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STEENROD SQUARES IN SPECTRAL SEQUENCES. II

BY

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ABSTRACT. We apply the results of the previous paper to three special cases. We obtain Steenrod operations on the change-of-rings spectral sequence, on the Eilenberg-Moore spectral sequence for the cohomology of classifying spaces, and on the Serre spectral sequence.

In the previous paper (hereafter called "I") we developed a theory of Steenrod operations on a general class of spectral sequences. Our object now is to compute these operations on E_2 in some interesting special cases. We will find it convenient to change the notation of I: we now deal with "vertical" Steenrod squares Sq_V^k and "diagonal" Steenrod squares Sq_D^k:

$$\mathrm{Sq}_V^k\colon E_2^{p,q} \longrightarrow E_2^{p,q+k} \qquad (0 \le k \le q) \tag{0.1}$$

$$\mathrm{Sq}_D^k\colon E_2^{p,q} \longrightarrow E_2^{p+k,2q} \qquad (0 \le k \le p). \tag{0.2}$$

They are defined by $\mathrm{Sq}_V^k = \mathrm{Sq}^k$ (with Sq^k as in (0.1) of I) and $\mathrm{Sq}_D^k = \mathrm{Sq}^{q+k}$ (with Sq^k as in (0.2) of I).

In our first application we suppose given an extension of cocommutative Hopf algebras $A \longrightarrow C \longrightarrow B$, as in [13], [14]; it may in general be twisted and noncentral as an extension of algebras, and twisted as an extension of coalgebras. Suppose given a commutative left C-coalgebra M and a commutative left C-algebra N. First we show, by analogy with the theory of groups, that the action of B on $\mathrm{Ext}_A(M, N)$ can be described directly in terms of an action of B upon A. In this we rely heavily on [13], [14], where it is demonstrated that an extension of Hopf algebras determines an action of base upon fiber by "conjugation". This result permits us to set up the change of rings spectral sequence converging to $\mathrm{Ext}_C(M, N)$ by analogy with Mac Lane's construction of the Lyndon spectral sequence [8, Chapter XI]. We then consider the Steenrod operations on E_2:

$$\mathrm{Sq}_V^k\colon \mathrm{Ext}_B^p(Z_2, \mathrm{Ext}_A^q(M, N)) \longrightarrow \mathrm{Ext}_B^p(Z_2, \mathrm{Ext}_A^{q+k}(M, N)), \tag{0.3}$$

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$$\mathrm{Sq}_D^k\colon \mathrm{Ext}_B^p(Z_2, \mathrm{Ext}_A^q(M, N)) \to \mathrm{Ext}_B^{p+k}(Z_2, \mathrm{Ext}_A^{2q}(M, N)), \tag{0.4}$$

and find that Sq_V^k is induced by the Steenrod square one ordinarily defines on the cohomology of the Hopf algebra A, whereas Sq_D^k coincides with the Steenrod square one usually defines on the cohomology of B.

In our second application we consider a topological group G and a principal G-bundle E. Then the spectral sequence described by Moore in [11] converges to $H^*(E/G)$ (coefficients in Z_2), and the Steenrod operations on E_2 look like

$$\mathrm{Sq}_V^k\colon \mathrm{Ext}_{H_*(G)}^{p,q}(H_*(E), Z_2) \to \mathrm{Ext}_{H_*(G)}^{p,q+k}(H_*(E), Z_2), \tag{0.5}$$

$$\mathrm{Sq}_D^k\colon \mathrm{Ext}_{H_*(G)}^{p,q}(H_*(E), Z_2) \to \mathrm{Ext}_{H_*(G)}^{p+k,2q}(H_*(E), Z_2). \tag{0.6}$$

We show that the vertical squares are induced by the topological squaring operations on $H_*(E)$ and $H_*(G)$; whereas the diagonal squares coincide with the algebraically defined operations on the cohomology of the Hopf algebra $H_*(G)$.

In our final application we use the derivation of the Serre spectral sequence given by Dress in [4], and recover the results of [1], [7], and [18].

We will find in each of our applications that the vertical squares satisfy the Cartan formula and the Adem relations (both with $\mathrm{Sq}^0 \neq 1$), and that the diagonal squares satisfy the Cartan formula and the Adem relations (both with $\mathrm{Sq}^0 \neq 1$). We also find that vertical and diagonal squares satisfy an interesting commutation relation

$$\mathrm{Sq}_V^k \mathrm{Sq}_D^j = \mathrm{Sq}_D^j \mathrm{Sq}_V^{k/2} \tag{0.7}$$

(read $\mathrm{Sq}_V^{k/2} = 0$ if k is odd). It is likely that these statements hold in general for the spectral sequence of a bisimplicial coalgebra.

The reader interested in the change of rings spectral sequence should cover the first five sections of this paper. The reader interested only in the Eilenberg-Moore spectral sequence is directed to §§1, 2, 6, and 7. The reader concerned with the Serre spectral sequence may proceed directly to §8.

1. **Actions of Λ on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$.** Our object in this section is to describe relationships among algebras Λ, Γ and left Γ-modules Σ, Ω, that permit one to define a Λ-action on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$.

Conventions concerning algebras, coalgebras and Hopf algebras over Z_2 are the same as in I, §1. Tensor products are over Z_2 unless otherwise stated. We continue to assume comultiplications commutative. A left action of an algebra Λ is written $\sigma(\Lambda, \Omega)\colon \Lambda \otimes \Omega \to \Omega$ and a right action is written $\rho(\Sigma, \Lambda)\colon \Sigma \otimes \Lambda \to \Sigma$.

We fix throughout this section a Hopf algebra Λ and a right Λ-algebra Γ.

Definition 1.1. By a mixed Λ-Γ module we mean a left Γ-module Σ that is also a right Λ-module in such a way that $\sigma(\Gamma, \Sigma)\colon \Gamma \otimes \Sigma \to \Sigma$ is a map of right Λ-modules (here Λ acts diagonally on $\Gamma \otimes \Sigma$). A map of mixed Λ-Γ modules must respect both left Γ-action and right Λ-action.

Proposition 1.2. *Suppose Γ a right Λ-Hopf algebra, and suppose Σ, Σ' mixed Λ-Γ modules. Then $\Sigma \otimes \Sigma'$ is a mixed Λ-Γ module, under diagonal left Γ-action and diagonal right Λ-action.*

Definition 1.3. By a left Λ-Γ module we mean a left Λ-module Ω that is also a left Γ-module in such a way that the diagram commutes:

$$\begin{tikzcd}[column sep=large]
\Gamma\otimes\Lambda\otimes\Omega \arrow[r, "\Gamma\otimes\sigma(\Lambda,\Omega)"] \arrow[d, "\Gamma\otimes\psi\otimes\Omega"] & \Gamma\otimes\Omega \arrow[rr, "\sigma(\Gamma,\Omega)"] & & \Omega \\
\Gamma\otimes\Lambda\otimes\Lambda\otimes\Omega \arrow[r, "\rho(\Gamma,\Lambda)\otimes\Lambda\otimes\Omega"] & \Gamma\otimes\Lambda\otimes\Omega \arrow[r, "{(2,1,3)}"] & \Lambda\otimes\Gamma\otimes\Omega \arrow[r, "\Lambda\otimes\sigma(\Gamma,\Omega)"] & \Lambda\otimes\Omega \arrow[u, "\sigma(\Lambda,\Omega)"']
\end{tikzcd} \tag{1.1}$$

(Here the notation $(2, 1, 3)$ is as in [5, p. 355]; it denotes the permutation of factors in a tensor product.) A map of left Λ-Γ modules must respect both left Γ-action and left Λ-action.

Proposition 1.4. *Suppose Γ a right Λ-Hopf algebra, and suppose Ω and Ω' left Λ-Γ modules. Then $\Omega \otimes \Omega'$ is a left Λ-Γ module under diagonal left Γ-action and diagonal left Λ-action.*

Suppose now we are given left Γ-modules Σ, Σ'; and a left Λ-Γ module Ω. Suppose further we are given a Z_2-homomorphism $\phi\colon \Sigma \otimes \Lambda \to \Sigma'$ satisfying

$$\begin{tikzcd}
\Gamma\otimes\Sigma\otimes\Lambda \arrow[r, "\sigma(\Gamma,\Sigma)\otimes\Lambda"] \arrow[d, "\Gamma\otimes\Sigma\otimes\psi"] & \Sigma\otimes\Lambda \arrow[r, "\phi"] & \Sigma' \\
\Gamma\otimes\Sigma\otimes\Lambda\otimes\Lambda \arrow[r, "{(1,3,2,4)}"] & \Gamma\otimes\Lambda\otimes\Sigma\otimes\Lambda \arrow[r, "\rho(\Gamma,\Lambda)\otimes\phi"] & \Gamma\otimes\Sigma' \arrow[u, "\sigma(\Gamma{,}\Sigma')"']
\end{tikzcd} \tag{1.2}$$

Then for any Z_2-homomorphism $f\colon \Sigma' \to \Omega$ we will write $\phi^*(f)\colon \Sigma \otimes \Lambda \to \Omega$ for the composition $\sigma(\Lambda, \Omega)(2, 1)(f \otimes \Lambda)(\phi \otimes \Lambda)(\Sigma \otimes \psi)$.

Proposition 1.5. *Suppose $f\colon \Sigma' \to \Omega$ is a map of left Γ-modules. Then for any $\lambda \in \Lambda$ the Z_2-homomorphism $\lambda(f)\colon \Sigma \to \Omega$ defined by $(\lambda(f))(x) = (\phi^*(f))(x \otimes \lambda)$ is in fact a Γ-homomorphism. Thus the correspondence $(\lambda \otimes f) \to \lambda(f)$ defines*

$$\sigma(\Lambda, \mathrm{Hom})\colon \Lambda \otimes \mathrm{Hom}_\Gamma(\Sigma', \Omega) \to \mathrm{Hom}_\Gamma(\Sigma, \Omega). \tag{1.3}$$

If Σ is a mixed Λ-Γ module, if $\Sigma' = \Sigma$, and if $\phi = \rho(\Sigma, \Lambda)\colon \Sigma \otimes \Lambda \to \Sigma$, then (1.3) *gives* $\mathrm{Hom}_\Gamma(\Sigma, \Omega)$ *the structure of a left Λ-module.*

Proof. That $\lambda(f)$ is a Γ-homomorphism is an easy diagram chase which uses (1.1), (1.2), and the commutativity of $\psi\colon \Lambda \to \Lambda \otimes \Lambda$. If Σ is a mixed Λ-Γ

module, if $\Sigma = \Sigma'$, and if $\phi = \rho(\Sigma, \Lambda)$, then the relations $\sigma(\Lambda, \Omega)(\mu \otimes \Omega) = \sigma(\Lambda, \Omega)(\Lambda \otimes \sigma(\Lambda, \Omega))$ and $\rho(\Sigma, \Lambda)(\Sigma \otimes \mu) = \rho(\Sigma, \Lambda)(\rho(\Sigma, \Lambda) \otimes \Lambda)$, and the Hopf condition $(\mu \otimes \mu)(1, 3, 2, 4)(\psi \otimes \psi) = \psi\mu\colon \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$ imply that $(\lambda_1\lambda_2)f = \lambda_1(\lambda_2 f)$.

If Σ is a mixed Λ-Γ module and Ω a left Λ-Γ module, then Proposition 1.5 gives a Λ-action on $\mathrm{Hom}_\Gamma(\Sigma, \Omega)$. We now use Proposition 1.5 to define a Λ-action on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$. Some conventions: an augmented complex of modules $\to \Delta_n \to \cdots \to \Delta_0 \to \Sigma \to 0$ will be denoted $\Delta \to \Sigma \to 0$. By a "projective" left Γ-module we mean an extended left Γ-module.

Proposition 1.6. *Suppose $\Delta \to \Sigma \to 0$ and $\Delta' \to \Sigma' \to 0$ are Γ-projective resolutions of the left Γ-modules Σ and Σ'. Suppose given a Z_2-homomorphism $\phi(\Sigma, \Lambda)\colon \Sigma \otimes \Lambda \to \Sigma'$ satisfying* (1.2) *(with $\phi = \phi(\Sigma, \Lambda)$). Then there exists a map of Z_2-chain complexes $\phi(\Delta, \Lambda)\colon \Delta \otimes \Lambda \to \Delta'$ which extends $\phi(\Sigma, \Lambda)$, and which satisfies* (1.2) *(with $\phi = \phi(\Delta, \Lambda)$, and Δ, Δ' replacing Σ, Σ'). Any two such $\phi(\Delta, \Lambda)$'s differ by a Z_2-chain homotopy $\xi(\Delta, \Lambda)\colon \Delta \otimes \Lambda \to \Delta'$ which satisfies* (1.2) *(with $\phi = \xi(\Delta, \Lambda)$, and Δ_n, Δ'_{n+1} replacing Σ, Σ').*

Proof. Suppose inductively $\phi(\Delta, \Lambda)$ is defined, with all desired properties, in dimensions $\leq n - 1$. Write $\Delta_n = \Gamma \otimes \underline{\Delta}_n$ as a left Γ-module, and choose any Z_2-homomorphism $\underline{\phi}\colon \underline{\Delta}_n \otimes \Lambda \to \Delta'_n$ satisfying $d\underline{\phi} = \phi(\Delta, \Lambda)(d \otimes \Lambda)$. Define $\phi(\Delta, \Lambda)$ in dimension n by setting it equal to the composition $\sigma(\Gamma, \Delta'_n)(\rho(\Gamma, \Lambda) \otimes \underline{\phi}) \cdot (1, 3, 2, 4)(\Gamma \otimes \underline{\Delta}_n \otimes \psi)\colon \Gamma \otimes \underline{\Delta}_n \otimes \Lambda \to \Delta'_n$. Then $\phi(\Delta, \Lambda)$ commutes with d. That it satisfies (1.2) (with $\phi = \phi(\Delta, \Lambda)$, and Δ_n, Δ'_n replacing Σ, Σ') follows from the fact that $\mu\colon \Gamma \otimes \Gamma \to \Gamma$ is a map of right Λ-modules. That any two such ϕ's differ by an appropriate chain homotopy follows from a similar argument.

Suppose now Σ a mixed Λ-Γ module and Ω a left Λ-Γ module. To define a Λ-action on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ we choose a Γ-projective resolution $\Delta \to \Sigma \to 0$, and a Z_2-chain map $\phi(\Delta, \Lambda)\colon \Delta \otimes \Lambda \to \Delta$ as in Proposition 1.6 (with $\Sigma' = \Sigma$, $\Delta' = \Delta$, $\phi(\Sigma, \Lambda) = \rho(\Sigma, \Lambda)$). Then we use Proposition 1.5 with $\phi = \phi(\Delta, \Lambda)$, and the functorial properties of the construction described there, to obtain a map of cochain complexes $\sigma(\Lambda, \mathrm{Hom})\colon \Lambda \otimes \mathrm{Hom}_\Gamma(\Delta, \Omega) \to \mathrm{Hom}_\Gamma(\Delta, \Omega)$. Passing to homology we obtain a Z_2-homomorphism for each $n \geq 0$:

$$\sigma(\Lambda, \mathrm{Ext})\colon \Lambda \otimes \mathrm{Ext}^n_\Gamma(\Sigma, \Omega) \to \mathrm{Ext}^n_\Gamma(\Sigma, \Omega). \tag{1.4}$$

Since Proposition 1.6 guarantees existence of chain homotopies we have

Proposition 1.7. $\sigma(\Lambda, \mathrm{Ext})$ *is independent of choice of Δ, and of the choice of $\phi(\Delta, \Lambda)$.*

It remains only to verify the relation $\sigma(\Lambda, \mathrm{Ext})(\mu \otimes \mathrm{Ext}) = \sigma(\Lambda, \mathrm{Ext}) \cdot (\Lambda \otimes \sigma(\Lambda, \mathrm{Ext}))$. (This is not immediate: $\phi(\Delta, \Lambda)$ of Proposition 1.6 has no associative property.) To this end,

Definition 1.8. By a projective mixed Λ-Γ module we mean one of the form $\Gamma \otimes \underline{\Delta}$. Here $\underline{\Delta}$ is a right Λ-module, Λ acts diagonally on the right of $\Gamma \otimes \underline{\Delta}$, and the left action of Γ is $\mu_\Gamma \otimes \underline{\Delta}: \Gamma \otimes \Gamma \otimes \underline{\Delta} \to \Gamma \otimes \underline{\Delta}$. By a Λ-Γ projective resolution of the mixed Λ-Γ module Σ we mean a complex over Σ of projective mixed Λ-Γ modules, that is split exact when regarded as a complex of right Λ-modules.

The existence of a Λ-Γ projective resolution of any Σ, and its uniqueness up to homotopy type, is proved by standard arguments in relative homological algebra (e.g. [8, Chapter IX]).

Proposition 1.7 enables us to use a Λ-Γ projective resolution to compute $\sigma(\Lambda, \mathrm{Ext})$: we just take $\phi(\Delta, \Lambda)$ of Proposition 1.6 equal to $\rho(\Delta, \Lambda)$. Since each Δ_n is now a mixed Λ-Γ module, we get from Proposition 1.5 the structure of a left Λ-module on $\mathrm{Hom}_\Gamma(\Delta_n, \Omega)$. Consequently

Proposition 1.9. *$\sigma(\Lambda, \mathrm{Ext})$ of (1.4) gives $\mathrm{Ext}^n_\Gamma(\Sigma, \Omega)$ the structure of a left Λ-module $(n \geq 0)$.*

The reader has noticed that we could have defined $\sigma(\Lambda, \mathrm{Ext})$ in the first place by using Λ-Γ projective resolutions. But Proposition 1.6 permits us to use resolutions having only a Γ-structure. The resulting flexibility will be useful in §§3 and 5 (see Proposition 3.11).

2. **Products and Steenrod squares.** In this section we find the relationship between the action of Λ on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$, and the products and Steenrod squares on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$. The main results are Propositions 2.5 and 2.7.

We denote the Steenrod algebra by the symbol $\mathfrak{A}$, and interpret the Adem relations as given on p. 2 of [17] with $\mathrm{Sq}^0 \neq 1$. Also we interpret the standard formula for the coproduct with $\mathrm{Sq}^0 \neq 1$. If we define an augmentation by $\epsilon(\mathrm{Sq}^0) = 1$, $\epsilon(\mathrm{Sq}^k) = 0$ $(k \neq 0)$ then $\mathfrak{A}$ is a Hopf algebra.

Suppose now Γ a Hopf algebra, Σ a left Γ-coalgebra, and Ω a commutative left Γ-algebra. We define products and Steenrod squares on $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ in the usual way: choose a Γ-projective resolution $\Delta \to \Sigma \to 0$; define a map of cochain complexes

$$(2.1) \qquad \phi: \mathrm{Hom}_\Gamma(\Delta, \Omega) \otimes \mathrm{Hom}_\Gamma(\Delta, \Omega) \to \mathrm{Hom}_\Gamma(\Delta \otimes \Delta, \Omega)$$

as in I, (2.2); and choose a sequence of Γ-homomorphisms $D_k: \Delta \to \Delta \otimes \Delta$ homogeneous of degree k and satisfying

$$(2.2) \qquad dD_k + D_k d = D_{k-1} + TD_{k-1}.$$

D_0 must be a chain map extending $\psi: \Sigma \to \Sigma \otimes \Sigma$. Then the cochain operations $\mu(x \otimes y) = D_0^* \phi(x \otimes y)$; $S^k x = D_{n-k}^* \phi(x \otimes x)$ $(n = \dim x)$ define products and Sq^k. Well-known arguments (e.g. [9]) show that under these definitions $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ becomes a commutative $\mathfrak{A}$-algebra.

We fix for the remainder of this section a Hopf algebra Λ, and a right Λ-Hopf algebra Γ

Definition 2.1. By a Λ-Γ coalgebra we mean a mixed Λ-Γ module Σ that is also a coalgebra in such a way that $\psi\colon \Sigma \to \Sigma \otimes \Sigma$ and $\epsilon\colon \Sigma \to Z_2$ are maps of mixed Λ-Γ modules. (Here $\Sigma \otimes \Sigma$ has the tensor product Λ-Γ structure, as in Proposition 1.2.)

Definition 2.2. By a Λ-Γ algebra we mean a left Λ-Γ module Ω that is also an algebra in such a way that $\mu\colon \Omega \otimes \Omega \to \Omega$ and $\eta\colon Z_2 \to \Omega$ are maps of left Λ-Γ modules. (Here $\Omega \otimes \Omega$ has the tensor product Λ-Γ structure, as in Proposition 1.4.)

We fix for the rest of this section a Λ-Γ coalgebra Σ and a commutative Λ-Γ algebra Ω. Now, a tensor product of projective Λ-Γ resolutions is again a complex of mixed Λ-Γ modules (Proposition 1.2) that is split over Λ. Consequently,

Proposition 2.3. *Suppose $\Delta \to \Sigma \to 0$ is a Λ-Γ projective resolution. Then the D_k's of (2.2) can be chosen Λ-Γ homomorphisms.*

Next we observe that if $\Delta \to \Sigma \to 0$ is a Λ-Γ projective resolution, then Proposition 1.5 gives to each side of (2.1) the structure of a left Λ-module. An easy diagram chase using the commutativity of $\psi\colon \Lambda \to \Lambda \otimes \Lambda$ and the fact that $\mu\colon \Omega \otimes \Omega \to \Omega$ is a Λ-morphism gives

Proposition 2.4. *ϕ of (2.1) is a homomorphism of left Λ-modules.*

From Propositions 2.3 and 2.4 follows immediately

Proposition 2.5. $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ *is a left Λ-algebra.*

To find the relationship between Steenrod squares and Λ-action we need a definition. Recall that if Φ is a commutative Z_2-algebra the Frobenius map $F\colon \Phi \to \Phi$ is defined by $F(x) = x^2$. Suppose now that Φ is a finite dimensional (commutative) coalgebra (or, if Φ is graded, finite dimensional in each degree). Define the "Verschiebung" $V\colon \Phi \to \Phi$ by $V = F^*\colon (\Phi^*)^* \to (\Phi^*)^*$. Then V is a morphism of coalgebras. If Φ happens to be a Hopf algebra V is a morphism of Hopf algebras. (One can define V without assuming Φ finite dimensional; Lemma 2.6 remains valid. We omit these details.)

Lemma 2.6. *Let Φ be a Z_2-coalgebra finite dimensional in each degree and let $\lambda \in \Phi$ be given. Then there exists an element $\theta \in \Phi \otimes \Phi$ satisfying $\psi(\lambda) = (1 + T)\theta + V\lambda \otimes V\lambda$.*

Proof. Using the commutativity of $\psi\colon \Phi \to \Phi \otimes \Phi$ one easily shows the existence of θ and w in $\Phi \otimes \Phi$ satisfying $\psi(\lambda) = (1 + T)\theta + w \otimes w$. Then for any σ in the dual algebra Φ^* we have $[V\lambda, \sigma] = [\lambda, F\sigma] = [\lambda, \mu(\sigma \otimes \sigma)] = [\psi\lambda, \sigma \otimes \sigma] = [w, \sigma] \cdot [w, \sigma] = [w, \sigma]$ (modulo 2). Thus $w = V\lambda$.

Proposition 2.7. *Suppose Σ a Λ-Γ coalgebra and Ω a commutative Λ-Γ algebra. Then for any $\lambda \in \Lambda$, $u \in \mathrm{Ext}_\Gamma^n(\Sigma, \Omega)$, and $k \geq 0$,*

$$\lambda(\mathrm{Sq}^k u) = \mathrm{Sq}^k(V(\lambda)u). \tag{2.3}$$

Proof. Choose a Λ-Γ projective resolution $\Delta \longrightarrow \Sigma \longrightarrow 0$, and choose a cocycle $x \in \mathrm{Hom}_\Gamma(\Delta_n, \Omega)$ to represent u. By Lemma 2.6 we can choose α_i, $\beta_i \in \Lambda$ so that

$$\psi(\lambda) = \sum_i (\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i) + V(\lambda) \otimes V(\lambda). \tag{2.4}$$

Now $\lambda(\mathrm{Sq}^k u)$ is represented by the cocycle $\lambda(D^*_{n-k}\phi(x \otimes x))$ in $\mathrm{Hom}_\Gamma(\Delta_{n+k}, \Omega)$. But if we choose the D_k's to be Λ-Γ morphisms (Proposition 2.3), D^*_{n-k} preserves Λ-action. Combining this observation with Proposition 2.4 and equation (2.4) we get

$$\begin{aligned} &\lambda(D^*_{n-k}\phi(x \otimes x)) \\ &\quad = D^*_{n-k}\phi(V(\lambda)x \otimes V(\lambda)x) + \sum_i D^*_{n-k}\phi(\alpha_i x \otimes \beta_i x + \beta_i x \otimes \alpha_i x). \end{aligned} \tag{2.5}$$

But (2.2) implies

$$\begin{aligned} D^*_{n-k}\phi(\alpha_i x \otimes \beta_i x + \beta_i x \otimes \alpha_i x) &= D^*_{n-k}(1 + T)\phi(\alpha_i x \otimes \beta_i x) \\ &= \delta D^*_{n-k+1}\phi(\alpha_i x \otimes \beta_i x) + D^*_{n-k+1}\phi\delta(\alpha_i x \otimes \beta_i x). \end{aligned} \tag{2.6}$$

But the second term on the right of (2.6) is zero (δ commutes with the action of Λ on $\mathrm{Hom}_\Gamma(\Delta, \Omega)$, and x is a cocycle); and the first term is a coboundary. Hence $\lambda(\mathrm{Sq}^k u)$ is represented by the cocycle $D^*_{n-k}\phi(V(\lambda)x \otimes V(\lambda)x)$. But this also represents $\mathrm{Sq}^k(V(\lambda)u)$.

The results of Propositions 1.9, 2.5, and 2.7 can be summarized in a useful way. We define a right action of $\mathfrak{A}$ on the Hopf algebra Λ by setting

$$\begin{aligned} \rho(\Lambda, \mathfrak{A})(\lambda \otimes \mathrm{Sq}^k) &= V(\lambda), \qquad k = 0, \\ &= \;0, \qquad\quad\; k > 0. \end{aligned} \tag{2.7}$$

One verifies that this rule is consistent with the Adem relations on $\mathfrak{A}$, and gives Λ the structure of a right $\mathfrak{A}$-Hopf algebra.

Proposition 2.8. *Suppose Σ a Λ-Γ coalgebra and Ω a commutative Λ-Γ algebra. Consider $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ as an $\mathfrak{A}$-algebra, and as a left Λ-module as in Proposition 1.9. Then $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ becomes a commutative $\mathfrak{A}$-Λ algebra.*

Proof. We need only check that $\mathrm{Ext}_\Gamma(\Sigma, \Omega)$ is a left $\mathfrak{A}$-Λ module. In fact, the commutativity of the diagram

$$
\begin{array}{ccccc}
\Lambda\otimes\mathfrak{A}\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega) & \xrightarrow{\Lambda\otimes\sigma(\mathfrak{A},\,\mathrm{Ext})} & \Lambda\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega) & \xrightarrow{\sigma(\Lambda,\mathrm{Ext})} & \mathrm{Ext}_\Gamma(\Sigma,\Omega) \\
 & & & & \uparrow{\scriptstyle \sigma(\mathfrak{A},\,\mathrm{Ext})} \\
\Big\downarrow{\scriptstyle \Lambda\otimes\psi\otimes\mathrm{Ext}} & & & & \mathfrak{A}\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega) \\
 & & & & \uparrow{\scriptstyle \mathfrak{A}\otimes\sigma(\Lambda,\,\mathrm{Ext})} \\
\Lambda\otimes\mathfrak{A}\otimes\mathfrak{A}\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega) & \xrightarrow{\rho(\Lambda,\mathfrak{A})\otimes\mathfrak{A}\otimes\mathrm{Ext}} & \Lambda\otimes\mathfrak{A}\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega) & \xrightarrow{(2,1,3)} & \mathfrak{A}\otimes\Lambda\otimes\mathrm{Ext}_\Gamma(\Sigma,\Omega)
\end{array}
\tag{2.8}
$$

is just a restatement of Proposition 2.7.

3. **Extensions of Hopf algebras.** In this section we show how an extension of Hopf algebras $A \to C \to B$ determines actions of C and B upon A by "conjugation". In particular we will find that A becomes a C-Hopf algebra. We will then use these results to discuss actions of C and B upon modules of the form $\mathrm{Hom}_A(P, Q)$.

We assume all algebras, coalgebras, Hopf algebras are graded and connected (G.C.). We do this in order to apply the results of [14], but we have no other use for this grading and will suppress it. If Σ is a G.C. coalgebra and Ω a G.C. algebra we write $\underline{\mathrm{Hom}}(\Sigma, \Omega)$ for the set of G.C. homomorphisms $\Sigma \to \Omega$; i.e., gradation preserving Z_2-maps which reduce in dimension zero to the identity $Z_2 \to Z_2$. Then $\underline{\mathrm{Hom}}(\Sigma, \Omega)$ is a group under "cup product", or "convolution" ([10, §8], [6, §1.5]). We use the notation of [6] and write $f * g$ for the convolution of f and g. We write f^{-1} for the inverse of f under convolution, but never for its inverse under composition. In sets such as $\underline{\mathrm{Hom}}(\Sigma \otimes \Sigma', \Omega \otimes \Omega')$ we define the convolution using tensor product comultiplication on $\Sigma \otimes \Sigma'$, and tensor product multiplication on $\Omega \otimes \Omega'$.

We suppose fixed an extension diagram

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B \tag{3.1}$$

of G.C. Hopf algebras, and suppose the multiplication on A commutative. "Extension" is as in Definition 2.1 of [14]: there exists a G.C. map k: $C \to A \otimes B$ that is simultaneously an isomorphism of left A-modules and right B-comodules. In this paper we are also assuming that comultiplications are commutative. Nevertheless the extension may have simultaneously nontrivial algebra and coalgebra "twistings" (τ_A and ϕ_B of [14]).

Let l: $A \otimes B \to C$ be the inverse of k under composition. As in Definition 2.2 of [14] we consider G.C. maps γ: $B \to C$, δ: $C \to A$ given by $\gamma = li_B$, $\delta = p_A k$ (here injections $i_B, \cdots$, and projections $p_A, \cdots$, are as in [14, §1]). We also consider G.C. maps ν_B: $B \otimes A \to C$, ν_C: $C \otimes A \to C$ given by

$$\nu_B = \gamma p_B * \alpha p_A * (\gamma p_B)^{-1}, \tag{3.2}$$

$$\nu_C = p_C * \alpha p_A * p_C^{-1}. \tag{3.3}$$

The map ν_B has been called σ in [14] and its properties have been investigated there. They are summarized in

Proposition 3.1. *There exists a unique G.C. map $\sigma(B, A)\colon B \otimes A \to A$ satisfying $\alpha\sigma(B, A) = \nu_B$. Both ν_B and $\sigma(B, A)$ are independent of the choice of k, and $\sigma(B, A)$ gives A the structure of a left B-Hopf algebra.*

Proof. Propositions 2.3, 2.5, 2.6 and 2.9 of [14]. (Since we assume here that the comultiplication on C is commutative, the "coaction" $\rho_B\colon B \to B \otimes A$ of [14] is the trivial map $(\eta \otimes \eta)\epsilon$. Hence Proposition 2.6 of [14] reduces to the statement that $\psi\colon A \to A \otimes A$ is a map of left B-modules.)

Proposition 3.2. *There exists a unique G.C. map $\sigma(C, A)\colon C \otimes A \to A$ satisfying $\alpha\sigma(C, A) = \nu_C$. Both ν_C and $\sigma(C, A)$ are independent of the choice of k, and $\sigma(C, A)$ gives A the structure of a left C-Hopf algebra.*

In fact Proposition 3.2 follows immediately from Proposition 3.1 and the following

Proposition 3.3.

$$\nu_C = \nu_B(\beta \otimes A). \tag{3.4}$$

In order to prove Proposition 3.3 we need two lemmas.

Lemma 3.4. *In $\underline{\mathrm{Hom}}(C, C)$, $\alpha\delta * \beta\gamma = C$.*

Proof. Use k to identify C with $A \otimes B$. Then we need only show that, in $\underline{\mathrm{Hom}}(A \otimes B, A \otimes B)$, $\mu(A \otimes \eta\epsilon \otimes \eta\epsilon \otimes B)\psi = A \otimes B$, where μ and ψ are given by (2.3) and (2.3)* of [14]. But this is easily checked.

Lemma 3.5. *In $\underline{\mathrm{Hom}}(C, C)$, $\gamma\beta * \alpha\theta = C$, where $\theta\colon C \to A$ is defined by $\theta = \sigma(B, A)(\beta^{-1} \otimes \delta)\psi_C$.*

Proof. Since the comultiplication on C is commutative, this follows easily from Lemma 2.10 of [14], Lemma 3.4 above, and our result that A is a B-module under $\sigma(B, A)$.

Proof of Proposition 3.3. Since $\beta \otimes A$ is a map of coalgebras, we have from (3.2)

$$\nu_B(\beta \otimes A) = \gamma\beta p_C * \alpha p_A * (\gamma\beta p_C)^{-1} \tag{3.5}$$

in $\underline{\mathrm{Hom}}(C \otimes A, C)$. But from Lemma 3.5 we have $\gamma\beta p_C = p_C * \alpha(\theta p_C)^{-1}$ so (3.5) becomes

$$(3.6) \qquad \nu_B(\beta \otimes A) = p_C * \alpha((\theta p_C)^{-1} * p_A * \theta p_C) * p_C^{-1}.$$

But $\mathrm{Hom}(C \otimes A, A)$ is an abelian group under convolution, so $(\theta p_C)^{-1} * p_A * \theta p_C = p_A$, and (3.6) reduces to (3.4). This completes the proof of Proposition 3.3; and so Proposition 3.2 is also proved.

Now let $\chi: C \to C$ be the antipode.

Definition 3.6. To an extension (3.1) we associate the structure of a right C-Hopf algebra on A, by letting $\rho(A, C) = \sigma(C, A)(2, 1)(A \otimes \chi)$. We refer to ρ as the action "opposite" to σ.

A simple diagram chase which uses the fact that $\chi: C \to C$ is an antiautomorphism of coalgebras, the relation $\chi\chi = C$, and the commutativity of $\psi_C: C \to C \otimes C$ shows that

$$(3.7) \qquad \alpha\rho(A, C) = p_C^{-1} * \alpha p_A * p_C.$$

Having given A the structure of a right C-Hopf algebra we proceed to describe certain mixed C-A modules and certain left C-A modules.

Proposition 3.7. *Suppose P a left C-module. If we regard P as a right C-module under the associated opposite action, and as a left A-module by restricting the original C-action, then P becomes a mixed C-A module. If P was originally a left C-coalgebra, it becomes after these constructions a C-A coalgebra.*

Proposition 3.8. *Suppose Q a left C-module. If we regard Q also as a left A-module by restricting the C-action, then Q becomes a left C-A module. If Q was originally a left C-algebra, it becomes after these constructions a C-A algebra.*

Both Propositions 3.7 and 3.8 follow easily from (3.7).

Combining Propositions 1.5, 3.7, 3.8 we find that for a pair P, Q of left C-modules we have defined an action

$$(3.8) \qquad \sigma(C, \mathrm{Hom}): C \otimes \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(P, Q).$$

The following two propositions all pertain to this action. They can be proved for Hopf algebras by analogy with well-known results in the theory of groups (e.g. [8, Chapter XI, §9]).

Proposition 3.9. *Under the action* (3.8), $\mathrm{Hom}_A(P, Q)$ *is annihilated by the augmentation ideal of A. Thus* $\sigma(C, \mathrm{Hom})$ *passes to an action of the quotient* $C//A = B$: $\sigma(B, \mathrm{Hom})$: $B \otimes \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(P, Q)$.

Proposition 3.10. *Suppose R a left B-module. Then the assignment to any B-homomorphism f: $R \to$* $\mathrm{Hom}_A(P, Q)$ *of the map $\underline{f}$: $R \otimes P \to Q$ given by $\underline{f}(r \otimes p)$*

$= f(r)(p)$ *is a* Z_2*-isomorphism* $\mathrm{Hom}_B(R, \mathrm{Hom}_A(P, Q)) \to \mathrm{Hom}_C(R \otimes P, Q)$ *(on the right-hand side* C *acts diagonally on* $R \otimes P$).

Suppose given left C-modules M and N, and a C-projective resolution $W \to M \to 0$. We regard M as a mixed C-A module, W as a complex of mixed C-A modules (Proposition 3.7), and N as a left C-A module (Proposition 3.8). Then W is an A-projective resolution of M, and is equipped with a map of chain complexes $W \otimes C \to W$ satisfying the conditions of Proposition 1.6. Therefore the action of C on $\mathrm{Ext}_A(M, N)$ can be computed as $\mathrm{Ext}_A(M, N) = H^*(\mathrm{Hom}_A(W, N))$, where the action of C on $\mathrm{Hom}_A(W, N)$ is as in (3.8). But the latter passes to an action of $C//A$, so we have defined

$$\sigma(B, \mathrm{Ext}): B \otimes \mathrm{Ext}_A(M, N) \to \mathrm{Ext}_A(M, N) \tag{3.9}$$

and proved

Proposition 3.11. $H^*(\mathrm{Hom}_A(W, N)) = \mathrm{Ext}_A(M, N)$ *as left* B*-modules, for any* C*-projective resolution* $W \to M \to 0$.

Proposition 3.11 is the motivation for our flexible definition of the action of C upon $\mathrm{Ext}_A(M, N)$, by way of Proposition 1.6. If we had given only the definition in terms of projective mixed C-A modules (Definition 1.8), we would have no proof for Proposition 3.12. A projective C-module, when regarded as a mixed C-A module, is not in general a projective mixed C-A module!

4. **Actions of $\mathfrak{A}$ on $\mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N))$.** We fix in this section an extension of Hopf algebras (3.1); A has commutative multiplication. We fix also a left C-coalgebra M and a commutative left C-algebra N. When appropriate we will regard M as a C-A coalgebra (Proposition 3.7) and N has a C-A algebra (Proposition 3.8). We will also regard B as a right Hopf algebra over the Steenrod algebra $\mathfrak{A}$, using (2.7) with $\Lambda = B$.

We apply Proposition 2.8 with $\Lambda = C$, $\Gamma = A$, $\Sigma = M$, $\Omega = N$, and find that (3.9) gives $\mathrm{Ext}_A(M, N)$ the structure of a commutative $\mathfrak{A}$-B algebra. Now we immediately make a second application of §§1 and 2, as summarized in Proposition 2.8, with $\Lambda = \mathfrak{A}$, $\Gamma = B$, $\Sigma = Z_2$, $\Omega = \mathrm{Ext}_A(M, N)$. We obtain a product

$$\mu: \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \otimes \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \to \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \tag{4.1}$$

and a vertical action

$$\sigma_V(\mathfrak{A}, \mathrm{Ext}) = \sigma(\Lambda, \mathrm{Ext}): \mathfrak{A} \otimes \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \to \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \tag{4.2}$$

under which Ext becomes an $\mathfrak{A}$-algebra; and a diagonal action

$$\sigma_D(\mathfrak{A}, \mathrm{Ext}) = \sigma(\mathfrak{A}, \mathrm{Ext}): \mathfrak{A} \otimes \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \to \mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) \tag{4.3}$$

under which Ext becomes an $\mathcal{A}$-algebra. Diagram (2.8) says that vertical and diagonal actions are related by

$$\mathrm{Sq}_V^k(\mathrm{Sq}_D^j u) = \mathrm{Sq}_D^j(\mathrm{Sq}_V^{k/2} u) \tag{4.4}$$

for any $u \in$ Ext. (Recall that the Verschiebung is determined on $\mathcal{A}$ by $V(\mathrm{Sq}^k) = \mathrm{Sq}^{k/2}$; e.g. [17, p. 24].) (Added in proof. Suppose the extension (3.1) is central; that is, the associated map $\sigma(B, A)\colon B \otimes A \to A$ is the trivial action. Then $\mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N)) = \mathrm{Ext}_B(Z_2, Z_2) \otimes \mathrm{Ext}_A(M, N)$, and the actions (4.2) and (4.3) reduce to $\mathrm{Sq}_V^k(x \otimes y) = \mathrm{Sq}^0 x \otimes \mathrm{Sq}^k y$, and $\mathrm{Sq}_D^k(x \otimes y) = \mathrm{Sq}^k x \otimes y^2$, for x in $\mathrm{Ext}_B(Z_2, Z_2)$, y in $\mathrm{Ext}_A(M, N)$.)

5. **Change of rings spectral sequence.** If Γ is a Hopf algebra and Σ a left Γ-coalgebra we write $W(\Gamma, \Sigma)$ for the acyclic bar construction over the left Γ-module Σ. We regard $W(\Gamma, \Sigma)$ as a simplicial object over the category of left Γ-coalgebras: if $W(\Gamma, \Sigma)_n = \Gamma^{\otimes(n+1)} \otimes \Sigma$ is the "inhomogeneous" representation, then the comultiplication is that of the $(n+2)$-fold tensor product.

Fix an extension of Hopf algebras (3.1). Fix a left C-coalgebra M and a commutative left C-algebra N. Define a bisimplicial left C-coalgebra X by setting $X_{p,q} = W(B)_p \otimes W(C, M)_q$ as a tensor product of left C-coalgebras (we abbreviate $W(B, Z_2) = W(B)$). X is equipped with an obvious augmentation (in the sense of I, §1) $\lambda\colon X \to W(C, M)$. Since C has an antipode, the tensor product of a projective left C-module with an arbitrary left C-module is again projective. So X can be regarded as a C-projective resolution of M, and we have:

Proposition 5.1. $\lambda^*\colon \mathrm{Ext}_C(M, N) \to H^*(\mathrm{Hom}_C(X, N))$ *is an isomorphism of Z_2-modules.*

Consequently the spectral sequence of the bisimplicial C-coalgebra X converges to $\mathrm{Ext}_C(M, N)$, and is equipped with Steenrod squares enjoying all the properties listed in I, §1. From Proposition 3.10 we have

$$E_0^{p,q} = \mathrm{Hom}_C(W(B)_p \otimes W(C, M)_q, N) = \mathrm{Hom}_B(W(B)_p, \mathrm{Hom}_A(W(C, M)_q, N)) \tag{5.1}$$

and since $W(B)_p$ is B-projective we get from Proposition 3.12

$$E_1^{p,q} = \mathrm{Hom}_B(W(B)_p, \mathrm{Ext}_A^q(M, N)) \tag{5.2}$$

from which follows

$$E_2^{p,q} = \mathrm{Ext}_B^p(Z_2, \mathrm{Ext}_A^q(M, N)). \tag{5.3}$$

Proposition 5.2. *The actions of the vertical Steenrod squares on E_2 are given by $\sigma_V(\mathcal{A}\,\mathrm{Ext})$ of (4.2).*

Proof. We take $E_2^{p,q}$ as given by (5.1) of I. Suppose u in $E_2^{p,q}$ is represented by the C-homomorphism $x: X_{p,q} \to N$. Let (D_k) be a special Eilenberg-Zilber map in the sense of I, §2. Then by Proposition 5.1 of I, $\mathrm{Sq}_V^k\, u = \mathrm{Sq}^k\, u \in E_2^{p,q+k}$ is represented by a C-homomorphism $X_{p,q+k} \to N$ given by

$$W(B)_p \otimes W(C, M)_{q+k} \xrightarrow{\psi\otimes\psi} W(B)_p \otimes W(B)_p \otimes W(C, M)_{q+k} \otimes W(C, M)_{q+k}$$

$$(5.4) \qquad \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes D^{q,q}_{q-k}} W(B)_p \otimes W(B)_p \otimes W(C, M)_q \otimes W(C, M)_q$$

$$\xrightarrow{(1,3,2,4)} W(B)_p \otimes W(C, M)_q \otimes W(B)_p \otimes W(C, M)_q \xrightarrow{x\otimes x} N \otimes N \xrightarrow{\mu} N.$$

We must compare this with the purely algebraic description of $\mathrm{Sq}_V^k\, u$ given in (4.2).

Under the isomorphism of Proposition 3.10, x corresponds to a B-homomorphism $x': W(B)_p \to \mathrm{Hom}_A(W(C, M)_q, N)$; which in turn represents an element $x'': W(B)_p \to \mathrm{Ext}_A^q(M, N)$ of $E_1^{p,q}$ (see (5.2)). Now observe that if B is regarded as a right $\mathfrak{A}$-Hopf algebra (as in (2.7), with $\Lambda = B$), then $W(B)$ is an $\mathfrak{A}$-B projective resolution of the mixed $\mathfrak{A}$-B module Z_2. (In fact, for any $b \in W(B)_p$ we set $(b)\mathrm{Sq}^k = 0$ if $k > 0$ and $(b)\mathrm{Sq}^0 = Vb$, where the Verschiebung is defined by virtue of the coalgebra structure on $W(B)$.) Therefore (Proposition 1.5) the algebraically defined $\mathrm{Sq}_V^k\, u$ is represented in $E_1^{p,q+k}$ by the composition

$$(5.5) \qquad W(B)_p \xrightarrow{V} W(B)_p \xrightarrow{x''} \mathrm{Ext}_A^q(M, N) \xrightarrow{\mathrm{Sq}^k} \mathrm{Ext}_A^{q+k}(M, N),$$

Now since (D_k) is an Eilenberg-Zilber map in the sense of I, §2, it is easy to see that the compositions

$$W(C, M) \xrightarrow{\psi} W(C, M) \times W(C, M) \xrightarrow{D_k} W(C, M) \otimes W(C, M)$$

form an Eilenberg-Zilber map in the sense of II, §2, equation (2.2). That is, Sq^k on $\mathrm{Ext}_A^q(M, N)$ can be computed by means of the cochain operation S^k: $\mathrm{Hom}_A(W(C, M)_q, N) \to \mathrm{Hom}_A(W(C, M)_{q+k}, N)$ given by

$$(5.6) \qquad S^k(x') = \psi^* D^*_{q-k}\phi(x' \otimes x').$$

Now, the value of the composition (5.5) upon an arbitrary $b \in W(B)_p$ is an element of $\mathrm{Ext}_A(M, N)$. Using (5.6) we can represent this element by a specific cochain $f(b) \in \mathrm{Hom}_A(W(C, M), N)$. $f(b)$ is

$$(5.7) \qquad W(C, M)_{q+k} \xrightarrow{\psi} W(C, M)_{q+k} \otimes W(C, M)_{q+k} \xrightarrow{D^{q,q}_{q-k}} W(C, M)_q \otimes W(C, M)_q$$

$$\xrightarrow{x''(V(b))\otimes x'(V(b))} N \otimes N \xrightarrow{\mu} N.$$

Using Lemma 2.6 write $\psi(b) = \Sigma\, b_i \otimes b_i' + \Sigma\, b_i' \otimes b_i + Vb \otimes Vb$, and define cochains $g_i(b) \in \mathrm{Hom}_A(W(C, M), N)$ by the compositions

$$(5.8)\qquad \begin{aligned} &W(C, M)_{q+k} \xrightarrow{\psi} W(C, M)_{q+k} \otimes W(C, M)_{q+k} \xrightarrow{D^{q,q}_{q-k}} W(C, M)_q \otimes W(C, M)_q \\ &\xrightarrow{x'(b_i)\otimes x'(b_i')+x'(b_i')\otimes x'(b_i)} N \otimes N \xrightarrow{\mu} N. \end{aligned}$$

By an argument similar to that used in the proof of Proposition 2.7, the identities (2.2) imply that all cochains $g_i(b)$ are cohomologous to zero. It follows immediately that (5.5) is represented on the E_0 level by the B-homomorphism h: $W(B)_p \to \mathrm{Hom}_A(W(C, M)_{q+k}, N)$ defined by $h(b) = f(b) + \Sigma_i\, g_i(b)$. But under the isomorphism $\mathrm{Hom}_B(W(B)_p, \mathrm{Hom}_A(W(C, M)_{q+k}, N)) = \mathrm{Hom}_C(W(B)_p \otimes W(C, M)_{q+k}, N)$, h corresponds exactly to (5.4). This completes the proof.

Proposition 5.3. *The actions of the diagonal Steenrod squares on* E_2 *are given by* $\sigma_D(\mathfrak{A}, \mathrm{Ext})$ *of* (4.3).

Proof. Suppose u in $E_2^{p,q}$ is represented by the C-homomorphism x: $X_{p,q} \to N$. Let (D_k) be a special Eilenberg-Zilber map. By Proposition 5.1 of I, $\mathrm{Sq}^k_D\, u = \mathrm{Sq}^{q+k}\, u \in E_2^{p+k,2q}$ is represented by the C-homomorphism

$$(5.9)\qquad \begin{aligned} &W(B)_{p+k} \otimes W(C, M)_{2q} \xrightarrow{\psi\otimes\psi} W(B)_{p+k} \otimes W(B)_{p+k} \otimes W(C, M)_{2q} \otimes W(C, M)_{2q} \\ &\xrightarrow{D^{p,p}_{p-k}\otimes D^{q,q}_0} W(B)_p \otimes W(B)_p \otimes W(C, M)_q \otimes W(C, M)_q \\ &\xrightarrow{(1,3,2,4)} W(B)_p \otimes W(C, M)_q \otimes W(B)_p \otimes W(C, M)_q \xrightarrow{x\otimes x} N \otimes N \xrightarrow{\mu} N. \end{aligned}$$

We must compare this with the purely algebraic description of $\mathrm{Sq}^k_D\, u$ given in (4.3).

Under the isomorphism of Proposition 3.10, x corresponds to a B-homomorphism x': $W(B)_p \to \mathrm{Hom}_A(W(C, M)_q, N)$; which in turn represents an element x'': $W(B)_p \to \mathrm{Ext}^q_A(M, N)$ of $E_1^{p,q}$ (see (5.2)). By the construction in §2 of the Steenrod squares on $\mathrm{Ext}_B(Z_2, \mathrm{Ext}_A(M, N))$, we find that $\mathrm{Sq}^k_D\, u$ is represented in the complex $\mathrm{Hom}_B(W(B), \mathrm{Ext}_A(M, N))$ by the cocycle

$$(5.10)\qquad \begin{aligned} &W(B)_{p+k} \xrightarrow{\psi} W(B)_{p+k} \otimes W(B)_{p+k} \xrightarrow{D^{p,p}_{p-k}} W(B)_p \otimes W(B)_p \\ &\xrightarrow{x''\otimes x''} \mathrm{Ext}^q_A(M, N) \otimes \mathrm{Ext}^q_A(M, N) \xrightarrow{\mu} \mathrm{Ext}^{2q}_A(M, N). \end{aligned}$$

But by the definition in §2 of the product on $\mathrm{Ext}_A(M, N)$, we find it can be represented by the cochain operation μ: $\mathrm{Hom}_A(W(C, M), N) \otimes \mathrm{Hom}_A(W(C, M), N) \to \mathrm{Hom}_A(W(C, M), N)$ given by $\mu(x' \otimes y') = \psi^* D_0^* \phi(x' \otimes y')$. Then the element of

$E_1^{p+k,2q}$ that is given by (5.10) can be represented on the E_0 level by

$$
(5.11)\quad \begin{aligned} &W(B)_{p+k} \xrightarrow{\psi} W(B)_{p+k} \otimes W(B)_{p+k} \xrightarrow{D^{p,p}_{p-k}} W(B)_p \otimes W(B)_p \\ &\xrightarrow{x' \otimes x'} \operatorname{Hom}_A(W(C, M)_q, N) \otimes \operatorname{Hom}_A(W(C, M)_q, N) \xrightarrow{\mu} \operatorname{Hom}_A(W(C, M)_{2q}, N). \end{aligned}
$$

But under the isomorphism $\operatorname{Hom}_B(W(B)_{p+k}, \operatorname{Hom}_A(W(C, M)_{2q}, N)) = \operatorname{Hom}_C(W(B)_{p+k} \otimes W(C, M)_{2q}, N)$, (5.11) corresponds exactly to (5.9). This completes the proof.

Proposition 5.4. *The product on E_2 is given by μ of* (4.1).

The proof is similar to what we have just done. One uses Proposition 5.1 of I.

6. **Actions of $\mathcal{A}$ on $\operatorname{Ext}_{H_*(G)}(H_*(E), Z_2)$.** Suppose G a topological group and E a G-space. Then $H_*(G)$ (coefficients in Z_2) is a Hopf algebra in the usual way, and the action $G \times E \to E$ defines a module structure $H_*(G) \otimes H_*(E) \to H_*(E)$ under which $H_*(E)$ becomes a left $H_*(G)$ coalgebra. Moreover the usual actions of $\mathcal{A}$ on the right of $H_*(G)$ and $H_*(E)$ give $H_*(G)$ the structure of a right $\mathcal{A}$-Hopf algebra, and $H_*(E)$ the structure of an $\mathcal{A}$-$H_*(G)$ coalgebra.

We now apply the results of §§1 and 2, as summarized in Proposition 2.8 with $\Lambda = \mathcal{A}$, $\Gamma = H_*(G)$, $\Sigma = H_*(E)$, $\Omega = Z_2$. We obtain a product on $\operatorname{Ext}_{H_*(G)}(H_*(E), Z_2)$; a vertical action

$$
(6.1)\quad \sigma_V(\mathcal{A}, \operatorname{Ext}) = \sigma(\Lambda, \operatorname{Ext}): \mathcal{A} \otimes \operatorname{Ext}_{H_*(G)}(H_*(E), Z_2) \to \operatorname{Ext}_{H_*(G)}(H_*(E), Z_2)
$$

under which Ext becomes an $\mathcal{A}$-algebra, and a diagonal action

$$
(6.2)\quad \sigma_D(\mathcal{A}, \operatorname{Ext}) = \sigma(\mathcal{A}, \operatorname{Ext}): \mathcal{A} \otimes \operatorname{Ext}_{H_*(G)}(H_*(E), Z_2) \to \operatorname{Ext}_{H_*(G)}(H_*(E), Z_2)
$$

under which Ext becomes an $\mathcal{A}$-algebra. The vertical and diagonal actions are related by

$$
(6.3)\quad \operatorname{Sq}_V^k(\operatorname{Sq}_D^j u) = \operatorname{Sq}_D^j(\operatorname{Sq}_V^{k/2} u)
$$

for any $u \in \operatorname{Ext}$ (Proposition 2.7).

7. **Eilenberg-Moore spectral sequence.** Suppose G a topological group. To any G-space E we assign a G-space $T(E) = G \times E$, where the action of G is given by $g(g', x) = (gg', x)$. T is a functor from the category of G-spaces to itself. Let I be the identity functor on the category of G-spaces. Define natural transformations $d: T \to I$, $s: T \to T^2$ by setting $d(g, x) = gx$; $s(g, x) = (g, e, x)$, where $e \in G$ is the identity element. Then (T, d, s) is a cotriple [2] on the

category of G-spaces. This cotriple gives rise in the usual way [2, p. 246] to a functor $\mathcal{T}$, which to any G-space E assigns a simplicial object $\mathcal{T}E$ over the category of G-spaces. In fact $(\mathcal{T}E)_p = G^{\times(p+1)} \times E$, and the face and degeneracy operations are as in [2]. Set $(\underline{\mathcal{T}}E)_p = (\mathcal{T}E)_p/G$. Then $\underline{\mathcal{T}}E$ is a simplicial object over the category of topological spaces. We let r: $(\underline{\mathcal{T}}E)_0 \to E/G$ be the projection $E \to E/G$.

Now let S be the "singular" functor from topological spaces to simplicial coalgebras: i.e., if K is a space, $(SK)_q$ is the free Z_2-module on the singular q-simplices Δ of K, and the coproduct is defined by $\psi(\Delta) = \Delta \otimes \Delta$. From $\underline{\mathcal{T}}E$ we obtain a bisimplicial coalgebra X by setting $X_{p,-} = S(\underline{\mathcal{T}}E)_p$. The map r passes to an augmentation (in the sense of I, §1) λ: $X \to S(E/G)$.

Proposition 7.1. *Suppose E a principal G-space, and $H_*(G)$ and $H_*(E)$ of finite type. Then λ^*: $H^*(E/G) \to H^*(\mathrm{Hom}_{Z_2}(CX, Z_2))$ is an isomorphism of Z_2-modules.*

Proof. This can be proved by an argument similar to that at the end of §5 of [12]. In fact, using the Eilenberg-Zilber theorem one shows the homology of the total complex CX isomorphic to the differential derived functor $\mathrm{Tor}_{CSG}(CSE, Z_2)$ of Eilenberg and Moore [11]. Theorem 3.1 of [11] then implies our result.

It follows from Proposition 7.1 that the spectral sequence of the bisimplicial Z_2-coalgebra X converges to $H^*(E/G)$, and is equipped with Steenrod squares enjoying all the properties of I, §1. Clearly, $E_2 = \mathrm{Ext}_{H_*(G)}(H_*(E), Z_2)$.

Proposition 7.2. *The actions of the vertical Steenrod squares on E_2 are given by $\sigma_V(\mathfrak{A}, \mathrm{Ext})$ of (6.1).*

Proposition 7.3. *The actions of the diagonal Steenrod squares on E_2 are given by $\sigma_D(\mathfrak{A}, \mathrm{Ext})$ of (6.2).*

The proofs of these results are similar to but less complicated than the proofs of Propositions 5.2 and 5.3, and we leave them to the reader. The main tool is Proposition 5.1 of I.

8. **The Serre spectral sequence.** Dress has shown in [4] how to associate to any Serre fibration f: $E \to B$ a bisimplicial coalgebra X (in Dress' notation, $X = K(f)$), and an augmentation λ: $X \to SE$ (here S is the singular functor from topological spaces to simplicial coalgebras) with the property that λ^*: $H^*(E) \to H^*(\mathrm{Hom}_{Z_2}(CX, Z_2))$ is an isomorphism. Using Dress' computation of the E_2 term of the spectral sequence, and the general results of I, the reader can recover the results of [1], [7], [18].

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