplices is given by $\left(\sum_{g\in\Gamma} \lambda_g g\right) \otimes \tilde{\sigma} \mapsto \sum_{g\in\Gamma} \lambda_g g\sigma$, where $\tilde{\sigma}$ is a simplex in $\tilde{X}$ (or, for $C_*(\tilde{X},\tilde{L})$ of $\tilde{X}\setminus\tilde{L}$) and $\sigma = p(\tilde{\sigma})$.

2.50 Remark. Note that this way, in particular we identify the L^2-chain- and cochain groups with each other (via an anti-linear isomorphism). However, this is nothing but the usual isomorphism of a Hilbert space with its dual. Note that this is not an isomorphism of chain complexes. Under the identifications, the chain- and cochain maps induced from the inclusion of $\overline{X}$ in $(\overline{X},\overline{L})$ become the obvious inclusion and orthogonal projection, respectively.

2.51 Lemma. *Under the identification of Convention 2.49 of the chain- and cochain complexes with spaces of L^2-functions on the sets of simplices of $\overline{X}$, cap-product with the fundamental class —defined on the (co)chain level using the Alexander-Whitney diagonal map— gives a map*

$$\mathfrak{g}\colon C^*_{(2)}(\overline{X},\overline{L}) \to C_*^{(2)}(\overline{X})$$

which sends an L^2-function a on the set of p-simplices in $\overline{X}\setminus\overline{L}$ to

$$\sum_{\overline{\sigma}\ n\text{-simplex of } \overline{X}} f_{n-p}(\overline{\sigma})\langle a, b_p(\overline{\sigma})\rangle_{l^2(\overline{X})}.$$

Here f_q, b_p are the front- and back-faces of the corresponding dimension, as usual in the Alexander-Whitney diagonal approximation. To be able to define this, we choose also a Γ-invariant local ordering of the vertices of $\overline{X}$, e.g. by lifting such a local ordering from X.

Proof. Using the notation introduced above, $C_*(X,\mathbb{C})$ can be identified with $\mathbb{C}\otimes_{\mathbb{Z}\pi} C_*(\tilde{X})$. For each simplex σ of X choose a lift $\tilde{\sigma}$ in $\tilde{X}$. Then the fundamental class of X can be written as $\sum_{\sigma\in X_n} 1\otimes\tilde{\sigma}$, where X_p denotes the collection of p-simplices in X.

The Alexander-Whitney cap-product of $a \in \hom_{\mathbb{Z}\pi}(C_p(\tilde{X}, \tilde{L}), l^2(\Gamma))$ with the fundamental class is then given by

$$\sum_{\sigma \in X_n} (a(b_p(\tilde{\sigma})))^* \otimes f_{n-p}(\tilde{\sigma}). \tag{2.52}$$

Here $\cdot^* \colon l^2(\Gamma) \to l^2(\Gamma)$ is the anti-linear isomorphism induced from $g \mapsto g^{-1}$ and from complex conjugation of the coefficients.

Now observe that the function $a = \sum_{\overline{\sigma} \in \overline{X}} a_{\overline{\sigma}} \overline{\sigma}$ is mapped to the cochain

$$\alpha \colon \tilde{\sigma} \mapsto \sum_{g \in \Gamma} \overline{a_{g^{-1}p(\tilde{\sigma})}} g,$$

and $\overline{a_{g^{-1}p(\tilde{\sigma})}} = \langle g^{-1}p(\tilde{\sigma}), a \rangle_{l^2}$.

By (2.52), capping this cochain α with the fundamental class gives the chain

$$\sum_{\sigma \in X_n} \sum_{g \in \Gamma} \overline{\langle g^{-1}p(b_p(\tilde{\sigma})), a \rangle} g^{-1} \otimes f_{n-p}(\tilde{\sigma}).$$

Under our identification, this chain becomes the function

$$\sum_{\overline{\sigma} \in \overline{X}_n} \langle a, b_p(\overline{\sigma}) \rangle f_{n-p}(\overline{\sigma}),$$

where we use the fact that the family $g^{-1}p(\tilde{\sigma})$ for $g \in \Gamma$ and $\sigma \in X_n$ is exactly the family of all n-simplices of $\overline{X}$, and the fact that the front- and back-face maps commute with the action of π (and Γ). □

2.53 Lemma. *Composing the cap-product with the fundamental class with the map induced from the inclusion $X \to (X, L)$ we get a map*

$$g_{\overline{X}} : C^*_{(2)}(\overline{X}, \overline{L}) \to C_*^{(2)}(\overline{X}, \overline{L})$$

which under the identifications of $C^*_{(2)}(\overline{X}, \overline{U})$ as well as $C_*^{(2)}(\overline{X}, \overline{U})$ with the space of L^2-functions on simplices in $\overline{X} \setminus \overline{L}$ maps such a function a (on p-simplices) to

$$\sum_{\overline{\sigma} \text{ } n\text{-simplex of } \overline{X}} f_{n-p}(\overline{\sigma}) \delta_{\overline{L}}(f_{n-p}(\overline{\sigma})) \langle a, b_p(\overline{\sigma}) \rangle_{l^2(\overline{X})}.$$

Here $\delta_{\overline{L}}(\overline{\sigma})$ is 1 if $\overline{\sigma}$ is not contained in $\overline{L}$, and is 0 if $\overline{\sigma} \in \overline{L}$.

Proof. This is an immediate consequence of the first lemma. □

2.54 Definition. Assume now that U is a compact subcomplex of $\overline{X}$ which has itself a subcomplex V (not necessarily contained in $\overline{L}$) such that (U, V) triangulates an oriented homology n-manifold with boundary (i.e. U is a codimension 0 submanifold with boundary). The above identifications and formulas apply to the chain- and cochain complexes of U and V with complex coefficients. Moreover, observe that these identifications give canonical embeddings P_U^* of $C^*(U, V, \mathbb{C})$ in $C^*_{(2)}(\overline{X}, \overline{L})$ and of $C_*(U, V; \mathbb{C})$ in $C_*^{(2)}(\overline{X}, \overline{L})$. The corresponding orthogonal projection is the adjoint P_U.

2.55 Proposition. *In the situation of Definition 2.54, with all the identifications described,*

$$g_U^{(2)} = P_U \circ g_{\overline{X}}^{(2)} \circ P_U^*.$$

In other words, the Poincaré duality operator on U is obtained from the one on $\overline{X}$ by compression onto the chain and cochain complex of U, considered as subcomplex of the ones of $\overline{X}$.

Proof. This is implied by the formula of Lemma 2.53. We only have to make the simple but key observation that a top-dimensional simplex of $\overline{X}$ which is not contained in U has no face contained in $U \setminus V$ (since in the star of an interior point, any two top-dimensional simplices can be joined by a sequence of top-dimensional simplices having pairwise a common face of codimension 1. Therefore the star in $\overline{X}$ of an interior point of U can not be bigger than the star in U). $\square$

Now we are ready to prove Theorem 0.5. We start with some auxiliary results we will use. As before, let $\mathcal{F}$ be a fundamental domain for the covering $\overline{X} \to X$. Remember that for each simplex σ in X we have chosen a lift $\overline{\sigma}$ in $\mathcal{F}$.

2.56 Definition. We write $l^2(\overline{X})$ for the space of L^2-functions on the *set* of simplices of $\overline{X}$ (with point measure), each simplex being one element. This way, we get an identification

$$C^*_{(2)}(\overline{X}) = l^2(\overline{X}) = \oplus_{\sigma \in X} l^2(\Gamma) \cdot \overline{\sigma}.$$

Assume that we have an exhaustion $X_1 \subset X_2 \subset \cdots \subset \overline{X}$ as in Definition 2.47. These define subspaces $l^2(X_k) \subset l^2(X)$. Let P_k^σ be the orthogonal projection

$$P_k^\sigma \colon l^2(\Gamma) \cdot \overline{\sigma} \to (l^2(\Gamma) \cdot \overline{\sigma}) \cap l^2(X_k) = l^2(X_k \cap \Gamma\overline{\sigma}). \tag{2.57}$$

Using the above identification, the orthogonal projection $P_k : l^2(\overline{X}) \to l^2(X_k)$ splits as $P_k = \mathrm{diag}_{\sigma \in X}(P_k^\sigma)$.

2.58 Definition. Fix an exhaustion $X_1 \subset X_2 \subset \cdots \subset \overline{X}$ as in Definition 2.47. For a Γ-equivariant operator $A \colon C^*(\overline{X}) \to C^*(\overline{X})$ (inducing the operator $A^{(2)}$ on $C^*_{(2)}$) define $A[k] := P_k A^{(2)} P_k$ (either considered as operator on $l^2(\overline{X})$ or on $l^2(X_k)$).

2.59 Remark. Fix an amenable exhaustion as in Definition 2.47. Observe that, if $c : C^*(\overline{X}) \to C^*(\overline{X})$ is the cellular cochain map with adjoint c^*, then $c[k]$ is the cochain map of X_k with adjoint $c^*[k]$. Note that the combinatorial Laplacian $\Delta[X_k] = c[k]c^*[k] + c^*[k]c[k]$ of X_k in general does not coincide with $\Delta[k]$ where $\Delta = cc^* + c^*c$ is the Laplacian of $\overline{X}$. By Proposition 2.55 for the Poincaré duality cochain operator we get $g_{X_k} = g[k]$

From this point, the proof of Theorem 0.5 is formally the same as the proof of Theorem 1.5 in Section 1 (the letter f there is replaced by g here): by Remark 2.59 we have a sequence of operators $g[k]$ and Laplacians $\Delta[X_k]$ and we have to prove that

$$\operatorname{tr}_k \chi_{(a,b)}(\Delta[X_k]g[k]\Delta[X_k]) \xrightarrow{k\to\infty} \operatorname{tr}_{\mathcal{N}\Gamma} \chi_{(a,b)}(\Delta^{(2)}\overline{g}\Delta^{(2)})$$

for $(a, b) = (-\infty, 0)$ and for $(a, b) = (0, \infty)$, where tr_k is defined in Definition 2.47.

To proceed, we only need the following ingredients, Lemmas 2.60 to 2.63 and Theorem 2.65, which replace Lemma 1.11, Lemma 1.10, Lemma 1.7, Lemma 1.12, and Theorem 1.15 in the covering situation, and the proof given in Section 1 goes through.

2.60 Lemma. *Use the Notation of Definitions 2.47 and 2.58.*

For Γ-equivariant linear operators $h_1, \dots, h_d : C^*(\overline{X}) \to C^*(\overline{X})$ (inducing operators $h_k^{(2)}$ on $C^*_{(2)}(\overline{X})$) and a polynomial $p(x_1, \dots, x_d)$ in non-commuting variables $x_1, \dots, x_d$ we have

$$\operatorname{tr}_{\mathcal{N}\Gamma}(p(h_1^{(2)}, \dots, h_d^{(2)})) = \lim_{k\to\infty} tr_k\,(p(h_1[k], \dots, h_d[k]))\,.$$

Proof. Because of linearity of the traces it suffices to study monomials $x_1 \dots x_d$. The lemma for $h_1 = h_2 = \dots h_d$ and slightly less general projections is due to Dodziuk-Mathai [9, Lemma 2.3]. An account (with yet another slightly different setting) can be found in [20, 4.6], and the proof given there carries over with no more than obvious changes to the more general situation we are considering here. □

2.61 Lemma. *Fix an amenable exhaustion as in Definition 2.47. For each simplex $\sigma \in X$*

$$\operatorname{tr}_k(P_k^\sigma) \xrightarrow{k\to\infty} 1,$$

where P_k^σ is defined in Definition 2.56.

Proof. This is just the definition of a balanced exhaustion. □

2.62 Lemma. *Fix an amenable exhaustion as in Definition 2.47. There is $K \geq 1$ such that for all $k \geq 1$*

$$\|\Delta^{(2)}\|, \|\Delta[X_k]\|, \|g^{(2)}\|, \|g[k]\| \leq K.$$

Proof. Recall from Remark 2.59 that

$$\Delta[X_k] = P_k c^{(2)} P_k (c^{(2)})^* P_k + P_k (c^{(2)})^* P_k c^{(2)} P_k; \qquad \text{and } g[k] = P_k g P_k.$$

The assertion follows now from sub-multiplicativity of the operator norm and the fact that $\|P_k\| \leq 1$ for each k, as these are orthogonal projections. $\square$

2.63 Lemma. *Fix an amenable exhaustion as in Definition 2.47. There is a constant $C_1 > 0$ (independent of k) such that for $0 < \epsilon < 1$ and $k \geq 1$*

$$\mathrm{tr}_k\left(\chi_{(0,\epsilon]}(\Delta_{2n}[X_k])\right) \leq \frac{C_1}{-\ln(\epsilon)}. \tag{2.64}$$

Proof. This can be proved as in [9, Lemma 2.5]. Alternatively, define

$$\ln \det\nolimits'_{(2)}(\Delta[X_k]) := \int_{0^+}^{\infty} \ln(\lambda)\, dF_{\Delta[X_k]}(\lambda)$$

with $F_{\Delta[X_k]}(\lambda) := \mathrm{tr}_k(\chi_{[0,\lambda]}(\Delta[k]))$, the spectral density function of the operator $\Delta[k]$ computed using tr_k instead of $\dim_{\mathbb{C}}$. Since $\Delta[X_k]$ is defined over $\mathbb{Z}$, $\ln \det' \Delta[X_k] \geq 0$ (compare [10, Theorem 3.4(1)]). Then the inequality follows from Lemma 1.31, Lemma 2.62, and Lemma 2.61. $\square$

2.65 Theorem. *Fix an amenable exhaustion as in Definition 2.47. The normalized sequence of Betti numbers converges, i.e. for each p*

$$\lim_{k\to\infty} \dim_k(\ker(\Delta_p[X_k])) = \dim_{\mathcal{N}\Gamma} \ker(\Delta_p^{(2)}).$$

Proof. This is essentially [9, Theorem 0.1]. Actually, our exhaustion is slightly more general than the ones considered there. But the proof only requires the assertions of Lemma 2.60, Lemma 2.61, Lemma 2.62, and Lemma 2.63, and so goes through without changes (compare [20, Section 6]). $\square$

Now the proof of the convergence assertion of Theorem 0.5 can be finished as described above. It only remains to prove the existence of the exhaustion $X_1 \subset X_2 \subset \cdots$ with the required properties.

2.3 Existence of balanced amenable exhaustions

Amenability of Γ in Theorem 0.5 is equivalent to the existence of Følner exhaustions by subcomplexes without additional structure (used e.g. in [9]). We will thicken them to get homology manifolds (with boundary), for which the signature is defined. We thank Steve Ferry who explained to us how to do this thickening.

We use the following notation:

2.66 Definition. Let K be a simplicial complex with a subcomplex X. We define $\mathrm{star}(X)$, the star of X, to be the union of the stars of all vertices in X, where the star of a vertex is the union of all closed simplices containing this vertex.

Denote the barycentric subdivision of K with K_b, of X with X_b.

We obviously have:

2.67 Lemma. *In the situation of Definition 2.66,* $\mathrm{star}(X)_b = \mathrm{star}(\mathrm{star}(X_b))$.

2.68 Lemma. *Let (K, L) be a triangulated homology manifold (not necessarily compact). Let $X' \subset K$ be a subcomplex. Then there exists a thickening $X \supset X'$ contained in the star of X', such that X is a subcomplex of the barycentric subdivision of K and such that X is a rational homology manifold with boundary Y. Here $X \cap L \subset Y$, but Y is not necessarily contained in L.*

Proof. Let $f : K \to \mathbb{R}$ be a piecewise linear map which is 1 on X' and 0 on the complement of the star of X'. Since $f^{-1}(0,1)$ does not contain a vertex, $1/2$ is a regular value, and therefore $X := f^{-1}([1/2, 1])$ will do the job. More specifically, $f^{-1}([1/2, 1))$ is homeomorphic to the product $f^{-1}(1/2) \times [1/2, 1)$, since there are no vertices. If $x \in X - f^{-1}(1/2)$, then it has a neighborhood which is open in X as well as in K, so it is a manifold point (and will be a boundary point whenever it belongs to L). All points $z \in f^{-1}(1/2)$ have the neighborhood $U := f^{-1}([1/2, 1)) = f^{-1}(1/2) \times [1/2, 1)$, and no matter how $f^{-1}(1/2)$ looks like, the inclusion $(U - \{y\}) \hookrightarrow U$ is a homotopy equivalence, so that (by excision) y is a boundary point of a rational homology manifold. It remains to observe that each path in X is homotopic to a path in $X - f^{-1}(1/2)$ such that $X - \partial X$ does not have more connected components than X.

Obviously, we can arrange for X to be a subcomplex of the barycentric subdivision of K. □

Now we go back to $\overline{X}$ and construct the exhaustions we can use. The covering group Γ being amenable means there is a Følner exhaustion $V_1 \subset V_2 \subset \dots \subset \Gamma$ with $\bigcup_{k\in\mathbb{N}} V_k = \Gamma$ by finite subsets V_k, i.e. $\lim_{k\to\infty} |U_R(\partial V_k)| \,/\, |V_k| = 0$. Remember that we have the fundamental domain $\mathcal{F}$ for the covering $\overline{X} \to X$. If we set $X'_k := V_k\mathcal{F}$, then X'_k is an exhaustion of $\overline{X}$ by finite subcomplexes as considered in [9]. It is standard that X'_k forms a balanced amenable exhaustion of $\overline{X}$. Let X_k be a thickening of X'_k as provided by Lemma 2.68. Since we want to deal with (simplicial) subcomplexes only, we replace X (and $\overline{X}$) by its barycentric subdivision. Our main observation is that X_k is contained in the star of X'_k. Fix $R > 0$ such that $\mathrm{star}(\mathrm{star}(\mathcal{F})) \subset U_R(\mathcal{F})$. By the Γ-invariance of the metric $X_k \subset U_R(X'_k)$. Since, on the other hand, $X'_k \subset X_k$, the sequence X_k forms a balanced amenable exhaustion of $\overline{X}$. This finishes the proof of Theorem 0.5.

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